# Extracting the Luttinger Parameter from a Single Wave Function

Bi-Yang Tan,[1,*] Yueshui Zhang,[2,*] Hua-Chen Zhang,[3] Wei Tang,[4] Lei Wang,[5,6] Hong-Hao Tu,[2,†] and Ying-Hai Wu[1,‡]

[1]School of Physics and Wuhan National High Magnetic Field Center, Huazhong University of Science and Technology, Wuhan 430074, China

[2]Faculty of Physics and Arnold Sommerfeld Center for Theoretical Physics, Ludwig-Maximilians-Universität München, 80333 Munich, Germany

[3]Department of Physics and Astronomy, Aarhus University, DK-8000 Aarhus C, Denmark

[4]Department of Physics and Astronomy, Ghent University, Krijgslaan 281, S9, B-9000 Ghent, Belgium

[5]Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China

[6]Songshan Lake Materials Laboratory, Dongguan, Guangdong 523808, China

(Received 5 April 2024; revised 15 October 2024; accepted 27 January 2025; published 19 February 2025)

The low-energy physics of Tomonaga-Luttinger liquids (TLLs) is controlled by the Luttinger parameter. We demonstrate that this parameter can be extracted from a single wave function for one-component TLLs with periodic boundary condition. This method relies on the fact that TLLs are described by conformal field theory in which crosscap states can be constructed. The overlaps between the crosscap states and the ground state as well as some excited states are proved to be universal numbers that directly reveal the Luttinger parameter. In microscopic lattice models, crosscap states are formed by putting each pair of antipodal sites into a maximally entangled state. Analytical and numerical calculations are performed in a few representative models to substantiate the conformal field theory prediction. The extracted Luttinger parameters are generally quite accurate in finite-size systems with moderate lengths, so there is no need to perform data fitting and/or finite-size scaling.

DOI: 10.1103/PhysRevLett.134.076501

*Introduction*—The theory of Tomonaga-Luttinger liquid (TLL) is a great triumph of strongly correlated physics [1]. Based on previous work of Tomonaga [2], Luttinger introduced a model to study interacting fermions in one dimension [3]. It clearly exemplifies the peculiarity of reduced dimensionality because infinitesimal interaction in a one-dimensional system drives it away from the Fermi liquid. In subsequent works [4–8], the bosonization framework was established to provide a unified treatment for many strongly correlated problems in one dimension. Spin chains are usually studied using the Jordan-Wigner transformation that converts spins to fermions. Gapless phases of bosons or fermions can be understood using free boson fields that correspond to density fluctuations. Instabilities of the TLL lead to symmetry-breaking ordered phases. It is not easy to realize one-dimensional systems, but experimental investigations have been carried out in solid state and cold atom platforms [9–18]. Another context in which TLL thrives is the edge of certain two-dimensional topological states [19–22].

The low-energy physics of one-component TLLs is captured by the Hamiltonian

$$H = \frac{v}{8\pi}\int_0^L \mathrm{d}x\left[(\partial_x\hat{\varphi})^2 + (\partial_x\hat{\theta})^2\right], \tag{1}$$

which is a system of compactified bosons in the language of conformal field theory (CFT). Here $L$ is the length of the system, $v$ is the characteristic velocity, $\hat{\varphi}$ is a compactified boson field with radius $R$ (i.e., $\hat{\varphi} \sim \hat{\varphi} + 2\pi R$), and $\hat{\theta}$ is the conjugate boson field of $\hat{\varphi}$ with radius $R' = 2/R$. The Luttinger parameter $K$ can be expressed using the radius as $K = R^2/4$. Its value is 1 for free fermions and deviates from 1 when interaction is turned on. For a given TLL, many properties are determined by $K$ so it is routinely computed in numerical studies. If the Hamiltonian has a microscopic U(1) symmetry (e.g., particle number conservation), $K$ can be obtained by comparing the ground-state energy of multiple systems with different U(1) charges [23]. This is feasible because $K$ can be interpreted as the stiffness of a superfluid. The powerlaw decay of certain correlation functions can be used to deduce $K$ when their bosonized expressions are known [1]. It has been found that $K$ also appears in the U(1) charge fluctuations of a subsystem [24,25], the U(1) symmetry-resolved entanglement spectrum [26,27], different types of entanglement entropy [28–32], and finite entanglement scaling analysis [33,34].

In this Letter, we propose that the Luttinger parameter in microscopic models can be extracted from the eigenstate

*These two authors contributed equally to this work.

†Contact author: h.tu@lmu.de

‡Contact author: yinghaiwu88@hust.edu.cn

 

crosscap overlap (ESCO). It has three favorable features: (i) a single wave function instead of multiple states is used; (ii) data fitting or finite-size scaling is not needed; (iii) microscopic U(1) symmetry is not required. To the best of our knowledge, no other method enjoys all three advantages simultaneously. Our method is motivated by classical results and recent progresses in CFT and integrable models. When a CFT is defined on the Klein bottle, important information can be inferred from the thermal entropy [35–47]. The Klein bottle can be viewed as a special cylinder with crosscaps at its ends. A simple circle with all pairs of antipodal points identified becomes a crosscap. It is possible to define a variety of crosscap states in the framework of boundary CFT [48]. For a rational CFT, the crosscap states have universal overlaps ("crosscap coefficients") with its ground state [49–51]. In some integrable models, the crosscap states have been identified as "integrable boundary states" [52–56]. The overlaps between all eigenstates of these integrable models and the crosscap states can be expressed as determinants, but their physical significance is not transparent. We aim to establish a general framework for studying crosscap states in CFT (without and with perturbations) such that physical properties can be deduced from their overlaps with eigenstates.

