Vortex and Translational Currents due to Broken Time-Space Symmetries

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We consider the classical dynamics of a particle in a ($d = 2, 3$)-dimensional space-periodic potential under the influence of time-periodic external fields with zero mean. We perform a general time-space symmetry analysis and identify conditions, when the particle will generate a nonzero averaged translational and vortex currents. We perform computational studies of the equations of motion and of corresponding Fokker-Planck equations, which confirm the symmetry predictions. We address the experimentally important issue of current control. Cold atoms in optical potentials and magnetic traps are among possible candidates to observe these findings experimentally.

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The idea of directed motion under the action of an external fluctuating field of zero mean goes back to Smoluchowski and Feynman [1]. It has been intensively studied in the past decades again [2]. It is believed to be connected with the functioning of molecular motors, and can be applied to transport phenomena which range from mechanical engines to an electron gas (see [3,4] and references therein).

The separation of the fluctuating fields into an uncorrelated white noise term and a time-periodic field was used to perform a symmetry analysis of the most simple case—a pointlike particle moving in a one-dimensional periodic potential [5]. It allowed to systematically choose space and time dependencies of potentials and ac fields such, that a nonzero dc current is generated. Various studies of the dynamical mechanisms of rectification have been reported (e.g., [6]). Among many experimental reports, we mention the successful testing of the above symmetry analysis using cold atoms in one-dimensional optical potentials [7]. By use of more laser beams, experimentalists can already fabricate two- and three-dimensional optical potentials, with different symmetries and shapes [8], with the aim of even more controlled stirring of cold atoms in these setups.

A particle which is moving in a $d = 2$, 3-dimensional periodic potential may contribute to a directed current along a certain direction. At the same time, the particle can perform vortex motion (which is not possible in a one-dimensional setting) generating a nonzero average of the angular momentum. Directed translational currents are supported by unbounded trajectories while vortex currents may be localized in a finite volume. The question is then, how can we control a type of the directed motion? To answer this question, we use the symmetry analysis which allows us to predict an appearance of certain directed currents. Namely, we identify the symmetries which ensure that either translational- or vortex components of the directed current are strictly zero. Breaking these symmetries one by one allows us to control the particle motion generating either directed, or vortex, currents, or both simultaneously. This is the main result of the present Letter.

We consider the dynamics of a classical particle (e.g., an atom of a cold dilute atom gas, loaded onto a proper optical lattice) exposed to an external potential field:

$$\mu\ddot{\mathbf{r}} + \gamma\dot{\mathbf{r}} = \mathbf{g}(\mathbf{r}, t) + \boldsymbol{\xi}(t), \qquad \mathbf{g}(\mathbf{r}, t) = -\boldsymbol{\nabla}U(\mathbf{r}, t). \quad (1)$$

Here $\mathbf{r} = \{x, y, z\}$ is the coordinate vector of the particle, the parameter $\gamma \geq 0$ characterizes the dissipation strength, and $\mu \geq 0$ defines the strength of the inertial term [9]. The force $\mathbf{g}(\mathbf{r}, t) = \{g_\alpha(\mathbf{r}, t)\}$, $\alpha = x, y, z$, is time and space periodic:

$$\mathbf{g}(\mathbf{r}, t) = \mathbf{g}(\mathbf{r}, t + T) = \mathbf{g}(\mathbf{r} + \mathbf{L}_\alpha, t), \qquad \alpha = x, y, z. \quad (2)$$

Here $\mathbf{L}_\alpha$ are the components of the basis of the unit cell. The absence of a dc bias implies

$$\langle \mathbf{g}(\mathbf{r}, t)\rangle_{\mathbf{L},T} \equiv \int_0^T \int_{\mathbf{L}} \mathbf{g}(\mathbf{r}, t)\, dt\, dx\, dy\, dz = 0 \quad (3)$$

where the spatial integration extends over one unit cell.

