

# Supersymmetric virial expansion for time-reversal invariant disordered systems

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## Abstract

We develop a supersymmetric virial expansion for two-point correlation functions of almost diagonal Gaussian random matrix ensembles (ADRMT) of the orthogonal symmetry. These ensembles have multiple applications in physics and can be used to study the universal properties of time-reversal invariant disordered systems which are either insulators or close to the Anderson localization transition. We derive a two-level contribution to the correlation functions of the generic ADRMT and apply these results to the critical (multifractal) power law banded ADRMT. Analytical results are compared with numerical ones.

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## 1. Introduction

Random matrix theories (RMT) have proven to be useful mathematical models to study quantum mechanical disordered or chaotic physical systems [1]. The Wigner–Dyson RMT (WDRMT), which is characterized by independent Gaussian probability distributions of the matrix elements with constant variances of the off-diagonal matrix entries, can be used to describe the statistics of electrons in small disordered samples in the universal (metallic) limit, as the supersymmetric  $\sigma$ -model obtained for the WDRMT is the same as that of electrons in a metallic dot with infinite conductance [2, 3]. The ensembles in the WDRMT can be classified by their symmetries, for example, the Gaussian orthogonal ensemble (GOE) consists of real symmetric matrices and can be used to describe time-reversal invariant systems; the Gaussian unitary ensemble (GUE) consists of Hermitian matrices with equal variances of the real and

imaginary parts of the off-diagonal matrix elements and can be used to describe systems with broken time-reversal symmetry [4].

Besides the universal metallic regime, physical applications require RMT models which can be used in cases of either insulators or critical systems at (or close to) the Anderson localization transition [5]. In such unconventional RMTs, the variance of the matrix elements depends on their distance to the diagonal. For example, the power law banded RMT (PLBRMT) is an unconventional RMT where the off-diagonal variances decrease as a power law  $r^{-2\alpha}$  of their distance to the diagonal  $r$  outside of a band of width  $B$  [6]. For  $\alpha = 1$ , the PLBRMT exhibits multifractal eigenfunctions and critical level statistics, a behavior different to the Poisson statistics of ideal insulators and the Wigner–Dyson statistics of ideal metals [5]. Numerical studies reveal an excellent agreement between different correlation functions of the critical PLBRMT and of the Anderson model at the transition point [7–9]. The bandwidth controls the fractality of the RMT eigenstates: their fractal dimensions are close to the space dimension,  $d$ , if the bandwidth is large and are much smaller than  $d$  in the case of the almost diagonal critical PLBRMT with small bandwidth. The latter regime of strong multifractality is relevant for the Anderson model in high dimensions.

The methods which can be used for an analytical study of the PLBRMT depend on the bandwidth. For large bandwidths, a field theoretical technique of the supersymmetric  $\sigma$ -model can be applied [6]. In the opposite case of the almost diagonal PLBRMT with small bandwidth, a complementary method, the supersymmetric virial expansion (VE), can be used. The VE is a regular expansion of the correlation functions in a number of interacting energy levels. In the supersymmetric version, it corresponds to the formal expansion in a number of independent supermatrices with broken supersymmetry. It is the supersymmetric analogy to the linked cluster expansion used to calculate thermodynamic quantities for classical imperfect gases [10].

The VE has been initially developed on the formal basis of the Trotter formula and a combinatorial analysis for a spectral form-factor of the generic case of almost diagonal random matrices (ADRMT) from the GUE symmetry class [11]. It has then been extended to calculate the density-of-states and the level compressibility [12, 13]. The supersymmetric (SuSy) VE has been developed later for unitary ADRMTs [14]. It is free of the complicated combinatorics and allows one to investigate not only the spectral statistics but also the statistics of the eigenfunctions. Recently, the SuSy VE has been used to study the dynamical scaling in multifractal systems [15].

To use the SuSy VE in the context of physical systems with time-reversal invariance, it is necessary to modify it such that it is applicable to orthogonal ADRMTs. Many experimentally relevant disordered quantum systems, e.g. cold quantum gases in a disordered optical lattice, belong to this class of systems [16, 17]. This paper fills this gap in the theory of the ADRMTs, namely, we extend the general formalism of the supersymmetric VE to orthogonal ADRMTs.

We organize the paper as follows: basic definitions of the almost diagonal RMT and of the two-point correlation functions are given in section 2; the supersymmetric representation of the correlation functions and of the supersymmetric VE are explained in sections 3 and 4, respectively; a parametrization of supermatrices is given in section 5; generic results for the leading terms of the VE are calculated in section 6 and are applied to study the local density-of-states (LDOS) of the almost diagonal critical PLBRMT in section 7. To the best of our knowledge, the LDOS–LDOS correlation function of the critical orthogonal PLBRMT has never been calculated before. We compare the analytical results with the numerical ones and conclude the paper with a brief discussion.