*General construction of crosscap states*—While the crosscap is easy to visualize, the problem of what kind of states qualify as conformal crosscap states in CFTs has not been completely settled [49–51]. We take the viewpoint that a conformal crosscap state should satisfy two sets of constraints. First, the field operators on antipodal spatial coordinates are identified with each other in some way. If a CFT has Lagrangian description, this can be made precise using the path integral formalism [57–59]. These requisites are physical manifestations of the geometric structure of crosscap and referred to as "'sewing conditions." Second, the partition functions on a cylinder with crosscap boundaries (tree channel) and on a Klein bottle (loop channel) should be equal. This "loop-tree channel correspondence" allows one to determine certain coefficients in crosscap states. As we turn to lattice models, the problem becomes even more complicated because the lattice counterpart of a field operator is generally difficult to find. It is thus unlikely that a conformal crosscap state can be converted rigorously to a lattice one. Based on the intuitive picture of identifying antipodal spatial points, recent works have studied lattice crosscap states in which antipodal sites form Bell states [52–56], but their relation to the conformal crosscap states has not been elucidated.

It is particularly instructive to study a specific lattice realization of the Hamiltonian in Eq. (1), namely, the modified Villain model [60] as illustrated in Fig. 1(a):

$$H_{\mathrm{mV}}=\sum_{j=1}^{N}\left[(2\pi R\hat{p}_j)^2+R'^2(\hat{X}_{j+1}-\hat{X}_j+2\pi\hat{n}_{j,j+1})^2\right]. \quad (2)$$

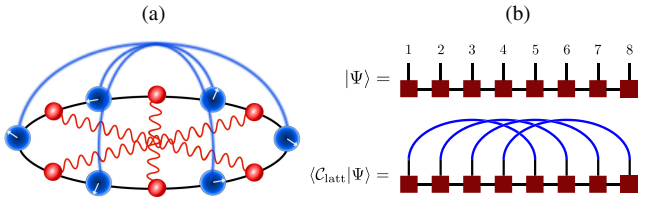


FIG. 1. (a) Schematics of the modified Villain model and its crosscap states $|\mathcal{C}_{\pm}\rangle$. (b) Graphic representations of an MPS with eight sites and its crosscap overlap.

There are $N$ sites in the chain (labeled by $j$), the lattice spacing is $a$, and periodic boundary condition is imposed. $\hat{X}_j$ ($\hat{E}_{j,j+1}$) is the site (link) variable at site $j$ [link $(j,j+1)$] and $\hat{p}_j$ ($\hat{n}_{j,j+1}$) is the associated canonical momentum, and they satisfy the standard commutation relations $[\hat{X}_j,\hat{p}_l]=[\hat{E}_{j,j+1},\hat{n}_{l,l+1}]=i\delta_{jl}$. The eigenbasis of these variables is denoted as $|X_j\rangle$ ($X_j\in\mathbb{R}$) and $|E_{j,j+1}\rangle$ [$E_{j,j+1}\in[0,2\pi)$], which span the local Hilbert space at site $j$ and link $(j,j+1)$, respectively. The tensor product of these local ones is the full Hilbert space. However, there are additional local operators $\hat{G}_j=e^{i(\hat{E}_{j,j+1}-\hat{E}_{j-1,j}-2\pi\hat{p}_j)}$ that commute with the Hamiltonian. We define the physical subspace $\mathcal{H}_{\mathrm{phy}}$ by imposing the Gauss law constraints $\hat{G}_j=1\ \forall\ j$. The projection operator onto the subspace $\mathcal{H}_{\mathrm{phy}}$ is denoted as $\mathcal{P}_G$. Using new variables $\hat{\theta}_j=R'(\hat{X}_j+2\pi\sum_{k=1}^{j}\hat{n}_{k-1,k})$ and $\hat{\varphi}_{j,j+1}=R\hat{E}_{j,j+1}$ [60], Eq. (2) can be rewritten as

$$H_{\mathrm{mV}}=\sum_{j=1}^{N}\left[(\hat{\varphi}_{j+1,j+2}-\hat{\varphi}_{j,j+1})^2+(\hat{\theta}_{j+1}-\hat{\theta}_j)^2\right]. \quad (3)$$

Its continuum limit is taken by sending $N\to\infty$ and $a\to 0$ with $L=Na$ kept unchanged. After the substitutions $\hat{\varphi}_{j+1,j+2}-\hat{\varphi}_{j,j+1}\to a\partial_x\hat{\varphi}(x)$ and $\hat{\theta}_{j+1}-\hat{\theta}_j\to a\partial_x\hat{\theta}(x)$, the TLL Hamiltonian emerges from Eq. (3). The operators $e^{i\hat{\theta}_j/R'}$ and $e^{i\hat{\varphi}_{j,j+1}/R}$ act within $\mathcal{H}_{\mathrm{phy}}$ thanks to the commutation relations $[e^{i\hat{\theta}_j/R'},\hat{G}_{j'}]=[e^{i\hat{\varphi}_{j,j+1}/R},\hat{G}_{j'}]=0\ \forall\ j,j'$. The compactification radii of $\hat{\theta}_j$ and $\hat{\varphi}_{j,j+1}$ are $R'$ and $R$, respectively.