The fluctuating force is modeled by a δ-correlated Gaussian white noise, $\boldsymbol{\xi}(t) = \{\xi_x, \xi_y, \xi_z\}$, $\langle \xi_\alpha(t)\xi_\beta(t')\rangle = 2\gamma D\delta(t - t')\delta_{\alpha\beta}$ (α, $\beta = x, y, z$). Here D is the noise strength. The statistical description of the system (1) at $\mu = 1$ is provided by the Fokker-Planck equation (FPE) [10]:

$$\{\partial_t + \boldsymbol{\nabla}_{\mathbf{r}} \cdot \mathbf{v} - \boldsymbol{\nabla}_{\mathbf{v}} \cdot [\gamma\mathbf{v} - \mathbf{g}(\mathbf{r}, t)] - \gamma D\Delta_{\mathbf{v}}\}P(\mathbf{r}, \mathbf{v}, t) = 0, \quad (4)$$

where $\mathbf{v} = \dot{\mathbf{r}}$. The respective FPE for $\mu = 0$ reads

$$\gamma\partial_t P(\mathbf{r}, t) = -[\boldsymbol{\nabla}_{\mathbf{r}} \cdot \mathbf{g}(\mathbf{r}, t) - D\Delta_{\mathbf{r}}]P(\mathbf{r}, t). \quad (5)$$

Each of the linear equations (4) and (5) has a unique attractor solution, $\hat{\mathbf{P}}$ which is space and time periodic [10].

Directed transport.—Let us consider the dc component of the directed current in terms of the attractor $\hat{\mathbf{P}}$:

$$\mathbf{J} = \langle \mathbf{v} \cdot \hat{\mathbf{P}}(\mathbf{r}, \mathbf{v}, t)\rangle_{T,\mathbf{L}}, \qquad \mu = 1, \tag{6}$$

$$\mathbf{J} = \gamma^{-1}\langle \mathbf{g}(\mathbf{r}, t) \cdot \hat{\mathbf{P}}(\mathbf{r}, t)\rangle_{T,\mathbf{L}}, \qquad \mu = 0. \tag{7}$$

The strategy is now to identify symmetry operations which invert the sign of $\mathbf{v}$, and, at the same time, leave Eq. (1) invariant. If such symmetries exist, the dc current $\mathbf{J}$ will strictly vanish. Sign changes of the current can be obtained by either inverting the spatial coordinates, or time (simultaneously allowing for shifts in the other variables). Below we list all operations together with the requirements the force $\mathbf{g}$ and the control parameters have to fulfill:

$$\hat{S}_1: \mathbf{r} \to -\mathbf{r} + \mathbf{r}', \; t \to t + \tau; \qquad \hat{S}_1(\mathbf{g}) \to -\mathbf{g}, \tag{8}$$

$$\hat{S}_2: \mathbf{r} \to \mathbf{r} + \Lambda, \; t \to -t + t'; \\ \hat{S}_2(\mathbf{g}) \to \mathbf{g} \;\; (\text{if } \gamma = 0); \qquad \hat{S}_2(\mathbf{g}) \to -\mathbf{g} \;\; (\text{if } \mu = 0). \tag{9}$$

Here t' and $\mathbf{r}'$ depend on the particular shape of $\mathbf{g}(\mathbf{r}, t)$ [11]. The system must be invariant under a spatial translation by the vector $2\mathbf{\Lambda}$ in space and 2τ in time, respectively. The vector $\mathbf{\Lambda}$ is therefore given by $\mathbf{\Lambda} = \sum_\alpha n_\alpha \mathbf{L}_\alpha/2$, $n_\alpha = 0$, 1, while $\tau = 0$, $T/2$. By a proper choice of $\mathbf{g}$ all relevant symmetries can be broken, and one can then expect the appearance of a nonzero dc current $\mathbf{J}$ [12].

To be more precise, we consider the case of a particle moving in a two-dimensional periodic potential and being driven by an external ac field: $\mathbf{g}(\mathbf{r}, t) = -\nabla V(\mathbf{r}) + \mathbf{E}(t) \equiv \mathbf{f}(\mathbf{r}) + \mathbf{E}(t)$. The symmetry $\hat{S}_1$ holds if the potential force is *antisymmetric*, $\mathbf{f}(-\mathbf{r} + \mathbf{r}') = -\mathbf{f}(\mathbf{r})$, and the driving function is *shift-symmetric*, $\mathbf{E}(t + T/2) = -\mathbf{E}(t)$. The symmetry $\hat{S}_2$ holds at the Hamiltonian limit, $\gamma = 0$, if the driving force is *symmetric*, $\mathbf{E}(-t + t') = \mathbf{E}(t)$.

