Non-Hermitian Luttinger liquids and vortex physics

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Abstract. – We study the effect of a single line defect on vortex filaments oriented parallel to the surface of a thin planar high-T_c superconductor. When the applied field is *tilted* relative to the line defect, the physics is described by a *non-Hermitian* Luttinger liquid of interacting quantum bosons in one spatial dimension with a point defect. With a combination of analytic and numerical methods we uncover a delicate interplay between enhancement of pinning due to Luttinger-liquid effects and depinning due to the tilted magnetic field. Interactions dramatically affect the transverse magnetization when the Luttinger-liquid parameter $g \leq 1$.

The past decade has seen considerable work on the statistical mechanics and dynamics of thermally excited vortices in type-II high-temperature superconductors [1]. The competition between interactions, pinning and thermal fluctuations gives rise to a wide range of novel phenomena, including a low-temperature *Bose glass* phase with vortices strongly pinned to a disordered array of columnar defects [2].

A convenient way of understanding interacting flux lines is provided by the formal mapping between the classical statistical mechanics of $(d + 1)$-dimensional directed flux lines and the nonrelativistic quantum mechanics of d-dimensional bosons. In this mapping, flux lines traversing the sample along the direction of the external magnetic field $\boldsymbol{H} = H\hat{\boldsymbol{z}}$ correspond to boson world lines propagating in imaginary time τ. The classical partition function of thermally excited vortex lines is proportional to a quantum-mechanical matrix element. The thickness of the sample in the z-direction, L_z, corresponds to the inverse temperature $\beta\hbar$ of the bosons, while thermal fluctuations of the vortices, due to finite $k_B T$, play the role of quantum fluctuations of the bosons, controlled by $\hbar$.

If the direction of the external magnetic field does *not* coincide with $\hat{\boldsymbol{z}}$, the direction of the columnar defects, it is convenient to separate the transverse component of the field $H_\perp$ from the parallel one $H_\parallel$ along $\hat{\boldsymbol{z}}$. When $H_\perp \ll H_\parallel$, the transverse component $H_\perp$ plays the role of a constant imaginary vector potential for the bosons [2, 3]. The corresponding quantum
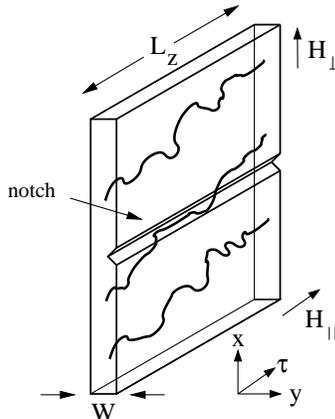

Fig. 1 – Schematic snapshot of vortex lines and a columnar pin realized by a "notch". Here, the z-axis is denoted by imaginary time τ.

Hamiltonian is *non-Hermitian*, with new and interesting properties. For $H_\perp$ less than a critical value $H_\perp^c$, the Bose glass phase mentioned above exhibits a "transverse Meissner effect", such that the vortex filaments remain pinned parallel to the columns even though the external field is tilted away from the column direction [1,2].

In this paper we study the effect of a *single* columnar pin (or an equivalent linear defect) on the statistical mechanics of thermally fluctuating vortex lines confined in a thin, superconducting slab (see fig. 1). Little has been done on vortex physics in the limit of a dilute concentration of columns (or twin planes). An effectively $(1+1)$-dimensional situation can be realized if the thickness W of the sample is comparable the London penetration depth λ. The resulting values of $W \approx 1\,\mu\text{m}$ for high-T_c cuprate compounds can easily be achieved with $\hat{y}$ parallel to the c-axis [1]. A further requirement for reduced dimensionality is that the average vortex spacing a_0 should be larger than the sample thickness, leading to typical fields of order $100\,\text{G}$.

The feasibility of studying vortex physics in samples which are effectively $(1+1)$-dimensional was demonstrated by Bollé *et al.* [4] in thin samples of NbSe$_2$, where the effect of point disorder on interacting vortices near H_{c1} was observed. Similar experiments might be possible on thin high-T_c samples where a single columnar defect could be implemented mechanically by cutting a thin "notch", as shown in fig. 1. A closely related problem in $(2+1)$ dimensions concerns the effect of an isolated twin plane or grain boundary on vortex matter, where point disorder leads to algebraic decay of density correlations [5] of the Abrikosov flux lattice similar to the Luttinger-liquid correlations discussed below. A single such plane has a similar pinning effect on bulk flux lines as does a linear defect in $(1+1)$-dimensional systems subject only to thermal fluctuations.