## 2. Main definitions

We consider a Gaussian real symmetric ADRMT defined by the following statistics of matrix entries:

$$\langle H_{ij} \rangle = 0, \quad \langle H_{ii}^2 \rangle = 1, \quad \langle H_{i \neq j}^2 \rangle = B^2 F(|i - j|) \equiv b_{ij} \ll 1, \quad (1)$$

where  $F(x)$  is a real decaying function and the small parameter  $B \ll 1$  reflects that the ensemble is almost diagonal. The matrices of the ensemble are of size  $N \times N$  with  $N \gg 1$ . We introduce the retarded and advanced Green's functions by

$$\hat{G}^{R/A}(E) \equiv \frac{1}{E - \hat{H} \pm i\eta}, \quad \eta \rightarrow +0, \quad (2)$$

with  $\hat{H}$  being an arbitrary matrix of the ensemble and we define the two-point Green's function at the band center  $E = 0$  as

$$\mathcal{G}_{pq}(\omega) \equiv \hat{G}_{pp}^R\left(\frac{\omega}{2}\right) \hat{G}_{qq}^A\left(-\frac{\omega}{2}\right), \quad (3)$$

using the matrix elements of the Green's functions  $\hat{G}_{nn}^{R/A} = \langle n | \hat{G}^{R/A} | n \rangle \langle n |$  with  $n = 1, \dots, N$  is a canonical basis vector in the vector space of the RMT.

We will consider the disordered average two-point correlation functions, namely, the spectral correlation function and the diagonal LDOS correlation function, which are defined by

$$R_2(\omega) \equiv \Delta^2 \sum_{i,j=1}^N \left\langle \left\langle \delta\left(E_i + \frac{\omega}{2}\right) \delta\left(E_j - \frac{\omega}{2}\right) \right\rangle \right\rangle \quad (4)$$

and

$$C_2(\omega) \equiv \Delta^2 \sum_{p=1}^N \sum_{i,j=1}^N \left\langle \left\langle |\psi_i(p)|^2 |\psi_j(p)|^2 \delta\left(E_i + \frac{\omega}{2}\right) \delta\left(E_j - \frac{\omega}{2}\right) \right\rangle \right\rangle. \quad (5)$$

Here,  $\Delta$  is the mean level spacing of the RMT;  $E_{i,j}$  are the eigenvalues of the random matrices;  $\langle \langle ab \rangle \rangle \equiv \langle ab \rangle - \langle a \rangle \langle b \rangle$  and  $\langle \dots \rangle$  denotes the average over the ensemble of random matrices. For orthogonal almost diagonal RMT, the mean level spacing is known to be [11]

$$\Delta|_{B \ll 1} \approx \frac{1}{N} (\sqrt{2\pi} + O(B^2)). \quad (6)$$

The spectral and the LDOS correlation functions can be expressed by the two-point Green's function:

$$R_2(\omega) = \frac{\Delta^2}{2\pi^2} \sum_{p,q=1}^N \text{Re}[\langle \langle \mathcal{G}_{pq}(\omega) \rangle \rangle], \quad (7)$$

$$C_2(\omega) = \frac{\Delta^2}{2\pi^2} \sum_{p=1}^N \text{Re}[\langle \langle \mathcal{G}_{pp}(\omega) \rangle \rangle]. \quad (8)$$

Thus, one needs  $\langle \langle \mathcal{G}_{pq}(\omega) \rangle \rangle$  to study the correlation functions (4, 5).

### 3. Supersymmetric representation of the two-point Green's function

The ensemble averaged two-point Green's function can be obtained with the help of a supersymmetric field theory [14]. Let us introduce  $2N$  super vectors of the form

$$\psi_i^{R/A} \equiv \begin{pmatrix} S_i^{R/A} \\ \chi_i^{R/A} \\ (S_i^{R/A})^* \\ (\chi_i^{R/A})^* \end{pmatrix}, \quad (9)$$

$$(\psi_i^{R/A})^\dagger \equiv ((S_i^{R/A})^*, (\chi_i^{R/A})^*, S_i^{R/A}, -\chi_i^{R/A}), \quad (10)$$

where  $i = 1, \dots, N$ ,  $S_i^{R/A}$  are commuting and  $\chi_i^{R/A}$  are anti-commuting Grassmannian variables and the indices  $R/A$  stand for the retarded/advanced sectors. We use the outer product of these super vectors to define the supermatrices:

$$Q_i \equiv \frac{1}{2} \begin{pmatrix} \psi_i^R \otimes (\psi_i^R)^\dagger & -\psi_i^R \otimes (\psi_i^A)^\dagger \\ \psi_i^A \otimes (\psi_i^R)^\dagger & -\psi_i^A \otimes (\psi_i^A)^\dagger \end{pmatrix}. \quad (11)$$

