Renormalised Interactions of Strongly Correlated Systems via Composite Fields

Oleksandr Sulyma

Master's thesis
at the Faculty of Physics,
Ludwig-Maximilians-Universität München

Supervisor: Prof. Dr. Jan von Delft
Co-supervisor: Benedikt Schneider

Munich, 28 August 2025

Renormalisierte Wechselwirkungen stark korrelierter Systeme über zusammengesetzte Felder

Oleksandr Sulyma

Masterarbeit
an der Fakultät für Physik,
Ludwig-Maximilians-Universität München

Betreuer: Prof. Dr. Jan von Delft
Mitbetreuer: Benedikt Schneider

München, 28. August 2025

Abstract

The renormalized interactions of particles in quantum many-body systems and general field theories are described by one-particle-irreducible (1PI) vertices. The numerical calculation and treatment of these objects are challenging because of their high dimensionality and complicated frequency and momentum structure. To address these problems, multiple solutions have been proposed: a frequency parametrization using asymptotic classes; the single-boson exchange (SBE) formalism, which uses only physical correlation functions, thereby avoiding vertex divergencies in the parquet formalism; and symmetric estimators which avoid the amputation of Green's functions, to name a few. In this work we present a unified framework based on the inverse Legendre transform of the composite field effective action, that generalizes asymptotic classes, symmetric improved estimators, the SBE and the parquet formalism. We demonstrate that these representations of the four-point vertex correspond to different choices of composite fields and naturally extend to more general theories and any-order vertices via simple tree diagrams.

Acknowledgments

First of all, I would like to thank my supervisor, Professor Jan von Delft, for giving me the opportunity to join his group and complete my master's degree on a topic I am truly passionate about. I received many useful comments from him that significantly improved this work. After presenting these results at the Institute for Theoretical Physics in Cologne, I realized that the outcome could have been much worse without Prof. Jan's critical feedback and valuable suggestions.

My second big thank-you goes to Benedikt Schneider, without whom the significance of my field-theoretical results might not have been recognized. He managed to push the project further until simplicity emerged — and it worked! Only recently did I learn that Benedikt had worked on related topics before me, and I consider myself lucky to have met him. I am grateful for the many ideas he proposed, as well as his efforts in proof-checking the calculations and polishing the thesis.

I also wish to thank Nepomuk Ritz and Marc Ritter for stimulating discussions and helpful comments on my presentation. In addition, I am especially grateful to Nepomuk for proofreading my thesis.

I would further like to thank Professor Valery Gusynin for directing me to the literature on functional Legendre transforms.

Finally, I would like to thank John from Lost Weekend for creating a productive atmosphere during my afternoon work sessions, and Omar from Coffee Fellows for providing a quiet environment for my morning studies.

Contents

Chapter 1

Introduction

Strongly correlated electron systems exhibit a wide range of emergent phenomena, such as unconventional superconductivity, quantum magnetism, and metal-insulator transitions. These phenomena arise from the intricate interplay of electronic degrees of freedom, which generally cannot be adequately described by non-interacting or mean-field theories. The essential physics of these systems is encoded in the renormalized interaction vertex, which captures the effects of many-body correlations beyond single-particle properties. Understanding and accurately computing the vertex function is therefore central to the theoretical description of strongly correlated matter.

Recent advances in many-body theory have led to the development of powerful new methods for computing vertex functions. These include, for example, fully frequency- and momentum-dependent parquet approaches [1–7], functional renormalization group methods [8], and dynamical mean-field theory (DMFT) extensions such as the Dynamical Vertex Approximation (DΓA) [9, 10] and the DMFT+fRG (DMF^{2}RG)[11–13], as well as methods like TRILEX [14, 15] and QUADRILEX [16]. However, the full frequency and momentum dependence of the vertices presents significant computational challenges, which limit the applicability of these advanced numerical methods.

To address these challenges, several innovative approaches have been proposed in recent years, including frequency asymptotic parameterization [17], the single boson exchange formalism [18–23] and the (a)-symmetric estimators for vertex functions [24–26]. While these approaches have been developed from different perspectives, they share a common functional foundation.

The frequency asymptotic parameterization addresses the challenge of interpolating and extrapolating the frequency (and momentum) dependence of vertex functions to high frequencies (or momenta) by classifying them into asymptotic classes whose behavior at high frequencies (or momenta) can be predicted.

The single boson exchange formalism reduces computational complexity by reconstructing higher-order vertex functions from lower-order objects, effectively approximating multi-frequency dependencies through simpler building blocks. It also circumvents vertex divergences that can arise in numerical parquet calculations in the strong coupling regime [27].

Symmetric estimators eliminate numerical artifacts in vertex calculations by providing explicit expressions that avoid problematic disconnected contributions and external leg amputations.

Each method addresses different problems that arise during vertex calculations: frequency extrapolation, computational scaling and numerical stability, respectively.

Although these three approaches are often used in combination, their derivation is based on seemingly distinct theoretical paradigms. The frequency asymptotic parameterization categorizes diagrams by the connectivity of their external arguments to bare interaction vertices; the single-boson exchange approach is formulated in terms of U-reducibility [18]; and the concept of (a)-symmetric estimators builds on the use of equations of motion to rewrite vertex functions in terms of Green's functions.

In this thesis, we present a unified framework based on functional methods that links asymptotic classes, symmetric estimators, single-boson exchange and parquet equations, and generalizes them to vertex functions of arbitrary order and for a broad class of underlying theories. This unification is made possible by generalizing the tree expansion and the simple structure of the one-particle-irreducible (1PI) effective action when expressed via nPI effective action using an inverse Legendre transform.

Throughout this work we employ a superfield notation. To make the derivations simpler to follow, we choose to do the calculations using a fundamental field with fixed statistics.

The thesis is organized as follows. Ch. 2 reviews the fundamental results for renormalised interactions, also known as 1PI vertices. In Ch. 3, we explain how an inverse Legendre transformation of the composite field effective action can be used to study 1PI vertices non-perturbatively. As an example, we consider the three-particle irreducible (3PI) effective action to derive self-energy estimators, Bethe-Salpeter equations, as well as parquet and asymptotic class decompositions of the four-point vertex. Subsequent chapters develop these results in different ways and can be read independently. In Ch. 4, we show how renormalised interactions can be decomposed via composite fields using simple tree diagrams. When these fields are connected correlation functions we get a decomposition in terms of multi-particle Green's functions, recreating the parquet equations, while for local (in time) averaged bilinear fields we get a decomposition in terms of the exchange of single bosons. Ch. 5 follows a different path to study 1PI vertices, using two- and three-point Green's functions as composite fields in a theory with up to quartic interactions. We show how to express asymptotic classes and 1PI vertices in terms of Green's functions of composite field operators.

Chapter 2

1PI effective action

Analogous to how bare interactions are defined through derivatives of the classical action, renormalized interactions are determined by the quantum effective action. Perturbatively, the latter is represented as a sum of connected one-particle-irreducible (1PI) diagrams in the presence of external fields. In this chapter, we adopt the non-perturbative definition of the effective action via a Legendre transform and review a standard result of the 1PI formalism: the tree expansion, which expresses arbitrary connected Green's functions in terms of derivatives of the 1PI effective action. The material presented here can be found in many textbooks, such as [8, 28, 29].

Consider a theory described by the classical action $S[\varphi]$, where the field φ has fixed statistics. We explicitly retain a factor ζ with $\zeta = 1$ for bosonic systems and $\zeta = -1$ for fermionic systems.

The generating functional W of the connected Green's functions is defined as $e^{-W[J]} = \int D\varphi e^{-S[\varphi]-J_a\varphi^a}$. Here, the Einstein summation convention is used for each contraction of one lower and one upper index. We use DeWitt's notation [28]: each index a is a multi-index that can consist of multiple discrete and continuous quantum numbers. For example: $J_a\varphi^a = \sum_i \int d^4x J_i(x)\varphi^i(x)$. From W we can calculate connected correlation functions of the fundamental field φ^a via functional derivatives

$$G^{a_1\dots a_n} \equiv \frac{\delta}{\delta J_{a_1}} \dots \frac{\delta}{\delta J_{a_n}} W[J] = (-1)^{n-1}\langle\varphi^{a_1}\dots\varphi^{a_n}\rangle_c, \tag{2.1}$$

where the average of an arbitrary operator $A[\varphi]$ built out of the φ^a is defined as

$$\langle A[\varphi]\rangle \equiv e^W \int D\varphi A[\varphi] e^{-S[\varphi]-J_a\varphi^a}, \tag{2.2}$$

and the subscript c in $\langle\dots\rangle_c$ indicates that only the connected part is taken.

For the two-point correlation function, it is convenient to use lowercase notation, $g^{ab} \equiv G^{ab}$. Calculating the right side of Eq. (2.1) for g^{ab}, we obtain

$$g^{ab} \equiv G^{ab} = \bar{\varphi}^a\bar{\varphi}^b - \langle\varphi^a\varphi^b\rangle, \tag{2.3}$$

where $\bar{\varphi}^a \equiv \frac{\delta W}{\delta J_a} = \langle\varphi^a\rangle$.

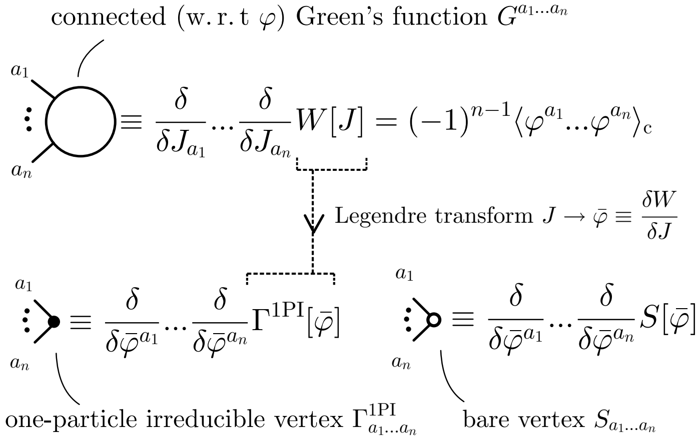

Figure 2.1: Diagrammatic definitions of connected Green's function $G^{a_1 \cdots a_n}$ (see Eq. (2.1)) and 1PI vertex $\Gamma^{1PI}_{a_1 \ldots a_n}$ (see Eq. (2.6)).

.

The 1PI effective action $\Gamma^{1PI}[\bar{\varphi}]$ is defined as the Legendre transformation of W with respect to the source J_a [1],

$$\Gamma^{1PI}[\bar{\varphi}] = W - J_a \bar{\varphi}^a, \tag{2.4}$$

where J_a as a functional of $\bar{\varphi}^a$ is the solution of $\frac{\delta W}{\delta J_a} = \bar{\varphi}^a$.

It is well-known [29] that Γ^{1PI} is given by a sum of all connected one-particle irreducible (1PI) diagrams with external lines contracted by $\bar{\varphi}^a$, where *one-particle irreducible* diagram means that one cut of any internal line cannot make the diagram disconnected. The fundamental result of the 1PI effective action formalism is that any connected correlation function $G^{a_1 \cdots a_n}$ of order $n > 2$ can be written as a sum of tree diagrams, where the internal lines are one-particle propagators $g^{a_1 a_2}$ and the vertices are functional derivatives of Γ^{1PI} (called 1PI vertices). This result is the so-called *tree expansion*. Although the derivation of the tree expansion is textbook knowledge (see [8], for example), we include it here to emphasize the recursive structure generating the expansion, that we use in the next chapters.

We start with the equation of motion that follows from Eq. (2.4), the definition of Γ^{1PI},

$$\Gamma^{1PI}_{,a} \equiv \frac{\delta}{\delta \bar{\varphi}^a} \Gamma^{1PI} = -\zeta J_a, \tag{2.5}$$

where ζ comes from commuting $J_a \bar{\varphi}^a = \zeta \bar{\varphi}^a J_a$. Comma notation is used for derivatives of the functional with respect to its variables. In particular, the general 1PI vertex is defined as

$$\Gamma^{1PI}_{,a \ldots b} \equiv \frac{\delta}{\delta \bar{\varphi}^a} \cdots \frac{\delta}{\delta \bar{\varphi}^b} \Gamma^{1PI}. \tag{2.6}$$

Note that the order of indices is the same on both sides. It is convenient to drop the commas by defining the corresponding vertex. For example, $\Gamma^{1PI}_{a \ldots b} \equiv \Gamma^{1PI}_{,a \ldots b}$.

[1] We could also use $W' = -W$ and define Green's functions as derivatives of W'. Definition $\Gamma^{1PI} = J_a \bar{\varphi}^a - W'$ then leads to additional minus signs in the tree expansion for our main results (Fig. 6.1).

Diagrammatic definitions of the bare and 1PI vertices as well as the connected Green's function are shown in Fig. 2.1.

The transformation matrix $\frac{\delta J_a}{\delta \bar{\varphi}^b}$ and its inverse $\frac{\delta \bar{\varphi}^a}{\delta J_b}$ (in which $\bar{\varphi}^a$ is a functional of J_b) give the identity $\frac{\delta \bar{\varphi}^b}{\delta J_a} \frac{\delta J_c}{\delta \bar{\varphi}^b} = \delta^a_c$, where δ^a_c is a Kronecker-delta for discrete indices and a Dirac-delta for continuous indices. Using $g^{ab} = \frac{\delta \bar{\varphi}^b}{\delta J_a}$ and the equation of motion (2.5), the identity becomes

$$g^{ab}\Gamma^{\mathrm{1PI}}_{bc} = -\zeta \delta^a_c. \tag{2.7}$$

As another consequence of $g^{ab} = \frac{\delta \bar{\varphi}^b}{\delta J_a}$, the chain rule $\frac{\delta}{\delta J_a} = \frac{\delta \bar{\varphi}^b}{\delta J_a} \frac{\delta}{\delta \bar{\varphi}^b}$ can be written as

$$\frac{\delta}{\delta J_a} = g^{ab} \frac{\delta}{\delta \bar{\varphi}^b}. \tag{2.8}$$

Tree expansion

Now, let us find expressions for higher order connected correlation functions $G^{a_1 \ldots a_n}$ in terms of 1PI vertices. To illustrate the recursive structure of the derivation, it is sufficient to consider the bosonic case. Therefore, we set $\zeta = 1$ for the rest of this chapter.

It is convenient to further condense our notations and write, instead of the sequence of indices $a_1 \ldots a_n$, only their number n (and omit them completely for φ^a and g^{ab}), so, in particular, $g^{ab}\Gamma^{\mathrm{1PI}}_{bc} = g\Gamma^{\mathrm{1PI}}_2$. To reduce ambiguity, we place contracting indices as close together as possible. For example, $G^3 = (g)^2\Gamma^{\mathrm{1PI}}_3 g$ stands for $G^{abc} = g^{aa'}g^{bb'}\Gamma^{\mathrm{1PI}}_{a'b'c'}g^{c'c}$ [2]. Equations without indices effectively describe a zero-dimensional system, however recovering the proper index structure in the end is trivial.

This notation allows us to write Eq. (2.1) for $n > 2$ as

$$G^n = \left(\frac{\delta}{\delta J_1}\right)^{n-2} g = \left((-\Gamma^{\mathrm{1PI}}_2)^{-1} \frac{\delta}{\delta \bar{\varphi}}\right)^{n-2} (-\Gamma^{\mathrm{1PI}}_2)^{-1}, \tag{2.9}$$

where for the second equality we used Eq. (2.8) and

$$g = (-\Gamma^{\mathrm{1PI}}_2)^{-1} \tag{2.10}$$

(as follows from Eq. (2.7) above for $\zeta = 1$). For $n = 3$ one finds

$$G^3 = (g)^3\Gamma^{\mathrm{1PI}}_3. \tag{2.11}$$

Apply Eq. (2.8) one more time to get the four-point correlation function

$$G^4 = (g)^4\Gamma^{\mathrm{1PI}}_4 + P_3(g)^2\Gamma^{\mathrm{1PI}}_3 g\Gamma^{\mathrm{1PI}}_3(g)^2, \tag{2.12}$$

where P_3 indicates three terms coming from the derivative of $(g)^3$. More specifically, P_n denotes a sum over all n distinct permutations of the external indices in the associated term (with a factor ζ inserted if permutation involves interchange of an odd number of indices). As the simplest example, consider $P_2\varphi^a\varphi^b = \varphi^a\varphi^b + \zeta\varphi^b\varphi^a$.

[2] Here we implicitly used that G^3 is connected, so $g^{ab}g^{b'a'}\Gamma^{\mathrm{1PI}}_{a'b'c'}g^{c'c}$ is not possible.

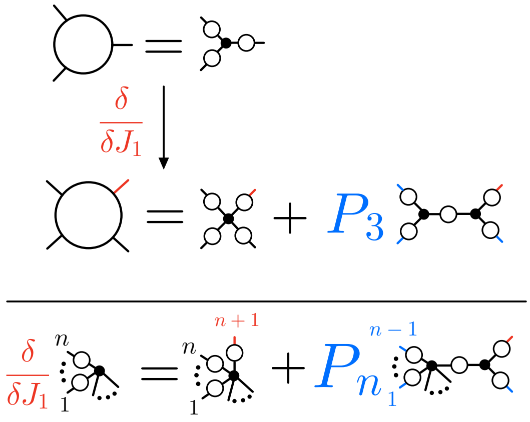

Figure 2.2: Tree expansions of the connected Green's functions G^3 and G^4 (see Eqs. (2.11) and (2.12)), using the recursive rule in Eq. (2.13).

Although there are 4! possible permutations of the four external indices in Eq. (2.12), symmetries of the tensors involved reduce this number to only three.

As shown in Fig. 2.2, Eqs. (2.11), (2.12) can be represented as tree diagrams where an open circle with two legs denotes g and a black circle with n legs represents the vertex $\Gamma_n^{1\text{PI}}$. To perform further differentiations with respect to J_1, it is convenient to employ the following recursive rules: the derivative of an internal propagator (i.e., one without external indices) yields $\frac{\delta}{\delta J_1} g = (g)^3 \Gamma_3^{1\text{PI}}$ (recall Eq. (2.11)) and the derivative of a product $(g)^n \Gamma_{n+m}^{1\text{PI}}$ with n external indices in $(g)^n$ and m internal indices (i.e., ones that contract with internal propagators) gives

$$\frac{\delta}{\delta J_1}[(g)^n \Gamma_{n+m}^{1\text{PI}}] = (g)^{n+1} \Gamma_{1+n+m}^{1\text{PI}} + P_n (g)^{n-1} \Gamma_{n+m}^{1\text{PI}} g \Gamma_3^{1\text{PI}} (g)^2, \tag{2.13}$$

where P_n comes from the derivative of $(g)^n$. A diagrammatic representation of Eq. (2.13) is given in the bottom of Fig. 2.2. The figure highlights that differentiation of a diagram with respect to J_1 corresponds to the insertion of an external line with propagator in all possible ways both at already existing vertices and into each propagator, thereby yielding a new three-point vertex attached to that propagator. Iterating these rules, starting with Eq. (2.11), shows that every correlation function is expressible as a sum of all tree graphs with a fixed number of external lines, where the indices at the free ends are permuted (via P_n) such that the sum is (anti)-symmetric with respect to the exchange of any two external indices. The result for Green's functions up to fifth order is presented in the top row of Fig. 6.1.

This concludes the derivation of the 1PI tree expansion. In chapter 4, we show how the same 1PI vertices can be further expressed using a similar tree expansion but in terms of derivatives of the composite field effective action. This analogy is illustrated in Fig. 4.1. Before that, however, Chapter 3 discusses the connection between 1PI and composite field effective actions via the inverse Legendre transform.