Motivated by the intuitive picture mentioned above and previous works [49–56], we construct the *Ansätze*

$$|\mathcal{C}_{\pm}\rangle=\mathcal{P}_G\prod_{j=1}^{N/2}|\mathcal{B}^{\mathrm{V}}_{\pm}\rangle_{j,j+N/2}\otimes|\mathcal{B}^{\mathrm{L}}_{\mp}\rangle_{j,j+N/2} \quad (4)$$

for crosscap states by putting antipodal links and sites in the generalized Bell pairs

$$|\mathcal{B}^{\mathrm{L}}_{\mp}\rangle_{j,j+N/2}=\int_0^{2\pi}\mathrm{d}E|E\rangle_{j,j+1}|\mp E\rangle_{j+N/2,j+N/2+1},$$
$$|\mathcal{B}^{\mathrm{V}}_{\pm}\rangle_{j,j+N/2}=\int_{-\infty}^{\infty}\mathrm{d}X|X\rangle_j|\pm X\rangle_{j+N/2}, \quad (5)$$

respectively. It can be verified that the relations

$$e^{i(\hat{\theta}_j \mp \hat{\theta}_{j+N/2})/R'}|\mathcal{C}_\pm\rangle = |\mathcal{C}_\pm\rangle,$$
$$e^{i(\hat{\varphi}_{j,j+1} \pm \hat{\varphi}_{j+N/2,j+N/2+1})/R}|\mathcal{C}_\pm\rangle = |\mathcal{C}_\pm\rangle \tag{6}$$

are fulfilled for all $j$, which in the continuum limit becomes

$$e^{i[\hat{\theta}(x) \mp \hat{\theta}(x+L/2)]/R'}|\mathcal{C}_\pm\rangle = |\mathcal{C}_\pm\rangle,$$
$$e^{i[\hat{\varphi}(x) \pm \hat{\varphi}(x+L/2)]/R}|\mathcal{C}_\pm\rangle = |\mathcal{C}_\pm\rangle. \tag{7}$$

These identities are sewing conditions in the sense that field operators on antipodal spatial coordinates are combined to form certain operators under which the crosscap states are invariant. It is amusing that they can be derived rigorously using our formalism.

For subsequent calculations, we introduce the mode expansions [61]

$$\hat{\theta}(x) = \tilde{\varphi}_0 + \frac{4\pi x}{L}\pi_0 + \sum_{k\neq 0}\frac{i}{k}\left(e^{i\frac{2\pi k}{L}x}a_k - e^{-i\frac{2\pi k}{L}x}\bar{a}_k\right),$$
$$\hat{\varphi}(x) = \varphi_0 + \frac{4\pi x}{L}\tilde{\pi}_0 + \sum_{k\neq 0}\frac{i}{k}\left(e^{i\frac{2\pi k}{L}x}a_k + e^{-i\frac{2\pi k}{L}x}\bar{a}_k\right), \tag{8}$$

where the zero modes and oscillatory modes satisfy $[\varphi_0, \pi_0] = [\tilde{\varphi}_0, \tilde{\pi}_0] = i$ and $[a_k, a_l] = [\bar{a}_k, \bar{a}_l] = k\delta_{k+l,0}$, respectively. $\varphi_0$ ($\tilde{\varphi}_0$) is an angular variable with radius $R$ ($R'$), and $\pi_0$ ($\tilde{\pi}_0$) is the associated canonical momentum. The eigenvectors of $\pi_0$ and $\tilde{\pi}_0$ that can be annihilated by all bosonic annihilation operators $a_{k>0}$ and $\bar{a}_{k>0}$ are denoted as $|n, m\rangle$ ($n, m \in \mathbb{Z}$); they are highest weight states (Virasoro primary states) of the CFT and satisfy $\pi_0|n, m\rangle = (n/R)|n, m\rangle$ and $\tilde{\pi}_0|n, m\rangle = (m/R')|n, m\rangle$. The TLL Hamiltonian in Eq. (1) becomes

$$H = \frac{2\pi v}{L}\left[\pi_0^2 + \tilde{\pi}_0^2 + \sum_{k=1}^{\infty}(a_{-k}a_k + \bar{a}_{-k}\bar{a}_k) - \frac{1}{12}\right], \tag{9}$$

whose energy eigenstates are just Virasoro primary states $|n, m\rangle$ and the descendant states obtained by applying the creation operators $a_{k<0}$ and $\bar{a}_{k<0}$ on top of $|n, m\rangle$. The crosscap states can be expressed as

$$|\mathcal{C}_+\rangle = \kappa_+ \exp\left[\sum_{k=1}^{\infty}\frac{(-1)^k}{k}a_{-k}\bar{a}_{-k}\right]\sum_{n\in\mathbb{Z}}|2n, 0\rangle,$$
$$|\mathcal{C}_-\rangle = \kappa_- \exp\left[-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}a_{-k}\bar{a}_{-k}\right]\sum_{m\in\mathbb{Z}}|0, 2m\rangle. \tag{10}$$

The overall factors are not fixed by normalization but should be chosen as $\kappa_+ = \sqrt{R'}$ and $\kappa_- = \sqrt{R}$ to fulfill the loop-tree channel correspondence [62].

The conformal crosscap states in Eq. (10) are scale-invariant (i.e., they contain no scale) and cannot be normalized. Their lattice counterparts may be normalizable [Eq. (14) for spin-1 = 2 chains defined below] or not normalizable [Eq. (4) for the modified Villain model], depending on the local Hilbert space dimension. In fact, for a given lattice model whose low-energy theory is the compactified boson CFT, the crosscap states together with the coefficients $\kappa_\pm$ in Eq. (10) should be understood as the continuum description of their lattice counterparts $|\mathcal{C}^\pm_{\text{latt}}\rangle$. This means that they are fully determined by the overlaps of lattice crosscap states with low-lying eigenstates in the continuum limit: $\langle\mathcal{C}_\pm|n, m; \{n_k\}, \{\bar{n}_k\}\rangle = \lim_{N\to\infty}\langle\mathcal{C}^\pm_{\text{latt}}|\psi_{n,m;\{n_k\},\{\bar{n}_k\}}\rangle$, where $|\psi_{n,m;\{n_k\},\{\bar{n}_k\}}\rangle$ is the lattice counterpart of the CFT eigenstate $|n, m; \{n_k\}, \{\bar{n}_k\}\rangle$.