These definitions allow one to write the ensemble averaged two-point Green's function as

$$\langle \mathcal{G}_{pq}(\omega) \rangle = \int D\{Q\} \mathcal{R}_p \mathcal{A}_q \left( \prod_{i=1}^N e^{S_0[Q_i]} \right) \left( \prod_{i \neq j}^N e^{S_p[Q_i, Q_j]} \right). \quad (12)$$

Here,  $\int D\{Q\} \equiv \prod_{i=1}^N D\{Q_i\}$  is the measure of integration over the supermatrices  $Q_i$  and  $\mathcal{R}_p, \mathcal{A}_q$  are the symmetry-breaking factors between the commuting and anti-commuting variables in the retarded and advanced sectors, respectively. They can be chosen as follows:

$$\mathcal{R}_p = \chi_p^R (\chi_p^R)^*, \quad \mathcal{A}_q = \chi_q^A (\chi_q^A)^*. \quad (13)$$

The action of the field integral is split into two parts:  $S_0$  depends on a single supermatrix,

$$S_0[Q_i] \equiv -\frac{1}{2} (\text{Str}[Q_i])^2 + i \frac{\omega + i\eta}{2} \text{Str}[\Lambda Q_i], \quad \Lambda \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad (14)$$

$S_p$  depends on the product of two supermatrices and is proportional to the off-diagonal variances,

$$S_p[Q_i, Q_j] \equiv -b_{ij} \text{Str}[Q_i Q_j]. \quad (15)$$

### 4. The virial expansion

The two-point Green's function (12) of the ADRMT can be approximately calculated using the virial expansion [11–14]. The basic concept of the VE is similar to the idea of resonant level interactions which has been used in a semi-empirical renormalization group approach [18, 19]: a small probability for energy levels of the ADRMT to interact in the energy space allows one to develop a regular perturbative expansion in the number of interacting levels. Each term in the VE reflects a contribution of a certain number of simultaneously interacting levels. In contrast to the semi-empirical renormalization group approach, the VE is a regular perturbative expansion, thus allowing an arbitrary number of resonant level interactions and a controlled estimate of the contribution of the unused resonances to be included.

The mathematical expressions of the virial expansion can be obtained by rewriting the term containing the off-diagonal variances in the superintegral (12)

$$\prod_{i \neq j}^N e^{S_p[Q_i, Q_j]} \equiv \mathcal{V}^D + \sum_{m=2}^{\infty} \mathcal{V}^{(m)}, \quad (16)$$

$$\mathcal{V}^D = 1, \quad \mathcal{V}^{(2)} = \sum_{i > j=1}^N (e^{2S_p[Q_i, Q_j]} - 1) \quad (17)$$

and so on (see details in [14]).

The VE is similar to the cluster expansion in the classical gas theory [10]; however, the SuSy representation of the resulting terms ensures that, in contrast to the cluster expansion in the classical gas theory, no further reordering of the terms is necessary. Thus,  $\mathcal{V}^D$  corresponds to the contributions of the diagonal part of the RMT,  $\mathcal{V}^{(2)}$  reflects the contributions of two-level interactions and so on. Using equations (16)–(17), we can formally expand the averaged two-point Green’s function:

$$\langle \mathcal{G}_{pq} \rangle = \langle \mathcal{G}_{pq}^D \rangle + \langle \mathcal{G}_{pq}^{(2)} \rangle + \dots$$

After integrating out all supermatrices  $Q_j$  which are neither included in the SuSy breaking factor ( $j \neq p, q$ ) nor linked to this factor through  $\mathcal{V}^{(m)}$  functions [14], the first two terms of the VE read

$$\langle \mathcal{G}_{pq}^D \rangle = \int D\{Q_p\} D\{Q_q\} \mathcal{R}_p \mathcal{A}_q e^{S_0[Q_p] + S_0[Q_q]}, \quad (18)$$

$$\langle \mathcal{G}_{p \neq q}^{(2)} \rangle = \int D\{Q_p\} D\{Q_q\} \mathcal{R}_p \mathcal{A}_q (e^{S_0[Q_p] + S_0[Q_q]})(e^{2S_p[Q_p, Q_q]} - 1), \quad (19)$$

$$\langle \mathcal{G}_{pp}^{(2)} \rangle = \sum_{n \neq p}^N \int D\{Q_p\} D\{Q_n\} \mathcal{R}_p \mathcal{A}_p (e^{S_0[Q_p] + S_0[Q_n]})(e^{2S_p[Q_p, Q_n]} - 1). \quad (20)$$