Chapter **3**

Inverse Legendre transform of composite field effective action to 1PI action

The Legendre transformation is a powerful technique in classical mechanics and field theory for constructing a function (or functional) that describes a system in terms of its relevant degrees of freedom. It allows to define a quantum analog of the classical action, the 1PI effective action, which is a functional of the fundamental field average $\bar{\varphi}^a$. By introducing source terms coupled to nonlinear combinations of fundamental fields (i.e., $\varphi^a\varphi^b$), we can define functionals of composite fields $\boldsymbol{\psi}$, such as the full propagator g [1]. This can be useful in two distinct but related ways.

First, we can classify contributions to 1PI vertices in terms of objects with a certain *irreducibility* that depends on the choice of the composite fields $\boldsymbol{\psi}$. These are then typically used to justify approximations. For example, as was first shown by Eckhardt et al. [31], we can use $\bar{\varphi}$ and a full propagator g as variables to reproduce the well-known parquet decomposition, which was previously only derived using diagrammatic arguments [2, 30]. It groups contributions to the 1PI four-point vertex in terms of two-particle reducible diagrams, which can be disconnected by cutting two internal propagator lines, and fully two-particle irreducible diagrams, which are commonly approximated in various practical schemes (Parquet Approximation, DΓA, ...)

Second, appropriate composite fields allow us to reveal non-perturbative structures of the composite field effective actions themselves. As will be demonstrated in this chapter, these structures can be related to the 1PI effective action through an inverse Legendre transformation. To achieve this, it is essential to retain $\bar{\varphi}$ as one of the variables.

As an important example, consider the choice $\psi^a = \bar{\varphi}^a$ together with the connected Green's functions $\psi^{ab} = g^{ab}$ and $\psi^{abc} = G^{abc}$. The corresponding functional $\Gamma[\bar{\varphi}, g, G^3]$ is known as three-particle irreducible (3PI) effective action, and is formally defined (in section 3.2) as a Legendre transform similarly to 1PI action. For non-relativistic systems with cubic and quartic interactions the 3PI action was first

[1] For a historical sketch on development of functional Legendre transforms including composite fields, see the end of the section 6.2.1 in [30]

investigated by De Dominicis and Martin in [1] and [2]. Their results can be extended [32, 33] to general classical actions of the form

$$S[\varphi]=\sum_{n=2}^{N+1}\frac{1}{n!}\varphi^{a_n}...\varphi^{a_1}S_{a_1...a_n}[0], \tag{3.1}$$

where $S_{a_1...a_n}\equiv S_{,a_1...a_n}$ [2]. For a theory with cubic and quartic interactions ($N=3$) there is a simple dependence of the effective action $\Gamma[\bar\varphi,g,G^3]$ on $\bar\varphi$ [2, 33]

$$\Gamma[\bar\varphi,g,G^3]=S[\bar\varphi]+\sum_{n=2}^{3}G^nF_n[\bar\varphi]+\Lambda[g,G^3], \tag{3.2}$$

$$F_{a_1...a_n}\equiv S_{a_n...a_1}(-1)^{n-1}/n!, \tag{3.3}$$

where G^nF_n stands for $G^{a_1\cdots a_n}F_{a_1...a_n}$. The same F_n holds for the choice $\psi^a=\bar\varphi^a$ and $\psi^{ab}=g^{ab}$ in case only cubic interactions are present (for $S_3[0]\neq 0$ the system with fixed statistics can be only bosonic, $\zeta=1$).

Eq. (3.2) can be viewed as the definition for $\Lambda[g,G^3]$. It was shown in [2] that Λ consists of $\frac{-1}{2}\ln\det g$ plus a sum of all possible 3PI diagrams[3]. In that sense Λ modulo the $\ln\det g$ part, can be seen as a generalization of the Luttinger-Ward functional and can be exactly identified with the Luttinger-Ward functional for the choice $\psi^a=\bar\varphi^a$ and $\psi^{ab}=g^{ab}$. For completeness we derive Eq. (3.2) from the path integral definition of the theory in Appendix A.1.

$\Gamma[\bar\varphi,g,G^3]$ can be related to the 1PI effective action $\Gamma^{1\mathrm{PI}}[\bar\varphi]$ by an inverse Legendre transform. We find the remarkably simple result

$$\boxed{\Gamma^{1\mathrm{PI}}[\bar\varphi]=S[\bar\varphi]+\Omega[g_{\bar\varphi},G^3_{\bar\varphi}]}\ , \tag{3.4}$$

where

$$\Omega[g,G^3]=\Lambda[g,G^3]-\sum_{n=2}^{3}G^n\frac{\delta\Lambda}{\delta G^n} \tag{3.5}$$

is the Legendre transform of Λ with respect to g and G^3. The subscript $\bar\varphi$ in $G^n_{\bar\varphi}$ means that the latter are functionals of $\bar\varphi$ obtained from the equations of motion $\frac{\delta\Gamma}{\delta g}=0$ and $\frac{\delta\Gamma}{\delta G^3}=0$. This will be explained in detail in section 3.2. We see from Eq. (3.4) that $\Gamma^{1\mathrm{PI}}$ expressed in terms of $\bar\varphi$ and Green's functions has an even simpler dependence on $\bar\varphi$ than $\Gamma[\bar\varphi,g,G^3]$.

In what follows, we use this new insight to demonstrate how properties of $\Gamma[\bar\varphi,g,G^3]$ can be transferred to the 1PI effective action by explicitly calculating the first four derivatives of Eq. (3.4). These results motivate further developments in the next chapters. To make it easier to follow, we restrict ourselves to bosonic systems and postulate functional identities for Γ. These relations will be derived rigorously in section 3.2 within our general formalism. Readers preferring a more self-contained treatment may wish to skip ahead and return to the following section afterward.

[2]The order of indices in Eq. (3.1) can be confirmed by taking functional derivatives of both sides w.r.t. φ and then setting $\varphi=0$.

[3]Such graphs can be disconnected by cutting three lines (represented by a full propagator) into only two parts, one of which must be a three-point 1PI vertex.

3.1 Preliminary results: self-energy estimator, Bethe-Salpeter equations, parquet and asymptotic class decompositions from 3PI effective action

Consider a bosonic theory with up to quartic interactions, so that the formulas (3.2) and (3.4) apply. The functional $\Gamma[\bar{\varphi}, g, G^3]$ possesses two properties that lead to useful non-perturbative results for 1PI vertices, as derived in this section.

First, three-particle irreducibility of the functional $\Gamma[\bar{\varphi}, g, G^3]$ allows us to classify contributions to the four-point 1PI vertex and derive Bethe-Salpeter-type equations for its reducible part. For an even theory (i.e., one with vanishing odd-order Green's and vertex functions) we recover the well-known parquet formalism [2]. This motivates us, in the next chapter 4, to decompose 1PI vertices of arbitrary order using general composite fields.

Second, the structure (3.2) allows us to classify contributions to 1PI vertices by the connectivity of their external legs to the bare interactions. These groupings are referred to as asymptotic classes [17], as they govern the high-frequency behavior of the vertex functions. In chapter 5 this result is used to derive representations for asymptotic classes and 1PI vertices via Green's functions of composite field operators (the so-called a-symmetric estimators). Here, we reproduce such a formula for the self-energy.

Step 1: First functional derivative of the effective action

Let us start by taking the first functional derivative of Eq. (3.4) with respect to $\bar{\varphi}$

$$\Gamma_1^{1\text{PI}}[\bar{\varphi}] = S_1[\bar{\varphi}] + \sum_{n=2}^{3} G^n_{\bar{\varphi},1} \frac{\delta\Omega}{\delta G^n} = S_1[\bar{\varphi}] - \sum_{n,m=2}^{3} g_{\bar{\varphi}}^{-1} G^{1n}_{\bar{\varphi}} G^m_{\bar{\varphi}} \frac{\delta^2\Lambda}{\delta G^m \delta G^n}, \tag{3.6}$$

where we used $G^n_{\bar{\varphi},1} = g_{\bar{\varphi}}^{-1} G^{1n}_{\bar{\varphi}}$ with the help of Eq. (2.8). The dependence of $G^4_{\bar{\varphi}}$ on $\bar{\varphi}$ is defined via the tree expansion formula (2.12). All derivatives of Λ are evaluated at $G^n = G^n_{\bar{\varphi}}$.

To proceed, we need functional relations that connect derivatives of Γ (or Λ) to Green's functions, similar to Eq. (2.7). As shown in Sec. 3.2 we can write

$$g^{-1} \sum_{n=2}^{3} G^{1n} \frac{\delta^2\Lambda}{\delta G^n \delta G^m} = -F_{m,1}. \tag{3.7}$$

Eq. (3.6) becomes

$$\Gamma_1^{1\text{PI}} = S_1 + \sum_{m=2}^{3} G^m_{\bar{\varphi}} F_{m,1}. \tag{3.8}$$

The last term in Eq. (3.8) contains an implicit $\bar{\varphi}$-dependence in $G^n_{\bar{\varphi}}$ as well as an explicit one in $F_{n,1}[\bar{\varphi}]$ (see Eq. (3.3)).

By taking derivatives $\frac{\delta}{\delta\bar{\varphi}} = g^{-1}\frac{\delta}{\delta J_1}$ of Eq. (3.8), we reproduce Schwinger-Dyson (SD) equations. Indeed, the first differentiation gives

$$-g_{\bar{\varphi}}^{-1} = S_2 - \frac{1}{2} g_{\bar{\varphi}} S_4 + g_{\bar{\varphi}}^{-1} \sum_{m=2}^{3} G^{1m}_{\bar{\varphi}} F_{m,1}, \tag{3.9}$$

where we used Eq. (2.5) and $G^{ba}F_{ba,cd} = -\frac{1}{2}g^{ba}S_{abcd}$ in condensed form (note that $F_{m,2}$ vanishes for $m > 2$).

Step 2: Chain rule and general formula for np 1PI vertex

Another way to generate 1PI vertices (first proposed in [34]) is to use the chain rule $\frac{\delta}{\delta\bar{\varphi}} = \frac{\delta}{\delta\psi} + \sum_{l=2}^{3} G^{l}_{\bar{\varphi},1}\frac{\delta}{\delta G^l}$. We use ψ in place of $\bar{\varphi}$ to distinguish $\bar{\varphi}$, the variable of $\Gamma^{1\text{PI}}[\bar{\varphi}]$, from ψ, the variable of the effective action $\Gamma[\psi, g, G^3]$ (which is the same functional as (3.2)).

Eq. (3.7) gives for $G^{l}_{\bar{\varphi},1} = g_{\bar{\varphi}}^{-1}G^{1l}_{\bar{\varphi}}$,

$$G^{l}_{\bar{\varphi},1} = \sum_{k=2}^{3} F_{k,1}K^{k|l}, \tag{3.10}$$

where we defined $K^{k|l}$ as the inverse of $(-\frac{\delta^2\Lambda}{\delta G^l \delta G^m})$:

$$\sum_{l=2}^{3} K^{k|l}\frac{\delta^2\Lambda}{\delta G^l \delta G^m} = -\delta^k_m. \tag{3.11}$$

With the help of Eq. (3.10), the chain rule becomes

$$\frac{\delta}{\delta\bar{\varphi}} = \frac{\delta}{\delta\psi} + \sum_{k,l=2}^{3} F_{k,1}K^{k|l}\frac{\delta}{\delta G^l}. \tag{3.12}$$

To calculate $\Gamma^{1\text{PI}}_n$, we differentiate Eq. (3.8) $(n-1)$ times using Eq. (3.12) for all terms except S_1,

$$\Gamma^{1\text{PI}}_n = S_n + (\frac{\delta}{\delta\psi} + \sum_{k,l=2}^{3} F_{k,1}K^{k|l}\frac{\delta}{\delta G^l})^{n-1}\sum_{m=2}^{3} G^m F_{m,1}. \tag{3.13}$$

After taking all functional derivatives, we substitute $G^n = G^n_{\bar{\varphi}}$.

Evaluating only the ψ-derivatives in Eq. (3.13) generates groups of terms with different connectivities of the external legs to the bare vertices (contained in $F_{n,m}[\bar{\varphi}]$); these groups can be identified with asymptotic classes. To see this, we calculate Eq. (3.13) with $n = 2, 3, 4$ in the next steps.

Step 3: Second derivative of the effective action - Self-Energy estimator

For $n = 2$ we get

$$\Gamma^{1\text{PI}}_2 = S_2 + g_{\bar{\varphi}}F_{2,2} + \sum_{k,l=2}^{3} F_{k,1}K^{k|l}F_{l,1}. \tag{3.14}$$

For an even theory, $S_2 = S_2[0]$ and the last term simplifies to $F_{3,1}K^{3|3}F_{3,1}$. When expressed in terms of composite field operators, Eq. (3.14) then reduces to the symmetric estimator for the self-energy established by Kugler [25]. To show this, we need another functional relation (proved in section 3.2)

$$K^{3|3} = -\langle\varphi^3\varphi^3\rangle - \langle\varphi^3\varphi\rangle g^{-1}\langle\varphi\varphi^3\rangle. \tag{3.15}$$

The contraction $F_{3,1}\langle\varphi^3...\rangle$ in Eq. (3.14) defines the following composite operator in correlation functions: $F_{bcd,a}\varphi^b\varphi^c\varphi^d = \frac{\delta}{\delta\varphi^a}S_{\rm int}$, where $S_{\rm int}$ is the interacting (quartic) part of the action in an even theory. Applying this identity to expression (3.14) yields the self-energy estimator

$$-g^{-1} - S_2[0] = \langle\frac{\delta^2 S_{\rm int}}{\delta\varphi^2}\rangle - \langle\frac{\delta S_{\rm int}}{\delta\varphi}\frac{\delta S_{\rm int}}{\delta\varphi}\rangle - \langle\frac{\delta S_{\rm int}}{\delta\varphi}\varphi\rangle g^{-1}\langle\varphi\frac{\delta S_{\rm int}}{\delta\varphi}\rangle, \tag{3.16}$$

where we also used Eqs. (2.7) and $gF_{2,2} = \frac{1}{2}\langle\varphi^2\rangle S_4 = \langle\frac{\delta^2 S_{\rm int}}{\delta\varphi^2}\rangle$. Equation (3.16) corresponds to Eq. (12) of [25].

Step 4: Derivative of $\sum_{l=2}^3 K^{k|l}F_{l,1}$ - Bethe-Salpeter equations

For $n > 2$, the combination $\sum_{l=2}^3 K^{k|l}F_{l,1}$ appears frequently in Eq. (3.13). It is convenient to evaluate its derivative as follows

$$\begin{aligned}\frac{\delta}{\delta\bar\varphi}\sum_{l=2}^3 K^{k|l}F_{l,1} &= \left(\frac{\delta}{\delta\psi} + \sum_{n=2}^3 G^n_{\bar\varphi,1}\frac{\delta}{\delta G^n}\right)\sum_{l=2}^3 K^{k|l}F_{l,1}\\ &= \sum_{l=2}^3 K^{k|l}F_{l,2} + \sum_{l,n=2}^3 G^n_{\bar\varphi,1}\left(\frac{\delta}{\delta G^n}K^{k|l}\right)F_{l,1}\\ &= \sum_{l=2}^3 K^{k|l}I_{l|2},\end{aligned} \tag{3.17}$$

$$I_{l|2} \equiv F_{l,2} + \sum_{n,m=2}^3 G^n_{\bar\varphi,1}\frac{\delta^3\Lambda}{\delta G^n\delta G^l\delta G^m}G^m_{\bar\varphi,1}. \tag{3.18}$$

where we used Eq. (3.10).

Now, consider Eq. (3.10) for $l = 2$. Differentiating both sides with respect to $\bar\varphi$, and using Eq. (3.17), we obtain

$$G^2_{\bar\varphi,2} = \sum_{k=2}^3 K^{2|k}I_{k|2}, \tag{3.19}$$

where the right side is a functional of $\bar\varphi$ after substituting $G^n = G^n_{\bar\varphi}$ (recall that $K^{m|n}$ and $I_{k|2}$ are defined in terms of derivatives of $\Gamma[\bar\varphi, g, G^3]$).

For an even theory, Eq. (3.19) is precisely the Bethe-Salpeter equation for $I_{2|2}$ after we relate $K^{2|2}$ to composite field Green's functions as in Eq. (3.15). In that case, the expression for $K^{2|2}$,

$$K^{ab|cd} = g^{aa'}g^{bb'}\Gamma^{\rm 1PI}_{a'b'c'd'}g^{c'c}g^{d'd} - g^{ac}g^{bd} - g^{ad}g^{bc}, \tag{3.20}$$

is known as the two-particle Green's function. Using Eqs. (2.8) and (2.12), the left side of Eq. (3.19) becomes $G^4_{\bar\varphi}g^{-2}_{\bar\varphi} = g_{\bar\varphi}g_{\bar\varphi}\Gamma^{\rm 1PI}_4$. We get

$$g_{\bar\varphi}g_{\bar\varphi}\Gamma^{\rm 1PI}_4 = K^{2|2}I_{2|2}. \tag{3.21}$$

By taking a further derivative, we can obtain an equation that describes three-particle scattering processes (as will be shown in the next chapter 4). In this way, we recover Bethe-Salpeter-type equations first introduced by Weinberg in [35].

Step 5: Calculation of the 4p 1PI vertex - Parquet decomposition

Now, differentiate Eq. (3.14) using Eq. (3.19) to get the three-point 1PI vertex

$$\Gamma_3^{1\mathrm{PI}} = S_3 + P_3(g_{\bar{\varphi},1}F_{2,2}) + \sum_{n,m,l=2}^{3} G^n_{\bar{\varphi},1}G^m_{\bar{\varphi},1}\frac{\delta^3\Lambda}{\delta G^n\delta G^m\delta G^l}G^l_{\bar{\varphi},1}, \tag{3.22}$$

where we used $\sum_{n'=2}^{3} K^{n|n'}F_{n',1} = G^n_{\bar{\varphi},1}$.

Next, we evaluate the four-point 1PI vertex by taking the derivative of Eq. (3.22). All terms involving derivatives of $G^n_{\bar{\varphi},1} = \sum_{n'=2}^{3} K^{n|n'}F_{n',1}$ (including $g_{\bar{\varphi},1} = G^2_{\bar{\varphi},1}$) are evaluated using Eq. (3.19); their sum yields

$$P_3\sum_{k,l=2}^{3}(F_{k,2} + \sum_{n,m=2}^{3} G^n_{\bar{\varphi},1}G^m_{\bar{\varphi},1}\frac{\delta^3\Lambda}{\delta G^n\delta G^m\delta G^k})K^{k|l}I_{l|2} = P_3\sum_{k,l=2}^{3} I_{k|2}K^{k|l}I_{l|2} \tag{3.23}$$

We get

$$\Gamma_4^{1\mathrm{PI}} = I_4 + P_3\sum_{k,l=2}^{3} I_{k|2}K^{k|l}I_{l|2}, \tag{3.24}$$

$$I_4 \equiv S_4 + \sum_{n,m,l,k=2}^{3} G^n_{\bar{\varphi},1}G^m_{\bar{\varphi},1}\frac{\delta^4\Lambda}{\delta G^n\delta G^m\delta G^l\delta G^k}G^l_{\bar{\varphi},1}G^k_{\bar{\varphi},1} \tag{3.25}$$

For even theories, one recalls the Bethe-Salpeter equation (3.21) to obtain $I_{2|2}K^{2|2}I_{2|2} = I_{2|2}g_{\bar{\varphi}}g_{\bar{\varphi}}\Gamma_4^{1\mathrm{PI}}$. Eq. (3.24) then reduces to the parquet decomposition [2]. Note that it has the tree expansion form given by Eq. (2.12). This result will be extended to any order vertex and using arbitrary composite fields in the next chapter 4.