Finally, the symmetry $\hat{S}_2$ holds at the overdamped limit, $\mu = 0$, if the potential force is shift-symmetric, $\mathbf{f}(\mathbf{r} + \mathbf{\Lambda}) = -\mathbf{f}(\mathbf{r})$ and the driving force is antisymmetric, $\mathbf{E}(t + t') = -\mathbf{E}(-t)$.

In order to break the above symmetries, we choose

$$V(\mathbf{r}) = V(x, y) = \cos(x)[1 + \cos(2y)], \tag{10}$$

$$E_{x,y}(t) = E^{(1)}_{x,y} \sin t + E^{(2)}_{x,y} \sin(2t + \theta). \tag{11}$$

The potential (10) is shift-symmetric, $\mathbf{\Lambda} = \{\pm\pi, 0\}$. The symmetry $\hat{S}_1$ is broken since E is not shift-symmetric. Therefore in general we expect $\mathbf{J} \neq \mathbf{0}$.

In Fig. 1 we show the computational evaluation of equations (4) and (5). We confirm the presence of a nonzero dc current. Applying operations $\hat{S}_1$ and $\theta \to \theta + \pi$ we conclude $\mathbf{J}(\theta + \pi) = -\mathbf{J}(\theta)$, which allows for an easy inversion of the current direction, as also confirmed by the data in Fig. 1(a). In the overdamped limit $\mu = 0$, $\hat{S}_2$ is restored for $\theta = 0$, $\pm\pi$, and therefore $\mathbf{J}(-\theta) = -\mathbf{J}(\theta)$ [thick lines in Fig. 1(a)]. Upon approaching the Hamiltonian limit, $\gamma \to 0$, the points where $\mathbf{J} = \mathbf{0}$ shift from $\theta = 0$, π to $\theta = \pm\pi/2$ where the symmetry $\hat{S}_2$ is restored [thin lines in Fig. 1(a)]. In the underdamped regime, the dc current can be approximated as $J_\alpha \propto J^{(0)}_\alpha \sin[\theta - \theta^{(0)}_\alpha(\gamma)]$, $\alpha = x$, y. The phase lag is equal to $\theta^{(0)}_{x,y} = \pi/2$ and $\theta^{(0)}_{x,y} = 0$ in the Hamiltonian and overdamped limits, respectively [Fig. 1(c)] [13].

Even more control over the current direction is possible, by imposing the symmetry conditions (8) and (9) on each component $g_\alpha(\mathbf{r}, t)$ independently. For (10) and (11) with $E^{(2)}_x = E^{(1)}_y = 0$, $\theta = 0$ the symmetry transformation $\hat{S}_c$: $x \to -x$, $y \to y$, $t \to t + \pi$ implies that the current along the x direction is absent, $J_x = 0$, and directed transport is happening along the y axis [see Fig. 1(b), curve (i)]. We may conclude, that the symmetry analysis turns out to be a powerful tool of predicting and controlling directed currents of particles which move in $d = 2$, 3-dimensional potentials under the influence of external ac fields. Note that dynamical mechanisms of current rectification of a

