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Spin-spin correlation functions for a spin-boson model with a structured bath

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Recently a new strategy for performing measurements on solid state (Josephson) qubits was proposed, in which the qubit is coupled to the measurement device through a single damped harmonic oscillator.[1)] Due to this coupling, the measurement comes arbitrarily close to a von-Neumann measurement. This system was also discussed in connection with electron transfer processes[2)] and decoherence control of two-level atoms in lossy cavities.[3)]

In this short note we discuss the dynamics of the system shown in Fig. 1, namely a two-level system coupled to an harmonic oscillator, which is coupled to a bath of harmonic oscillators. This system can be mapped to a standard model for dissipative quantum systems, namely the *spin-boson model*.[2)] In this case the spectral function governing the dynamics of the spin will have a resonance peak. We diagonalize the model by means of infinitesimal unitary transformations (*flow equations*),[4)] thereby decoupling the small quantum system from its environment. We calculate the renormalized tunneling matrix element for different coupling strengths and compare our results with an adiabatic renormalization calculation. The renormalization of the tunneling matrix element plays an important role for performing quantum measurements on qubits as it tells how close a measurement comes to a von-Neumann measurement.[1)] We also calculate spin-spin correlation functions for several instructive parameter choices. Spin-spin correlation functions can be used to calculate dephasing and relaxation times for measurements on qubits.[1)]

The system shown in Fig. 1, namely

$$\tilde{\mathcal{H}} = -\frac{\Delta_0}{2}\sigma_x + \Omega B^\dagger B + g(B^\dagger + B)\sigma_z + \sum_k \tilde{\omega}_k \tilde{b}_k^\dagger \tilde{b}_k + (B^\dagger + B)\sum_k \kappa_k(\tilde{b}_k^\dagger + \tilde{b}_k) + (B^\dagger + B)^2 \sum_k \frac{\kappa_k^2}{\tilde{\omega}_k},$$

with the spectral function $J(\omega) \equiv \sum_k \kappa_k^2 \delta(\omega - \tilde{\omega}_k) = \Gamma\omega$ can be mapped to a spin-boson model[2)]

$$\mathcal{H} = -\frac{\Delta_0}{2}\sigma_x + \frac{1}{2}\sigma_z \sum_k \lambda_k(b_k^\dagger + b_k) + \sum_k \omega_k b_k^\dagger b_k, \quad (1)$$

where the dynamics of the spin depends only on the spectral function $J(\omega) \equiv \sum_k \lambda_k^2 \delta(\omega - \omega_k)$ given by

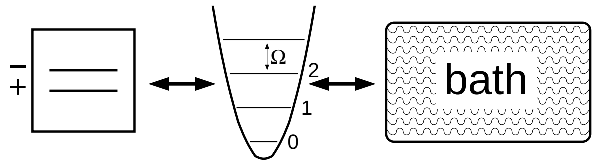

Fig. 1. A two-level system is coupled to an harmonic oscillator with frequency Ω, which is coupled to a bath of harmonic oscillators.

$$J(\omega) = \frac{2\alpha\omega\Omega^4}{(\Omega^2 - \omega^2)^2 + (2\pi\Gamma\omega\Omega)^2} \text{ with } \alpha = \frac{8\Gamma g^2}{\Omega^2}. \quad (2)$$

Using the flow equation technique we approximately diagonalize the Hamiltonian $\mathcal{H}$ [Eq.(1)] by means of infinitesimal unitary transformations $U(l)$:

$$\mathcal{H}(l) = U(l)\mathcal{H}U^\dagger(l), \quad (3)$$

where l is the so-called flow parameter. Here $\mathcal{H}(l=0) = \mathcal{H}$ is the initial Hamiltonian and $\mathcal{H}(l=\infty)$ is the final diagonal Hamiltonian. In a differential formulation

$$\frac{d\mathcal{H}(l)}{dl} = [\eta(l), \mathcal{H}(l)] \quad \text{with} \quad \eta(l) = \frac{dU(l)}{dl}U^{-1}(l). \quad (4)$$

Using the flow equation approach one can decouple system and bath by diagonalizing $\mathcal{H}(l=0)$:[4,5)]

$$\mathcal{H}(l=\infty) = -\frac{\Delta_\infty}{2}\sigma_x + \sum_k \omega_k b_k^\dagger b_k. \quad (5)$$

