Algebraic Bethe ansatz for a discrete-state BCS pairing model

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We show in detail how Richardson's exact solution of a discrete-state BCS (DBCS) model can be recovered as a special case of an algebraic Bethe-ansatz solution of the inhomogeneous XXX vertex model with twisted boundary conditions: by implementing the twist using Sklyanin's **K**-matrix construction and taking the quasiclassical limit, one obtains a complete set of conserved quantities $\mathbf{H}_i$ from which the DBCS Hamiltonian can be constructed as a second order polynomial. The eigenvalues and eigenstates of the $\mathbf{H}_i$ (which reduce to the Gaudin Hamiltonians in the limit of infinitely strong coupling) are exactly known in terms of a set of parameters determined by a set of on-shell Bethe ansatz equations, which reproduce Richardson's equations for these parameters. We thus clarify that the integrability of the DBCS model is a special case of the integrability of the twisted inhomogeneous XXX vertex model. Furthermore, by considering the twisted inhomogeneous XXZ model and/or choosing a generic polynomial of the $\mathbf{H}_i$'s as Hamiltonian, more general exactly solvable models can be constructed. To make the paper accessible to readers that are not Bethe-ansatz experts, the introductory sections include a self-contained review of those of its feature which are needed here.

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I. INTRODUCTION AND SUMMARY

In a series of pioneering experiments in the mid-1990's, Ralph, Black, and Tinkham observed a spectroscopic gap indicative of pairing correlations in Al nanograins[1] that were so small that their electronic excitation spectra were discrete. Their results inspired a growing number of theoretical studies of superconducting pairing correlations in nanograins with a fixed number electrons (see Refs. 2,3 for recent reviews). These works are based on a model, to be called discrete-state BCS (DBCS) model below, described by a reduced BCS Hamiltonian for a discrete set of doubly degenerate energy levels, with a pairing interaction that scatters pairs of electrons from one level to the next. The DBCS model was solved exactly by Richardson in a series of papers starting in 1963:[4] he explicitly constructed all eigenstates and eigenenergies of the DBCS Hamiltonian in terms of a set of energy parameters whose values are determined by (numerically) solving a set of algebraic equations, to be called ''Richardson's equations.'' Though his work had, for a long time, been overlooked by the condensed matter community, it has recently received increasing attention in the context of studying pairing correlations in nanoscale Al grains, where the existence of an exact solution has turned out to be as useful as it had been unexpected.

The existence of an exact solution to a nontrivial model of course immediately raises the question whether it is related to any of the standard ways of exactly solving solvable models. The goal of this paper is to show that this is indeed the case: *Richardson's solution of the DBCS model is a special case of an algebraic Bethe-ansatz solution of the so-called inhomogeneous XXX vertex model with twisted boundary conditions.*

This insight builds upon a series of recent observations regarding exact properties of the DBCS model: In 1997, Cambiaggio, Rivas, and Saraceno[5] showed (though unaware of Richardson's work) that the DBCS model was integrable, and explicitly constructed all the constants of the motion [see Eq. (10) below]. In 2000, Amico, Falci, and Fazio[6] realized that the DBCS integrals of motion are in fact very similar to the integrals of motion of the XXX Gaudin model [see Eq. (11) below], differing from the latter only by an additional S_z term, and that Richardson's equations are very similar to the so-called Gaudin equations, differing from the latter only by an additional constant term. Now, it has long been known (see, e.g., Chap. 13.2 of Gaudin's book[7]) that the Gaudin model can be derived from the inhomogeneous XXX vertex (IXXX) model with periodic boundary conditions, by taking the so-called quasiclassical limit, and that, correspondingly, the Gaudin equations can be derived by taking the quasiclassical limit of Bethe-ansatz equations of the IXXX model. Since Richardson's ansatz satisfies the Gaudin equations modified by the additional constant term, Amico, Falci, and Fazio[6] referred to Richardson's ansatz as an ''off-shell Bethe ansatz,'' i.e., an ansatz *not* satisfying the Bethe equations of the original XXX model, but of a modified version thereof. (The off-shell Bethe ansatz was originally introduced by Babujian and Flume in a context quite different than finding eigenstates and eigenvalues of integral models, namely, to solve Knizhnik-Zamolodchikov differential equations arising in conformal Wess-Zumino models.[8])