## 5. Parametrization of the supermatrices

A convenient parametrization of the  $Q$ -matrices for the SuSy VE has been first suggested in [20]. For each supermatrix, one introduces two positive variables  $\lambda_i^{R/A} \in [0, \infty]$ , two angles  $\phi_i^{R/A} \in [0, 2\pi]$  and four anti-commuting Grassmann variables  $\eta_i^{R/A}, (\eta_i^{R/A})^*$  ( $i = 1, \dots, N$ ) in the following way:

$$S_i^{R/A} = \lambda_i^{R/A} (1 - \frac{1}{2}(\eta_i^{R/A})^* \eta_i^{R/A}) e^{i\phi_i^{R/A}}, \quad \chi_i^{R/A} = \lambda_i^{R/A} \eta_i^{R/A} e^{i\phi_i^{R/A}}. \quad (21)$$

The integration measure in this parametrization is given by

$$D\{Q\} = \prod_{i=1}^N \frac{d\lambda_i^R d\phi_i^R (d\eta_i^R)^* d\eta_i^R}{\pi \lambda_i^R} \frac{d\lambda_i^A d\phi_i^A (d\eta_i^A)^* d\eta_i^A}{\pi \lambda_i^A}, \quad (22)$$

and the single matrix part of the action reads

$$S_0[Q_i] = -\frac{1}{2}((\lambda_i^R)^2 - (\lambda_i^A)^2)^2 + i \frac{\omega + i\eta}{2} ((\lambda_i^R)^2 + (\lambda_i^A)^2). \quad (23)$$

Using rotations in superspace, we can find a convenient representation of the two matrix part of the action in the new variables (see appendix A for details):

$$S_p[Q_i, Q_j] = -b_{ij} (\lambda_i^R \lambda_j^R \cos(\theta_{ij}^R) (1 - \frac{1}{2}(\alpha_{ij}^R)^* \alpha_{ij}^R) - \lambda_i^A \lambda_j^A \cos(\theta_{ij}^A) (1 - \frac{1}{2}(\alpha_{ij}^A)^* \alpha_{ij}^A))^2, \quad (24)$$

where  $\alpha_{ij}^{R/A} \equiv \eta_i^{R/A} - \eta_j^{R/A}$  and

$$\theta_{ij}^{R/A} \equiv \phi_j^{R/A} - \phi_i^{R/A} - \frac{i}{2}((\eta_i^{R/A})^* \eta_j^{R/A} - (\eta_j^{R/A})^* \eta_i^{R/A}). \quad (25)$$

As a direct consequence of the underlying symmetries of the ensembles, the supermatrices defined in equation (11) are twice the size of the supermatrices which have been used for the SuSy VE of the unitary ADRMT (cf equations (A.1), (A.2) [14]). Parametrizing the supermatrices as defined in equation (21), this yields a different amount of effective variables necessary for the integral representation of the two-point Green's function; the two matrix part of the action contains one effective angle in the case of the GUE (cf equation (A.10) [14]), whereas for the GOE two effective angles ( $\theta_{ij}^R$  and  $\theta_{ij}^A$  in (24)) appear. For the further calculations, we will use other commuting variables which allow one to simplify the single matrix part of the action:

$$R_i \equiv (\lambda_i^R)^2 - (\lambda_i^A)^2, \quad S_i \equiv (\lambda_i^R)^2 + (\lambda_i^A)^2; \quad (26)$$

where  $-S_i \leq R_i \leq S_i$ ,  $S_i \in [0, \infty]$ . The integration measure changes to

$$D\{Q\} = \prod_{i=1}^N \frac{dS_i dR_i d\phi_i^R d\phi_i^A (d\eta_i^R)^* d\eta_i^R (d\eta_i^A)^* d\eta_i^A}{2\pi^2(S_i^2 - R_i^2)}. \quad (27)$$

## 6. Generic results for the first two terms of the virial expansion

Integration of the superintegral in (18) and calculation of the diagonal contribution of the product of single averaged Green's functions  $\langle \hat{G}^R(\omega/2) \rangle \langle \hat{G}^A(-\omega/2) \rangle$  yield

$$\langle \langle \mathcal{G}_{p \neq q}^D(\omega) \rangle \rangle = 0, \quad \langle \langle \mathcal{G}_{pp}^D(\omega) \rangle \rangle \Big|_{N \gg 1} \approx \frac{2N i}{s + \frac{i\eta}{\Delta}} = 2\pi N \delta(s) + \frac{2N i}{s}, \quad (28)$$

where  $s = \omega/\Delta$ .