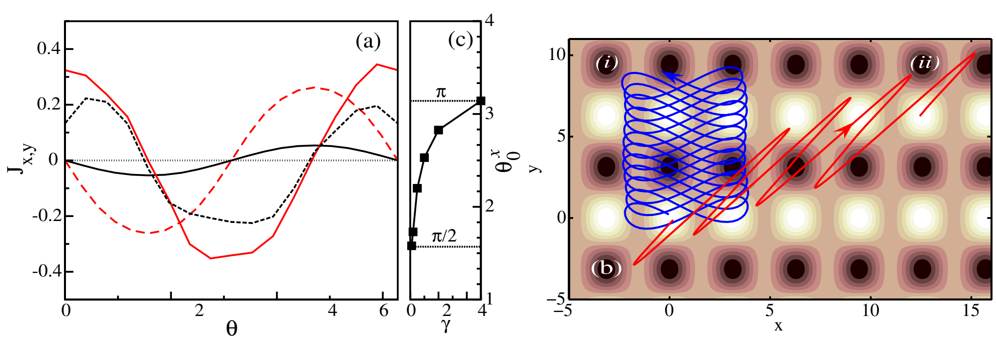

FIG. 1 (color online). (a) Dependence of the current components, J_x(solid line) and J_y(dashed line), on θ for (1), (10), and (11) with $D = 1$, $E^{(1)}_x = -E^{(2)}_x = 2$, $E^{(1)}_y = -E^{(2)}_y = 2.5$. Data are for the overdamped ($\mu = 0$, $\gamma = 1$ thick lines) and underdamped ($\mu = 1$, $\gamma = 0.1$ thin lines) cases, respectively; (b) the time evolution of the mean particle position, $\bar{\mathbf{r}}(t) = \int \mathbf{r} P(\mathbf{r}, \mathbf{v}, t)\, d\mathbf{r}\, d\mathbf{v}$, for $\theta = 0$. The trajectories are superimposed on the contour plot of the potential (10). Curve (i) corresponds to $E^{(1)}_x = 3$, $E^{(2)}_x = E^{(1)}_y = 0$, $E^{(2)}_y = 3.5$ and curve (ii)—to the parameters of panel (a). The other parameters are $D = \mu = 1$, $\gamma = 0.1$; (c) the phase lag $\theta^{(0)}_x$ as a function of the dissipation strength γ.

two-dimensional deterministic tilting ratchet were discussed in Ref. [14].

Vorticity.—At variance to the one-dimensional case, particles in two and three dimensions can perform vortex motion, thereby generating ring currents, or nonzero angular momentum. First of all we note that the particle dynamics is not confined to one spatial unit cell of the periodic potential $U(\mathbf{r}, t)$. Even in the case when a directed current is zero due to the above symmetries, $\mathbf{J} = 0$, the particle can perform unbiased diffusion in coordinate space. In order to distinguish between directed transport and spatial diffusion on one side, and rotational currents on the other side, we use the angular velocity [15]

$$\mathbf{\Omega}(t) = [\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)]/\dot{\mathbf{r}}^2(t), \qquad \mathbf{J}_{\mathbf{\Omega}} = \langle \mathbf{\Omega}(t) \rangle_t, \qquad (12)$$

as a measure for the particle rotation, where $\langle \ldots \rangle_t = \lim_{t\to\infty} \frac{1}{t} \int_0^t \ldots dt'$. $\mathbf{\Omega}(t)$ is invariant under translations in space and time. It describes the speed of rotation with which the velocity vector $\dot{\mathbf{r}}$ [the tangential vector to the trajectory $\mathbf{r}(t)$] encompasses the origin.

Using the above strategy, we search for symmetry operations that leave the equations of motion invariant, but do change the sign of the angular velocity. If such symmetries exist, rotational currents strictly vanish on average. The sign of $\mathbf{\Omega}$ can be inverted by either (i) time inversion $t \to -t$ together with an optional space inversion $\mathbf{r} \to \pm\mathbf{r}$, or (ii) the permutation of any two variables, e.g., $\hat{\mathcal{P}}_{xy}$: $\{x, y, z\} \to \{y, x, z\}$. That leads to the following possible symmetry transformations:

$$\hat{R}_1\colon \mathbf{r} \to \hat{\mathcal{P}}\mathbf{r} + \mathbf{r}', \; t \to t + \tau; \qquad \hat{R}_1(\mathbf{g}) \to \mathbf{g}, \qquad (13)$$

$$\begin{aligned} &\hat{R}_2\colon \mathbf{r} \to \pm\mathbf{r} + \mathbf{\Lambda}, \; t \to -t + t', \\ &\hat{R}_2(\mathbf{g}) \to \mathbf{g} \;\; (\text{if } \gamma = 0); \\ &\hat{R}_2(\mathbf{g}) \to -\mathbf{g} \;\; (\text{if } \mu = 0). \end{aligned} \qquad (14)$$

Here $\hat{\mathcal{P}}$ stands for any of the following operations: $\hat{\mathcal{P}}_{xy}$, $\hat{\mathcal{P}}_{yz}$, or $\hat{\mathcal{P}}_{zx}$ and t' and $\mathbf{r}'$ again depend on the particular shape of $\mathbf{g}(\mathbf{r}, t)$.

