Selective control of the symmetric Dicke subspace in trapped ions

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We propose a method of manipulating selectively the symmetric Dicke subspace in the internal degrees of freedom of N trapped ions. We show that the direct access to ionic-motional subspaces, based on a suitable tuning of motion-dependent ac Stark shifts, induces a two-level dynamics involving previously selected ionic Dicke states. In this manner, it is possible to produce, sequentially and unitarily, ionic Dicke states with increasing excitation number. Moreover, we propose a probabilistic technique to produce directly any ionic Dicke state assuming suitable initial conditions.

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I. INTRODUCTION

Multipartite entangled states play a fundamental role in quantum information, where these states are used for different applications including the improvement of spectroscopy towards the Heisenberg limit [1]. In this sense, general sequential techniques for building entangled multipartite states have been recently proposed [2]. In Ref. [3], an experiment is described where the robust one-excitation symmetric Dicke states [4], called W, of $N \leqslant 8$ ions are prepared in their electronic levels with the aid of N entangling pulses. Also, a maximally entangled [Greenberger-Horne-Zeilinger (GHZ)] state with six ions has been experimentally realized [5]. From a theoretical point of view, adiabatic ground-state transitions were proposed for generating GHZ states and symmetric Dicke states with $N/2$ excitations in N ions [6]. More recently, a method for generating multiqubit entangled states via global addressing of an ion chain in the frame of the Tavis-Cummings model has been discussed [7]. A four-qubit W state with two excitations has already been realized in linear optics [8], which may present astonishing multipartite properties [9], and more general proposals may be considered [10]. It is well-established that a physical system must fulfill several requirements in order to qualify as a potential candidate for quantum computing tasks [11]. Among them, overcoming decoherence and scalability considerations may require not only efficient single- and two-qubit gates but also the availability of collective multipartite operations in suitable subspaces.

In this paper, we consider a system composed of N trapped ions addressed collectively by two laser fields in a global lambda-type excitation scheme. We will introduce a method for tailoring the Hilbert space in order to restrict the quantum dynamics to the symmetric Dicke subspace. As we show below, this method allows a different and useful way to manipulate selectively the collective ionic-motional system. In particular, these multipartite selective interactions will permit the generation of ionic Dicke states with any number of excitations in a sequential manner or, through a probabilistic technique, in a single-shot measurement. This method is based on global selective interactions characterized by a proper tuning of collective motion-dependent Stark shifts. Selective interactions with a single atom have been proposed in the realm of cavity QED [12] and trapped ions [13,14]. Furthermore, it has been demonstrated that they also allow the generation of arbitrary harmonic oscillator states [15] and their measurement via instantaneous interactions [16].

II. MODEL

Let us consider a Raman laser excitation of N three-level trapped ions as shown in Fig. 1. We will make use of these internal levels and the collective center-of-mass motional mode associated with the frequency ν. A traveling-wave field excites the transition between the states $|g_j\rangle \leftrightarrow |c_j\rangle$, with coupling strength $\Omega_{2j}=\Omega_{2j}(\vec{r}_j)$ and detuning $\Delta(\Delta \gg \Omega_{2j})$. Similarly, a standing-wave field excites off resonantly the transition between the electronic internal states $|e_j\rangle \leftrightarrow |c_j\rangle$, with position-dependent coupling strength $\Omega_{1j}=\Omega_{1j}(\vec{r}_j)$ and detuning $\Delta+\nu \gg \Omega_{1j}$. This scenario is described, after a first optical rotating-wave approximation (RWA), by the Hamiltonian

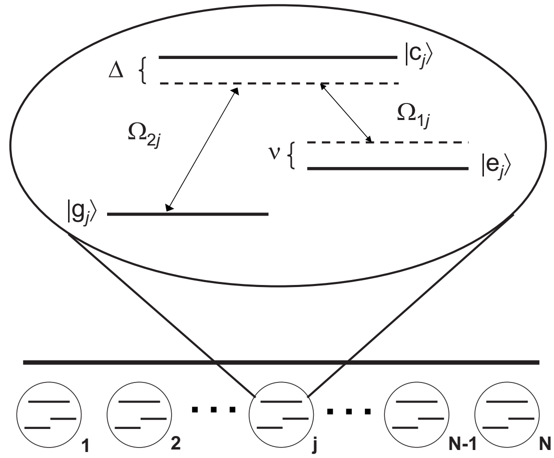

FIG. 1. N three-level ions in a linear Paul trap where the energy diagram of the jth ion is displayed.