Here Δ_∞ is the renormalized tunneling frequency. For the generator of the flow we choose the Ansatz:[5)]

$$\eta = \sum_k \left(i\sigma_y \Delta(b_k + b_k^\dagger) + \sigma_z \omega_k (b_k - b_k^\dagger)\right) \frac{\lambda_k}{2}\left(\frac{\Delta - \omega_k}{\Delta + \omega_k}\right) + \frac{\Delta}{2}\sum_{q,k} \lambda_k \lambda_q I(\omega_k, \omega_q, l)(b_k + b_k^\dagger)(b_q - b_q^\dagger), \quad (6)$$

$$\text{with} \quad I(\omega_k, \omega_q, l) = \frac{\omega_q}{\omega_k^2 - \omega_q^2}\left(\frac{\omega_k - \Delta}{\omega_k + \Delta} + \frac{\omega_q - \Delta}{\omega_q + \Delta}\right).$$

The flow equations for the effective Hamiltonian [Eq. (5)] then take the following form:

$$\frac{\partial J(\omega, l)}{\partial l} = -2(\omega - \Delta)^2 J(\omega, l) \quad (7)$$

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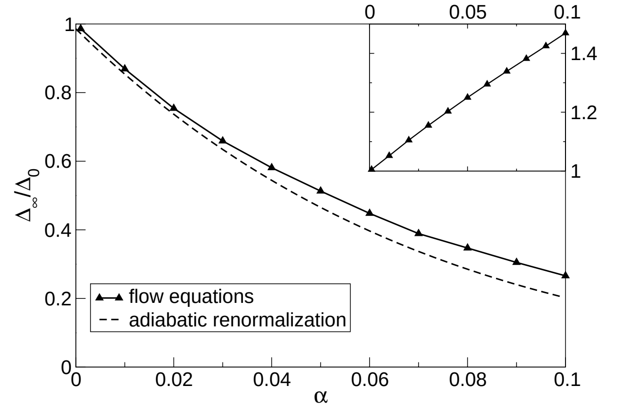

Fig. 2. The renormalized tunneling frequency Δ_∞ as function of α calculated using the flow equation approach and adiabatic renormalization (with $p = 1$) for $\Delta_0 = 1.0$, $\Gamma = 0.02$ and $\Delta_0/\Omega = 0.1$. The inset shows the flow equation result for $\Delta_0 = 1.0$, $\Gamma = 0.06$ and $\Delta_0/\Omega = 1.1$. The maximum α corresponds to a coupling between two-level system and harmonic oscillator of $g \approx 7.9/0.4$.

$$+ 2\Delta J(\omega, l) \int \mathrm{d}\omega' J(\omega', l) I(\omega, \omega', l),$$

$$\frac{d\Delta}{dl} = -\Delta \int \mathrm{d}\omega J(\omega, l) \frac{\omega - \Delta}{\omega + \Delta}. \tag{8}$$

The unitary flow diagonalizing the Hamiltonian generates a flow for the spin which takes the structure

$$\sigma_\mathrm{z}(l) = h(l)\sigma_\mathrm{z} + \sigma_\mathrm{x} \sum_\mathrm{k} \chi_\mathrm{k}(l)(b_\mathrm{k} + b_\mathrm{k}^\dagger), \tag{9}$$

where $h(l)$ and $\chi_\mathrm{k}(l)$ obey the differential equations

$$\frac{dh}{dl} = -\Delta \sum_\mathrm{k} \lambda_\mathrm{k} \chi_\mathrm{k} \frac{\omega_\mathrm{k} - \Delta}{\omega_\mathrm{k} + \Delta}, \tag{10}$$

$$\frac{d\chi_\mathrm{k}}{dl} = \Delta h \lambda_\mathrm{k} \frac{\omega_\mathrm{k} - \Delta}{\omega_\mathrm{k} + \Delta} + \sum_\mathrm{q} \chi_\mathrm{q} \lambda_\mathrm{k} \lambda_\mathrm{q} \Delta I(\omega_\mathrm{k}, \omega_\mathrm{q}, l). \tag{11}$$