To be concrete, we will again consider a particle moving in a two-dimensional periodic potential and being driven by an external ac field: $\mathbf{g}(\mathbf{r}, t) = -\nabla V(\mathbf{r}) + \mathbf{E}(t) \equiv \mathbf{f}(\mathbf{r}) + \mathbf{E}(t)$. For $d = 2$ there is an additional transformation due to a mirror reflection at any axis, $\hat{\Sigma}_x$: $\{x, y\} \to \{x, -y\}$ or $\hat{\Sigma}_y$: $\{x, y\} \to \{-x, y\}$,

$$\hat{R}_3\colon \mathbf{r} \to \hat{\Sigma}\mathbf{r}, \qquad t \to t + T/2; \qquad \hat{R}_3(\mathbf{g}) \to \mathbf{g}. \qquad (15)$$

Symmetry $\hat{R}_1$ can be satisfied for the *Hamiltonian*, *underdamped*, and *overdamped* cases if $\hat{\mathcal{P}}\mathbf{f}(\hat{\mathcal{P}}\mathbf{r} + \mathbf{r}') = \mathbf{f}(\mathbf{r})$ and $\hat{\mathcal{P}}\mathbf{E}(t + t') = \mathbf{E}(t)$. The symmetry $\hat{R}_2$ apply for the same cases as their counterparts $\hat{S}_2$ if there is no space inversion. In the presence of space inversion they can be satisfied both in the Hamiltonian [if $\mathbf{f}(-\mathbf{r}) = -\mathbf{f}(\mathbf{r})$ and $\mathbf{E}(t + t') = -\mathbf{E}(-t)$] and overdamped [if $\mathbf{f}(-\mathbf{r}) = \mathbf{f}(\mathbf{r})$ and $\mathbf{E}(t + t') = \mathbf{E}(-t)$] limits. The symmetry $\hat{R}_3$ is relevant if $\hat{\Sigma}_x$ can be applied: $f_x(x, -y) = f_x(x, -y)$, $f_y(x, -y) = -f_y(x, -y)$, $E_x(t + T/2) = E_x(t)$, and $E_y(t + T/2) = -E_y(t)$. Similar conditions can be found for $\hat{\Sigma}_y$.

We performed numerical integrations of the equation of motion (1) with the following potential and driving force:

$$V(x, y) = [-3(\cos x + \cos y) + \cos x \cos y]/2, \qquad (16)$$

$$E_x(t) = E_x^{(1)} \cos t, \qquad E_y(t) = E_y^{(1)} \cos(t + \theta). \qquad (17)$$

Averaging was performed over $N = 10^5$ different stochastic realizations [16]. Figure 2 shows the dependence of the rotational current (12) on the relative phase θ. The system is invariant under the transformation $\hat{S}_1$ (8), therefore the directed current $\mathbf{J} = \mathbf{0}$. However, for the underdamped case, $\gamma \neq 0$, all the relevant symmetries (13)–(15) are violated, and the resulting rotational current (12) is nonzero, and depends on the phase θ [Fig. 2(a)]. Note that symmetry $\hat{R}_2$ is restored when $\theta = 0, \pm\pi$, thus the current disappears in the Hamiltonian and overdamped limits for these values of the phase. The left upper inset in Fig. 2 shows the actual trajectory of a given realization, confirming that the particle is acquiring an average nonzero angular momentum, while not leaving a small finite volume due to slow diffusion and absence of directed currents.