Although point disorder can be important in $(1+1)$-dimensional geometries [2,6], it can be neglected compared to thermal fluctuations in certain regimes for high-T_c samples. Consider the pinning energy (per length) due to point pins as experienced by a single vortex near a columnar defect [2,7]: $U_{\text{point}} \approx \Delta/a_0 T$ and $U_{\text{columnar}} \approx (\Phi_0/4\pi\lambda)^2$, where Δ is the disorder correlator. In high-T_c compounds, U_{columnar} dominates by several orders of magnitude. A more delicate analysis shows that for clean samples or temperatures close to T_c point disorder can also be neglected away from the pin on the length scales considered here [7,8].

In the following we will consider a single columnar defect or "notch" in a system of in-

teracting flux lines in $(1 + 1)$ dimensions. Although results can also be obtained using a flux-line–related phonon formalism [9], we here found it convenient to work with an equivalent quantum Hamiltonian [1,3]:

$$\hat{H} = -\frac{(k_{\mathrm{B}}T)^2}{2m} \int \mathrm{d}x \Psi^\dagger(x)\left(\frac{\mathrm{d}}{\mathrm{d}x} - h\right)^2 \Psi(x) +$$

$$+ \frac{1}{2} \int \mathrm{d}x\,\mathrm{d}y\, n(x)V(|x - y|)n(y) - \epsilon_0\, n(0), \tag{1}$$

where $V(|x|)$ is a short-range repulsive vortex interaction potential, $\Psi(x)$ annihilates a bosonic flux line, $n(x) = \Psi^\dagger(x)\Psi(x)$ is the boson number density and m is the vortex tilt modulus. The imaginary vector potential $h = \Phi_0 H_\perp/(4\pi T)$ arises due to the tilted magnetic field and ϵ_0 is the strength of the defect modelled by a δ-potential at the origin. In the following we set $k_{\mathrm{B}}T = 1$ (*i.e.* $\hbar = 1$ in the quantum model).

Without the local potential and the non-Hermitian term, this model has been well studied [10,11]. In particular, Haldane [11] has shown that this spinless Luttinger liquid exhibits a line of critical points with continuously varying exponents. His calculation is based on the bosonization technique, where the boson field

$$\Psi^\dagger \sim \sqrt{n_0 + \frac{\mathrm{d}u}{\mathrm{d}x}} \sum_{m=-\infty}^{\infty} e^{i2\pi m(n_0 x + u(x))} e^{i\phi(x)} \tag{2}$$

is represented in terms of a boson phase operator $\phi(x)$ and (dimensionless) phonon operator $u(x)$. The two fields satisfy the commutation relation $[\phi(x), u(y)] = (i/2)\,\mathrm{sgn}(x - y)$. With applications to vortex physics in mind, we have extended the bosonization approach and work on quantum impurities [12,13] to the non-Hermitian case $h > 0$ and calculated asymptotic low-energy properties for the model (1).

In addition, we have performed a non-perturbative numerical analysis using the Density-Matrix Renormalization Group (DMRG) [14] for a discretized version of the Hamiltonian (1)

$$H = \sum_{i=0}^{L}\left[-t\left(b_i^\dagger b_{i+1}e^h + b_{i+1}^\dagger b_i e^{-h}\right) + \frac{U}{2}n_i(n_i - 1) + V n_i n_{i+1}\right] - \epsilon_0 b_0^\dagger b_0 \tag{3}$$

corresponding to a non-Hermitian *Bose-Hubbard* model where $n_i = b_i^\dagger b_i$ and the hopping is $t = 1/2m$ (for unit lattice constant). In the following we set $t = 1$. We work in the canonical ensemble, fixing the density of bosons per site n_0. We have retained an onsite and a next-neighbor interaction, which turn out to be sufficient to qualitatively describe the full phase diagram. Furthermore, for computational purposes we allow at most 2 bosons per site, which effectively renormalizes the on-site repulsion. The lattice model (3) is a good approximation to (1) for small filling n_0 (average number of bosons per site). Our calculation is based on an extension of the DMRG to non-Hermitian systems with complex eigenvalues and eigenvectors (for details see [15]).