Step 6: Asymptotic class decomposition of the 4p 1PI vertex

Finally, let us expand Eq. (3.24), using definition (3.18), as follows,

$$\Gamma_4^{1\mathrm{PI}} = S_4 + P_3\mathcal{K}_4^1 + P_6\mathcal{K}_4^2 + \mathcal{K}_4^3, \tag{3.26}$$

where we defined asymptotic classes

$$\mathcal{K}_4^1 \equiv F_{2,2}K^{2|2}F_{2,2} \tag{3.27}$$

$$\mathcal{K}_4^2 \equiv F_{2,2}\sum_{n,m,l=2}^{3} K^{2|n}\frac{\delta^3\Lambda}{\delta G^n\delta G^m\delta G^l}G^m_{\bar{\varphi},1}G^l_{\bar{\varphi},1} \tag{3.28}$$

$$\begin{aligned}\mathcal{K}_4^3 \equiv \sum_{n,m,l,k=2}^{3} G^n_{\bar{\varphi},1}G^m_{\bar{\varphi},1}G^l_{\bar{\varphi},1}G^k_{\bar{\varphi},1}\Big(&\frac{\delta^4\Lambda}{\delta G^n\delta G^m\delta G^l\delta G^k}\\ &+ P_3\sum_{n',m'=2}^{3}\frac{\delta^3\Lambda}{\delta G^n\delta G^m\delta G^{n'}}K^{n'|m'}\frac{\delta^3\Lambda}{\delta G^{m'}\delta G^l\delta G^k}\Big).\end{aligned} \tag{3.29}$$

In Eq. (3.29), P_3 permutes derivatives $\frac{\delta}{\delta G^{\cdots}}$ with external indices[4]. The upper index m in $\mathcal{K}_4^m$ denotes the number of independent time (or frequency) arguments, assuming time-translational invariance and that the action S is local in time. For example, in the class $\mathcal{K}_4^2$, two indices are contained in $F_{2,2} = -\frac{1}{2}S_4$, which forces their time components to be equal; the total number of independent time arguments is then $4 - 1 - 1 = 2$, where the second reduction by one comes from time-translational invariance.

Equations (3.27) and (3.28) correspond to Eqs. (B1) and (B2) of [17], respectively[5].

One may notice that the asymptotic classes $\mathcal{K}_4^2$ and $\mathcal{K}_4^3$ contain the tree expansion (see Eqs. (2.11) and (2.12)) with derivatives of Λ as vertices and $K^{n|m}$ as internal lines. This structure allows us to express asymptotic classes in terms of Green's functions of composite field operators (like $\frac{\delta S_{\mathbf{int}}}{\delta \varphi}$ in Eq. (3.16)), similarly to how the tree expansion in derivatives of $\Gamma^{1\text{PI}}$ yields Green's functions of fundamental field φ. In chapter 5, we demonstrate this result for general asymptotic classes and apply it to 1PI vertices.

Before developing these ideas further, we clarify and extend two key concepts. First, we introduce the composite-field effective action and derive the functional relations it satisfies. Then, we perform its inverse Legendre transform to recover the 1PI effective action. These steps are carried out in the following section.

3.2 General formulation with condensed notations

Let us define the functional for connected Greens functions $W[J_{\mathbf{a}}]$ more generally from

$$e^{-W[J_{\mathbf{a}}]} = \int D\varphi e^{-S[\varphi] - J_{\mathbf{b}}\phi^{\mathbf{b}}[\varphi] - \mathcal{S}[J_{\mathbf{a}}]}, \tag{3.30}$$

by adding source terms with two main requirements: 1) sources $J_{\mathbf{a}}$ couple linearly to the composite fields $\phi^{\mathbf{a}}[\varphi]$; 2) the field-independent part $\mathcal{S}[J_{\mathbf{a}}]$ and its first derivative vanish for $J_{\mathbf{a}} = 0$[6]. With this definition, one can calculate the connected correlation function of the fundamental field φ as well as the composite ones,

$$G^{\mathbf{a_1}\ldots\mathbf{a_n}} \equiv \frac{\delta}{\delta J_{\mathbf{a_1}}} \ldots \frac{\delta}{\delta J_{\mathbf{a_n}}} W[J_{\mathbf{a}}] = (-1)^{n-1} \langle \phi^{\mathbf{a_1}} \ldots \phi^{\mathbf{a_n}} \rangle_c + \mathcal{S}_{\mathbf{a_1}\ldots\mathbf{a_n}}[J_{\mathbf{a}}]. \tag{3.31}$$

We similarly define $\bar{\phi}^{\mathbf{a}} \equiv \frac{\delta W}{\delta J_{\mathbf{a}}} \overset{J_{\mathbf{b}}\to 0}{\longrightarrow} \langle \phi^{\mathbf{a}} \rangle$. The bold index $\mathbf{a}$ means a vector of fundamental field index a and multi-indices of the composite fields. We leave them unspecified for the moment and denote by the placeholder index $(\bullet)$, so $\mathbf{a} = \big(a, (\bullet)\big)$. The contraction $J_{\mathbf{a}}\bar{\phi}^{\mathbf{a}}$ can be written as

$$J_{\mathbf{a}}\bar{\phi}^{\mathbf{a}} = J_a \bar{\varphi}^a + J_{(\bullet)}\bar{\phi}^{(\bullet)} \tag{3.32}$$

[4] It agrees with the definition of P_n if we consider external indices in $\frac{\delta}{\delta G^{\cdots}}$ as a single composite external index. Another way to make this precise is to expand the brackets in Eq. (3.29) and then apply P_3 to the external indices in products of $G^{\cdots}_{\bar{\varphi},1}$.

[5] Note that definition (3.18) and BS equation (3.21) imply $\mathcal{K}_4^2 = F_{2,2} gg \Gamma_4^{1\text{PI}} - F_{2,2} K^{2|2} F_{2,2}$ for an even theory. The last term in Eq. (3.29) is denoted as $\mathcal{R}$ in [17] (see also Fig. 5).

[6] The term $\mathcal{S}[J_{\mathbf{a}}]$ gives more freedom in constructing effective actions; it is needed to recover the single-boson exchange decomposition in section 4.2.

As an example, let us consider two composite fields $\phi^{(a_1...a_n)} = \varphi^{a_1}...\varphi^{a_n}$ for $n = 2, 3$. In this case $(\bullet)$ is a vector of (ab) and (abc) (in condensed notations (2) and (3) respectively), and the contraction becomes $J_{(\bullet)}\bar{\phi}^{(\bullet)} = J_{(ab)}\bar{\phi}^{(ab)} + J_{(abc)}\bar{\phi}^{(abc)} = J_{(2)}\bar{\phi}^{(2)} + J_{(3)}\bar{\phi}^{(3)}$. It is convenient to use condensed notation $\boldsymbol{\phi}$ for $\phi^{\mathbf{a}}$ (to avoid confusion with $\phi = \varphi$ for $\phi^a = \varphi^a$) and similarly for the sources.

The effective action $\Gamma[\bar{\boldsymbol{\phi}}]$ is defined via generalized Legendre transformation

$$\Gamma[\bar{\boldsymbol{\phi}}] = W - J_{\mathbf{a}}\bar{\phi}^{\mathbf{a}}, \tag{3.33}$$

where $J_{\mathbf{a}}$ as a functional of $\bar{\phi}^{\mathbf{a}}$ is the solution of $\frac{\delta W}{\delta J_{\mathbf{a}}} = \bar{\phi}^{\mathbf{a}}$.

Vertices are defined as derivatives of the effective action $\Gamma[\bar{\boldsymbol{\phi}}]$:

$$\Gamma_{\mathbf{a_1}...\mathbf{a_n}} \equiv \Gamma_{,\mathbf{a_1}...\mathbf{a_n}} = \frac{\delta}{\delta\bar{\phi}^{\mathbf{a_1}}}...\frac{\delta}{\delta\bar{\phi}^{\mathbf{a_n}}}\Gamma. \tag{3.34}$$

$\Gamma[\bar{\boldsymbol{\phi}}]$ obeys relations similar to those of the 1PI effective action $\Gamma^{1\text{PI}}[\bar{\varphi}]$. In particular, the equation of motion becomes

$$\Gamma_{,\mathbf{a}} \equiv \frac{\delta}{\delta\bar{\phi}^{\mathbf{a}}}\Gamma = -\gamma_{\mathbf{a}}^{\mathbf{b}}J_{\mathbf{b}}\ , \tag{3.35}$$

where we interchanged $J_{\mathbf{a}}\bar{\phi}^{\mathbf{a}} = \bar{\phi}^{\mathbf{a}}\gamma_{\mathbf{a}}^{\mathbf{b}}J_{\mathbf{b}}$ using the matrix[7]

$$\gamma_{\mathbf{a}}^{\mathbf{b}} = \zeta^{N_{\mathbf{a}}}\delta_{\mathbf{a}}^{\mathbf{b}}, \tag{3.36}$$

where $N_{\mathbf{a}}$ is the number of indices in $\mathbf{a}$ (recall that ζ keeps track of fermionic signs). Another relation, similar to Eq. (2.7), is

$$G^{\mathbf{ab}}\Gamma_{\mathbf{bc}} = -\gamma_{\mathbf{c}}^{\mathbf{a}}. \tag{3.37}$$

From the definition (3.36) we have $\gamma_{\mathbf{b}}^{\mathbf{a}}\gamma_{\mathbf{c}}^{\mathbf{b}} = \delta_{\mathbf{c}}^{\mathbf{a}}$. We can then multiply Eq. (3.37) from the left by $\gamma_{\mathbf{a}}^{\mathbf{d}}$ to get $(\gamma_{\mathbf{a}}^{\mathbf{d}}G^{\mathbf{ab}})\Gamma_{\mathbf{bc}} = -\delta_{\mathbf{c}}^{\mathbf{d}}$, or more explicitly for $\Gamma_{\mathbf{bc}}$

$$\begin{pmatrix} \Gamma_2 & \Gamma_{1(\bullet)} \\ \Gamma_{(\bullet)1} & \Gamma_{(\bullet)(\bullet)} \end{pmatrix} = -\begin{pmatrix} \zeta g & \zeta G^{1(\bullet)} \\ \gamma_{(\bullet)}^{(\bullet)}G^{(\bullet)1} & \gamma_{(\bullet)}^{(\bullet)}G^{(\bullet)(\bullet)} \end{pmatrix}^{-1}. \tag{3.38}$$

To avoid ambiguities, contracted bullet indices (one lower and one upper) are placed as close together as possible.

Using a block-matrix identity for the right side of Eq. (3.38)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} ... & ... \\ ... & (D - CA^{-1}B)^{-1} \end{pmatrix}, \tag{3.39}$$

we get

$$(-\Gamma_{(\bullet)(\bullet)})^{-1} = \gamma_{(\bullet)}^{(\bullet)}(G^{(\bullet)(\bullet)} - G^{(\bullet)1}g^{-1}G^{1(\bullet)}), \tag{3.40}$$

which is the analogue of Eq. (2.10).

[7] Our notation for the composite index $\mathbf{a}$ and matrix $\gamma_{\mathbf{b}}^{\mathbf{a}}$ is taken from [36].

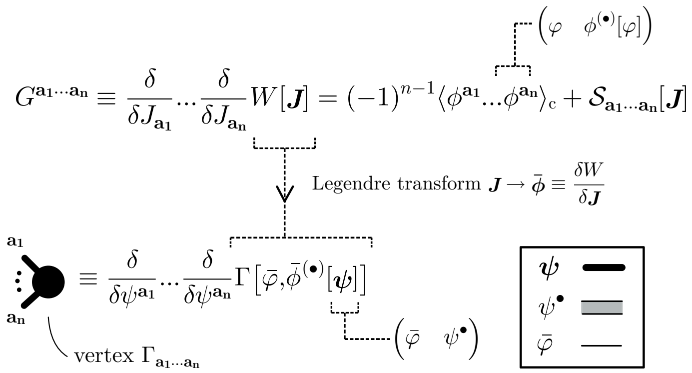

Figure 3.1: Diagrammatic definitions of connected composite field Green's function $G^{\mathbf{a_1 \ldots a_n}}$ (see Eq. (3.31)) and corresponding vertex $\Gamma_{\mathbf{a_1 \ldots a_n}}$ (see Eq. (3.34)).

We may want to use other variables $\psi^{\mathbf{a}}[\bar{\boldsymbol{\phi}}]$ instead of $\bar{\boldsymbol{\phi}} \equiv \big(\bar{\varphi}, \bar{\phi}^{(\bullet)}\big)$, while keeping the fundamental field $\bar{\varphi}^a$ as one of them. It means that $\psi^a = \bar{\phi}^a = \bar{\varphi}^a$, and for other components we choose $\psi^{\bullet}$, where the bullet index $\bullet$ takes the same values as in $\phi^{(\bullet)}$. For example, when the composite fields are connected correlation functions, we take $\psi^{\bullet} = G^{\bullet}$ with $\bullet = \big(2, 3\big)$, which corresponds to $(\bullet) = \big((2),(3)\big)$ in $\phi^{(\bullet)}$. The chain rules are $\frac{\delta}{\delta \bar{\phi}^{\mathbf{a}}} = \psi^{\mathbf{b}}_{,\mathbf{a}} \frac{\delta}{\delta \psi^{\mathbf{b}}}$ and $\frac{\delta}{\delta \bar{\phi}^{(\bullet)}} = \psi^{\bullet}_{,(\bullet)} \frac{\delta}{\delta \psi^{\bullet}}$, using $\frac{\delta \psi}{\delta \bar{\phi}^{(\bullet)}} = 0$. Placing brackets around $\bullet$ allows us to distinguish the field $\bar{\phi}^{(\bullet)}$ from $\psi^{\bullet}$ in the comma notation, as in $\psi^{\bullet}_{,(\bullet)}$ in the second chain rule.

Expressing $\bar{\phi}^{(\bullet)}$ in $\Gamma\big[\bar{\varphi}, \bar{\phi}^{(\bullet)}\big]$ in terms of $\boldsymbol{\psi} \equiv \big(\bar{\varphi}, \psi^{\bullet}\big)$, we get a new functional $\Gamma[\boldsymbol{\psi}] \equiv \Gamma\big[\bar{\varphi}, \bar{\phi}^{(\bullet)}[\boldsymbol{\psi}]\big]$ which we denote with the same symbol. The definition and graphical representation of the composite field effective action is summarized in Fig. 3.1.

Functional relation (3.40) can be expressed via derivatives of $\Gamma[\bar{\varphi}, \psi^{\bullet}]$, using $\frac{\delta}{\delta \bar{\phi}^{(\bullet)}} = \psi^{\bullet}_{,(\bullet)} \frac{\delta}{\delta \psi^{\bullet}}$. We have

$$K^{\bullet|\bullet} \equiv (-\Gamma_{\bullet\bullet})^{-1} = \psi^{\bullet}_{,(\bullet)} (G^{(\bullet)(\bullet)} - G^{(\bullet)1} g^{-1} G^{1(\bullet)}) \psi^{\bullet}_{,(\bullet)}, \tag{3.41}$$

where the last equality holds for either $J_{(\bullet)} = 0$ or linear relation between $\psi^{\bullet}$ and $\bar{\phi}^{(\bullet)}$.

Inverse Legendre transformation to 1PI effective action

Since we are interested in 1PI vertices – defined as functional derivatives of $\Gamma^{\mathrm{1PI}}[\bar{\varphi}]$ – we need to make a connection between $\Gamma^{\mathrm{1PI}}[\bar{\varphi}]$ and the composite field effective action $\Gamma[\bar{\boldsymbol{\phi}}]$. We can obtain the 1PI effective action $\Gamma^{\mathrm{1PI}}[\bar{\varphi}]$ from $\Gamma[\bar{\boldsymbol{\phi}}]$ by doing an inverse Legendre transformation from $\bar{\phi}^{(\bullet)}$ to $J_{(\bullet)}$ and then setting $J_{(\bullet)} = 0$. This procedure is possible only if we retain $\bar{\varphi}$ as one of the variables in $\bar{\boldsymbol{\phi}}$.

Let us show this more explicitly by keeping $J_{(\bullet)}$ nonzero. Substituting W from the definition of $\Gamma[\bar{\boldsymbol{\phi}}]$ (Eq. (3.33)) into the definition of the 1PI effective action (Eq. (2.4)), one performs an inverse Legendre transform of the composite field effective action $\Gamma[\bar{\varphi}, \bar{\phi}^{(\bullet)}]$ to 1PI

$$\Gamma^{\mathrm{1PI}}[\bar{\varphi}] = \Gamma[\bar{\varphi}, \bar{\phi}^{(\bullet)}_{\bar{\varphi}}] + J_{(\bullet)} \bar{\phi}^{(\bullet)}_{\bar{\varphi}}. \tag{3.42}$$

Here, $\bar{\phi}^{(\bullet)}_{\bar{\varphi}}$ is the solution of $\frac{\delta\Gamma[\bar{\varphi},\bar{\phi}^{(\bullet)}_{\bar{\varphi}}]}{\delta\bar{\phi}^{(\bullet)}_{\bar{\varphi}}} = \Gamma_{(\bullet)} = -\gamma^{(\bullet)}_{(\bullet)}J_{(\bullet)}$ for $\bar{\phi}^{(\bullet)}$; it is still a functional of $\bar{\varphi}$, which we denote as a subscript, and $J_{(\bullet)}$. Note that the dependence of $\Gamma^{1\mathrm{PI}}[\bar{\varphi}]$ on $J_{(\bullet)}$ is not explicitly shown.

Let us return to the example $\phi^{(n)}[\varphi] = \varphi^n$ and $\psi^n = G^n$ with $n = 2, 3$. There exists a linear relation between $G^\bullet$ and $\bar{\phi}^{(\bullet)}$ (see Eqs. (A.1) and (A.2))

$$\bar{\phi}^{(\bullet)} = \phi^{(\bullet)}[\bar{\varphi}] + G^\bullet\bar{\phi}^{(\bullet)}_{,\bullet}. \tag{3.43}$$

Eq. (3.43) together with $J_{(\bullet)} = -\gamma^{(\bullet)}_{(\bullet)}\Gamma_{(\bullet)}$ then give $J_{(\bullet)}\bar{\phi}^{(\bullet)}[\bar{\varphi}, G^\bullet_{\bar{\varphi}}] = J_{(\bullet)}\phi^{(\bullet)}[\bar{\varphi}] - G^\bullet_{\bar{\varphi}}\bar{\phi}^{(\bullet)}_{,\bullet}\Gamma_{(\bullet)}$ [8] and allow us to change basis in $\bar{\phi}^{(\bullet)}_{,\bullet}\Gamma_{(\bullet)} = \Gamma_\bullet$, such that Eq. (3.42) becomes

$$\Gamma^{1\mathrm{PI}}[\bar{\varphi}] = \Gamma - G^\bullet_{\bar{\varphi}}\Gamma_\bullet + J_{(\bullet)}\phi^{(\bullet)}[\bar{\varphi}], \tag{3.44}$$

where $\Gamma = \Gamma[\bar{\varphi}, G^\bullet_{\bar{\varphi}}]$. Using Eq. (3.2) for the first two terms on the right side and setting $J_{(\bullet)} = 0$ (so the last term vanishes), we arrive at the result given in Eq. (3.4).

Generally, the 1PI effective action can be calculated from $\Gamma[\bar{\varphi}, \psi^\bullet_{\bar{\varphi}}]$ by substituting $\bar{\phi}^{(\bullet)}_{\bar{\varphi}} = \bar{\phi}^{(\bullet)}[\bar{\varphi}, \psi^\bullet_{\bar{\varphi}}]$ in Eq. (3.42), where $\psi^\bullet_{\bar{\varphi}}$ is the solution of $\Gamma_\bullet = -\phi^{(\bullet)}_{,\bullet}\gamma^{(\bullet)}_{(\bullet)}J_{(\bullet)}$ (as follows from $\Gamma_\bullet = \phi^{(\bullet)}_{,\bullet}\Gamma_{(\bullet)}$ and $\Gamma_{(\bullet)} = -\gamma^{(\bullet)}_{(\bullet)}J_{(\bullet)}$). In this case, Eq. (3.42) becomes

$$\Gamma^{1\mathrm{PI}}[\bar{\varphi}] = \Gamma[\bar{\varphi}, \psi^\bullet_{\bar{\varphi}}] + J_{(\bullet)}\bar{\phi}^{(\bullet)}[\bar{\varphi}, \psi^\bullet_{\bar{\varphi}}], \tag{3.45}$$

which is an inverse Legendre transform of the composite field effective action $\Gamma[\bar{\varphi}, \psi^\bullet]$ to the 1PI.