The superintegrals (19), (20) can be calculated using a saddle-point approach (cf section 4.1 in [14]). Rescaling all  $S_i$  and  $R_i$  variables with  $\bar{S}_i = \sqrt{b_{p\alpha}} S_i$ ,  $\bar{R}_i = \sqrt{b_{p\alpha}} R_i$ , where  $\alpha \in \{n, q\}$ , we obtain a large negative factor of order  $O(1/B^2)$  in front of  $R_i^2$  in the single matrix part of the action:

$$S_0[Q_i] = -\frac{1}{2b_{p\alpha}} \bar{R}_i^2 + i \frac{\omega + i\eta}{2\sqrt{b_{p\alpha}}} \bar{S}_i, \quad b_{p\alpha} \ll 1. \quad (29)$$

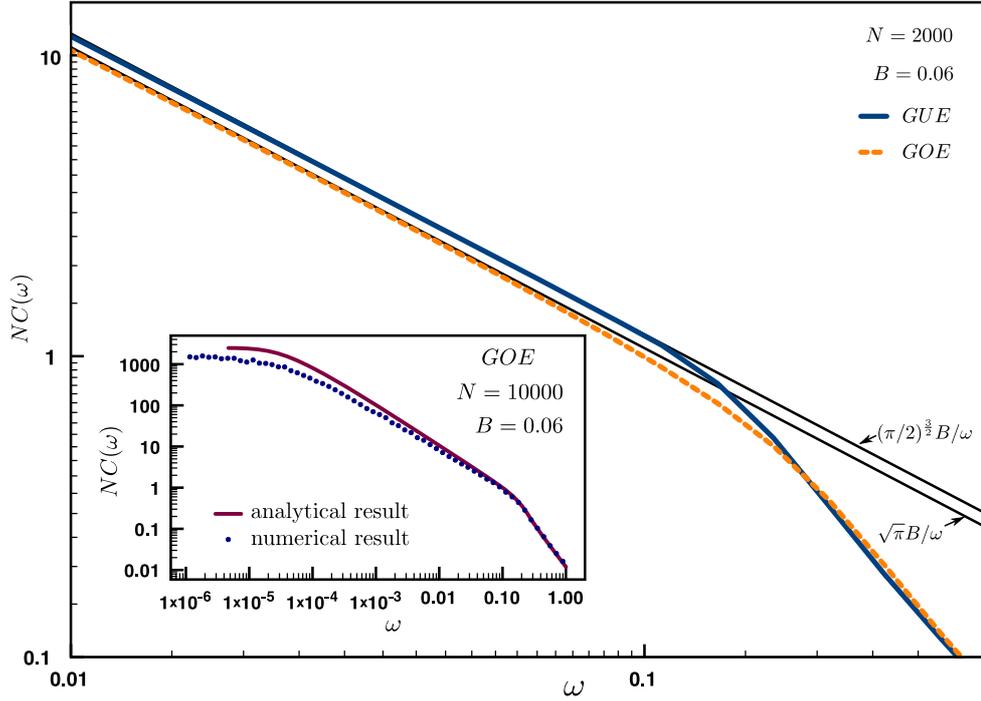
This allows us to employ a saddle-point approach in the  $R_i$ :

$$\int_0^\infty d\bar{S}_{p,\alpha} \int_{-\bar{S}_{p,\alpha}}^{\bar{S}_{p,\alpha}} d\bar{R}_{p,\alpha} \frac{\mathcal{R}_p \mathcal{A}_\alpha f(\bar{R}_{p,\alpha}, \bar{S}_{p,\alpha})}{(\bar{S}_p^2 - \bar{R}_p^2)(\bar{S}_\alpha^2 - \bar{R}_\alpha^2)} \left( e^{-\frac{1}{2b_{p\alpha}}(\bar{R}_p^2 + \bar{R}_\alpha^2) + i \frac{\omega + i\eta}{2\sqrt{b_{p\alpha}}}(\bar{S}_p + \bar{S}_\alpha)} \right) \\ \approx 2\pi b_{p\alpha} \left( \int_0^\infty d\bar{S}_{p,\alpha} \frac{\mathcal{R}_p \mathcal{A}_\alpha f(0, \bar{S}_{p,\alpha})}{\bar{S}_p^2 \bar{S}_\alpha^2} \left( e^{i \frac{\omega + i\eta}{2\sqrt{b_{p\alpha}}}(\bar{S}_p + \bar{S}_\alpha)} \right) + O(\sqrt{b_{p\alpha}}) + O\left(\frac{\bar{R}_t}{\bar{S}_t}\right) \right),$$

where  $f(\bar{R}_{p,\alpha}, \bar{S}_{p,\alpha}) \equiv (e^{2S_p[Q_p, Q_n]} - 1)$  and  $\bar{R}_t, \bar{S}_t$  are the typical values where the integrals over  $\bar{R}$  and  $\bar{S}$  converge. Based on rough estimates, one obtains two requirements for the validity of the saddle-point approximation [14]:

$$\omega \leq B \ll 1. \quad (30)$$

However, the analytical results derived from the saddle-point approximation and the results of the numerical diagonalization are in a very good agreement even in the regime  $\omega \simeq 1$ ; see an example of the LDOS correlation function  $C(\omega)$ , equation (42): figure 1 of the present paper



**Figure 1.** LDOS correlation function for the critical ADRMT with  $B = 0.06$ . *Main panel:* comparison of the analytical results for the cases of GOE and GUE,  $N = 2000$ . Black solid lines indicate a reference slope predicted by the two matrix approximation of the VE, equation (45). *Inset:* comparison of the analytical and numerical results for the case of GOE,  $N = 10000$ . Note that the difference between the numerically and the analytically found slope in the intermediate regime  $B\Delta \ll \omega \ll E_0$  is due to the accuracy of the two matrix approximation, see detailed discussion in the main text.