$$\hat{H} = \hbar\nu\hat{a}^\dagger\hat{a} + \hbar\omega_e\sum_{j=1}^{N}|e_j\rangle\langle e_j| + \hbar\omega_c\sum_{j=1}^{N}|c_j\rangle\langle c_j| + \hbar\Bigg[\cos(k_1\hat{z})e^{i\omega_1 t}\sum_{j=1}^{N}\Omega_{1j}|e_j\rangle\langle c_j| + e^{-i(k_2\hat{z}-\omega_2 t)}\sum_{j=1}^{N}\Omega_{2j}|g_j\rangle\langle c_j| + \text{H.c.}\Bigg]. \quad (1)$$

We go then to an interaction picture inside the Lamb-Dicke regime: $\eta_i\sqrt{\bar{n}}\ll 1$, where $\bar{n}$ is the average phonon number and $\eta_i \equiv k_i\sqrt{\hbar/2m\nu}$ are the Lamb-Dicke parameters. In this way, we can adiabatically eliminate levels $|c_j\rangle$, obtaining the blue-sideband second-order effective Hamiltonian

$$\hat{H}_{\text{eff}} = -\hbar\hat{\Delta} + \hbar(\hat{a}^\dagger\hat{\tilde{J}}^+ + \hat{a}\hat{\tilde{J}}^-), \quad (2)$$

where $\hat{\tilde{J}}^+ = \sum_{j=1}^{N}\Omega_{\text{eff}}^j\hat{\sigma}_j^\dagger$, with $\Omega_{\text{eff}}^j = 2i\eta_2\Omega_{1j}\Omega_{2j}^*/\Delta$, $\hat{\sigma}_j^\dagger = |e_j\rangle\langle g_j|$, and

$$\hat{\Delta} = \frac{1}{\Delta}\sum_{j=1}^{N}[1-\eta_1^2(2\hat{a}^\dagger\hat{a}+1)]|\Omega_{1j}|^2|g_j\rangle\langle g_j| + \frac{1}{\Delta}\sum_{j=1}^{N}|\Omega_{2j}|^2|e_j\rangle\langle e_j| \quad (3)$$

is the motion-dependent ac Stark shift. In this case, we can discard terms involving level $|c_j\rangle$ by assuming no initial population. The phonon-number dependence of the Stark shift (3) is due to the standing-wave Raman laser, which together with the traveling wave produce the dynamics of Eq. (2). Note that ac Stark shifts have already been used for experimental realization of two-qubit gates and multipartite entanglement [17].

The detuning $\hat{\Delta}$ can be corrected by a fixed position-dependent quantity δ_0^j via dc Stark shift or retuning of the lasers frequencies. In this manner, Hamiltonian (2) can be written as

$$\hat{H}_{\text{eff}} = -\hbar\sum_{j=1}^{N}\Omega_0^j(\hat{n}-\delta_0^j)|g_j\rangle\langle g_j| + \hbar(\hat{a}^\dagger\hat{\tilde{J}}^+ + \hat{a}\hat{\tilde{J}}^-), \quad (4)$$

where $\Omega_0^j = 2\eta_1^2|\Omega_{1j}|^2/\Delta$. It will be convenient to rewrite the Hamiltonian of Eq. (4) in the interaction picture with respect to the first term, where it reads

$$\hat{H}_{\text{eff}}^I = \hbar\sum_{j=1}^{N}\Omega_{\text{eff}}^j\hat{a}^\dagger\hat{\sigma}_j^\dagger e^{-i[\Omega_0^j(\hat{n}-\delta_0^j)+\sum_{k\neq j}\Omega_0^k|g_k\rangle\langle g_k|]t} + \text{H.c.} \quad (5)$$