Now we can derive the second functional relation, in addition to (3.41), and prove the identities used in the previous section. To this end, consider equation of motion $\Gamma_\bullet = 0$ at $\psi^\bullet = \psi^\bullet_{\bar{\varphi}}$ for $J_{(\bullet)} = 0$. Applying derivative $\frac{\delta}{\delta\bar{\varphi}}$ to both sides of this equation and multiplying from the right by $K^{\bullet|\bullet} = -(\Gamma_{\bullet\bullet})^{-1}$, we get for $J_{(\bullet)} = 0$

$$\psi^\bullet_{\bar{\varphi},1} = \Gamma_{1\bullet}K^{\bullet|\bullet}. \tag{3.46}$$

For the choice $\psi^\bullet = G^\bullet$, Eqs. (3.46) and (3.41) coincide with Eqs. (3.7) and (3.15), respectively, provided we use the structure (3.2) and $G^3_{,(3)} = \delta^3_3$ for a bosonic theory with $\mathcal{S}[\boldsymbol{J}] = 0$. In Appendix A.2, we present an efficient method to express $K^{\bullet|\bullet}$ in terms of connected Green's functions, thereby recovering Eq. (3.20).

The derivation of 1PI effective action from $\Gamma[\bar{\varphi}, \psi^\bullet_{\bar{\varphi}}]$ via Eq. (3.45) will be used in the next chapter 4 to decompose 1PI vertices into contributions whose irreducibility is determined by the choice of composite fields $\psi^\bullet$, and in chapter 5, focusing on $\psi^\bullet = G^\bullet$ with $\bullet = (2, 3)$, to derive a representation of 1PI vertices via Green's functions of composite field operators.

[8] When we move $\gamma^{(\bullet)}_{(\bullet)}\Gamma_{(\bullet)}$ to the right through $G^\bullet_{\bar{\varphi}}\bar{\phi}^{(\bullet)}_{,\bullet}$ another $\gamma^{(\bullet)}_{(\bullet)}$ appears due to the total exchange of contracted $(\bullet)$ indices. The product of two $\gamma^{(\bullet)}_{(\bullet)}$ then gives $\delta^{(\bullet)}_{(\bullet)}$, so we have $\gamma^{(\bullet)}_{(\bullet)}\Gamma_{(\bullet)}G^\bullet_{\bar{\varphi}}\bar{\phi}^{(\bullet)}_{,\bullet} = G^\bullet_{\bar{\varphi}}\bar{\phi}^{(\bullet)}_{,\bullet}\Gamma_{(\bullet)}$

Chapter **4**

Decomposition of 1PI vertices via composite fields

There are two widely used approaches to deriving equations of motion for one-particle-irreducible (1PI) vertices and closing the Schwinger–Dyson (SD) hierarchy. The diagrammatic approach classifies vertex contributions by irreducibility, as in the parquet [2] or single-boson exchange [22] decompositions. The n-particle irreducible (nPI) approach [37] obtains them from the stationarity of the nPI effective action.

These methods can be combined: 1PI vertices can be decomposed using derivatives of the nPI effective action. Eckhardt et al. [31] demonstrated this for the four-point vertex using the 2PI action, and we found analogous results in section 3.1 with the 3PI action. This yields two advantages: (i) vertex components are rigorously defined via effective-action derivatives, and (ii) the framework extends systematically to arbitrary-order vertices and general composite fields $\psi^{\bullet}$, as shown below.

In this chapter, we differentiate Eq. (3.45) and obtain a simple rule for decomposing a general n-point 1PI vertex (see Fig. 4.1): sum over all possible tree diagrams built from vertices I_n and $I_{n|\bullet|\ldots|\bullet}$ ($n > 1$), connected by $K^{\bullet|\bullet}$ as internal lines. I_n and $I_{n|\bullet|\ldots|\bullet}$ are defined through derivatives of $\Gamma[\bar{\varphi}, \psi^{\bullet}]$ (see Eqs. (4.4), (4.6) and (3.46)). Bethe-Salpeter-type equations for these new vertices follow from analogous differentiation of Eqs. (3.46) and (3.41). The parquet and single-boson exchange decompositions are recovered in the first two sections. In the final section, we present an efficient method to calculate composite field effective action and prove its irreducibility using these decompositions.

For simplicity, we assume that $\phi^{(\bullet)}$ and $\psi^{\bullet}$ are bosonic fields, so that $\gamma^{(\bullet)}_{(\bullet)} = \delta^{(\bullet)}_{(\bullet)}$ and $\gamma^{\bullet}_{\bullet} = \delta^{\bullet}_{\bullet}$. For fermionic systems ($\zeta = -1$), we further assume that only even correlation and vertex functions are nonzero.

We begin by differentiating Eq. (3.45) with respect to $\bar{\varphi}^a$:

$$\Gamma^{1\text{PI}}_a = \psi^{\mathbf{a}}_{\bar{\varphi},a}(\Gamma_{\mathbf{a}} + J_{(\bullet)}\bar{\phi}^{(\bullet)}_{,\mathbf{a}}), \tag{4.1}$$

where $\psi^{\mathbf{a}}_{\bar{\varphi}} \equiv \begin{pmatrix}\bar{\varphi} & \psi^{\bullet}_{\bar{\varphi}}\end{pmatrix}$ and we have used $\frac{\delta J_{(\bullet)}}{\delta \bar{\varphi}} = g^{-1}\frac{\delta J_{(\bullet)}}{\delta J_1} = 0$. The matrix $\psi^{\bullet}_{\bar{\varphi},1}$ is related to derivatives of $\Gamma[\bar{\varphi}, \psi^{\bullet}]$ via Eq. (3.46).

For $\mathbf{a} = \bullet$, the bracketed term in Eq.(4.1) vanishes because $\Gamma_{\bullet} = -J_{(\bullet)}\bar{\phi}^{(\bullet)}_{,\bullet}$. Consequently, when differentiating Eq.(4.1) with respect to $\bar{\varphi}^b$, the term proportional to $\psi^{\mathbf{a}}_{,ba} = \psi^{\bullet}_{,ba}\delta^{\mathbf{a}}_{\bullet}$ drops out. We thus obtain:

$$\Gamma^{1\mathrm{PI}}_{ba} = \psi^{\mathbf{b}}_{\bar{\varphi},b}\psi^{\mathbf{a}}_{\bar{\varphi},a}(\Gamma_{\mathbf{ba}} + J_{(\bullet)}\bar{\phi}^{(\bullet)}_{,\mathbf{ba}}), \tag{4.2}$$

where in the fermionic case, we treat $\psi^{\mathbf{a}}_{\bar{\varphi},a}$ and $\psi^{\mathbf{b}}_{\bar{\varphi},b}$ as bosonic because they are nonzero only when the indices **a** and **b** are not composite.

The bracketed term in Eq. (4.2) vanishes whenever one external index is composite and the other is contracted with the matrix $\boldsymbol{\psi}_{\bar{\varphi},1}$. To see this, we differentiate the equation of motion $\Gamma_{\bullet} = -J_{(\bullet)}\bar{\phi}^{(\bullet)}_{,\bullet}$ at $\psi^{\bullet} = \psi^{\bullet}_{\bar{\varphi}}$ with respect to $\bar{\varphi}$ (analogous to the proof of Eq. (3.46)), noting that $\frac{\delta J_{(\bullet)}}{\delta\bar{\varphi}} = 0$. As a result, differentiating Eq. (4.2) with respect to $\bar{\varphi}$ acts only on the bracketed terms (as in Eq. (4.1)), yielding

$$\Gamma^{1\mathrm{PI}}_{a_1\ldots a_n} = I_{a_1\ldots a_n}, \qquad n = 1, 2, 3, \tag{4.3}$$

where $I_{a_1\ldots a_n}$, the *fully irreducible vertex* with respect to the composite fields, is defined as

$$I_{a_1\ldots a_n} = \psi^{\mathbf{a_1}}_{\bar{\varphi},a_1}\ldots\psi^{\mathbf{a_n}}_{\bar{\varphi},a_n}(\Gamma_{\mathbf{a_1\ldots a_n}} + J_{(\bullet)}\bar{\phi}^{(\bullet)}_{,\mathbf{a_1\ldots a_n}}). \tag{4.4}$$

Applying $\frac{\delta}{\delta\bar{\varphi}}$ to Eq. (4.3) for $n = 3$ yields the four-point 1PI vertex:

$$\Gamma^{1\mathrm{PI}}_{4} = I_4 + P_3\psi^{\bullet}_{\bar{\varphi},2}I_{2|\bullet} \tag{4.5}$$

where $I_{n|\bullet|\ldots|\bullet}$ is defined as

$$I_{a_1\ldots a_n|\bullet|\ldots|\bullet} = \psi^{\mathbf{a_1}}_{\bar{\varphi},a_1}\ldots\psi^{\mathbf{a_n}}_{\bar{\varphi},a_n}(\Gamma_{\mathbf{a_1\ldots a_n}\bullet\ldots\bullet} + J_{(\bullet)}\bar{\phi}^{(\bullet)}_{,\mathbf{a_1\ldots a_n}\bullet\ldots\bullet}), \tag{4.6}$$

with the same number of external bullet indices on both sides. We recall that a bullet index $\bullet$ denotes differentiation with respect to $\psi^{\bullet}$. We refer to $I_{n|\bullet}$ simply as the *irreducible vertex* with respect to the composite fields.

To simplify the analysis, we set $J_{(\bullet)} = 0$ (the sources $J_{(\bullet)}$ will be retained only in section 4.3). The four point vertex decomposition, Eq. (4.5), can then be cast into a tree expansion form analogous to Eq. (2.12). For this, we need Bethe-Salpeter-type equation for $I_{2|\bullet}$:

$$\psi^{\bullet}_{\bar{\varphi},2} = I_{2|\bullet}K^{\bullet|\bullet}, \tag{4.7}$$

derived in Appendix B.1 from Eq. (3.46).

When we apply $\frac{\delta}{\delta\bar{\varphi}}$ on $I_{n\ldots}$, the derivative acts on the bracketed term (producing $I_{n+1\ldots}$) and on n factors of $\psi^{\bullet}_{\bar{\varphi},1}$, giving

$$\frac{\delta}{\delta\bar{\varphi}}I_{n\ldots} = I_{n+1\ldots} + P_n\psi^{\bullet}_{\bar{\varphi},2}I_{n-1|\bullet\ldots}. \tag{4.8}$$

Using Eq. (4.7), we obtain

$$\frac{\delta}{\delta\bar{\varphi}}I_{n\ldots} = I_{n+1\ldots} + P_nI_{2|\bullet}K^{\bullet|\bullet}I_{n-1|\bullet|\ldots}. \tag{4.9}$$

Together with $\frac{\delta}{\delta\bar{\varphi}}K^{\bullet|\bullet} = K^{\bullet|\bullet}I_{1|\bullet|\bullet}K^{\bullet|\bullet}$, Eq. (4.9) generates the tree expansion (see Fig. 4.1), in direct analogy to Eq. (2.13).

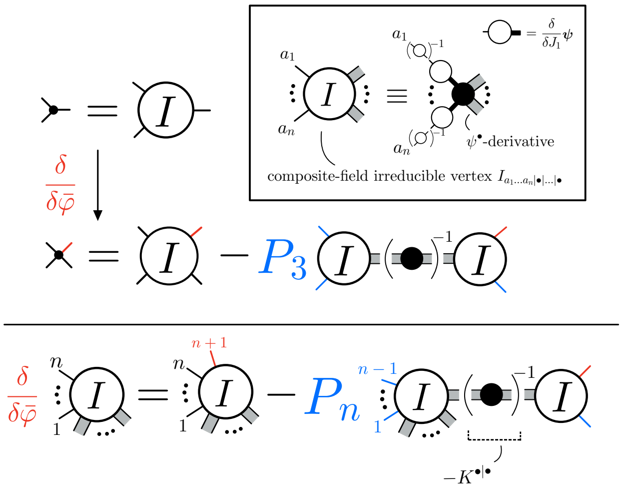

Figure 4.1: Tree expansions of the 1PI vertices $\Gamma_3^{1\text{PI}}$ and $\Gamma_4^{1\text{PI}}$ (see Eqs. (4.3) and (4.10)), using the rule in Eq. (4.9), with internal lines given by $K^{\bullet|\bullet} = -(\Gamma_{\bullet\bullet})^{-1}$. The definition of $I_{a_1...a_n|\bullet|...|\bullet}$ is provided in Eq. (4.6).

Applying the rule (4.9) to $\Gamma_n^{1\text{PI}} = (\frac{\delta}{\delta\bar{\varphi}})^{n-3} I_3$ (see Eq. (4.3)) shows that an n-point 1PI vertex can be expressed as the sum of all tree diagrams constructed from I_n and $I_{n|\bullet|...|\bullet}$, with internal bullet indices connected by $K^{\bullet|\bullet}$.

For $n = 4$, this casts the decomposition (4.5) into the symmetric form

$$\Gamma_4^{1\text{PI}} = I_4 + P_3 I_{2|\bullet} K^{\bullet|\bullet} I_{2|\bullet}. \tag{4.10}$$

For $n = 5$, one finds

$$\Gamma_5^{1\text{PI}} = I_5 + P_{10} I_{3|\bullet} K^{\bullet|\bullet} I_{2|\bullet} + P_{15} I_{2|\bullet} K^{\bullet|\bullet} I_{1|\bullet|\bullet} K^{\bullet|\bullet} I_{2|\bullet}, \tag{4.11}$$

where we have also used $\frac{\delta}{\delta\bar{\varphi}} K^{\bullet|\bullet} = K^{\bullet|\bullet} I_{1|\bullet|\bullet} K^{\bullet|\bullet}$.

As a side note, applying the rule (4.8) to $\psi^\bullet_{\bar{\varphi},n} = (I_{2|\bullet} K^{\bullet|\bullet})_{,n-2}$ (see Eq. (4.7)) yields Bethe-Salpeter-type equations for $I_{n|\bullet}$. For $n = 3$, this gives:

$$\psi^\bullet_{\bar{\varphi},3} = (I_{3|\bullet} + P_3 \psi^\bullet_{\bar{\varphi},2} I_{1|\bullet|\bullet}) K^{\bullet|\bullet}. \tag{4.12}$$

In the following sections, we consider two important choices for $\psi^\bullet$:

1. Connected correlation functions $G^\bullet$: In this case, I_n acquires an interpretation in terms of nPI diagrams, and $K^{\bullet|\bullet}$ becomes a multi-particle Green's function.

2. Bilinear fields local in time: Here, I_n corresponds to the sum of composite-particle–irreducible diagrams, and $K^{\bullet|\bullet}$ becomes the propagator of the composite boson particle.

4.1 1PI vertices via two-particle scattering

It is natural to decompose 1PI vertices either in terms of 1PI vertices themselves or via connected Green's functions. These two choices for composite fields are equivalent, as they are directly related through the tree expansion (see Eqs. (2.7), (2.11) and (2.12)). The functional $\Gamma[\bar{\varphi}, \psi^\bullet]$ is then known as the nPI effective action.

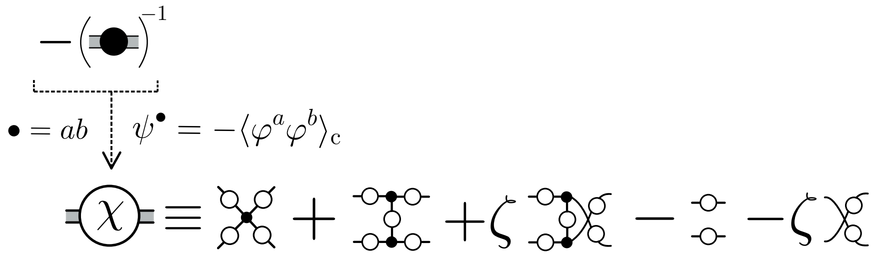

Figure 4.2: Depiction of the two-particle Green's function, also known as the generalized susceptibility for an even theory, as given by Eq. (4.13).

In this section, we consider the simplest choice, $\psi^{\bullet} = g$ and $\phi^{(\bullet)} = \varphi^2$ for $\bullet = 2$, in order to recover the parquet decomposition [2, 31]. We further set $\mathcal{S}[\boldsymbol{J}] = 0$ and $J_{(\bullet)} = 0$.

In the vertex decompositions, each internal line $K^{\bullet|\bullet} = (-\Gamma_{\bullet\bullet})^{-1}$ becomes a two-particle Green's function $\chi^{\bullet|\bullet}$ (see Fig. 4.2):

$$K^{ab|cd} = \chi^{ab|cd} \equiv g^{aa'} g^{bb'} g^{cd}_{\bar{\varphi},a'b'} - g^{ad} g^{bc} - \zeta g^{ac} g^{bd}, \tag{4.13}$$

$$g^{cd}_{\bar{\varphi},ab} = (\Gamma^{1\text{PI}}_{abhk} + \Gamma^{1\text{PI}}_{ahf} g^{fe} \Gamma^{1\text{PI}}_{ebk} + \zeta \Gamma^{1\text{PI}}_{abf} g^{fe} \Gamma^{1\text{PI}}_{ehk}) g^{hc} g^{kd}, \tag{4.14}$$

as previously noted in Eq. (3.20) for an even theory. This follows from Eqs. (3.41) by expressing $G^{(\bullet)(\bullet)}$ and $G^{(\bullet)1}$ in terms of connected correlation functions, using the higher-order analogue of Eq. (2.3). Formula (4.13) is closely related to the linked-cluster decomposition principle [29]. An efficient method for decomposing $K^{\bullet|\bullet}$ is presented in Appendix A.2.

For an even theory, Eqs. (4.14) and (2.12) give

$$g_{\bar{\varphi},2} g^{-2} = \Gamma^{1\text{PI}}_4, \tag{4.15}$$

and $\chi^{ab|cd} = G^{abcd} - g^{ad} g^{bc} - \zeta g^{ac} g^{bd}$ is also known as the generalized susceptibility [7].

When the four-point vertex $\Gamma^{1\text{PI}}_4$ cannot be treated perturbatively, we decompose it using Eq. (4.5):

$$\Gamma^{1\text{PI}}_4 = I_4 + P_3 I_{2|\bullet} g^{\bullet}_{\bar{\varphi},2}. \tag{4.16}$$

The Bethe-Salpeter equation (4.7), together with Eq. (4.13), can be written as

$$g_{\bar{\varphi},2} g^{-2} = 2 I_{2|2} + g^{\bullet}_{\bar{\varphi},2} I_{2|\bullet}. \tag{4.17}$$

Equations (4.16), (4.17) and (4.15) constitute the well-known parquet equations[1]. A less general functional derivation of them was already given by Eckhardt et al. [31] [2].

[1] To bring Eq. (4.17) into the conventional form, one can redefine $I'_{2|2} \equiv 2 I_{2|2}$, giving $g_{\bar{\varphi},2} g^{-2} = I'_{2|2} + \frac{1}{2} g^{\bullet}_{\bar{\varphi},2} I'_{2|\bullet}$.

[2] In [31], the parquet decomposition of the four-point vertex (our Eq. (4.16), their Eq. (33)) was obtained for an even theory and dealing directly with derivatives of the 2PI effective action. In our approach, the new vertices (4.4) and (4.6) allow for extensions to other choices of composite fields and vertices of arbitrary order. Agreement with their results requires $I_{2|2} = g^{-2} + 2\Gamma_{\bullet\bullet}$, which follows by multiplying Eq. (4.13) from the left by $\Gamma_{\bullet\bullet}$ and using $\Gamma_{\bullet\bullet} g_{\bar{\varphi},2} = -I_{2|\bullet}$ from the Bethe-Salpeter equation (4.7).