An appealing property of $|\mathcal{C}_\pm\rangle$ is that their overlaps with certain highest weight states $|n, m\rangle$ are universal numbers depending only on the compactification radius (equivalently, the Luttinger parameter):

$$|\langle\mathcal{C}_+|2n, 0\rangle|^2 = R' = \frac{1}{\sqrt{K}}, \qquad n \in \mathbb{Z},$$
$$|\langle\mathcal{C}_-|0, 2m\rangle|^2 = R = 2\sqrt{K}, \qquad m \in \mathbb{Z}. \tag{11}$$

It was found in Ref. [39] that the Luttinger parameter appears in the thermal entropy of compactified boson CFTs on the Klein bottle. In view of the spacetime symmetry of CFTs, it is natural to expect that the same variable can also be found using the ground state alone. Our derivation not only puts this expectation on a firm ground but also provides further insights into crosscap states. Some excited states are incorporated in the present formalism, and perturbations to a pure CFT can be investigated [66].

*Spin chains with microscopic U(1) symmetry*—To apply the general theory in specific lattice models, the crosscap states should be expressed in terms of the microscopic degrees of freedom. It is helpful to begin with spin-1/2 chains. The first one is the XXZ model,

$$H_{\text{XXZ}} = \sum_{j=1}^{N}(S^x_jS^x_{j+1} + S^y_jS^y_{j+1} + \Delta S^z_jS^z_{j+1}), \tag{12}$$

with $K = \pi/[2(\pi - \cos^{-1}\Delta)]$ for $\Delta \in (-1, 1]$ [1]. The second one is the Haldane-Shastry (HS) model,

$$H_{\text{HS}} = \left(\frac{\pi}{N}\right)^2\sum_{1\le j<l\le N}\frac{\mathbf{S}_j\cdot\mathbf{S}_l}{\sin^2\left[\frac{\pi}{N}(j-l)\right]}, \tag{13}$$

with $K = 1/2$ [67,68]. Both models have a U(1) symmetry associated with the conservation of the $z$ component of the total spin.

It is natural to speculate that

$$|\mathcal{C}_{\rm latt}\rangle = \prod_{j=1}^{N/2} (|\uparrow\rangle_j|\uparrow\rangle_{j+N/2} + |\downarrow\rangle_j|\downarrow\rangle_{j+N/2}) \quad (14)$$

is a crosscap state when $N$ is even [52–56], but there is one subtlety: it corresponds to $|\mathcal{C}_+\rangle$ only if $\mathrm{mod}(N,4)=0$ but does not for the cases with $\mathrm{mod}(N,4)=2$. This property can be traced back to the constraint $(S^z_j - S^z_{j+N/2})|\mathcal{C}_{\rm latt}\rangle = 0$. The $S^z_j$ operator in bosonization reads $S^z_j \sim \partial_x\hat{\theta}(x) + (-1)^j\alpha\cos[\hat{\theta}(x)/R']$ for XXZ and HS chains ($\alpha$ is a model-dependent constant). For $\mathrm{mod}(N,4)=0$, the lattice constraint $(S^z_j - S^z_{j+N/2})|\mathcal{C}_{\rm latt}\rangle = 0$ is hence consistent with Eq. (7).

The validity of Eq. (14) in the XY model (i.e., $\Delta = 0$ in the XXZ model) and the HS model can be substantiated by analytical calculations. In the hardcore boson basis, the Virasoro primary states $|n,0\rangle$ have the wave functions $\Psi^\lambda(x_1,\ldots,x_M) \propto (-1)^{\sum_l x_l}\prod_{j<k}|\sin[\pi(x_j-x_k)/N]|^\lambda$, where $1 \le x_1 < \cdots < x_M \le N$ denote the positions of spin-$\uparrow$ sites (occupied by hardcore bosons), and $\lambda = 1$ ($\lambda = 2$) for the XY (HS) chain [69]. The number of bosons $M$ is related to the U(1) charge $n$ via $M = N/2 + n$. It is important to note that the cases with $\mathrm{mod}(N,4)=2$ and $\mathrm{mod}(N,4)=0$ are different. Because $|\mathcal{C}_{\rm latt}\rangle$ only contains states with even $M$, $\langle\mathcal{C}_{\rm latt}|\Psi^\lambda(M)\rangle$ vanishes identically for odd $M$. We also have $\langle\mathcal{C}_{\rm latt}|\Psi^\lambda(M = N/2+2n)\rangle = 0$ when $\mathrm{mod}(N,4)=2$, so $|\mathcal{C}_{\rm latt}\rangle$ does not correspond to $|\mathcal{C}_\pm\rangle$ in these cases. In contrast, it can be proven that

$$|\langle\mathcal{C}_{\rm latt}|\Psi^{\lambda=1}(M)\rangle|^2 = 1,$$
$$|\langle\mathcal{C}_{\rm latt}|\Psi^{\lambda=2}(M)\rangle|^2 = \frac{2^M(M!)^3}{[(M/2)!]^2(2M)!} = \sqrt{2} + \mathcal{O}(1/M) \quad (15)$$

if $\mathrm{mod}(N,4)=0$ and $M = N/2+2n$ [62]. This is a perfect agreement with the field theory prediction Eq. (11), giving a strong indication that $|\mathcal{C}_{\rm latt}\rangle$ corresponds to $|\mathcal{C}_+\rangle$ for $\mathrm{mod}(N,4)=0$.