The exact overdamped limit, $\mu = 0$, is singular for the definition (12) since the velocity of a particle, $\dot{\mathbf{r}}(\mathbf{t})$, is a nowhere differentiable function. The overdamped limit can be approached by increasing γ at a fixed $\mu = 1$. Alternatively, one may remove the restriction on μ allowing for an infinitesimal value $0 < \mu \ll 1$ at a fixed dissipation strength $\gamma = 1$. Both parameter choices equally regularize (12). Numerical simulations for the former way of regularization show that if $\gamma/\mu \geq 5$ then the rotational current completely reflects the symmetries corresponding

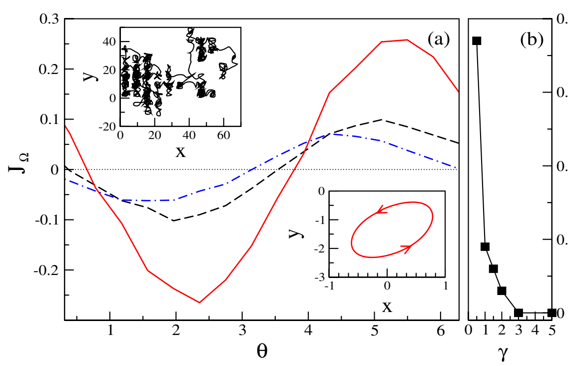

FIG. 2 (color online). (a) Dependence $J_\Omega(\theta)$, Eq. (12), for (1), (16), and (17), with $\mu = 1$, $D = 0.5$, $E_x^{(1)} = 0.4$, $E_y^{(1)} = 0.8$ and $\gamma = 0.2$ (solid line), $\gamma = 0.05$ (dashed line), and $\gamma = 2$ (dash-dotted line). Insets: the trajectory (left inset) and the corresponding attractor solution, $\bar{\mathbf{r}}(t)$, (right inset) for the case $\gamma = 0.2$ and $\theta = \pi/2$. We have used $N = 10^5$ independent stochastic realizations to perform the noise averaging; (b) the phase lag θ_0 as a function of the dissipation strength γ.

to the overdamped case [see dependence $\theta_0(\gamma)$ in Fig. 2(b)].

Let us discuss the relation of our results to the case of multidimensional stochastic tilting ratchets under the influence of a colored noise studied previously [17]. Since equivalent (in a statistical sense) stochastic processes, $\boldsymbol{\xi}(t)$, have been used as driving forces, the symmetry $\hat{R}_1$ (13) can be violated only by an asymmetric potential. But all potentials considered in Refs. [17] are invariant under the permutation transformation $\hat{\mathcal{P}}$. As a consequence, vortex structures for a *local velocity field* presented in Refs. [17] are completely symmetric (clockwise vortices are mapped into counterclockwise ones by $\hat{\mathcal{P}}$) and, therefore, the average rotation for any trajectory equals zero.

The phase space dimension is five for $d = 2$ and seven for $d = 3$. Therefore, in the Hamiltonian limit ($\gamma = 0$), Arnold diffusion [18] takes place. The particle dynamics is no longer confined within chaotic layers of finite width. That leads to unbounded, possibly extremely slow, diffusion in the momentum subspace via a stochastic web [18]. Therefore a direct numerical integration of the equations of motion may lead to incorrect conclusions.

Our method of directed current control can be applied to a rich variety of physical settings, such as cold atoms in two- and three-dimensional potentials (optical guiding) [19], colloidal particles on magnetic bubble lattices [20], ferrofluids [21], and vortices in superconducting films with pinning sites [22], to name a few. We also expect that our theory can lead to an enhancement of particle separation in laser beams of complex geometry [23].

To conclude, we formulated conditions for the absence of both translational and rotational components of the directed current generated by particles moving in spatially periodic potentials, under the influence of external ac fields. Proper choices of these potentials and fields allow us to break the above symmetries, and therefore provide necessary conditions for a generation of certain directed currents. As long as the symmetry conditions for the absence of translational and vortex currents are independent one can tune the experimental setup to obtain either translational or rotational currents. Numerical studies supplement the symmetry analysis and confirm the conclusions.

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