For an even theory, Eq. (4.4) with $\psi^{\mathbf{a}}_{,a} = \delta^{\mathbf{a}}_a$ yields

$$I_4 = \Gamma_4[\bar{\varphi}, g]. \tag{4.18}$$

Here, I_4 coincides with the fully (two-particle) irreducible vertex [2] of standard parquet theory, whose diagrammatic expansion consists of all two-particle-irreducible diagrams with four external legs and full propagators as internal lines. This also follows directly from Eq. (4.18) together with the well-established fact that the interacting part of $\Gamma[\bar{\varphi}, g]$, the Luttinger-Ward functional, contains only 2PI diagrams.

4.2 1PI vertices via exchange of composite boson particles

Choosing Green's functions as composite fields to decompose 1PI vertices is computationally expensive. Although $I_{n|\bullet}$ can be viewed as $(n+1)$-point vertex, it is of higher order because $\bullet$ contains at least two indices and is therefore nonlocal in time. A way to reduce the order of these vertices is to choose a composite field local in time, while still capturing the non-trivial $\bar{\varphi}$-dependence of the 1PI effective action (as in Eq. (3.4) for Green's functions).

In a theory with quartic interactions only, such a choice can be motivated by the Schwinger-Dyson (SD) equation (3.9), which relates connected Green's functions to $\bar{\varphi}$ (similarly as $G^n_{\bar{\varphi}}$ in Eq. (3.4)). We can introduce the local composite field $\frac{1}{2}S_4\langle\varphi^2\rangle$ to incorporate some of the $\bar{\varphi}$-dependence. Indeed, in Eq. (3.9), the first two terms $(S_2 - \frac{1}{2}S_4 g)$ can be written as $S_2[0] + \frac{1}{2}S_4\langle\varphi^2\rangle$, using Eq. (2.3). The SD equation (3.9) then becomes

$$-g^{-1} = S_2[0] + \frac{1}{2}S_4\langle\varphi^2\rangle + g^{-1}\sum_{n=2}^{3} G^{1n}F_{n,1} + J_{(\bullet)}\phi^{(\bullet)}_{,2}, \tag{4.19}$$

where we have recovered the source term by adding $J_{(\bullet)}\phi^{(\bullet)}_{,2}[\bar{\varphi}]$ to $S_2[\bar{\varphi}]$. In what follows, we define $\frac{1}{2}S_4\langle\varphi^2\rangle$ as a composite field via a Legendre transform with this source term, and then study the decomposition (4.10).

To make the discussion concrete, consider a theory of fermions $f_i(\tau)$ and $f^\dagger_i(\tau)$ with $i = \vec{x}, \sigma$. We take $a = \tau n i$, where $n = f, f^\dagger$ denotes the field type, and set $\varphi^{\tau f i} \equiv f_i(\tau)$ and $\varphi^{\tau f^\dagger i} \equiv f^\dagger_i(\tau)$.

The interacting part of the action is $S_{\text{int}} = \int d\tau L_{\text{int}}$, with

$$L_{\text{int}} = -\frac{1}{4}\sum_{ijkl}\mathcal{U}_{ijkl} f^\dagger_i(\tau) f^\dagger_j(\tau) f_k(\tau) f_l(\tau). \tag{4.20}$$

Because S_{int} is local in time, the field $S_4\varphi^2$ contains a Dirac delta function (which we do not want to include in our composite field). There are three distinct components of $\frac{1}{2}S_4\varphi^2$, corresponding to index types ff, $f^\dagger f^\dagger$ or $f^\dagger f$ (also denoted by pp, $\overline{pp}$ and $\overline{ph}$ respectively). To make this explicit in S_4 while avoiding redundant delta functions, we set $\bullet = \tau i j \mu$ with $\mu = \begin{pmatrix} ff & f^\dagger f^\dagger & f^\dagger f\end{pmatrix}$ and define $U^{\bullet|\bullet}$ as

$$U^{\tau_1 ij\mu|\tau_2 kl\nu} \equiv \delta(\tau_1 - \tau_2)\begin{pmatrix} 0 & \mathcal{U}_{ijkl} & 0 \\ \mathcal{U}_{klij} & 0 & 0 \\ 0 & 0 & 2\mathcal{U}_{ikjl}\end{pmatrix}^{\mu\nu}. \tag{4.21}$$

We then express $\frac{1}{2}S_4\varphi\varphi$ as

$$\frac{1}{2}S_{abcd}\varphi^d\varphi^c = \gamma_{ab|\bullet}\phi^{(\bullet)},$$
$$\phi^{(\bullet)}[\varphi] \equiv \frac{1}{2}\varphi^a\varphi^b\gamma_{ab|\bullet}U^{\bullet|\bullet}, \tag{4.22}$$

where the metric

$$\gamma_{ab|\tau ijnm} \equiv \frac{1}{2}(\delta_a^{\tau in}\delta_b^{\tau jm} + \zeta\delta_b^{\tau in}\delta_a^{\tau jm}) \tag{4.23}$$

ensures proper contraction. For example, for $\mu = ff$ we have a gap field $\phi^{(\tau ij\mu)} = \frac{1}{2}\sum_{kl}\mathcal{U}_{ijkl}f_k(\tau)f_l(\tau)$.

The source term, $J_{(\bullet)}\phi^{(\bullet)}_{,2}[\bar{\varphi}] = \gamma_{2|\bullet}U^{\bullet|\bullet}J_{(\bullet)}$, in the Schwinger-Dyson equation allows us to rigorously introduce $\psi^\bullet$, the variable of effective action, in place of $\frac{1}{2}S_4\langle\varphi^2\rangle$. We define

$$\psi^\bullet \equiv \frac{1}{2}\langle\varphi^2\rangle\gamma_{2|\bullet}U^{\bullet|\bullet} - J_{(\bullet)}U^{\bullet|\bullet}, \tag{4.24}$$

so $\frac{1}{2}S_4\langle\varphi^2\rangle + J_{(\bullet)}\phi^{(\bullet)}_{,2} = \gamma_{2|\bullet}\psi^\bullet$ in the SD equation, which becomes

$$-g_{\boldsymbol{\psi}}^{-1} = S_2[0] - \gamma_{2|\bullet}\psi^\bullet + g_{\boldsymbol{\psi}}^{-1}\sum_{n=2}^{3} G_{\boldsymbol{\psi}}^{1n}F_{n,1}. \tag{4.25}$$

Together with the tree expansions (2.11) and (2.12), and the decompositions (4.3) and (4.10), Eq. (4.25) defines the functionals $g_{\boldsymbol{\psi}}$ and $G^n_{\boldsymbol{\psi}}$ for $n = 2, 3$.

To obtain the simple relation $\psi^\bullet = \bar{\phi}^{(\bullet)}$, we choose

$$\mathcal{S}[\boldsymbol{J}] = -\frac{1}{2}J_{(\bullet)}U^{\bullet|\bullet}J_{(\bullet)}. \tag{4.26}$$

The right side of Eq. (4.24) then equals to $\bar{\phi}^{(\bullet)} = \frac{\delta W}{\delta J_{(\bullet)}}$. This way, we achieve Hubbard-Stratonovich (HS) bosonization. In the next section, we calculate the effective action and show this explicitly. For now, we focus on the decomposition (4.10).

We have for the internal line, using Eqs. (3.41) (3.31) and $\bar{\phi}^{(\bullet)}_{,\bullet} = \delta^\bullet_\bullet$,

$$K^{\bullet|\bullet} = \frac{1}{4}U^{\bullet|\bullet}\gamma_{ab|\bullet}\chi^{ab|cd}\gamma_{cd|\bullet}U^{\bullet|\bullet} - U^{\bullet|\bullet}, \tag{4.27}$$

where $\chi^{2|2} = -\langle\varphi^2\varphi^2\rangle_c - \langle\varphi^2\varphi\rangle_c g^{-1}\langle\varphi\varphi^2\rangle_c$ (see Eq. (4.13)).

$K^{\bullet|\bullet}$ is interpreted as the propagator of the composite field $\phi^{(\bullet)}$ [3]. It is an internal line for the reducible parts of the vertex. Removing a bare vertex $U^{\bullet|\bullet}$ disconnects such contributions, implying that, besides the reducible parts $\Gamma_{1\bullet}K^{\bullet|\bullet}$, the vertex I_n contains only U-irreducible diagrams (those that remain connected even if a bare vertex is removed). A complete proof is given in the next section.

[3] For an even theory, $K^{\bullet|\bullet} = \bar{\phi}^{(\bullet)}\bar{\phi}^{(\bullet)} - \langle\phi^{(\bullet)}\phi^{(\bullet)}\rangle$ is analogous to $g = \bar{\varphi}\bar{\varphi} - \langle\varphi\varphi\rangle$.

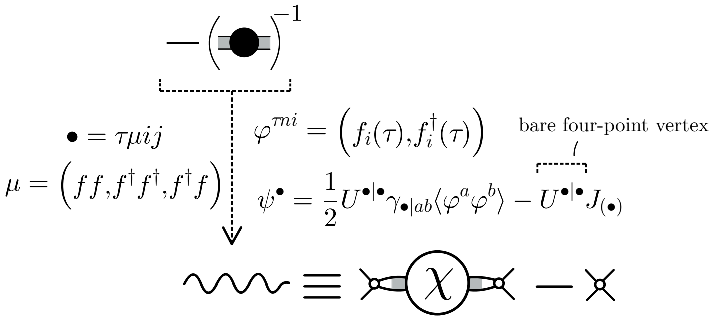

Figure 4.3: Depiction of the propagator of a composite boson particle, as given by Eq. (4.27).

To obtain an equation for $I_{2|\bullet}$ from Eq. (4.7), we first compute

$$\psi^\bullet_{\bar\varphi,2} = \frac{1}{2}(\bar\varphi^2 - g_{\bar\varphi})_{,2}\gamma_{2|\bullet}U^{\bullet|\bullet} = -\frac{1}{2}g^{-2}\chi^{2|2}\gamma_{2|\bullet}U^{\bullet|\bullet}, \tag{4.28}$$

using definition (4.24) and $\frac{\delta}{\delta J_1}J_{(\bullet)} = 0$. Combining Eqs. (4.7), (4.27) and (4.28) (see Appendix B.2) yields

$$I_{ab|\bullet} = -\gamma_{ab|\bullet} + \frac{1}{2}(g^{cd}_{\bar\varphi,ab} - I_{ab|\bullet}K^{\bullet|\bullet}I_{c'd'|\bullet}g^{c'c}g^{d'd})\gamma_{cd|\bullet}. \tag{4.29}$$

This shows the partial irreducibility of $I_{ab|\bullet}$: removing a bare vertex cannot disconnect the two external indices a, b from the composite index $\bullet$.

Equations (4.10), (4.27) and (4.29) correspond to the single-boson exchange (SBE) equations (8), (6) and (B3) of [22], respectively[4].

4.3 Calculation of the effective action

While rules for the diagrammatic content of the Luttinger-Ward functional in terms of the full propagator g are well established, the corresponding rules for $\Gamma[\bar\varphi, \psi^\bullet]$ for a general $\psi^\bullet$ are less straightforward. For the choice $\psi^\bullet = G^\bullet$, the nPI community has made considerable efforts to derive loop expansions of the effective action using successive Legendre transformations [1, 37].

In our approach, we obtain the same result by relating the Schwinger-Dyson (SD) equations and the decompositions for 1PI vertices to the equations of motion $J_{(\bullet)} = -\Gamma_{(\bullet)}$.

To illustrate the idea, let us use the 1PI effective action. The SD equation (3.8) already has the form $\Gamma^{1\mathrm{PI}}_1 = -\zeta J_1$. It is fully expressed in terms of derivatives of $\Gamma^{1\mathrm{PI}}$, if we use $g_{\bar\varphi} = (-\zeta\Gamma^{1\mathrm{PI}}_2)^{-1}$ and $G^3_{\bar\varphi} = g^3_{\bar\varphi}\Gamma^{1\mathrm{PI}}_3$. Iterating the SD equation (3.8), starting from $\Gamma^{1\mathrm{PI}} \approx S$, yields $\Gamma^{1\mathrm{PI}}$ as a sum of only 1PI diagrams [28].

To construct the composite-field effective action in an analogous way, we must recover the sources $J_{(\bullet)}$ in the SD equations and substitute $J_{(\bullet)} = -\Gamma_\bullet\psi^\bullet_{,(\bullet)}$. This can be done by applying the simple replacement rule

$$S_n[\bar\varphi] \to S_n[\bar\varphi] + J_{(\bullet)}\phi^{(\bullet)}_{,n}[\bar\varphi]. \tag{4.30}$$

[4] Our I_4, $K^{\bullet|\bullet}$ and $I_{2|\bullet}$ correspond to $\varphi^{\mathrm{firr},\alpha} - 2U^\alpha$, ω^α and λ^α of [22], respectively.

In what follows, we use this procedure to calculate the effective actions employed in Secs. 4.1 and 4.2.

Two-particle irreducibility

To calculate the 2PI effective action $\Gamma[\bar{\varphi}, g]$, we set $\mathcal{S}[\boldsymbol{J}] = 0$ and choose $\psi^\bullet = g$ with $\phi^{(2)} = \varphi^2$. The sources in the SD equation (3.9) are recovered via the replacement $S_2 \to S_2 + J_{(\bullet)}\phi^{(\bullet)}_{,2}$.

We obtain

$$\Gamma_\bullet = -\frac{1}{2}S_2 - \frac{1}{2}g^{-1} + \frac{1}{4}gS_4 - \frac{1}{2}\sum_{m=2}^{3} g^{-1}G^{1m}_{\psi}F_{m,1}, \tag{4.31}$$

using $\phi^{(2)}_{,2} = 2\delta^2_2$ and $J_{(\bullet)} = \Gamma_\bullet$, which follows from $J_{(\bullet)} = -\Gamma_{(\bullet)} = -\Gamma_\bullet \psi^\bullet_{,(\bullet)}$ and $\psi^\bullet_{,(\bullet)} = -\delta^\bullet_\bullet$. The functionals G^n_ψ are defined via the tree expansions (2.11) and (2.12), and the decompositions (4.3) and (4.10). To close the system, we still need an expression for Γ_1 from the SD equation (3.8).

For a bosonic ($\zeta = 1$) theory with only cubic interactions, this last step is unnecessary: the $\bar{\varphi}$-dependence is trivial due to the structure (3.2). In this case $\Gamma_2 = S_2$, $\Gamma_{1\bullet} = -\frac{1}{2}S_3$ and $\Gamma_\bullet = -\frac{1}{2}S_2 + \Lambda_\bullet$, so Eq. (4.31) reduces to

$$\Lambda_\bullet = -\frac{1}{2}g^{-1} + \frac{1}{4}S_3 gg\Gamma^{1\mathrm{PI}}_3, \tag{4.32}$$

where $\Gamma^{1\mathrm{PI}}_3$ as a functional of g follows from the decomposition (4.3) for $n = 3$. Using $\bar{\phi}^{(\bullet)}_{,\mathbf{abc}} = 0$ and the structure (3.2), one finds

$$\Gamma^{1\mathrm{PI}}_3 = S_3 + \Lambda_{\bullet\bullet\bullet}(gg\Gamma^{1\mathrm{PI}}_3)^3. \tag{4.33}$$

Equations (4.32) and (4.33) non-perturbatively define $\Lambda[g]$ through a functional integro-differential equation depending on g and S_3. While these equations cannot be solved exactly, they admit an iterative scheme generating the perturbative expansion of $\Lambda[g]$. Starting with $\Lambda \approx -\frac{1}{2}\ln\det g$ and $\Gamma^{1\mathrm{PI}}_3 \approx S_3$, the first iteration yields $\Lambda \approx -\frac{1}{2}\ln\det g + \frac{1}{12}S_3 g^3 S_3$ and $\Gamma^{1\mathrm{PI}}_3 \approx S_3 - (S_3 g)^3$. The next iteration gives the three-loop approximation

$$\Lambda \approx -\frac{1}{2}\ln\det g + \frac{1}{12}S_3 g^3 S_3 + \frac{1}{24}(S_3)^4 g^6. \tag{4.34}$$

This reproduces the well-known perturbative expansion of the Luttinger-Ward functional [38].

U-irreducibility

Let us return to the fermionic system introduced in Sec. 4.2. We set $\mathcal{S}[\boldsymbol{J}] = -\frac{1}{2}J_{(\bullet)}U^{\bullet|\bullet}J_{(\bullet)}$ and choose $\psi^\bullet = \bar{\phi}^{(\bullet)}$, where $\phi^{(\bullet)}[\varphi]$ is defined in Eq. (4.22).

Multiplying Eq. (4.24) by $(U^{\bullet|\bullet})^{-1}$ and using $J_{(\bullet)} = -\Gamma_\bullet$, we obtain $(U^{\bullet|\bullet})^{-1}\psi^\bullet = \frac{1}{2}\langle\varphi^2\rangle\gamma_{2|\bullet} + \Gamma_\bullet$. With $\langle\varphi^2\rangle = \bar{\varphi}^2 - g$, this becomes

$$\Gamma_\bullet = (U^{\bullet|\bullet})^{-1}\psi^\bullet - \frac{1}{2}\bar{\varphi}^2\gamma_{2|\bullet} + \frac{1}{2}g_\psi\gamma_{2|\bullet}, \tag{4.35}$$

where g_{ψ} is defined in Eq. (4.25) using the tree expansions for the Green's functions (2.11) and (2.12), together with the 1PI vertices (4.3) and (4.10). Note that $\psi^{\bullet}_{\bar{\varphi},1} = \Gamma_{1\bullet} K^{\bullet|\bullet}$ remains valid even for nonzero sources $J_{(\bullet)}$ (recall the derivation of Eq. (4.3)).

To determine the $\bar{\varphi}$-dependence of $\Gamma[\bar{\varphi}, \psi^{\bullet}]$, we recall the equations of motion $\Gamma_a = -J_{\mathbf{a}} \bar{\phi}^{\mathbf{a}}_{,a} = -J_a \zeta$ and $-\zeta J_1 = \Gamma_1^{\mathrm{1PI}}$, which imply $\Gamma_1 = \Gamma_1^{\mathrm{1PI}}$. We recover the source in the SD equation (3.8) by adding $J_{(\bullet)} \phi^{(\bullet)}_{,1} = (\frac{1}{2}\langle \varphi^2 \rangle S_4 - \gamma_{2|\bullet} \psi^{\bullet}) \bar{\varphi}$. This leads to

$$\Gamma_1 = S_1 - \frac{1}{2} \bar{\varphi}^3 S_4 - \bar{\varphi} \gamma_{2|\bullet} \psi^{\bullet} + \frac{1}{6} S_4 G^3_{\psi}, \tag{4.36}$$

where $G^3_{\psi} = (g_{\psi})^3 \Gamma_3^{\mathrm{1PI}}$.

To compute $\Gamma[\bar{\varphi}, \psi^{\bullet}]$, we iterate Eqs. (4.25), (4.35) and (4.36) using the decompositions for the first two 1PI vertices. In the first-order approximation, $g_{\psi} \approx (S_2[0] + \gamma_{2|\bullet} \psi^{\bullet})^{-1}$ and $\Gamma_1 \approx S_1 - \frac{1}{2} \bar{\varphi}^3 S_4 - \bar{\varphi} \gamma_{2|\bullet} \psi^{\bullet}$. This, together with Eq. (4.35), yields

$$\Gamma[\bar{\varphi}, \psi^{\bullet}] \approx S - 3S_{\mathrm{int}} - \frac{1}{2} \bar{\varphi}^2 \gamma_{2|\bullet} \psi^{\bullet} + \frac{1}{2} \psi^{\bullet} (U^{\bullet|\bullet})^{-1} \psi^{\bullet} + \frac{1}{2} \ln \det \big(S_2[0] + \gamma_{2|\bullet} \psi^{\bullet} \big), \tag{4.37}$$

which is the one-loop 1PI effective action for the theory bosonized via the Hubbard-Stratonovich transformation [39].