One can show that the function $h(l)$ decays to zero as $l \to \infty$. Therefore the observable σ_z decays completely into bath operators.[5] We integrated the flow equations numerically in order to calculate the renormalized *tunneling matrix element*, Δ_∞, for different values of α. In Fig. 2 we compare our result with an *adiabatic renormalization*[6] calculation for the spectral function of Eq.(2):

$$\frac{\Delta_\infty}{\Delta_0} = \left(\frac{p^2\Delta_\infty^2 - \omega_2^2}{p^2\Delta_\infty^2} \right)^{-\frac{\alpha\Omega^2}{8\pi\Gamma\sqrt{\pi^2\Gamma^2 - 1}} \frac{1}{\omega_2^2}} \times \left(\frac{p^2\Delta_\infty^2 - \omega_1^2}{p^2\Delta_\infty^2} \right)^{-\frac{\alpha\Omega^2}{8\pi\Gamma\sqrt{\pi^2\Gamma^2 - 1}} \frac{1}{\omega_1^2}} \tag{12}$$

with

$$\omega_{1/2}^2 = \Omega^2 \left[1 - (2\pi\Gamma)^2/2 \pm 2\pi\Gamma\sqrt{\pi^2\Gamma^2 - 1} \right], \tag{13}$$

where p is an unspecified constant. The main plot in Fig. 2 shows Δ_∞/Δ_0 for $\Delta_0 \ll \Omega$. In this limit both methods yield qualitatively the same dependence. For $\Delta_0 > \Omega$ (see inset of Fig. 2) the flow equation result shows the expected level repulsion, namely $\Delta_\infty > \Delta_0$ for $\alpha > 0$. For conceptual reasons, adiabatic renormalization can not work in this limit. We also calculated

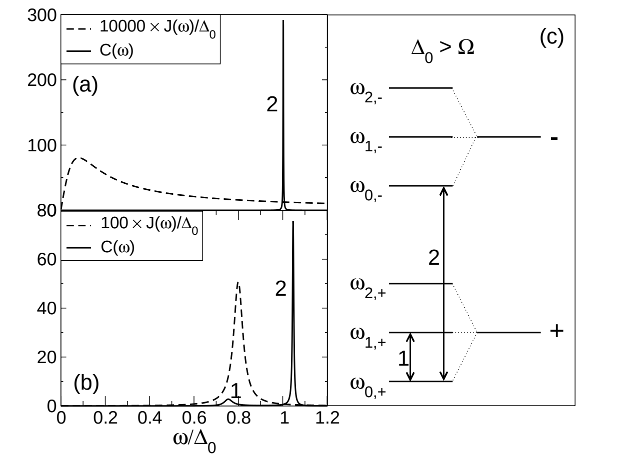

Fig. 3. (a) $C(\omega)$ for $\Delta_0 = 1.0$, $\Omega = 0.8$, $\Gamma = 1.6$ and $g^2 = 0.005$ (corresponds to $\alpha = 0.1$). (b) $C(\omega)$ for $\Delta_0 = 1.0$, $\Omega = 0.8$, $\Gamma = 0.01$ and $g = 0.1$ (corresponds to $\alpha = 0.00125$). (c) Term scheme of a two-level system coupled to an harmonic oscillator for $\Delta_0 > \Omega$.

the Fourier transform, $C(\omega)$, of the *spin-spin correlation function*

$$C(t) \equiv \frac{1}{2} \langle \sigma_\mathrm{z}(t)\sigma_\mathrm{z}(0) + \sigma_\mathrm{z}(0)\sigma_\mathrm{z}(t) \rangle. \tag{14}$$

Fig. 3 shows $C(\omega)$ for two instructive limits, namely
(a) $\Gamma \approx \Delta_0 \approx \Omega$, $g \ll \Gamma$: Due to the small coupling between two-level system and harmonic oscillator, we expect the tunneling matrix element not to be altered very much, $\Delta_\infty \approx \Delta_0$, and $C(\omega)$ is expected to show a single peak at Δ_∞.
(b) $\Omega \approx \Delta_0$, $g \ll (\Omega, \Delta_0)$, $\Gamma \ll g$: In this case we expect a double peak structure, which can be understood from the term scheme shown in Fig. 3(c). A second order perturbation calculation for the coupled two-level-harmonic oscillator system yields the following two frequencies for the peak position of $C(\omega)$, corresponding to transitions 1 and 2 in Fig. 3(c)

$$\omega_{1,+} - \omega_{0,+} = \Omega - g^2 2\Delta_0/(\Delta_0^2 - \Omega^2)$$

$$\omega_{0,-} - \omega_{0,+} = \Delta_0 + g^2 2\Delta_0/(\Delta_0^2 - \Omega^2).$$

This is consistent with Fig. 3(b).

In summary, we investigated the renormalization of the tunneling matrix element and calculated spin-spin correlation functions for the system depicted in Fig. 1, using a flow equation approach.

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