In the Hermitian case, $h = 0$ without impurity, we have first calculated the Luttinger-liquid parameter g which governs the long-wavelength behavior of correlation functions and is important to understand the response at finite tilt. We adapt the DMRG work of ref. [16] to *periodic* boundary conditions, essential for the study of persistent currents (*i.e.* arrays of tilted vortex lines) discussed below. We focus exclusively on the superfluid (Luttinger liquid) phase. Via DMRG we have calculated the boson correlation function, which from conformal field theory is expected to behave as $\langle\Psi^\dagger(x)\Psi(0)\rangle \sim |L\sin(\pi x/L)|^{-1/2g}$. We have verified this

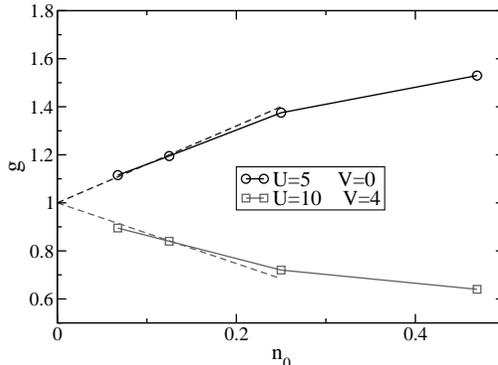Fig. 2 – Luttinger-liquid parameter g *vs.* density. The dashed lines show the analytic result at low densities.

behavior numerically with high accuracy and have extracted g by a fit to the data (see fig. 2). We also determined g from the compressibility and the finite-size dependence of the ground-state energy [17] with excellent agreement between the values of g obtained by both methods. For arbitrary short-range potentials (in continuum or lattice models) we have derived the general low-density result $g \approx 1 - 2an_0 + O(a^2 n_0^2)$, where a is the two-particle scattering length. For our lattice Hamiltonian (3) we find $a = -(8t^2 - 4tV - UV)/(2tU + UV + 4tV)$. As shown in fig. 2, this asymptotic result is in good agreement with the numerical data.

We now include the pinning term proportional to ϵ_0 in eq. (1). In order to determine the relevance of this term at long wavelengths we have performed a perturbative renormalization group (RG) analysis. We obtain the following renormalization flow of the pinning strength:

$$\epsilon_0(l) = \epsilon_0(l_0)\left(\frac{l_0}{l}\right)^{g-1}, \tag{4}$$

where l is an effective length scale or inverse cutoff momentum. For $g > 1$, the renormalized coupling flows to zero at long length scales while for $g < 1$ it diverges.

Remarkably, while in fermionic systems with (generic) repulsive interactions one always has $g < 1$, the *bosonic* Luttinger liquid studied here can be tuned to either regime. This can be easily seen by setting $U = \infty$, $V = 0$ in (3): Since hard-core bosons in 1d are equivalent to noninteracting spinless fermions, we obtain $g = 1$. Smaller U increases g from 1, while additional next-neighbor interactions $V > 0$ lead to $g < 1$. In the following we will denote the special situation $g = 1$ as the *free-fermion limit*, for which we have replaced the DMRG by computationally less expensive exact diagonalization.

The irrelevance/relevance of the pin can be clearly observed in the Friedel oscillations of the boson density $\Delta n(x) \equiv \langle n(x)\rangle - n_0$, for which we find the analytic result

$$\Delta n \;\propto\; \frac{\cos(2\pi n_0|x|)}{|x|^\alpha} \quad (a \ll |x| \ll \xi_\perp),$$
$$\propto\; \cos(2\pi n_0|x|)\exp[-|x|/\xi_\perp] \quad (|x| \gg \xi_\perp), \tag{5}$$