Now let us show that $\Gamma[\bar{\varphi}, \psi^{\bullet}]$, when expanded in powers of $\bar{\varphi}$, contains only diagrams that cannot be disconnected by removing a bare vertex $U^{\bullet|\bullet}$. To this end, consider Eqs. (4.35), (4.36), (4.27) and (4.29).

Assume that Γ at some iteration is given by a sum of U-irreducible diagrams with external legs contracted by $\bar{\varphi}$. This implies that I_n and $I_{n|\bullet}$ can be disconnected into two parts by removing a bare interaction, provided that each part contains at least one external index from n [5]. When $g_{\psi} = (I_2)^{-1}$ is expanded in powers of $\bar{\varphi}$, this property is preserved: g_{ψ} can still be disconnected (by removing a bare vertex) into two parts, each containing one external index. In the next approximation for Γ obtained from Eqs. (4.35) and (4.36), new contributions arise from $\int g_{\psi} \gamma_{2|\bullet} d\psi^{\bullet}$ and $\int d\bar{\varphi} S_4 (g_{\psi})^3 I_3$. From the properties of g_{ψ} and I_3 stated above, it follows that these contributions are U-irreducible [6]. Since the lowest-order approximation of Γ, Eq. (4.37), is U-irreducible, it follows by induction that all higher-order corrections preserve this property. Therefore, Γ is given by a sum of U-irreducible diagrams only.

[5] For $I_{2|\bullet}$ this follows directly from Eqs. (4.29) and (4.27). For I_n, the result follows from the fact that $\Gamma_{1\bullet} K^{\bullet|\bullet}$ is a sum of connected but U-reducible diagrams. For I_2 this agrees with Eq. (4.25).

[6] The key point is that both g_{ψ} and $(g_{\psi})^3 I_3$ disconnect, after removing bare vertex, into two parts each containing an external index. Contraction of their external indices with a bare vertex makes them U-irreducible.

Chapter 5

1PI vertices via Green's functions of composite field operators

The standard way to nonperturbatively compute the 1PI vertices is to calculate them in terms of the amputated correlation functions using the tree expansion (see Eqs. (2.11)-(2.13)). When one computes a 1PI vertex in this manner, the uncertainty in the inversion of g arising from non-exact numerical data contributes for each external index. To reduce the effect of the amputations, one may instead use an alternative formula (called estimator) for the 1PI vertex expressed solely in terms of the correlation functions and the self-energy [24–26]. At the end of this chapter, we present a simple rule (Fig. 5.2) to get such an estimator from the expression of the 1PI vertex in terms of the amputated correlation functions.

For theories with only cubic interactions, such as QED, this result is well-known[1]: when computing the connected part of the scattering matrix involving photons, one can use a current operator in place of the product of the inverse bare propagator (with an external index) and a photon field operator in the connected correlation function. For theories with quartic interactions involving fermions, the corresponding result is known as the symmetric improved estimators [26] (shown in Fig. 5.2 for the first two vertices).

To express the 1PI vertex in terms of Green's functions of composite field operators, we use

$$\Gamma_n^{\mathrm{1PI}} = S_n + (\frac{\delta}{\delta\psi} + F_{\bullet,1} K^{\bullet|\bullet} \frac{\delta}{\delta G^{\bullet}})^{n-1} F_{\bullet,1} \gamma_{\bullet}^{\bullet} G^{\bullet}, \tag{5.1}$$

which is the formula (3.13) written in condensed form and generalized to fermions. It holds for a theory with cubic and quartic interactions. To simplify the derivations, we assume that for fermionic systems ($\zeta = -1$) only even correlation and vertex functions are nonzero.

The right side of Eq. (5.1) can be evaluated as follows. First, compute all derivatives $\frac{\delta}{\delta\psi}$, which act only on $F_{\bullet,1}$ containing the external index (since $\frac{\delta}{\delta\psi}\Gamma_{\bullet\bullet\ldots} = \frac{\delta}{\delta\psi}\Lambda_{\bullet\bullet\ldots} = 0$). The remaining derivatives are of the form $F_{\bullet,n} K^{\bullet|\bullet} \frac{\delta}{\delta G^{\bullet}}$. Analogously

[1] It can be found in many QFT textbooks; for example, see Chapter 14.8.3 in [40].

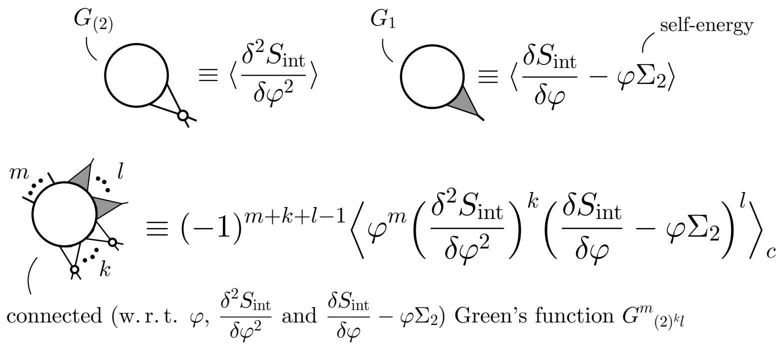

Figure 5.1: Definitions of composite field Green's functions as given by Eqs. (5.3), (5.7) and (5.8)

to $g\frac{\delta}{\delta\bar\varphi} = \frac{\delta}{\delta J_1}$, we have from Eq. (3.41)

$$K^{\bullet|\bullet}\frac{\delta}{\delta G^\bullet} = G^\bullet_{,(\bullet)}\Big(\frac{\delta\bar\phi^{(\bullet)}}{\delta J_{(\bullet)}} - G^{(\bullet)1}g^{-1}\frac{\delta\bar\phi^{(\bullet)}}{\delta J_1}\Big)\frac{\delta}{\delta\bar\phi^{(\bullet)}}$$
$$= G^\bullet_{,(\bullet)}\Big(\frac{\delta}{\delta J_{(\bullet)}} - G^{(\bullet)1}g^{-1}\frac{\delta}{\delta J_1}\Big), \tag{5.2}$$

where we added a zero term $G^\bullet_{,(\bullet)}(\frac{\delta\bar\varphi}{\delta J_{(\bullet)}} - G^{(\bullet)1}g^{-1}\frac{\delta\bar\varphi}{\delta J_1})\frac{\delta}{\delta\bar\varphi}$ to the expression and used $\frac{\delta\bar\phi^{\mathbf{a}}}{\delta J_{\mathbf{b}}}\frac{\delta}{\delta\bar\phi^{\mathbf{a}}} = \frac{\delta}{\delta J_{\mathbf{b}}}$ (recall that $\phi^a = \varphi^a$).

Contracting (5.2) with $F_{\bullet,n}$ naturally introduces the following composite field operators and their corresponding Green's functions.

The first one is $\frac{\delta}{\delta\varphi^{a_1}}...\frac{\delta}{\delta\varphi^{a_n}}S_{\rm int}$, where $S_{\rm int}[\varphi]$ is the interacting part of the action $S[\varphi]$. This operator corresponds to the lower index $(a_1...a_n)$ (or just (n) if condensed) in the correlation function

$$G_{(a_1...a_n)} \equiv F_{\bullet,a_1...a_n}G^\bullet_{,(\bullet)}\frac{\delta}{\delta J_{(\bullet)}}W = \langle S_{{\rm int},a_1...a_n}[\varphi]\rangle, \tag{5.3}$$

where we evaluated $F_{\bullet,a_1...a_n}G^\bullet_{,(b_1...b_m)} = \frac{1}{m!}S_{a_1...a_n b_m...b_1}[0]$, using Eqs. (3.3), (A.1) and (A.2). In a theory with cubic and quartic interactions, $G_{(n)}$ vanishes for $n > 2$.

We extend the definition of the Green's function (2.1) to include the lower index (n), with the order of indices matching the order of operators in the expectation value. For example, $G^a{}_{(b)} = \zeta G_{(b)}{}^a = -\langle\varphi^a\frac{\delta S_{\rm int}}{\delta\varphi^b}\rangle_c$. Using this, the SD equation (3.9) can be rewritten as

$$\Sigma_2 = G_{(1)}{}^1 g^{-1} = -\langle\frac{\delta S_{\rm int}}{\delta\varphi}\varphi\rangle_c g^{-1}, \tag{5.4}$$

where the self-energy is defined as $\Sigma_2 \equiv -g^{-1} - \zeta S_2[0]$. This formula is known as the asymmetric estimator [24]; and for completeness we derive it in Appendix C.

When contracted with $F_{\bullet,1}$, the derivative (5.2) becomes

$$\frac{\delta}{\delta J^1} \equiv F_{\bullet,1}G^\bullet_{,(\bullet)}\Big(\frac{\delta}{\delta J_{(\bullet)}} - G^{(\bullet)1}g^{-1}\frac{\delta}{\delta J_1}\Big) \tag{5.5}$$
$$= F_{\bullet,1}G^\bullet_{,(\bullet)}\frac{\delta}{\delta J_{(\bullet)}} - \Sigma_2\frac{\delta}{\delta J_1}, \tag{5.6}$$

where we used Eq. (5.4). The first term on the right side generates the operator $\frac{\delta S_{\rm int}}{\delta\varphi}$ when applied to W. This motivates the introduction of the second composite field operator $(\frac{\delta S_{\rm int}}{\delta\varphi^a} - \Sigma_{ab}\varphi^b)$, which corresponds to the lower index a in the correlation function

$$G_a \equiv \frac{\delta}{\delta J^a}W = \langle\frac{\delta S_{\rm int}}{\delta\varphi^a} - \Sigma_{ab}\varphi^b\rangle. \tag{5.7}$$

We further extend the Green's function definition (2.1) to include the lower indices (n) and 1, with the order of indices again corresponding to the operator ordering in the expectation value. General connected (w.r.t. φ, $\frac{\delta^2 S_{\rm int}}{\delta\varphi^2}$ and $\frac{\delta S_{\rm int}}{\delta\varphi} - \varphi\Sigma_2$) Green's function $G^m{}_{(2)^k l}$ is

$$G^m{}_{(2)^k l} \equiv (-1)^{m+k+l-1}\Big\langle\varphi^m\Big(\frac{\delta^2 S_{\rm int}}{\delta\varphi^2}\Big)^k\Big(\frac{\delta S_{\rm int}}{\delta\varphi} - \varphi\Sigma_2\Big)^l\Big\rangle_c. \tag{5.8}$$

Diagrammatic definitions are summarized in Fig. 5.1.

For example, using Eq. (5.4), we get

$$G_2 = -\langle\frac{\delta S_{\rm int}}{\delta\varphi}\frac{\delta S_{\rm int}}{\delta\varphi}\rangle_c - \langle\frac{\delta S_{\rm int}}{\delta\varphi}\varphi\rangle_c g^{-1}\langle\varphi\frac{\delta S_{\rm int}}{\delta\varphi}\rangle_c. \tag{5.9}$$

This coincides with the right side of Eq. (3.16). The symmetric estimator for the self-energy can then be written as (see Appendix C)

$$\zeta\Sigma_{ab} = S_{abc}[0]\bar{\varphi}^c + G_{(ab)} + G_{ab}. \tag{5.10}$$

In what follows, we present two approaches for obtaining estimators for 1PI vertices. In section 5.1, following the idea outlined above, we first evaluate the ψ-derivatives in Eq. (5.1), yielding the asymptotic class decomposition. We then derive simple rules to express each class in terms of Green's functions of composite field operators. In section 5.2, we show how to get estimators for 1PI vertices directly from their representation in terms of amputated Green's functions, as illustrated in Fig. 5.2.

5.1 Estimators for asymptotic classes

Generalizing Eqs. (3.22) and (3.26), the 1PI vertices can be parametrized as

$$\Gamma^{\rm 1PI}_{m>2} = S_m + \sum_{k=0}^{\lfloor m/2\rfloor} P_{c^m_k}\mathcal{K}^{m-k-1}_m, \tag{5.11}$$

where $\mathcal{K}^n_m$ is an asymptotic class depending on n external frequencies and momenta, assuming frequency- and momentum- independent bare interactions $S_3[0]$ and $S_4[0]$. The coefficient $c^m_k = \frac{m!}{2^k k!(m-2k)!}$ counts the number of ways to choose k unordered pairs from n indices, and $\lfloor x\rfloor$ denotes the greatest integer less than x. We have

$$\mathcal{K}^{m-k-1}_m = (F_{\bullet,2})^k(F_{\bullet,1})^{m-2k}K^{\bullet^{m-k}} \tag{5.12}$$

$$\begin{aligned} K^{\bullet^n} &= (K^{\bullet|\bullet}\frac{\delta}{\delta G^\bullet})^n\Omega[G^\bullet] \\ &= (K^{\bullet|\bullet}\frac{\delta}{\delta G^\bullet})^{n-2}K^{\bullet|\bullet}\gamma^\bullet_\bullet, \end{aligned} \tag{5.13}$$

where the order of external (composite) indices is the same on both sides of Eqs. (5.12)-(5.13); and the order of bullet indices in $K^{\bullet^{m-k}}$ is the same as order of products of $F_{\bullet,n}$. For example, $\mathcal{K}^1_{abcd} = \sum_{n,m=2}^{3} F_{e_1\ldots e_n,ab} F_{f_1\ldots f_m,cd} K^{e_1\ldots e_n|f_1\ldots f_m} \zeta^m$.

Eq. (5.13) has a structure analogous to $G^n_{\bar\varphi} = (g_{\bar\varphi}\frac{\delta}{\delta\bar\varphi})^{n-2} g_{\bar\varphi}$ (see Eq. (2.9)). Thus, $K^{\bullet^n}$ admits its own tree expansion. For example, if all external bullet indices are bosonic ($\bullet = 2$), so that no factors of ζ appear, the first two expansions are

$$K^{\bullet^3} = (K^{\bullet|\bullet})^3 \Gamma_{\bullet^3} \tag{5.14}$$

$$K^{\bullet^4} = (K^{\bullet|\bullet})^4 \Gamma_{\bullet^4} + P_3 (K^{\bullet|\bullet})^2 \Gamma_{\bullet^3} K^{\bullet|\bullet} \Gamma_{\bullet^3} (K^{\bullet|\bullet})^2. \tag{5.15}$$

To express $\mathcal{K}^m_n$ in terms of Green's functions of composite field operators, we start from its definition (5.12) and use Eq. (5.2) to calculate $K^{\bullet^m}$:

$$K^{\bullet^m} = (G^{\bullet}_{,(\bullet)})^{m-2} (\frac{\delta}{\delta J_{(\bullet)}} - G^{(\bullet)1} g^{-1} \frac{\delta}{\delta J_1})^{m-2} K^{\bullet|\bullet} \gamma^{\bullet}_{\bullet}. \tag{5.16}$$

To evaluate derivatives in (5.16), we will derive below a simple rule (5.22) that generates the tree expansion, in analogy with Eq. (2.13). It is convenient to focus on

$$\mathcal{K}^{n-1}_n = (F_{\bullet,1})^n K^{\bullet^n}, \tag{5.17}$$

from which $K^{\bullet^n}$ can be obtained by removing $F_{\bullet,1}$ for each external index (this step will be explained shortly).

Using Eqs. (5.16) and (5.5), we find

$$\mathcal{K}^{n-1}_n = (\frac{\delta}{\delta J^1})^{n-2} K^{\bullet|\bullet} (F_{\bullet,1})^2. \tag{5.18}$$

From Eq. (3.40) and (5.18) for $n = 2$, we obtain

$$\mathcal{K}^1_2 = K^{\bullet|\bullet} (F_{\bullet,1})^2 = G_{1(1)}. \tag{5.19}$$

Using $G^1_1 = \frac{\delta\bar\varphi}{\delta J^1} = 0$, we find $G_2 = G_{1(1)}$, hence $\mathcal{K}^1_2 = G_2$. Equation (5.18) then reduces to

$$\mathcal{K}^{n-1}_n = (\frac{\delta}{\delta J^1})^{n-2} G_2. \tag{5.20}$$

Since $G^1_1 = 0$, we have $\frac{\delta}{\delta J^1} G_2 = G_3$, and therefore

$$\mathcal{K}^2_3 = G_3. \tag{5.21}$$

We obtain the rule for $\frac{\delta}{\delta J^1} G^{\cdots}_n$:

$$\frac{\delta}{\delta J^1} {G_n}^m = {G_{n+1}}^m - P_n \frac{\delta}{\delta J^1} ({G_{(1)}}^1 g^{-1}) {G^1}_{n-1}{}^m = {G_{n+1}}^m - P_n {G_2}^1 g^{-1} {G^1}_{n-1}{}^m, \tag{5.22}$$

where only lower external indices are permuted by P_n and we used

$$\frac{\delta}{\delta J^1} (F_{\bullet,1} G^{\bullet}_{,(\bullet)}) = 0. \tag{5.23}$$

Noticing that the recursive relation (5.22) is effectively the same as Eq. (2.13), we conclude that $\mathcal{K}_n^{n-1}$ is given by the sum of all tree diagrams constructed from ${G^m}_l$, where internal upper indices contracted using $(-g^{-1})$.

For example, for $\mathcal{K}_3^4$ one finds:

$$\mathcal{K}_4^3 = G_4 - P_3 {G_2}^1 g^{-1} {G^1}_2. \tag{5.24}$$

From the rule (5.22), it follows that each external index in $\mathcal{K}_n^{n-1}$ will appear in a Green's function of the form ${G^{\cdots}}_{\ldots 1}$, or more precisely in $F_{\bullet,1}$ (see Eqs. (5.7), (5.4) and (5.3)). To obtain $K^{\bullet^n}$ from $\mathcal{K}_n^{n-1}$, we remove one contraction $F_{\bullet,1}$ for each external index.

For example, from $\mathcal{K}_2^1 = G_2 = K^{\bullet|\bullet}(F_{\bullet,1})^2$ we remove two contractions $F_{\bullet,1}$, recovering $K^{\bullet|\bullet}$.

The general asymptotic class $\mathcal{K}_m^{n-1}$ can be obtained from $K^{\bullet^m}$ (see definition (5.12)), or more simply from $\mathcal{K}_m^{m-1}$ by making the necessary replacements $F_{\bullet,1} \to F_{\bullet,2}$. For each such replacement in ${G_1}^{\cdots}$, formulas (5.3)-(5.7) imply

$${G_1}^{\cdots} \to {G_{(2)}}^{\cdots} - {G_{(2)}}^1 g^{-1} G^{1\cdots}. \tag{5.25}$$

For example, starting from $\mathcal{K}_2^1 = G_2 = K^{\bullet|\bullet}(F_{\bullet,1})^2$, replacing one $F_{\bullet,1} \to F_{\bullet,2}$ (equivalently, applying (5.25) to one lower index) yields:

$$G_2 \to G_{(2)1} - {G_{(2)}}^1 g^{-1} {G^1}_1 = G_{(2)1}, \tag{5.26}$$

where we used $G_1^1 = 0$. Thus

$$\mathcal{K}_3^1 = G_{(2)1}. \tag{5.27}$$

Similarly:

$$\mathcal{K}_4^1 = G_{(2)(2)} - {G_{(2)}}^1 g^{-1} {G^1}_{(2)}, \tag{5.28}$$

$$\mathcal{K}_4^2 = G_{(2)2} - {G_{(2)}}^1 g^{-1} {G^1}_2. \tag{5.29}$$

To obtain the Green's function representation of the 1PI vertices in terms of composite field operators, we substitute these results for asymptotic classes into the decomposition (5.11).