For the XXZ model with $\Delta \neq 0$, the ground state can be computed numerically using the density matrix renormalization group (DMRG) [70]. When the ground state $|\Psi\rangle$ is expressed as a matrix product state (MPS) [71,72], its crosscap overlap $\langle\mathcal{C}_{\rm latt}|\Psi\rangle$ would be the tensor contraction shown in Fig. 1(b). Numerical results at many different $N$'s and $\Delta$'s are displayed in Fig. 2. The MPS bond dimension is 2000 for $N = 100$ and 6000 for $N = 400$. The accuracy is very good in a wide range of $\Delta$ for a moderate $N = 100$. The relative error of $K$ is of the order $10^{-4}$ or smaller when $\Delta \in [-0.95, 0.8]$. As $\Delta$ approaches 1, clear deviation from exact values is observed [not visible in Fig. 2(a) due to its scale] and the relative error reaches 7.20% at $\Delta = 1$. This

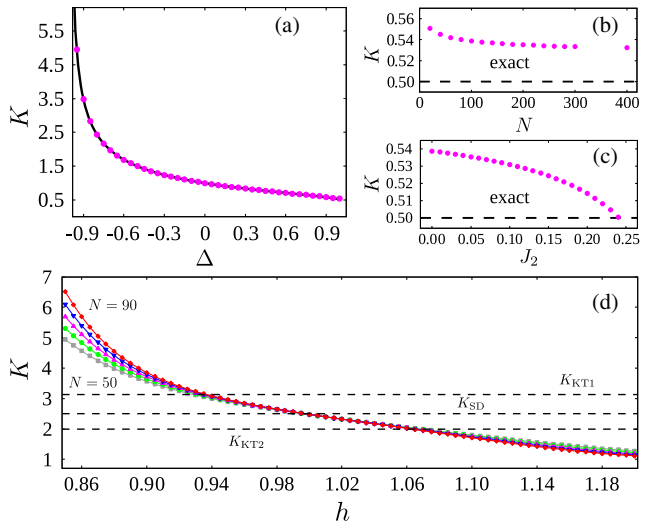


FIG. 2. Numerically extracted Luttinger parameters in three models. Exact values are indicated using solid or dashed black lines for comparison. (a) The XXZ model with $N = 100$ and various $\Delta$. (b) The XXZ model with $\Delta = 1$ and various $N$. (c) The next-nearest-neighbor model with $N = 100$ and various $J_2$. (d) The five-state clock model with various $h$ and $N$.

deviation is tentatively attributed to the marginal terms in the low-energy theory at $\Delta = 1$ [73], which usually cause strong finite-size effects. It does get smaller as $N$ increases but the convergence is very slow [see Fig. 2(b)]. To validate this conjecture, we study the model $H_{\rm next} = \sum_{j=1}^N(\mathbf{S}_j\cdot\mathbf{S}_{j+1} + J_2\mathbf{S}_j\cdot\mathbf{S}_{j+2})$ with a next-nearest-neighbor Heisenberg interaction, for which the marginal term disappears when $J_2 \approx 0.24$ [74]. The results for this model with $N = 100$ are presented in Fig. 2(c). As $J_2$ increases toward 0.24, the relative error of $K$ decreases and eventually becomes 0.11%. Further analysis based on conformal perturbation theory reveals that finite-size results for the XXZ model with $\Delta = 1$ should be fitted as $K = a[\ln(N)]^{-1} + b$ [62,66]. This leads to $b = 0.510$ using the data points with $N \in [100, 400]$ in Fig. 2(b).

*Spin chains without microscopic U(1) symmetry*—A fundamental improvement of our method is that microscopic U(1) symmetry is not required. To demonstrate this advantage, we consider the quantum $q$-state clock model [75–80]

$$H_{q\rm SC} = -\sum_{j=1}^N(\sigma_j^\dagger\sigma_{j+1} + \sigma_j\sigma_{j+1}^\dagger) - h\sum_{j=1}^N(\tau_j + \tau_j^\dagger), \quad (16)$$

where the $\mathbb{Z}_q$ spin operators are defined as $\sigma|\alpha\rangle = |\mathrm{mod}(\alpha - 1 + q, q)\rangle$ and $\tau|\alpha\rangle = e^{i2\pi\alpha/q}|\alpha\rangle$ with $\alpha = 1,\ldots,q$. This model is self-dual (SD) at $h = 1$. For $q \ge 5$ and intermediate $h$ (including the SD point), it hosts a Luttinger liquid that is separated by two Kosterlitz-Thouless (KT) transitions at $h_{\rm KT1}$ and $h_{\rm KT2}$ from gapped phases [81,82].

The transition points are related to each other by duality as $h_{\mathrm{KT1}}h_{\mathrm{KT2}} = 1$. There is no analytical expression for the Luttinger parameter, but its value at the SD point and the KT transition points was predicted to be $K_{\mathrm{KT1}} = q^2/8$, $K_{\mathrm{SD}} = q/2$, $K_{\mathrm{KT2}} = 2$ [40,77]. As in previous models, the *Ansatz*

$$|\mathcal{C}_{\mathrm{latt}}\rangle = \prod_{j=1}^{N/2}\sum_{\alpha=1}^{q}|\alpha\rangle_j|\alpha\rangle_{j+N/2} \tag{17}$$

is proposed as a crosscap state. Numerical results for the $q = 5$ case with $N = 50, \ldots, 90$ and various $h$ are presented in Fig. 2(d). The MPS bond dimension is 1500 for $N = 50$ and 2500 for $N = 90$. At the SD point, finite-size systems are described by CFT with irrelevant perturbations, and the Luttinger parameter has a relative error 0.79% at $N = 90$. In the Luttinger liquid regime, the crosscap overlap only exhibits weak dependence on the system size, so the data points for different $N$ almost coincide. For the small and large $h$ phases, the overlap strongly depends on $N$.