III. SELECTIVE CONTROL IN THE HOMOGENEOUS COUPLING CASE

A. Generalized selectivity

In order to illustrate how selectivity appears in the N-ion case, let us study the special situation of the N ions coupled homogeneously to the Raman lasers where $\Omega_{\text{eff}}^j \equiv \Omega_{\text{eff}} = 2i\eta_2\Omega_1\Omega_2^*/\Delta$, $\Omega_0^j \equiv \Omega_0 = 2\eta_1^2|\Omega_1|^2/\Delta$, and $\delta_0^j \equiv \delta_0$. In this case, the interaction part in Hamiltonian (4) corresponds to an anti-Tavis-Cummings model [18], a spin $j=N/2$ generalization of the Jaynes-Cummings model [19]. In this case, $\hat{\tilde{J}}^+ \to \Omega_{\text{eff}}\hat{J}^\pm$, and the new collective terms $\hat{J}^\pm$ can be considered as angular momentum operators, establishing a permutation symmetry on the ionic subsystem dynamics. That means that if the system is found at any time inside the symmetric Dicke subspace [4], associated with total angular momentum $j=N/2$, it will stay there along its evolution, reducing the Hilbert space dimension from 2^N to $N+1$. Under this plausible assumption, the collective operators $\hat{J}^\pm$ can be effectively and exclusively rewritten in the symmetric Dicke subspace via the following assignments:

$$\sum_{j=1}^{N}|g_j\rangle\langle g_j| \to \sum_{k=0}^{N-1}(N-k)|D_k\rangle\langle D_k|,$$

$$\hat{J}^+ \to \sum_{k=0}^{N-1}f_k|D_{k+1}\rangle\langle D_k|. \quad (6)$$

Here,

$$|D_k\rangle = \binom{N}{k}^{-1/2}\sum_k P_k(|g_1,g_2,\ldots,g_{N-k},e_{N-k+1},\ldots,e_N\rangle) \quad (7)$$

are the symmetric Dicke states with k excitations, $\{P_k\}$ is the set of all distinct permutations, and $f_k = \sqrt{(k+1)(N-k)}$. It is noteworthy to stress that in the assignments of Eq. (6) we have omitted the nonsymmetric components due to the assumed initial symmetric conditions. In this case, and under homogeneous driving, we can derive from Eq. (4) an analog to Eq. (5),

$$\hat{\tilde{H}}_{\text{eff}}^I = \hbar\hat{a}^\dagger\Omega_{\text{eff}}\sum_{k=0}^{N-1}f_k e^{i\Omega_0(-\hat{n}+N-1-k+\delta_0)t}|D_{k+1}\rangle\langle D_k| + \text{H.c.}, \quad (8)$$

a compact expression that will prove useful to study selective interactions inside the symmetric subspace. Let us consider the system prepared in the initial state $|N_0\rangle|D_{k_0}\rangle$. Then, the suitable choice of laser frequencies $\delta_0 = k_0+N_0-N+1$ yields a selective resonant coupling inside the subspace $\{|N_0\rangle|D_{k_0}\rangle, |N_0+1\rangle|D_{k_0+1}\rangle\}$. Moreover, provided that $\Omega_0 \gg \Omega_{\text{eff}}$, all other subspaces will remain off resonance obtaining an effective two-level dynamics. That is, by selecting a determined subspace the Hamiltonian (8) can be written as

$$\hat{\tilde{H}} = \hbar\sqrt{N_0+1}\,\Omega_{\text{eff}}f_{k_0}(\hat{\sigma}_{N_0}^+\hat{J}_{k_0}^+ + \hat{\sigma}_{N_0}^-\hat{J}_{k_0}^-), \quad (9)$$

where $\hat{J}_{k_0}^+ = |D_{k_0+1}\rangle\langle D_{k_0}|$ and $\hat{\sigma}_{N_0}^+ = |N_0+1\rangle\langle N_0|$ are effective spin-1/2 operators stemming from the reduced Hilbert space of the collective ionic state and the bosonic field, respectively. As we will se below, this selective global interaction will allow us to move comfortably inside the symmetric Dicke subspace with high precision [20].

Considering experimental parameters of ion experiments at NIST (Boulder, CO) [21], we could achieve an effective coupling $\Omega_{\text{eff}} \sim 10^5$ Hz, which produces population inversion in the subspace $\{|N_0\rangle|D_{k_0}\rangle, |N_0+1\rangle|D_{k_0+1}\rangle\}$ in a time $\tau \leqslant 0.1$ ms, shorter than the typical motional decoherence time $\tau_d \sim 10$ ms.