From $\Gamma_3^{\text{1PI}} = S_3 + P_3\mathcal{K}_3^1 + \mathcal{K}_3^2$, we find

$$\Gamma_3^{\text{1PI}} = S_3 + L_3, \tag{5.30}$$

where $L_3 \equiv P_3 G_{(2)1} + G_3$.

From Eq. (3.26) we obtain:

$$\Gamma_4^{\text{1PI}} = S_4 + L_4 - P_3 {L_2}^1 g^{-1} {L^1}_2, \tag{5.31}$$

where $L_4 = G_4 + P_6 G_{(2)2} + P_3 G_{(2)(2)}$ and ${L^1}_2 = {G^1}_2 + {G^1}_{(2)}$. For the even theory $({L^1}_2 = 0)$ with only quartic interactions, this result reduces to Eq. (131) of Lihm [26][2].

Estimator (5.31) has the tree expansion form as in Eq. (5.24) or, equivalently, for Γ_n^{1PI}, when expressed in terms of amputated correlation functions (see Fig. 5.2) via Eqs. (2.11)-(2.12). We will prove this result for the general vertex in the next section.

[2]Specifically, we choose fermionic system of section 4.2. The operators $\frac{\delta S_{\text{int}}}{\delta f_i}$ and $\frac{\delta S_{\text{int}}}{\delta f_i^\dagger}$ then correspond to $q_\sigma = [d_\sigma, H_{\text{int}}]$ and $(-q_\sigma^\dagger)$ of [26], respectively.

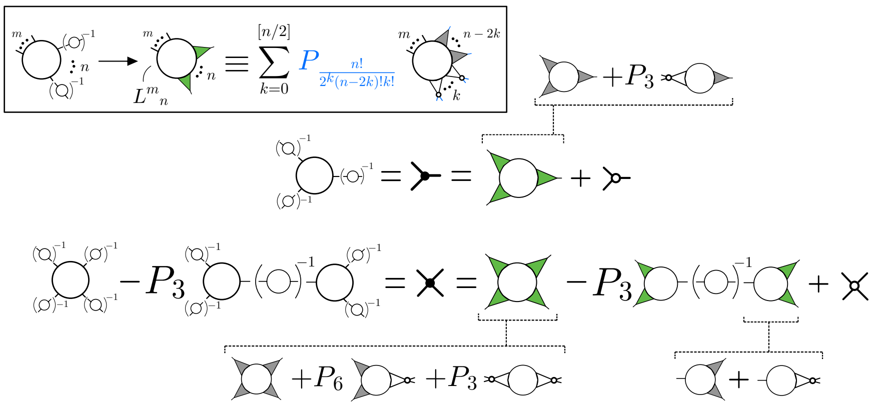

Figure 5.2: Three- and four-point 1PI vertices expressed in terms of the amputated correlation functions (left side) and their corresponding symmetric estimators (right side) for a theory with cubic and quartic interactions. To obtain the estimator for an np vertex from the left side expression, we add a bare vertex $S_n \equiv (\frac{\delta}{\delta\bar{\varphi}})^n S[\bar{\varphi}]$ and apply the rule in the left upper corner. Diagrammatic definitions are given in Fig. 5.1.

5.2 Estimators for 1PI vertex from its expression in terms of the amputated Green's functions

Here, we show that the symmetric estimator for the N-point 1PI vertex can be obtained directly from its representation in terms of amputated correlation functions by applying the following simple rules (see Fig. 5.2):

1. Add the bare vertex $S_N[\bar{\varphi}]$
2. Replace each amputated correlation function with n external and m internal indices, $G^{m+n}g^{-n}$, by $L^m{}_n$ defined as

$$L^m{}_n \equiv G^m{}_n + \sum_{k=1}^{\lfloor n/2 \rfloor} P_{c_k^n} G^m{}_{(2)^k n-2k}, \tag{5.32}$$

where $c_k^n = \frac{n!}{2^k(n-2k)!k!}$ is the same as in Eq. (5.11) and only lower indices are permuted by $P_{c_k^n}$. The sum in Eq. (5.32) runs over all possible unique pairings of n indices in $G^m{}_n$ into composite ones.

For an even theory, from the expression $\Gamma_6^{1\text{PI}} = g^{-6}G^6 - P_{10}g^{-6}(G^4g^{-1}G^4)$, we immediately obtain the estimator for the six-point 1PI vertex:

$$\Gamma_6^{1\text{PI}} = L_6 - P_{10}L_3{}^1 g^{-1} L^1{}_3, \tag{5.33}$$

where $L_6 = G_6 + P_{15}G_{(2)4} + P_{45}G_{(2)(2)2} + P_{15}G_{(2)(2)(2)}$ and $L^1{}_3 = G^1{}_3 + P_3G^1{}_{(2)1}$. In $P_3G^1{}_{(2)1}$, only lower indices are permuted by P_3.

We begin the proof with the symmetric estimator for the self-energy, Eq. (5.10), which can be written as

$$\Gamma_2^{1\text{PI}} = S_2 + L_2 - \tfrac{1}{2}S_4\bar{\varphi}^2, \tag{5.34}$$

where $L_2 = G_2 + G_{(2)}$.

We now seek a general formula for the evaluated derivative $\frac{\delta}{\delta\bar{\varphi}}L^m{}_n$. Here, "evaluation" (denoted by $\Longrightarrow$) means that $\frac{\delta}{\delta\psi}$ in $\frac{\delta}{\delta\bar{\varphi}} = \frac{\delta}{\delta\psi} + \frac{\delta}{\delta J^1}$ acts only on $F_{\bullet,1}$ containing the external index (see Eq. (5.1)).

By following a derivation similar to Eq. (5.22), we obtain:

$$\frac{\delta}{\delta\bar{\varphi}}L^m{}_n \Longrightarrow L^m{}_{n+1} - P_n L_2{}^1 g^{-1} L^{m+1}{}_{n-1}. \tag{5.35}$$

A more detailed derivation is given in Appendix C.

One can then check that the estimator for the three-point vertex, Eq. (5.30), follows from Eq. (5.34) by applying[3] (5.35) and using $L^1{}_1 = G^1{}_1 = 0$. Applying the rule a second time yields the estimator for Γ_4^{1PI}, Eq. (5.31).

For higher-order vertices, we need the derivative of the internal g^{-1}:

$$\frac{\delta}{\delta\bar{\varphi}}g_{\bar{\varphi}}^{-1} = -g^{-1}(\frac{\delta}{\delta J^1}g)g^{-1} \Longrightarrow -g^{-1}L_1{}^2 g^{-1}. \tag{5.36}$$

Using this and the rule (5.35), we can compute

$$\Gamma_n^{\mathrm{1PI}} = S_n + (\frac{\delta}{\delta\bar{\varphi}})^{n-3}L_3. \tag{5.37}$$

We find that the symmetric estimator for N-point 1PI vertex is obtained by adding to the bare vertex S_N all tree diagrams constructed from $L^m{}_n$, where internal upper indices are contracted with $(-g^{-1})$.

Thus, a 1PI vertex can be expressed in terms of correlation functions by summing all tree diagrams constructed either:

1. From $G^{m+n}g^{-n}$ only and $(-g^{-1})$ as internal line – this gives the standard representation in terms of amputated Green's functions; or

2. $L^m{}_n$ only and $(-g^{-1})$ as internal line – this gives the symmetric estimator.

This implies a simple transformation rule to obtain the estimator from the amputated representation: add the bare vertex S_N and replace each amputated correlation function $G^{m+n}g^{-n}$ (having n external and m internal indices) with $L^m{}_n$ given by Eq. (5.32).

For the first two vertices, Γ_3^{1PI} and Γ_4^{1PI}, this correspondence is illustrated in Fig. (5.2).

In a theory with only cubic interactions, $L^m{}_n = G^m{}_n$, so the substitution amounts to reducing each product of the field operator φ with the inverse bare propagator $(-S_2[0])$ by the composite operator $\frac{\delta S_{\mathrm{int}}}{\delta\varphi} = \frac{1}{2}\varphi^2 S_3[0]$ for each external amputation $g^{-1} = -S_2[0] - \Sigma_2$. For a gauge field, $\frac{\delta S_{\mathrm{int}}}{\delta\varphi}$ is the current to which the field couples.

When quartic interactions are present, this reduction also affects the already reduced external indices, yielding additional terms involving the composite operators $\frac{\delta^2 S_{\mathrm{int}}}{\delta\varphi^2}$.

[3] For the last term in Eq. (5.34), note that external indices are in S_4 and $\frac{\delta}{\delta J^1}\bar{\varphi} = 0$, so it drops out when one differentiates.

Chapter 6

Summary of main results

Our central result is that, similarly as connected Green's function are constructed from 1PI vertices via tree graphs, the vertex functions themselves admit tree expansions with respect to composite fields. This is shown in Fig. 6.1.

The single-boson exchange decomposition, symmetric estimator, the parquet decomposition and asymptotic classes are all special cases of this tree expansion for different choices of composite fields. Independent of the choice of composite indices, the tree-like structure of the resulting equations remains the same.

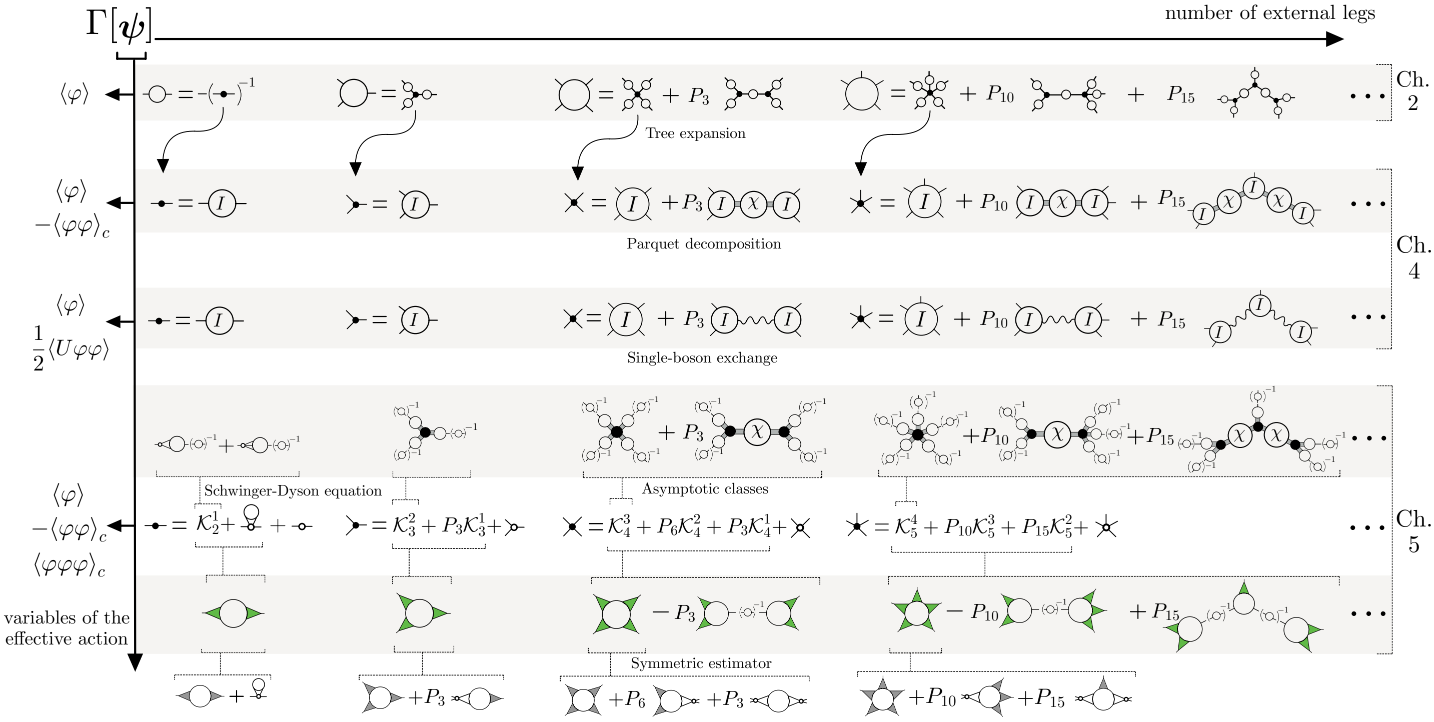

Figure 6.1: Overview of the main results. Rows show different tree expansions of n-point objects, with n increasing across the columns. While the structure of the decomposition remains the same, the details depend on the choice of independent variables $\boldsymbol{\psi}$ of the effective action $\Gamma[\boldsymbol{\psi}]$, which are given on the left. Row 1: Tree expansion of connected Green's functions in terms of 1PI vertices (Ch. 2). Row 2: Tree expansion of 1PI vertices in terms of 2PI vertices, yielding the parquet decomposition (Sec. 4.1). Row 3: Tree expansion of 1PI vertices in terms of U-reducible vertices, yielding the single-boson exchange decomposition (Sec. 4.2). Row 4: Tree expansion of asymptotic classes in terms of 3PI vertices (Sec. 5.1). Row 5: Decomposition of 1PI vertices in terms of asymptotic classes (Sec. 5.1). Row 6: Tree expansion of the non-bare part of 1PI vertices in terms of Green's functions of composite field operators, yielding symmetric-improved estimators (Sec. 5.2). Diagrammatic notations are explained in Figures. 2.1, 3.1, 4.2, 4.3 and 5.1.

Chapter 7

Conclusion

In this work, we have shown how non-perturbative equations for 1PI vertices in strongly correlated electron systems can be unified using the inverse Legendre transform of the composite field effective action.

In Sec. 3.1, starting from the 3PI effective action, we derived the symmetric self-energy estimator, the Bethe-Salpeter equations, the parquet formalism, and asymptotic class decomposition. In Sec. 4.2, we demonstrated that the single-boson exchange (SBE) decomposition can be also recovered using a local composite field. Then, in Sec. 5.2, we reproduced the symmetric estimator for the four-point 1PI vertex.

This unification enabled us to extend the results in two main directions:

1. Generalization to higher-order vertices, using simple tree diagrams (Fig. 6.1). In particular, we can obtain a decomposition for the six-point vertex, which plays a key role in nonlinear responses [41–43]. For such a vertex, combinatorial counting in diagrammatic approaches is challenging [44], making the functional formulation particularly attractive.

2. Extension to alternative decompositions – as shown in Ch. 4, the parquet and SBE formalisms can be generalized to composite fields, which may be highly non-linear in terms of field expectation values. Such choices arise naturally in situations with symmetry breaking. For example, in superconductors under external electromagnetic fields, the Goldstone mode can be introduced via a gauge transformation of the order parameter [39]. Using this mode as the composite field then suggests a new decomposition analogous to SBE.

   In QCD, the effective action defined via a Legendre transform in terms of both fundamental fields (quarks, gluons, ghosts) and non-linear composite fields (hadrons) has already been computed within the functional renormalization group (fRG) framework [45, 46]. A natural direction for future work is to extend the combined parquet–fRG framework [47, 48] to such non-linear composite fields.

We emphasize that many of the techniques used for correlated electron systems have direct analogues in other strongly interacting theories. Bethe–Salpeter-type equations, central to both parquet and SBE schemes, were first developed in particle physics to describe bound states [49] and multi-particle scattering [35]. Similarly, the study of infrared dynamics and out-of-equilibrium phenomena in QCD also demands non-perturbative methods, for which n-particle irreducible (nPI) techniques [37] provide controlled approximations. Finally, the rule for deriving symmetric estimators has an analogue in QED, where the photon field is replaced by the current it couples to [40]. Our results can be straightforwardly adapted to such systems with mixed statistics.

In summary, the combination of the inverse Legendre transform of composite-field effective actions with tree-expansion rules offers a unified functional framework for strongly interacting systems of widely different kinds—from QCD to correlated electrons. Its universality makes it possible to express diverse diagrammatic schemes, such as parquet, SBE, or symmetric estimators, in a single algebraic language. This opens the door to building dedicated computer algebra systems that operate directly on these functional objects, rather than on individual Feynman diagrams. Such a system could automate the derivation of flow equations, manage the combinatorics of high-rank tensors, and handle the Keldysh structure of vertices and Green's functions—tasks that become prohibitively tedious when done by hand. By embedding the functional non-perturbative rules into symbolic computation, one could streamline real-frequency calculations and make higher-order vertex treatments tractable across fields as diverse as condensed matter and high-energy physics[1].

[1]While tools like DoFun [50] exist in particle physics, no comparable project is available for strongly correlated electron systems

APPENDIX **A**

Structure of 3PI effective action and cluster decomposition formula

A.1 Proof of the structure (3.2)

To derive Eq. (3.2), we use $\phi^{(a_1 \ldots a_n)} \equiv \varphi^{a_1} \ldots \varphi^{a_n}$ for $n = 2, 3$ and set $\mathcal{S}[\boldsymbol{J}] = 0$ (see Eq. (3.30)). The two-point correlation function (Eq. (2.3)) can then be written as

$$g^{ab} = \bar{\varphi}^a \bar{\varphi}^b - \bar{\phi}^{(ab)}. \tag{A.1}$$

Similarly, for G^{abc} one finds

$$G^{abc} = 2\bar{\varphi}^a \bar{\varphi}^b \bar{\varphi}^c - P_3 \bar{\varphi}^a \bar{\phi}^{(bc)} + \bar{\phi}^{(abc)}, \tag{A.2}$$

where the definition of P_n is explained in Sec. 2 (see Eq. (2.12)). From relations (A.1)-(A.2), one can express $\bar{\phi}^{\mathbf{a}}$ as a functional of $\bar{\varphi}$ and G^n. This allows to verify Eq. (3.43) and $\bar{\phi}^{\mathbf{a}}_{,d} = \langle \phi^{\mathbf{a}}_{,d}[\varphi] \rangle$.

We also need the relation $\langle \phi^{\mathbf{a}}_{,d}[\varphi] \rangle \gamma^{\mathbf{b}}_{\mathbf{a}} J_{\mathbf{b}} = -\langle S_{,d}[\varphi] \rangle$, which follows from the trivial identity $\int D\varphi \frac{\delta}{\delta \varphi^d} (e^{-S[\varphi] - J_{\mathbf{a}} \phi^{\mathbf{a}}[\varphi]}) = 0$. We then obtain the Schwinger-Dyson equation (SDE)

$$\begin{aligned} \Gamma_{,d}[\bar{\varphi}, g, G^3] &= \bar{\phi}^{\mathbf{a}}_{,d} \Gamma_{,\mathbf{a}} = -\bar{\phi}^{\mathbf{a}}_{,d} \gamma^{\mathbf{b}}_{\mathbf{a}} J_{\mathbf{b}} \\ &= \langle S_{,d} \rangle = S_{,d}[\bar{\varphi}] - \frac{1}{2} g^{cb} S_{,bcd}[\bar{\varphi}] + \frac{1}{3!} G^{cba} S_{,abcd}[\bar{\varphi}]. \end{aligned} \tag{A.3}$$

Formally integrating this equation, with integration constant $\Lambda[g, G^3]$, yields Eq. (3.2).

A.2 Cluster decomposition formula

Let us choose $\psi^\bullet = g$ and $\phi^{(2)}[\varphi] = \varphi^2$ for $\bullet = 2$. Here, we show how to relate $K^{\bullet|\bullet}$ to connected Green's functions via Eq. (3.41). The resulting formula does not depend on the action, as it is merely a relation between Green's functions. It is convenient to perform the calculation in a theory with only cubic interactions, since in this case we can exploit the structure (3.2).