(This figure is in colour only in the electronic version)

and figure 3 in the paper [7] in the cases of the orthogonal and the unitary critical ADRMT, respectively. Corrections beyond the saddle-point approximation will be studied in details elsewhere [21].

After using the saddle-point approximation, we can integrate out the angles, the Grassmann variables and the remaining real variables. This yields

$$\langle\langle \mathcal{G}_{p \neq q}^{(2)}(\omega) \rangle\rangle \approx \sum_{k=1}^{\infty} (-1)^k \left( \frac{2\sqrt{2b_{pq}}}{i(\omega + i\eta)} \right)^{2k} \frac{\Gamma^2(k - \frac{1}{2})(2k - 1)(k - 1)}{4\Gamma(k + 1)}, \quad (31)$$

$$\langle\langle \mathcal{G}_{pp}^{(2)}(\omega) \rangle\rangle \approx \sum_{n \neq p}^N \sum_{k=1}^{\infty} (-1)^k \left( \frac{2\sqrt{2b_{pn}}}{i(\omega + i\eta)} \right)^{2k} \frac{\Gamma^2(k - \frac{1}{2})(2k - 1)}{4\Gamma(k)}. \quad (32)$$

These power series can be summed up by using a Fourier transform of the physically relevant real part of the two-point Green's function:

$$\text{Re}[\mathcal{G}_{pq}(t)] = \frac{1}{2\Delta} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} (\mathcal{G}_{pq}(\omega) + \mathcal{G}_{pq}^*(\omega)). \quad (33)$$

After applying the Fourier transform to equations (31), (32), we find the two-point Green's function in the time domain:

$$\text{Re}[\langle\langle \mathcal{G}_{p \neq q}^{(2)}(t) \rangle\rangle] = \frac{2\pi^2}{\Delta} b_{pq} |t| e^{-b_{pq} t^2 - \eta|t|} I_1(b_{pq} t^2), \quad (34)$$

$$\text{Re}[\langle\langle \mathcal{G}_{pp}^{(2)}(t) \rangle\rangle] = -\frac{2\pi^2}{\Delta} \sum_{n \neq p}^N b_{pn} |t| e^{-b_{pn} t^2 - \eta|t|} I_0(b_{pn} t^2). \quad (35)$$

The energy representation of the two-point Green's function can be obtained by the inverse Fourier transform. If  $\omega \neq 0$  we find

$$\text{Re}[\langle\langle \mathcal{G}_{p \neq q}^{(2)}(\omega) \rangle\rangle] = \left(\frac{\pi}{2}\right)^{\frac{3}{2}} |\bar{\omega}| e^{-\bar{\omega}^2} (3I_0(\bar{\omega}^2) + I_1(\bar{\omega}^2)) - \pi, \quad (36)$$

$$\text{Re}[\langle\langle \mathcal{G}_{pp}^{(2)}(\omega) \rangle\rangle] = \sum_{n \neq p}^N \left(\frac{\pi}{2}\right)^{\frac{3}{2}} |\bar{\omega}| e^{-\bar{\omega}^2} (I_0(\bar{\omega}^2) - I_1(\bar{\omega}^2)); \quad (37)$$

where  $\bar{\omega} = \omega/(4\sqrt{b_{p\alpha}})$ , and  $\alpha = q$  and  $n$  in equations (36) and (37), respectively. These are the generic results for the two matrix approximation of the two-point Green's function for ADRMTs of the GOE symmetry class.

## 7. Application to the critical PLBRMT

The critical PLBRMT is defined as follows:

$$\langle H_{ij} \rangle = 0, \quad \langle H_{ii}^2 \rangle = 1, \quad \langle H_{i \neq j}^2 \rangle = \frac{1}{2} \frac{B^2}{B^2 + |i - j|^2} \Big|_{B \ll 1} \approx \frac{1}{2} \frac{B^2}{|i - j|^2}. \quad (38)$$

The ensemble (38) is named 'critical' because it shows multifractal behavior of the eigenstates and critical level statistics at arbitrary  $B$  [6].