where a is a microscopic cut-off scale, the exponent $\alpha = \left\{{g \atop 2g-1}\right\}$ for $\left\{{g<1 \atop g>1}\right\}$ and the exponential is presumably corrected by a power law factor in the second line. $\xi_\perp \propto 1/h$ is the *decay length*.
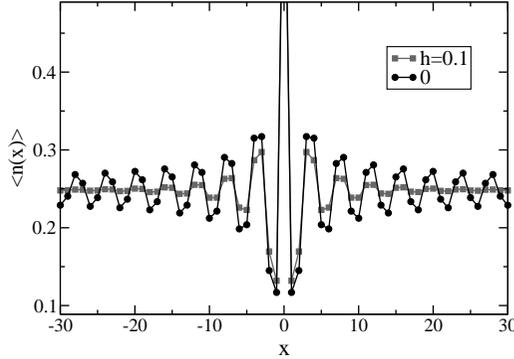
Fig. 3 – Friedel oscillations of the flux line density *vs.* distance x from the defect, calculated numerically for $\epsilon_0 = 2$, $U = 10$, $V = 4$, $L = 128$ and $n_0 = 0.25$, corresponding to $g \approx 0.72$.

As illustrated in fig. 3, the density oscillates with a phase set by the impurity position, and an algebraic envelope before exponential decay sets in for $x > \xi_\perp(h)$. In the vortex picture, configurations are dominated by parallel, tilted flux lines at distances larger than $\xi_\perp(h)$ from the pin. Closer in, vortices attempt to align with the maxima in the density oscillations present when $h = 0$. This alignment is limited by interactions as vortices enter and leave the aligned region with increasing imaginary time τ. The resulting vortex configurations resemble a symmetric traffic jam, with vortices queuing up (and occasionally changing places) in the vicinity of the columnar defect. With our conventions, the slope of the lines far from the pin is h/m, so new vortices enter the jam at imaginary time intervals $\xi_\parallel \approx m/hn_0$, where n_0 is the linear density of "bosons". If c is the Luttinger-liquid velocity we expect that $\xi_\perp(h) \propto c\xi_\parallel \propto 1/gh$, a diverging length scale we confirm with our analytic calculations. Note that the pinning strength is reduced dramatically for length scales $x > \xi_\perp(h)$ even for $g < 1$.

When $h > 0$, the non-hermiticity leads to a finite persistent current $J_{\rm b} = -\frac{i}{L}{\rm d}\langle\hat{H}\rangle/{\rm d}h$ in the ground state. This current is purely imaginary and corresponds to the transverse magnetization in the original flux line system [3]:

$$M_\perp \sim \Phi_0 \, {\rm Im} \, J_{\rm b} \,. \tag{6}$$

The defect reduces this current, due to flux lines pinned even in the presence of a tilted magnetic field. Although a single pin cannot modify the bulk current in the thermodynamic limit $L \to \infty$, it creates nontrivial finite-size effects. Since ${\rm Im} \, J_{\rm b} = hN_{\rm b}/mL$ in the absence of pinning (where $N_{\rm b} \equiv n_0 L$), it is convenient to define a "pinning number" $N_{\rm p}$ for vortices given by ${\rm Im} \, J_{\rm b} \equiv h(N_{\rm b} - N_{\rm p})/mL$. Because ${\rm Re} \, J_{\rm b} = 0$, this can be written

$$N_{\rm p} \equiv N_{\rm b}[J_{\rm b}(0) - J_{\rm b}(\epsilon_0)]/J_{\rm b}(0). \tag{7}$$

The quantity $N_{\rm p}$ may be readily calculated for the free-fermion case $g = 1$, where the ground-state energy is determined by filling up all the states below the "Fermi surface". The pinning number obtained in this way has the asymptotic behaviour ($L \to \infty$)

$$
\begin{aligned}
N_{\rm p} &\longrightarrow \frac{(m\epsilon_0)^2}{2\pi^2 n_0 h} \quad (m\epsilon_0 \ll h), \\
&\longrightarrow \frac{n_0}{h}\ln(|\epsilon_0|/n_0) \quad (m\epsilon_0 \gg h),
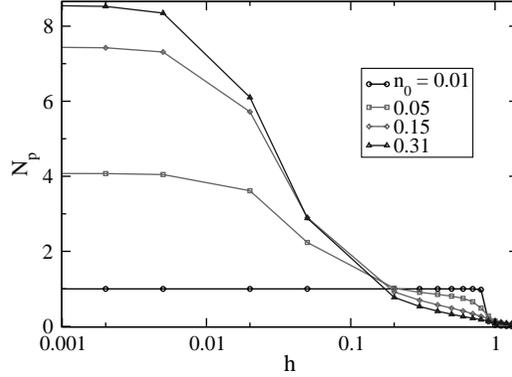\end{aligned}
\tag{8}
$$