*Summary and outlook*—To summarize, we have unveiled that the Luttinger parameter can be extracted from the overlaps between individual eigenstates and the crosscap states. The advantages of our method are corroborated by analytical and numerical results. This method may be applied to check if a trial wave function indeed describes the physics of TLL or other CFTs (see, e.g., Refs. [83,84]) in the absence of a Hamiltonian. From the theoretical perspective, our results open up an exciting new avenue and many interesting directions are yet to be explored. The crosscap state used here is constructed from a certain type of Bell states. For integrable spin chains, other types of crosscap states have been studied [53]. Do they have clear counterparts in CFTs? An extension of our method to multicomponent Luttinger liquids, such as the fermionic Hubbard model and SU(3) spin chains, is certainly very desirable. Finally, experimental protocols for measuring the crosscap overlap should be pursued. We hope that interesting results on these topics will be reported in the future.

*Acknowledgments*—We are grateful to Meng Cheng for stimulating discussions. B. Y. T. and Y. H. W. are supported by NNSF of China under Grant No. 12174130. Y. S. Z. is supported by the Sino-German (CSC-DAAD) Postdoc Scholarship Program. W. T. is supported by the Research Foundation Flanders (FWO) via Grant GOE1520N. L. W. is supported by the NNSF of China under Grants No. T2225018 and No. 92270107.

[1] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, Oxford, 2003).
[2] S.-i. Tomonaga, Remarks on Bloch's method of sound waves applied to many-fermion problems, Prog. Theor. Phys. **5**, 544 (1950).
[3] J. M. Luttinger, An exactly soluble model of a many-fermion system, J. Math. Phys. (N.Y.) **4**, 1154 (1963).
[4] D. C. Mattis and E. H. Lieb, Exact solution of a many-fermion system and its associated boson field, J. Math. Phys. (N.Y.) **6**, 304 (1965).
[5] D. C. Mattis, New wave-operator identity applied to the study of persistent currents in 1d, J. Math. Phys. (N.Y.) **15**, 609 (1974).
[6] A. Luther and I. Peschel, Single-particle states, Kohn anomaly, and pairing fluctuations in one dimension, Phys. Rev. B **9**, 2911 (1974).
[7] F. D. M. Haldane, Effective harmonic-fluid approach to low-energy properties of one-dimensional quantum fluids, Phys. Rev. Lett. **47**, 1840 (1981).
[8] J. von Delft and H. Schoeller, Bosonization for beginners—refermionization for experts, Ann. Phys. (Leipzig) **510**, 225 (1998).
[9] A. Yacoby, H. L. Stormer, N. S. Wingreen, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Nonuniversal conductance quantization in quantum wires, Phys. Rev. Lett. **77**, 4612 (1996).
[10] A. Schwartz, M. Dressel, G. Grüner, V. Vescoli, L. Degiorgi, and T. Giamarchi, On-chain electrodynamics of metallic $TMTSF_2x$ salts: Observation of Tomonaga-Luttinger liquid response, Phys. Rev. B **58**, 1261 (1998).
[11] M. Bockrath, D. H. Cobden, J. Lu, A. G. Rinzler, R. E. Smalley, L. Balents, and P. L. McEuen, Luttinger-liquid behaviour in carbon nanotubes, Nature (London) **397**, 598 (1999).
[12] Z. Yao, H. W. C. Postma, L. Balents, and C. Dekker, Carbon nanotube intramolecular junctions, Nature (London) **402**, 273 (1999).
[13] E. Levy, A. Tsukernik, M. Karpovski, A. Palevski, B. Dwir, E. Pelucchi, A. Rudra, E. Kapon, and Y. Oreg, Luttinger-liquid behavior in weakly disordered quantum wires, Phys. Rev. Lett. **97**, 196802 (2006).
[14] Y. Kono, T. Sakakibara, C. P. Aoyama, C. Hotta, M. M. Turnbull, C. P. Landee, and Y. Takano, Field-induced quantum criticality and universal temperature dependence of the magnetization of a spin-1/2 Heisenberg chain, Phys. Rev. Lett. **114**, 037202 (2015).
[15] S. Hofferberth, I. Lesanovsky, T. Schumm, A. Imambekov, V. Gritsev, E. Demler, and J. Schmiedmayer, Probing quantum and thermal noise in an interacting many-body system, Nat. Phys. **4**, 489 (2008).
[16] E. Haller, R. Hart, M. J. Mark, J. G. Danzl, L. Reichsöllner, M. Gustavsson, M. Dalmonte, G. Pupillo, and H.-C. Nägerl, Pinning quantum phase transition for a Luttinger liquid of strongly interacting bosons, Nature (London) **466**, 597 (2010).
[17] B. Yang, Y.-Y. Chen, Y.-G. Zheng, H. Sun, H.-N. Dai, X.-W. Guan, Z.-S. Yuan, and J.-W. Pan, Quantum criticality and the Tomonaga-Luttinger liquid in one-dimensional Bose gases, Phys. Rev. Lett. **119**, 165701 (2017).
[18] J. Vijayan, P. Sompet, G. Salomon, J. Koepsell, S. Hirthe, A. Bohrdt, F. Grusdt, I. Bloch, and C. Gross, Time-resolved observation of spin-charge deconfinement in fermionic Hubbard chains, Science **367**, 186 (2020).
[19] X.-G. Wen, Topological orders and edge excitations in fractional quantum Hall states, Adv. Phys. **44**, 405 (1995).

[20] A. M. Chang, Chiral Luttinger liquids at the fractional quantum Hall edge, Rev. Mod. Phys. **75**, 1449 (2003).

[21] C. Wu, B. A. Bernevig, and S.-C. Zhang, Helical liquid and the edge of quantum spin Hall systems, Phys. Rev. Lett. **96**, 106401 (2006).

[22] T. Li, P. Wang, H. Fu, L. Du, K. A. Schreiber, X. Mu, X. Liu, G. Sullivan, G. A. Csáthy, X. Lin, and R.-R. Du, Observation of a helical Luttinger liquid in InAs/GaSb quantum spin Hall edges, Phys. Rev. Lett. **115**, 136804 (2015).