The derivatives $K^{\bullet|\bullet} \frac{\delta}{\delta g}$ and $\frac{\delta}{\delta \bar{\varphi}} = \frac{\delta}{\delta \psi} + F_{\bullet,1} K^{\bullet|\bullet} \frac{\delta}{\delta g}$ commute, because $K^{\bullet|\bullet} = (-\Lambda_{\bullet\bullet})^{-1}$ is ψ-independent and $F_{2,1} = -\frac{1}{2} S_3[0]$ is constant. The two-point 1PI vertex $\Gamma_2^{\rm 1PI}$ can then be evaluated either as $(-g^{-1})$ from Eq. (2.7), or as $\frac{\delta^2}{\delta\bar{\varphi}^2}(S+\Omega)+\phi^{(2)}_{,2}J_{(2)}$ using Eq. (3.4), with the source recovered via $S\to S+\phi^{(\bullet)}J_{(\bullet)}$. Applying derivative $K^{\bullet|\bullet}\frac{\delta}{\delta g}$ to both representations yields

$$K^{\bullet|\bullet}\frac{\delta}{\delta g}(-g^{-1}) = (K^{\bullet|\bullet}\frac{\delta}{\delta g})\frac{\delta^2}{\delta\bar{\varphi}^2}(S+\Omega) - 2\delta_2^\bullet, \tag{A.4}$$

where we used $K^{\bullet|\bullet}\frac{\delta}{\delta g}(J_{(\bullet)}\phi_2^{(\bullet)}) = 2K^{\bullet|\bullet}\Gamma_{\bullet\bullet}\delta_2^\bullet = -2\delta_2^\bullet$ together with $J_{(\bullet)} = \Gamma_\bullet$.

Evaluating $\frac{\delta}{\delta g}$ gives $K^{\bullet|\bullet}g^{-2}$ on the left side. On the right side, we first commute the derivatives $K^{\bullet|\bullet}\frac{\delta}{\delta g}$ and $\frac{\delta^2}{\delta\bar{\varphi}^2}$, and then use $K^{\bullet|\bullet}\frac{\delta}{\delta g}\Omega = g^\bullet$ to identify $g_{\bar{\varphi},2}$. The relation (A.4) then simplifies

$$K^{\bullet|2}g^{-2} = g^\bullet_{\bar{\varphi},2} - 2\delta_2^\bullet, \tag{A.5}$$

which is precisely Eq. (4.13). In an even theory this further reduces to Eq. (3.20) with the help of Eqs. (2.8) and (2.12).

Appendix B

Bethe-Salpeter-type equations

In this Appendix we derive the Bethe-Salpeter equations (4.7) and (4.29).

B.1 Proof of Eq. (4.7)

We consider $\Gamma_{\bullet\bullet}$ and $\Gamma_{1\bullet}$ as functionals of $\bar{\varphi}$ at $\psi^{\bullet} = \psi^{\bullet}_{\bar{\varphi}}$.

From $K^{\bullet|\bullet} = (-\Gamma_{\bullet\bullet})^{-1}$ and Eq. (3.46), we obtain $K^{\bullet|\bullet}_{,1}\Gamma_{\bullet 1} = K^{\bullet|\bullet}(\Gamma_{\bullet\bullet})_{,1}\psi^{\bullet}_{\bar{\varphi},1}$. Substituting this result into $\psi^{\bullet}_{\bar{\varphi},2}$ yields the Bethe-Salpeter equation for $I_{2|\bullet}$, Eq. (4.7):

$$\psi^{\bullet}_{\bar{\varphi},2} = K^{\bullet|\bullet}[(\Gamma_{\bullet 1})_{,1} + (\Gamma_{\bullet\bullet})_{,1}\psi^{\bullet}_{\bar{\varphi},1}] = K^{\bullet|\bullet}I_{2|\bullet}, \tag{B.1}$$

where we have used the decomposition $I_{ab|\bullet} = \Gamma_{\bullet\mathbf{ab}}\psi^{\mathbf{a}}_{\bar{\varphi},a}\psi^{\mathbf{b}}_{\bar{\varphi},b} = \Gamma_{\bullet a\mathbf{b}}\psi^{\mathbf{b}}_{\bar{\varphi},b} + \Gamma_{\bullet\bullet\mathbf{b}}\psi^{\mathbf{b}}_{\bar{\varphi},b}\psi^{\bullet}_{\bar{\varphi},a}$.

B.2 Proof of Eq. (4.29)

Using $\psi^{\bullet}_{\bar{\varphi},2} = I_{2|\bullet}K^{\bullet|\bullet}$ and Eq. (4.28), we obtain $-\frac{1}{2}U^{\bullet|\bullet}\gamma_{2|\bullet}\chi^{2|2}g^{-2} = K^{\bullet|\bullet}I_{2|\bullet}$. Thus, Eq. (4.27) can equivalently be written as

$$K^{\bullet|\bullet} = -U^{\bullet|\bullet} - \frac{1}{2}K^{\bullet|\bullet}I_{\bullet c'd'}g^{c'c}g^{d'd}\gamma_{cd|\bullet}U^{\bullet|\bullet}. \tag{B.2}$$

Together with Eq. (4.28) this reduces the BS equation $\psi^{\bullet}_{\bar{\varphi},ab} = I_{ab|\bullet}K^{\bullet|\bullet}$ to

$$-\frac{1}{2}(g^{cd}_{\bar{\varphi},ab} - 2\delta^{cd}_{ab})\gamma_{cd|\bullet}U^{\bullet|\bullet} = -(I_{ab|\bullet} + \frac{1}{2}I_{ab|\bullet}K^{\bullet|\bullet}I_{\bullet c'd'}g^{c'c}g^{d'd}\gamma_{cd|\bullet})U^{\bullet|\bullet}. \tag{B.3}$$

Finally, multiplying both sides from the right by $(-U_{\bullet|\bullet})^{-1}$ yields Eq. (4.29).

Appendix C

Estimators for self-energy and general vertex

In this appendix, we derive estimators for the self-energy (Eqs. (5.4) and (5.10)) and then prove Eq. (5.35). We set $J_{(\bullet)} = 0$.

C.1 Self-energy estimators

Using relation (3.43), one can rewrite Eq. (3.8) as

$$\Gamma_1^{1\mathrm{PI}} = S_1 + G_{(1)} - \phi^{(\bullet)} G^{\bullet}_{,(\bullet)} F_{\bullet,1}, \tag{C.1}$$

which also holds for fermionic systems.

We apply $\frac{\delta}{\delta\bar{\varphi}}$ to both sides of Eq. (C.1):

$$\Gamma_2^{1\mathrm{PI}} = S_2 + \frac{\delta}{\delta\bar{\varphi}}(G_{(1)} - \phi^{(\bullet)} G^{\bullet}_{,(\bullet)} F_{\bullet,1}), \tag{C.2}$$

and, depending on how the last two terms are handled, derive the asymmetric (Eq. (5.4)) and symmetric (Eq. (5.10)) estimators for the self-energy Σ_2, defined as

$$\Sigma_2 \equiv -g^{-1} - \zeta S_2[0] = \zeta\big(\Gamma_2^{1\mathrm{PI}} - S_2[0]\big). \tag{C.3}$$

Asymmetric estimator

To obtain the asymmetric estimator, we use $G^{\bullet}_{,(\bullet)} F_{\bullet,1} = \frac{1}{2!}\delta^{(2)}_{(\bullet)} S_3[0] + \frac{1}{3!}\delta^{(3)}_{(\bullet)} S_4[0]$ (as in Eq. (5.3)) for each term in the brackets, which gives $\frac{\delta}{\delta\bar{\varphi}}(\phi^{(\bullet)} G^{\bullet}_{,(\bullet)} F_{\bullet,1}) = \bar{\varphi} S_3[0] + \frac{1}{2!}\bar{\varphi}^2 S_4[0]$. Using $\frac{\delta}{\delta\bar{\varphi}} = g^{-1}\frac{\delta}{\delta J_1}$, we get $\frac{\delta}{\delta\bar{\varphi}} G_{(1)} = g^{-1} G^{1}{}_{(1)}$. Substituting these results into Eq. (C.2) yields Eq. (5.4):

$$\zeta\Sigma_2 = g^{-1} G^{1}{}_{(1)}, \tag{C.4}$$

where for the left side of Eq. (C.2), we also used $\Gamma_2^{1\mathrm{PI}} = (-\zeta g)^{-1}$ (see Eq. (2.7)).

Symmetric estimator

To obtain the symmetric estimator, we write $\frac{\delta}{\delta\bar{\varphi}} = \frac{\delta}{\delta\psi} + \frac{\delta}{\delta J^1}$ for the last two terms in Eq. (C.2) and find

$$\Gamma_2^{\mathrm{1PI}} = S_2 + G_{(2)} + G_{1(1)} - \phi^{(\bullet)} G^{\bullet}_{,(\bullet)} F_{\bullet,2}, \tag{C.5}$$

where we used that $\frac{\delta}{\delta\psi}$ acts only on $F_{\bullet,1}$ with the external index (see Eq. (5.1)), and that $\frac{\delta}{\delta J^1}(F_{\bullet,1} G^{\bullet}_{,(\bullet)}) = 0$, so $\frac{\delta}{\delta J^1}(F_{\bullet,1} G^{\bullet}_{,(\bullet)} \phi^{(\bullet)}) = 0$ and $\frac{\delta}{\delta J^1} G_{(1)} = G_{1(1)}$.

Using $S_2 - F_{\bullet,2} G^{\bullet}_{,(\bullet)} \phi^{(\bullet)} = S_2[0] + \bar{\varphi} S_3[0]$ and the definitions (5.3) and (5.7), one then obtains the symmetric self-energy estimator, Eq. (5.10),

$$\Sigma_2 = S_3[0]\bar{\varphi} + G_{(2)} + G_{(1)(1)} - G_{(1)}{}^1 g^{-1} G^1{}_{(1)}. \tag{C.6}$$

C.2 Proof of Eq. (5.35)

We have for $\frac{\delta}{\delta\psi} G_n$:

$$\frac{\delta}{\delta\psi} G_n \implies P_n F_{\bullet,2} G^{\bullet}_{,(\bullet)} (G^{(\bullet)}{}_{n-1} - G^{(\bullet)1} g^{-1} G^1{}_{n-1})$$
$$= P_n G_{(2)n-1} - P_n G_{(2)}{}^1 g^{-1} G^1{}_{n-1}. \tag{C.7}$$

In total, using also Eq. (5.22),

$$\frac{\delta}{\delta\bar{\varphi}} G_n \implies G_{n+1} + P_n G_{(2)n-1} - P_n L_2{}^1 g^{-1} G^1{}_{n-1}, \tag{C.8}$$

where $L_2{}^1 = G_{(2)}{}^1 + G_2{}^1$. Note that this formula can easily be generalized to cases where additional upper indices or lower composite ones (2) are added to G_n, since the derivative $\frac{\delta}{\delta\psi}$ will not act on $F_{\bullet,2}$.

From Eq. (C.8), it is then straightforward to prove Eq. (5.35), using definition (5.32).

Bibliography

[1] C. De Dominicis and P. C. Martin, "Stationary Entropy Principle and Renormalization in Normal and Superfluid Systems. I. Algebraic Formulation", Journal of Mathematical Physics **5**, 14 (1964).

[2] C. De Dominicis and P. C. Martin, "Stationary Entropy Principle and Renormalization in Normal and Superfluid Systems. II. Diagrammatic Formulation", Journal of Mathematical Physics **5**, 31 (1964).

[3] H. Hafermann, E. G. van Loon, M. I. Katsnelson, A. I. Lichtenstein, and O. Parcollet, "Collective charge excitations of strongly correlated electrons, vertex corrections, and gauge invariance", Physical Review B **90**, 235105 (2014).

[4] J. Kuneš, "Efficient treatment of two-particle vertices in dynamical mean-field theory", Physical Review B **83**, 085102 (2011).

[5] N. Lin, E. Gull, and A. J. Millis, "Two-Particle Response in Cluster Dynamical Mean-Field Theory: Formalism and Application to the Raman Response of High-Temperature Superconductors", Physical Review Letters **109**, 106401 (2012).

[6] J. Otsuki, H. Hafermann, and A. Lichtenstein, "Superconductivity, antiferromagnetism, and phase separation in the two-dimensional Hubbard model: A dual-fermion approach", Physical Review B **90**, 235132 (2014).

[7] G. Rohringer, A. Valli, and A. Toschi, "Local electronic correlation at the two-particle level", Physical Review B **86**, 125114 (2012).

[8] P. Kopietz, L. Bartosch, and F. Schütz, *Introduction to the functional renormalization group*, Vol. 798 (Springer Science & Business Media, 2010).

[9] J.-M. Lihm, D. Kiese, S.-S. B. Lee, and F. B. Kugler, "The finite-difference parquet method: Enhanced electron-paramagnon scattering opens a pseudogap", arXiv preprint arXiv:2505.20116 (2025).

[10] A. Toschi, A. A. Katanin, and K. Held, "Dynamical vertex approximation: A step beyond dynamical mean-field theory", Phys. Rev. B **75**, 045118 (2007).

[11] A. A. Katanin, "Extended dynamical mean field theory combined with the two-particle irreducible functional renormalization-group approach as a tool to study strongly correlated systems", Phys. Rev. B **99**, 115112 (2019).

[12] C. Taranto, S. Andergassen, J. Bauer, K. Held, A. Katanin, W. Metzner, G. Rohringer, and A. Toschi, "From Infinite to Two Dimensions through the Functional Renormalization Group", Phys. Rev. Lett. **112**, 196402 (2014).

[13] D. Vilardi, C. Taranto, and W. Metzner, "Antiferromagnetic and d-wave pairing correlations in the strongly interacting two-dimensional Hubbard model from the functional renormalization group", Phys. Rev. B **99**, 104501 (2019).

[14] T. Ayral and O. Parcollet, "Mott physics and spin fluctuations: A unified framework", Phys. Rev. B **92**, 115109 (2015).

[15] T. Ayral and O. Parcollet, "Mott physics and spin fluctuations: A functional viewpoint", Phys. Rev. B **93**, 235124 (2016).

[16] T. Ayral and O. Parcollet, "Mott physics and collective modes: An atomic approximation of the four-particle irreducible functional", Phys. Rev. B **94**, 075159 (2016).

[17] N. Wentzell, G. Li, A. Tagliavini, C. Taranto, G. Rohringer, K. Held, A. Toschi, and S. Andergassen, "High-frequency asymptotics of the vertex function: Diagrammatic parametrization and algorithmic implementation", Phys. Rev. B **102**, 085106 (2020).

[18] F. Krien, "Efficient evaluation of the polarization function in dynamical mean-field theory", Phys. Rev. B **99**, 235106 (2019).

[19] F. Krien, A. Kauch, and K. Held, "Tiling with triangles: parquet and $GW\gamma$ methods unified", Phys. Rev. Res. **3**, 013149 (2021).

[20] F. Krien, A. I. Lichtenstein, and G. Rohringer, "Fluctuation diagnostic of the nodal/antinodal dichotomy in the Hubbard model at weak coupling: A parquet dual fermion approach", Phys. Rev. B **102**, 235133 (2020).

[21] F. Krien and A. Valli, "Parquetlike equations for the Hedin three-leg vertex", Phys. Rev. B **100**, 245147 (2019).

[22] F. Krien, A. Valli, and M. Capone, "Single-boson exchange decomposition of the vertex function", Phys. Rev. B **100**, 155149 (2019).

[23] F. Krien, A. Valli, P. Chalupa, M. Capone, A. I. Lichtenstein, and A. Toschi, "Boson-exchange parquet solver for dual fermions", Phys. Rev. B **102**, 195131 (2020).

[24] R. Bulla, A. C. Hewson, and T. Pruschke, "Numerical Renormalization Group Calculations for the Self-Energy of the Impurity Anderson Model", Journal of Physics: Condensed Matter **10**, 8365 (1998).

[25] F. B. Kugler, "Improved estimator for numerical renormalization group calculations of the self-energy", Phys. Rev. B **105**, 245132 (2022).

[26] J.-M. Lihm, J. Halbinger, J. Shim, J. von Delft, F. B. Kugler, and S.-S. B. Lee, "Symmetric improved estimators for multipoint vertex functions", Phys. Rev. B **109**, 125138 (2024).

[27] T. Schäfer, G. Rohringer, O. Gunnarsson, S. Ciuchi, G. Sangiovanni, and A. Toschi, "Divergent Precursors of the Mott-Hubbard Transition at the Two-Particle Level", Phys. Rev. Lett. **110**, 246405 (2013).

[28] B. S. DeWitt and A. A. Mullin, "Dynamical Theory of Groups and Fields", American Journal of Physics **34**, 1209 (1966).

[29] J. Negele and H. Orland, *Quantum Many-particle Systems*, Advanced Book Classics (Basic Books, 1988).

[30] A. Vasiliev, *Functional Methods in Quantum Field Theory and Statistical Physics* (Taylor & Francis, 1998).

[31] C. J. Eckhardt, P. Kappl, A. Kauch, and K. Held, "A functional-analysis derivation of the parquet equation", SciPost Phys. **15**, 203 (2023).

[32] J. M. Cornwall, R. Jackiw, and E. Tomboulis, "Effective action for composite operators", Phys. Rev. D **10**, 2428 (1974).

[33] V. Kazanskii, "Legendre transforms of the generating functionals in quantum field theory", Theoret. and Math. Phys., 875 (1972).

[34] A. N. Vasilev and A. K. Kazanskii, "Equations of motion for the Legendre transformation of an arbitrary order", Teor. Mat. Fiz. **14**, 289 (1973).

[35] S. Weinberg, "Systematic Solution of Multiparticle Scattering Problems", Phys. Rev. **133**, B232 (1964).

[36] J. M. Pawlowski, "Aspects of the functional renormalisation group", Annals of Physics **322**, 2831 (2007).

[37] J. Berges, "n-particle irreducible effective action techniques for gauge theories", Phys. Rev. D **70**, 105010 (2004).

[38] J. M. Luttinger and J. C. Ward, "Ground-State Energy of a Many-Fermion System. II", Phys. Rev. **118**, 1417 (1960).

[39] S. Weinberg, "Effective action and renormalization group flow of anisotropic superconductors", Nuclear Physics B **413**, 567 (1994).

[40] M. Schwartz, *Quantum Field Theory and the Standard Model*, Quantum Field Theory and the Standard Model (Cambridge University Press, 2014).

[41] P. Kappl, F. Krien, C. Watzenböck, and K. Held, "Nonlinear responses and three-particle correlators in correlated electron systems exemplified by the Anderson impurity model", Phys. Rev. B **107**, 205108 (2023).

[42] P. Kappl, T. Ribic, A. Kauch, and K. Held, *Ladder equation for the three-particle vertex and its approximate solution*, 2024, arXiv:2412.01848 [cond-mat.other].

[43] H. Rostami, M. I. Katsnelson, G. Vignale, and M. Polini, "Gauge invariance and Ward identities in nonlinear response theory", Annals of Physics **431**, 168523 (2021).

[44] T. Ribic, "Path integral methods for strong non-local correlations", PhD thesis (Technischen Universität Wien, 2018).

[45] J. Braun, L. Fister, J. M. Pawlowski, and F. Rennecke, "From quarks and gluons to hadrons: Chiral symmetry breaking in dynamical QCD", Phys. Rev. D **94**, 034016 (2016).

[46] K. Fukushima, J. M. Pawlowski, and N. Strodthoff, "Emergent hadrons and diquarks", Annals of Physics **446**, 169106 (2022).

[47] J.-P. Blaizot, J. M. Pawlowski, and U. Reinosa, “Functional renormalization group and 2PI effective action formalism”, Annals of Physics **431**, 168549 (2021).

[48] F. B. Kugler and J. von Delft, “Multiloop functional renormalization group for general models”, Phys. Rev. B **97**, 035162 (2018).

[49] E. E. Salpeter and H. A. Bethe, “A Relativistic Equation for Bound-State Problems”, Phys. Rev. **84**, 1232 (1951).

[50] M. Q. Huber, A. K. Cyrol, and J. M. Pawlowski, “DoFun 3.0: Functional equations in mathematica”, Computer Physics Communications **248**, 107058 (2020).

Declaration

I hereby declare that this thesis is my own work, and that I have not used any sources and aids other than those stated in the thesis.

München, 28. August 2025

Oleksandr Sulyma