Fig. 4 – Pinning number in the free-fermion limit $g = 1$ for $L = 100$ and $\epsilon_0 = 2$. Note the "step" due to single-vortex depinning and the strong enhancement at small tilt h.

results valid provided $h \ll n_0 = N_b/L$. Remarkably, N_p diverges as $h \to 0$. When the pinning is strong, the functional form (8) can be understood in terms of the aligned local density wave which extends out to a distance $\xi_\perp$. The $N_p \approx \xi_\perp n_0 \approx n_0/h$ vortices entrained in this "traffic jam" do not contribute to the current. To check this divergence, we have also calculated the pinning number numerically within the lattice model (3), both in the free-fermion limit ($U = \infty$, $V = 0$) and for general interactions. Results are shown in fig. 4. A clearly visible feature is the "step" at intermediate tilt for low boson densities, corresponding to the single-vortex depinning transition at $h_c \approx m\epsilon_0$ [3]. Most prominent, however, is the dramatic increase in the number of pinned vortices at small tilt. We find similar results with DMRG for $g < 1$. In the *linear-response* limit ($hL \to 0$) we find more generally that $N_p \sim L^{3-2g}$ for an irrelevant defect ($g > 1$), while in the relevant case almost all vortices are pinned, *i.e.* $N_p \to N_b$ for large system size L. The *residual* current for a relevant pin has the linear-response form $J_b(h)|_{h\to 0} \sim hL^{1-1/g}$, which vanishes as $L \to \infty$. A similar result for fermions was obtained by Gogolin and Prokov'ev [18] in the case of a real vector potential.
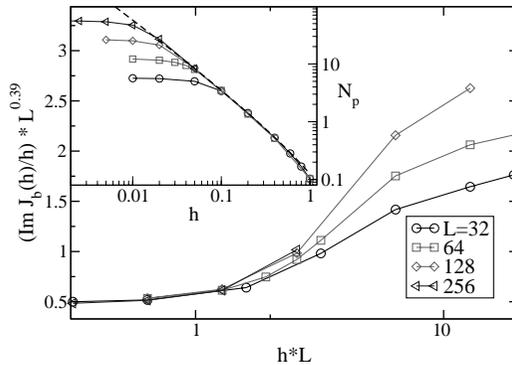

Fig. 5 – Main plot: finite-size scaling of the current (DMRG results) for filling $n_0 = 0.25$, $\epsilon_0 = 2$ and a relevant pin ($g \approx 0.72$). Notice the data collapse in the linear-response regime $hL \to 0$. Inset: pinning number N_p as a function of h for the same parameters. The dashed line gives the logarithmic behavior in eq. (9) with an offset of const $= 0.5$.

We have verified this finite-size dependence of the current with good accuracy in the DMRG calculation (see fig. 5).

The equivalence to a real vector potential breaks down at finite h. Our results are consistent with two qualitatively different types of behaviour depending on the value of g. While for an irrelevant pin with $\left\{ \begin{smallmatrix} 1<g<3/2 \\ 3/2<g \end{smallmatrix} \right\}$ a simple power law scaling ansatz of the type $N_{\mathrm{p}}(h) = \left\{ \begin{smallmatrix} h^{-3+2g}\Phi(hL) \\ \Phi(hL) \end{smallmatrix} \right\}$ works, our analytic work suggests a nontrivial logarithmic correction for $g < 1$:

$$N_{\mathrm{p}}(h) = (n_0/h)(-(1/g - 1)\ln(h) + \mathrm{const}). \tag{9}$$

This equation is valid for $hL \gg 1$ and $h \ll n_0$. The DMRG data (see inset of fig. 5) are consistent with this conjecture.

In conclusion, we have studied the effect of a single columnar defect on a sea of interacting vortices in $1+1$ dimensions, in the presence of a tilted magnetic field. The physics is described in terms of the ground state of a non-Hermitian Luttinger liquid. Our calculations demonstrate that repulsive interactions can lead to a dramatic enhancement in the number of pinned flux lines for $g < 1$ and thus to a strong transverse Meissner effect controlled by $\xi_\perp(h)$. Details of our analytic and numerical work will appear later [8].

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