[23] F. C. Alcaraz, M. N. Barber, and M. T. Batchelor, Conformal invariance and the spectrum of the XXZ chain, Phys. Rev. Lett. **58**, 771 (1987).

[24] H. F. Song, S. Rachel, and K. Le Hur, General relation between entanglement and fluctuations in one dimension, Phys. Rev. B **82**, 012405 (2010).

[25] S. Nishimoto, Tomonaga-Luttinger-liquid criticality: Numerical entanglement entropy approach, Phys. Rev. B **84**, 195108 (2011).

[26] A. M. Läuchli, Operator content of real-space entanglement spectra at conformal critical points, arXiv:1303.0741.

[27] L. Vanderstraeten, B. Vanhecke, A. M. Läuchli, and F. Verstraete, Approaching the Kosterlitz-Thouless transition for the classical *XY* model with tensor networks, Phys. Rev. E **100**, 062136 (2019).

[28] P. Calabrese, J. Cardy, and E. Tonni, Entanglement entropy of two disjoint intervals in conformal field theory, J. Stat. Mech. (2009) P11001.

[29] C. M. Herdman and A. Del Maestro, Particle partition entanglement of bosonic Luttinger liquids, Phys. Rev. B **91**, 184507 (2015).

[30] H. Barghathi, E. Casiano-Diaz, and A. Del Maestro, Particle partition entanglement of one dimensional spinless fermions, J. Stat. Mech. (2017) 083108.

[31] A. Bastianello, Rényi entanglement entropies for the compactified massless boson with open boundary conditions, J. High Energy Phys. 10 (2019) 141.

[32] B. Estienne, Y. Ikhlef, A. Rotaru, and E. Tonni, Entanglement entropies of an interval for the massless scalar field in the presence of a boundary, J. High Energy Phys. 05 (2024) 236.

[33] Y. He, D. Pekker, and R. S. K. Mong, One-dimensional repulsive Hubbard model with mass imbalance: Orders and filling anomaly, Phys. Rev. B **104**, 195126 (2021).

[34] A. A. Eberharter, L. Vanderstraeten, F. Verstraete, and A. M. Läuchli, Extracting the speed of light from matrix product states, Phys. Rev. Lett. **131**, 226502 (2023).

[35] H.-H. Tu, Universal entropy of conformal critical theories on a Klein bottle, Phys. Rev. Lett. **119**, 261603 (2017).

[36] W. Tang, L. Chen, W. Li, X. C. Xie, H.-H. Tu, and L. Wang, Universal boundary entropies in conformal field theory: A quantum Monte Carlo study, Phys. Rev. B **96**, 115136 (2017).

[37] L. Chen, H.-X. Wang, L. Wang, and W. Li, Conformal thermal tensor network and universal entropy on topological manifolds, Phys. Rev. B **96**, 174429 (2017).

[38] H.-X. Wang, L. Chen, H. Lin, and W. Li, Topological and geometric universal thermodynamics in conformal field theory, Phys. Rev. B **97**, 220407(R) (2018).

[39] W. Tang, X. C. Xie, L. Wang, and H.-H. Tu, Klein bottle entropy of compactified boson conformal field theory, Phys. Rev. B **99**, 115105 (2019).

[40] Z.-Q. Li, L.-P. Yang, Z. Y. Xie, H.-H. Tu, H.-J. Liao, and T. Xiang, Critical properties of the two-dimensional $q$-state clock model, Phys. Rev. E **101**, 060105(R) (2020).

[41] W. Tang, H.-H. Tu, and L. Wang, Continuous matrix product operator approach to finite temperature quantum states, Phys. Rev. Lett. **125**, 170604 (2020).

[42] H. García-Compeán and N. Quiroz, On the topological and crosscap entropies in non-oriented RCFTs, Eur. Phys. J. Plus **136**, 881 (2021).

[43] Y. Chen, K. Ji, Z. Y. Xie, and J. F. Yu, Cross derivative of the Gibbs free energy: A universal and efficient method for phase transitions in classical spin models, Phys. Rev. B **101**, 165123 (2020).

[44] H. Li and L.-P. Yang, Tensor network simulation for the frustrated $J_1 - J_2$ Ising model on the square lattice, Phys. Rev. E **104**, 024118 (2021).

[45] R. Vanhove, L. Lootens, H.-H. Tu, and F. Verstraete, Topological aspects of the critical three-state Potts model, J. Phys. A **55**, 235002 (2022).

[46] Y. Zhang, A. Hulsch, H.-C. Zhang, W. Tang, L. Wang, and H.-H. Tu, Universal scaling of Klein bottle entropy near conformal critical points, Phys. Rev. Lett. **130**, 151602 (2023).

[47] H. Shimizu and A. Ueda, Tensor network simulations for non-orientable surfaces, arXiv:2402.15507.

[48] N. Ishibashi, The boundary and crosscap states in conformal field theories, Mod. Phys. Lett. A **04**, 251 (1989).

[49] D. Fioravanti, G. Pradisi, and A. Sagnotti, Sewing constraints and non-orientable open strings, Phys. Lett. B **321**, 349 (1994).

[50] G. Pradisi, A. Sagnotti, and Y. S. Stanev, Completeness conditions for boundary operators in 2d conformal field theory, Phys. Lett. B **381**, 97 (1996).

[51] J. Fuchs, L. Huiszoon, A. Schellekens, C. Schweigert, and J. Walcher, Boundaries, crosscaps and simple currents, Phys. Lett. B **495**, 427 (2000).

[52] J. Caetano and S. Komatsu, Crosscap states in integrable field theories and spin chains, J. Stat. Phys. **187**, 30 (2022).