### 7.1. The spectral and the LDOS correlation functions

Inserting the variances (38) into equations (7), (8), (36) and (37), we obtain the spectral and the LDOS correlation functions for the critical ADRMT of the orthogonal symmetry:

$$R_2(\omega) = \frac{1}{N^2} \sum_{n \neq p=1}^N (\sqrt{2\pi} |\tilde{\omega}| |p - n| e^{-\tilde{\omega}^2 |p - n|^2} I_0(\tilde{\omega}^2 |p - n|^2) - 1), \quad (39)$$

$$C_2(\omega) = \frac{1}{2N^2} \sum_{n \neq p=1}^N \sqrt{\frac{\pi}{2}} |\tilde{\omega}| |p - n| e^{-\tilde{\omega}^2 |p - n|^2} (I_0(\tilde{\omega}^2 |p - n|^2) - I_1(\tilde{\omega}^2 |p - n|^2)), \quad (40)$$

where  $\tilde{\omega} = \omega/(\sqrt{8B})$ . Let us introduce a shifted (positive) spectral correlator:

$$R_2^+(\omega) \equiv \frac{1}{N^2} \sum_{n \neq p=1}^N \sqrt{2\pi} |\tilde{\omega}| |p - n| e^{-\tilde{\omega}^2 |p - n|^2} I_0(\tilde{\omega}^2 |p - n|^2), \quad (41)$$

and scale the LDOS correlation function:

$$C(\omega) \equiv \frac{C_2(\omega)}{R_2^+(\omega)}. \quad (42)$$

This scaling eliminates the level repulsion effect at  $\omega < B\Delta$ ; therefore,  $C(\omega)$  is convenient to study the correlations of the eigenfunctions taken at the same space point and with a given energy difference. The unitary case has been explored in [7]. The qualitative behavior of  $C(\omega)$  for the orthogonal case is similar to the unitary one, see figure 1. There are three regions which reflect different physical phenomena:

- (1)  $\omega > E_0 \propto B$ : the energy difference is larger than the hopping band width of the ADRMT which results in anticorrelations of the eigenstates and in a generic power law behavior  $C(\omega) \propto \omega^{-2}$  (see [7] for details).
- (2)  $0 < \omega < B\Delta$ : the energy separation is very small so that the behavior of  $C_2$  and  $R_2$  is governed by the level repulsion in the energy space; therefore, the ratio  $C = C_2/R_2$  saturates to a constant. Note that the analytically obtained constant is a bit larger than the numerically found value because of the reasons explained below.
- (3)  $B\Delta < \omega < E_0$ : this is the most interesting region where strong spatial correlations of the fractal eigenstates must result in a scaling  $C(\omega) \propto \omega^\mu$ ,  $\mu = d_2/d - 1$ ,  $d_2$  and  $d$  are the second fractal dimension and the space dimension, respectively [22, 23]. This property is especially nontrivial in the case of strong multifractality  $d_2 \ll d$  ( $d = 1$  for RMT) where the fractal eigenstates are sparse, nevertheless, they are strongly correlated in space. The fractal dimension  $d_2$  depends on the symmetry class and on the parameter  $B$ ; for example, the second fractal dimension for the critical ADRMT is equal to

$$\text{GOE} : d_2 \simeq \sqrt{2}B; \quad \text{GUE} : d_2 \simeq \pi B/\sqrt{2} \quad (43)$$

(cf [24, 25]). Thus, the critical ADRMT corresponds to the case of the strong multifractality. The parameter  $E_0$  can be referred to as ‘the upper cut-off of the multifractality’. This energy scale also depends on the symmetry class and on the parameter  $B$ ; using equation (40) and results of the paper [7], one can show that the upper cut-off of the multifractality for the critical ADRMT is equal to

$$\text{GOE} : E_0 \simeq \sqrt{\pi}B; \quad \text{GUE} : E_0 \simeq \left(\frac{\pi}{2}\right)^{\frac{3}{2}} B. \quad (44)$$

If  $d_2$  is small, the leading term of the VE cannot distinguish the exponents  $\mu = d_2 - 1$  and  $\mu \simeq -1$ . One has to take into account the interaction of a larger number of energy levels to find the correct exponent [15]. That is why the analytical result (40) shows the scaling

$$NC(B\Delta < \omega < E_0) \simeq \frac{E_0}{\omega}, \quad (45)$$

see figure 1, and the analytically calculated constant for  $C(\omega < B\Delta)$  is a bit larger as compared with the numerically obtained value. As  $E_{0|\text{GUE}} > E_{0|\text{GOE}}$ , the GUE curve in figure 1 is shifted upward compared to the GOE curve.

## 8. Conclusions

In this paper we have developed the supersymmetric virial expansion for the two-point correlation functions of the almost diagonal Gaussian Random Matrix ensembles of the orthogonal symmetry class. We have derived the generic results for the two matrix approximation of the two-point Green’s function in the time (equations (34), (35)) and energy (equations (36), (37)) representations. This contribution to the Green’s function results from an interaction of two energy levels in the energy space and can be used to study the statistical properties of time-reversal invariant disordered systems which are either insulators or close to the point of the Anderson localization transition.