B. Applications of generalized selectivity

We discuss now some applications of our method for selectively manipulating the Dicke subspace. Let us consider the initial state $|\Psi(0)\rangle = |0\rangle|g\cdots g\rangle \equiv |0\rangle|D_0\rangle$. Tuning into resonance the subspace transition $\{|0\rangle|D_0\rangle, |1\rangle|D_1\rangle\}$, the evolution of this state is given by

$$|\Psi(t)\rangle = \cos(\sqrt{N}|\Omega_{\text{eff}}|t)|0\rangle|D_0\rangle - ie^{i\phi}\sin(\sqrt{N}|\Omega_{\text{eff}}|t)|1\rangle|D_1\rangle, \tag{10}$$

where $\Omega_{\text{eff}} = |\Omega_{\text{eff}}|e^{-i\phi}$. The one-excitation Dicke state $|D_1\rangle$ is also a W state

$$|W_N\rangle = \frac{1}{\sqrt{N}}(|eg\cdots g\rangle + |geg\cdots g\rangle + \cdots + |g\cdots ge\rangle). \tag{11}$$

This N-partite entangled state has great importance in quantum information theory due to its persistent entanglement properties, as long as more operational effort is needed to disentangle this state [22]. If this interaction is turned on for a time $2\sqrt{N}|\Omega_{\text{eff}}|t = \pi$ and $\phi = \pi/2$, then Eq. (10) becomes

$$|\Psi(t)\rangle = |1\rangle|D_1\rangle \equiv |1\rangle|W_N\rangle, \tag{12}$$

yielding state $|W_N\rangle$ in the metastable N two-level ions. If the system evolves for a time such that $\cos(\sqrt{N}|\Omega_{\text{eff}}|t) = 1/\sqrt{N+1}$, then

$$|\Psi(t)\rangle = |W_{N+1}\rangle, \tag{13}$$

where the $(N+1)$th qubit is the reduced bosonic spin-1/2 system.

Once the system is prepared in the state given in Eq. (12), and tuning to resonance the red-sideband subspace transition $\{|1\rangle|D_1\rangle, |0\rangle|D_2\rangle\}$, a pulse with Rabi angle $2\sqrt{N}|\Omega_{\text{eff}}|t_2 = \pi$ will lead to

$$|\Psi(t_2)\rangle = |0\rangle|D_2\rangle. \tag{14}$$

In this manner, it is clear that a successive application of collective blue- and red-sideband interactions can produce deterministically and sequentially all symmetric Dicke states $|D_k\rangle$ with number of excitations k.

Another interesting application of multipartite selective interactions is the possibility to discriminate between ionic states with a different number of excitations. Suppose we have an ionic state prepared in a superposition of states with a different number of excitations $\Sigma_{k=0}^{N} c_k|D_k\rangle$, with $\Sigma_{k=0}^{N}|c_k|^2 = 1$. For example, this state can correspond to an atomic coherent state [23] given by $\exp(i\theta\hat{J}_x)|g\cdots g\rangle$. Note that an interaction proportional to $\hat{J}_x$ can be generated by applying a Raman laser field tuned to the carrier transition on the N ions collectively and homogeneously. The center-of-mass mode is initialized in the state $|N_0\rangle$ and we consider an (additional) ancillary qubit in the ground state $|g\rangle_A$. We tune then to resonance the collective blue-sideband subspace $\{|N_0\rangle|D_{k_0-1}\rangle, |N_0+1\rangle|D_{k_0}\rangle\}$, where $|D_{k_0}\rangle$ is the state with k_0 excitations we want to discriminate. In this way, after a collective π-pulse on the ions, we obtain a state of the form

$$|\Psi_1\rangle = \left(c_{k_0-1}|N_0+1\rangle|D_{k_0}\rangle + |N_0\rangle\sum_{k\neq k_0-1}^{N} c_k|D_k\rangle\right)|g\rangle_A. \tag{15}$$

Now, a π-pulse with the laser field tuned to the first red sideband on the ancillary qubit leads to $|\Psi_2\rangle = (c_{k_0-1}|D_{k_0}\rangle|e\rangle_A + \Sigma_{k\neq k_0-1}^{N} c_k|D_k\rangle|g\rangle_A)|N_0\rangle$. Then, if we measure the ancilla in the excited state $|e\rangle_A$, the collective ionic state will collapse into the Dicke state $|D_{k_0}\rangle$ with k_0 excitations. Remark that the projection on ancillary state $|e\rangle_A$, that should happen with a probability $|c_{k_0-1}|^2$, can be done with high precision via well-established electron-shelving techniques.