[53] C. Ekman, Crosscap states in the XXX spin-1/2 spin chain, arXiv:2207.12354.

[54] T. Gombor, Integrable crosscap states in $\mathfrak{gl}(n)$ spin chains, J. High Energy Phys. 10 (2022) 096.

[55] T. Gombor, Integrable crosscaps in classical sigma models, J. High Energy Phys. 03 (2023) 146.

[56] M. He and Y. Jiang, Integrable crosscap states: From spin chains to 1d Bose gas, J. High Energy Phys. 08 (2023) 079.

[57] R. Blumenhagen and E. Plauschinn, *Introduction to Conformal Field Theory* (Springer, Berlin, Heidelberg, 2009), 10.1007/978-3-642-00450-6.

[58] I. Brunner and K. Hori, Notes on orientifolds of rational conformal field theories, J. High Energy Phys. 07 (2004) 023.

[59] G. Y. Cho, C.-T. Hsieh, T. Morimoto, and S. Ryu, Topological phases protected by reflection symmetry and crosscap states, Phys. Rev. B **91**, 195142 (2015).

[60] M. Cheng and N. Seiberg, Lieb-Schultz-Mattis, Luttinger, and 't Hooft—anomaly matching in lattice systems, SciPost Phys. **15**, 051 (2023).
[61] P. D. Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer-Verlag, New York, 1997), 10.1007/978-1-4612-2256-9.
[62] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.134.076501 for details about the derivation of crosscap states, the analytical calculation of the crosscap overlap for certain Jastrow-type eigenstates of the spin-1/2 XY and Haldane-Shastry chains, and the logarithmic correction to the crosscap overlap of the spin-1/2 Heisenberg chain, which include Refs. [63–65].
[63] H.-C. Zhang, Y.-H. Wu, T. Xiang, and H.-H. Tu, Chiral conformal field theory for topological states and the anyon eigenbasis on the torus, Nucl. Phys. **B976**, 115712 (2022).
[64] H. Saleur and C. Itzykson, Two-dimensional field theories close to criticality, J. Stat. Phys. **48**, 449 (1987).
[65] J. L. Cardy, Logarithmic corrections to finite-size scaling in strips, J. Phys. A **19**, L1093 (1986).
[66] Y. Zhang, Y.-H. Wu, L. Wang, and H.-H. Tu, Crosscap states and duality of Ising field theory in two dimensions, arXiv:2409.11046.
[67] F. D. M. Haldane, Exact Jastrow-Gutzwiller resonating-valence-bond ground state of the spin-1/2 antiferromagnetic Heisenberg chain with $1/r^2$ exchange, Phys. Rev. Lett. **60**, 635 (1988).
[68] B. S. Shastry, Exact solution of an $s = 1/2$ Heisenberg antiferromagnetic chain with long-ranged interactions, Phys. Rev. Lett. **60**, 639 (1988).
[69] Y. Kuramoto and Y. Kato, *Dynamics of One-Dimensional Quantum Systems: Inverse-Square Interaction Models* (Cambridge University Press, Cambridge, England, 2009), 10.1017/CBO9780511596827.
[70] S. R. White, Density matrix formulation for quantum renormalization groups, Phys. Rev. Lett. **69**, 2863 (1992).
[71] S. Östlund and S. Rommer, Thermodynamic limit of density matrix renormalization, Phys. Rev. Lett. **75**, 3537 (1995).
[72] U. Schollwöck, The density-matrix renormalization group in the age of matrix product states, Ann. Phys. (Amsterdam) **326**, 96 (2011).
[73] I. Affleck, D. Gepner, H. J. Schulz, and T. Ziman, Critical behaviour of spin-s Heisenberg antiferromagnetic chains: Analytic and numerical results, J. Phys. A **22**, 511 (1989).
[74] I. Affleck, Exact critical exponents for quantum spin chains, non-linear $\sigma$-models at $\theta = \pi$ and the quantum Hall effect, Nucl. Phys. **B265**, 409 (1986).
[75] S. Elitzur, R. B. Pearson, and J. Shigemitsu, Phase structure of discrete Abelian spin and gauge systems, Phys. Rev. D **19**, 3698 (1979).
[76] J. Fröhlich and T. Spencer, The Kosterlitz-Thouless transition in two-dimensional Abelian spin systems and the Coulomb gas, Commun. Math. Phys. **81**, 527 (1981).
[77] G. Ortiz, E. Cobanera, and Z. Nussinov, Dualities and the phase diagram of the p-clock model, Nucl. Phys. **B854**, 780 (2012).
[78] W. Li, S. Yang, H.-H. Tu, and M. Cheng, Criticality in translation-invariant parafermion chains, Phys. Rev. B **91**, 115133 (2015).
[79] G. Sun, T. Vekua, E. Cobanera, and G. Ortiz, Phase transitions in the $\mathbb{Z}_p$ and U(1) clock models, Phys. Rev. B **100**, 094428 (2019).
[80] D. Berenstein and P. N. T. Lloyd, One dimensional staggered bosons, clock models and their non-invertible symmetries, arXiv:2311.00057.
[81] P. B. Wiegmann, One-dimensional Fermi system and plane *XY* model, J. Phys. C **11**, 1583 (1978).
[82] H. Matsuo and K. Nomura, Berezinskii-Kosterlitz-Thouless transitions in the six-state clock model, J. Phys. A **39**, 2953 (2006).
[83] J. I. Cirac and G. Sierra, Infinite matrix product states, conformal field theory, and the Haldane-Shastry model, Phys. Rev. B **81**, 104431 (2010).
[84] P. Patil, Y. Tang, E. Katz, and A. W. Sandvik, Indicators of conformal field theory: Entanglement entropy and multiple-point correlators, Phys. Rev. B **96**, 045140 (2017).