To demonstrate how the method works, we have applied the generic results of the two matrix approximation to the Gaussian orthogonal critical power law banded RMT with small bandwidth, equation (38). We have obtained for this ensemble the spectral correlation function (equation (39)) and the local-density-of-states correlation function (equations (40), (42)); the latter has to the best of our knowledge not been derived before. A comparison of the analytically obtained function  $C(\omega)$  with the results of the direct numerical diagonalization shows an excellent agreement at  $\omega \geq B$ , see the inset in figure 1. In the region  $B\Delta < \omega < B$ , the two matrix approximation yields qualitatively correct results: we find a scaling  $C(\omega) \propto \omega^{-1}$  in this parametrically large energy window; however, the exponent is different from the exponent of the expected critical scaling  $C(\omega) \propto \omega^{d_2-1}$  [22]. This is because the second fractal dimension is small in the case of the ADRMT,  $d_2 \sim B \ll 1$  [24] and the leading term of the VE is unable to distinguish exponents  $1 - d_2$  and 1 [7]. One has to take into account the interaction of a larger number of the energy levels to find the correct exponent [15]. We note that the scaling relations obtained from the first term of the VE can be rewritten as follows:  $C(\omega) \simeq d_2/(2s)$ , where  $s = \omega/\Delta$  is the dimensionless energy, and the values of  $d_2$  and  $\Delta$  correspond to either GOE or GUE, see equations (43)–(45) and the paper [11]. This gives a hint that even the two matrix approximation for  $C(\omega > B\Delta)$  reflects multifractality of the critical eigenstates. Multifractality and critical exponents of the critical ADRMT are beyond the scope of the present paper and they will be described in detail in a forthcoming paper [15].

A Fourier transform of  $C(\omega)$  yields the return probability for an initially localized wave packet. Therefore, the results of section 7 might be relevant to study the fractal properties of cold atoms in a disordered optical potential which does not break the time-reversal symmetry [16, 17].

A generalization of the current results for the three matrix approximation is straightforward (cf [14]). In the future, we plan to use a SuSy field theory, which is the starting point of the perturbative VE (equation (12)), to obtain nonperturbative results accounting for the interaction of all energy levels at least with an accuracy of the first loop of the Renormalization Group procedure.

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## Appendix A. Simplification of the two matrix part of the action

The supermatrices (11) written in the parametrization (21) can be diagonalized in each sector by the following transformation:

$$D = \sigma^{-1} U^\dagger Q U \sigma^{-1}, \quad Q = U \sigma D \sigma U^\dagger. \quad (\text{A.1})$$

Here,  $U$  and  $U^\dagger$  are unitary block diagonal matrices:

$$U \equiv \begin{pmatrix} U_R e^{i\phi^R} & 0 & 0 & 0 \\ 0 & U_R^* e^{-i\phi^R} & 0 & 0 \\ 0 & 0 & U_A e^{i\phi^A} & 0 \\ 0 & 0 & 0 & U_A^* e^{-i\phi^A} \end{pmatrix}_{RA}, \quad (\text{A.2})$$

$$U_{R/A} \equiv U(\eta^{R/A}) \equiv \begin{pmatrix} (1 - \frac{1}{2}(\eta^{R/A})^* \eta^{R/A}) & -(\eta^{R/A})^* \\ \eta^{R/A} & (1 + \frac{1}{2}(\eta^{R/A})^* \eta^{R/A}) \end{pmatrix}; \quad (\text{A.3})$$

$\sigma$  is a symmetric orthogonal matrix:

$$\sigma \equiv \begin{pmatrix} \sigma' & 0 \\ 0 & \sigma' \end{pmatrix}, \quad \sigma' \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad (\text{A.4})$$

and  $D$  is a matrix which is diagonal in each sector:

$$D \equiv \begin{pmatrix} \lambda_R \lambda_R & -\lambda_R \lambda_A \\ \lambda_A \lambda_R & -\lambda_A \lambda_A \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.5})$$

In this representation,  $R$  labels the retarded and  $A$  the advanced sectors of the supermatrix. Each of the  $4 \times 4$  submatrices of the sectors in equations (A.4), (A.5) contains bosonic and fermionic sectors in a pattern determined by the outer product of the super vectors defined in equation (10). The supertrace in the two matrix part of the action can be simplified by using the invariance of the supertrace under cyclic permutations:

$$\text{Str}[Q_i Q_j] = \text{Str}[U_i \sigma D_i \sigma U_i^\dagger U_j \sigma D_j \sigma U_j^\dagger] = \text{Str}[\sigma U_j^\dagger U_i \sigma D_i \sigma U_i^\dagger U_j \sigma D_j], \quad (\text{A.6})$$

and the following property of  $2 \times 2$  supermatrices (cf [26]):

$$U^\dagger(\eta_i) U(\eta_j) = U(\eta_j - \eta_i) e^{\frac{1}{2}(\eta_i^* \eta_j - \eta_j^* \eta_i)}. \quad (\text{A.7})$$

This allows one to obtain equation (24).

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