On the other hand, it has been shown that the use of selective interactions in a single trapped ion can lead to deterministic and universal manipulation of the motional state [15]. Along these lines, similar manipulation could be implemented here to grant access to arbitrary states inside the symmetric Dicke subspace. In this case, the motional Fock states would be replaced by symmetric states in the internal ionic degrees of freedom with a fixed number of excitations.

IV. SELECTIVE CONTROL IN THE INHOMOGENEOUS COUPLING CASE

In the more general case of ions interacting inhomogeneously with Raman lasers, we cannot discriminate preselected symmetric Dicke states. However, multipartite selectivity will still allow us to manipulate *ionic number states*, that is, ionic states with a determined number of excitations but not necessarily symmetric. For example, if laser fields interact inhomogeneously with initially deexcited trapped ions in a carrierlike excitation of the form $U = \exp(-i\theta\hat{\tilde{J}}_x)$, where $\hat{\tilde{J}}_x = \hat{\tilde{J}}^+ + \hat{\tilde{J}}^-$, this will not lead to a superposition of symmetric Dicke states. On the opposite, this will lead to a superposition of nonsymmetric collective number states arising from the action of the operators $\hat{\tilde{J}}^+$ and $\hat{\tilde{J}}^-$ on the collective ionic states. It is known that to deal with the unitary evolution of high-dimensional inhomogeneously coupled systems is extremely difficult [23–25]. In this case, instead of writing the Hamiltonian (4) in the basis of the symmetric Dicke states, as in Eq. (8), we should write it in the corresponding basis of nonsymmetric collective number states $|\tilde{D}_k^{\ell}\rangle$ with k excitations. In this way, we may look for conditions to set into resonance a determined subspace. States $|\tilde{D}_k^{\ell}\rangle$ appear naturally from successive applications of $\hat{\tilde{J}}^+$ and $\hat{\tilde{J}}^-$ on a given initial collective state. The index ℓ accounts for the fact that, depending on the number of ionic excitations, there could exist more than one nonsymmetric collective state with a determined number of excitations. In the same spirit of Eq. (2), we can write the associated Hamiltonian

$$\hat{H}_{\text{eff}}^{\text{I}} = -\hbar \sum_{k,\ell} \langle \tilde{D}_k^{\ell} | \hat{\Delta} | \tilde{D}_k^{\ell} \rangle | \tilde{D}_k^{\ell} \rangle \langle \tilde{D}_k^{\ell} | + \hbar \hat{a}^{\dagger} \sum_{k,\ell} \tilde{\Omega}_{\text{eff}}^{k,\ell} | \tilde{D}_{k+1}^{\ell} \rangle \langle \tilde{D}_k^{\ell} | + \text{H.c.} \tag{16}$$

Here, $\tilde{\Omega}_{\text{eff}}^{\ell,k}$ is the new effective coupling constant, which in the homogeneous case corresponds to Ω_{eff}. As in the homogeneous case, if $\Omega_0 \gg \tilde{\Omega}_{\text{eff}}^{\ell,k}$, we can tune to resonance a determined subspace, for example, the inhomogeneous blue-sideband doublet $\{|N_0\rangle|\tilde{D}_{k_0}^{\ell}\rangle, |N_0+1\rangle|\tilde{D}_{k_0+1}^{\ell}\rangle\}$. In this case, from Hamiltonian (16) in the interaction picture, we can derive that the condition to tune to resonance this subspace is $\langle \tilde{D}_{k_0+1}^{\ell} | \hat{\Delta}_{N_0+1} | \tilde{D}_{k_0+1}^{\ell} \rangle - \langle \tilde{D}_{k_0}^{\ell} | \hat{\Delta}_{N_0} | \tilde{D}_{k_0}^{\ell} \rangle = 0$. This condition can be fulfilled by compensating the detuning $\hat{\Delta}$ through shifts in the lasers frequencies for fixed values of δ_0^j, depending on the subspace we want to select. This procedure is similar to the homogeneous case, but now δ_0^j will be inhomogenously distributed, that is, different for each ion.

V. CONCLUSIONS

In conclusion, we have introduced a selective technique that allows a collective manipulation of the ionic degrees of freedom inside the symmetric Dicke subspace. We have studied the homogeneous and inhomogeneous cases, showing applications in both cases, mainly related to the generation and control of number states in the ionic external and internal degrees of freedom. We believe that the introduced concepts may inspire similar physics in other quantum-optical setups with diverse applications, and that they might even be helpful to transfer collective atomic number states to propagating fields.

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