In this paper, we address the following question: can one construct a vertex model, integrable by the algebraic Bethe ansatz (ABA), whose quasiclassical limit *directly* gives the DBCS model, in other words, which is directly solved by a normal ''on-shell'' Bethe ansatz? The answer is positive: we show that the sought-after model is an IXXX model with *twisted* (instead of periodic) boundary conditions, which we shall call the TIXXX model; its transfer matrix yields, in the quasiclassical limit, a complete set of conserved quantities, $\mathbf{H}_i$, from which the DBCS Hamiltonian can be constructed as a second order polynomial. Our emphasis on twisted boundary conditions, in order to arrive at on-shell Bethe ansatz equations, is the main difference between our work and that

of Refs. 6,9. We implement the twist using the boundary **K**-matrix construction of Sklyanin, which he introduced while developing his method of separation of variables,[10–12] an alternative (and in some cases more powerful) way to the ABA for constructing wave functions. In fact, Sklyanin himself mentioned in a side remark in Ref. 11 that the quasiclassical limit of the IXXX model with twisted boundary conditions (using a diagonal **K**-matrix) produces a modified version of the Gaudin model (though he was not aware, at the time, of the connection of the latter to the DBCS model).

We hope that our work fully clarifies the origin of the integrability of the DBCS model by explicitly constructing the integrable TIXXX model from which the latter can be derived. Moreover, by this construction we pave the way for using the powerful algebraic Bethe-ansatz machinery to calculate various quantities that have not yet been studied for the DBCS model. For example, there has recently been great progress in using the ABA to calculate matrix elements (or form factors) and correlation functions in vertex models (see Ref. 13, and Ref. 14 for a more recent development). By building upon our work, it should now be possible to fruitfully apply these results to the DBCS model, too.[15]

Our work also suggests ways for constructing integrable generalizations of the DBCS model, by considering other vertex models with twisted boundary conditions. In fact, one such generalization, which is Bethe-ansatz solvable, has recently been found independently by Amico, Di Lorenzo, and Osterloh.[9] They showed that by a slight generalization of the integrals of motion of the DBCS model, another integrable model is obtained. We shall show that the latter can be obtained by taking the quasiclassical limit of the inhomogeneous XXZ vertex model with twisted boundary conditions (TIXXZ model), in complete ananology to the derivation of the DBCS model from the TIXXX model.

Another interesting direction in which our work could be pursued, is to consider boundary conditions with *nondiagonal* **K**-matrices. These generally lead to models which are not solvable by the ABA. However, their eigenstates and eigenvalues can, in many cases, nevertheless be found using Sklyanin's method of separation of variables.

The paper is intended to be accessible also to readers that are not thoroughly familiar with the details of the algebraic Bethe ansatz; those of its features which are needed here are therefore introduced and reviewed in pedagogical detail. The structure of the paper is as follows. In Sec. II we introduce the DBCS and Gaudin models, recall how their integrals of motion are constructed, and give the equations (Richardson's or Gaudin's) that have to be satisfied in order to obtain eigenstates and eigenvectors. Sections III and IV contain a review of well-known material, in a form that is useful for the novel developments of subsequent sections: they give a self-contained introduction to the ABA method, as applied to the XXX and XXZ vertex models. Since both are special cases of the so-called six-vertex model, we shall actually begin by discussing the latter in full generality, before specializing later on. In Sec. III we explain how the Yang-Baxter equations satisfied by the **R**-matrices of local Boltzmann weights lead to the exchange relations for the components of the Monodromy matrix $\mathcal{T}$. Furthermore, we derive the fact that the transfer matrices for different spectral parameters commute, which is the underlying reason for the integrability of the model. In Sec. IV we exploit the exchange relations of the components of the Monodromy matrix to construct the eigenstates and eigenvalues of the model. In Sec. V, we explain how the results of Secs. III and IV can be straightforwardly generalized to the case of twisted boundary conditions using Sklyanin's **K**-matrix. Section VI contains our new results: we show that by taking the quasiclassical limit of the TIXXZ model, one recovers a generalized version of the DBCS model. We also show that if one specializes these results to the TIXXX model, one recovers the DBCS model. Section VII contains some brief conclusions and an outlook for future applications of our results.

II. THE DBCS AND GAUDIN MODELS

The DBCS model that is commonly used[2,3] to describe superconducting pairing correlations in nanoscale metallic grains is defined as follows: one consideres a reduced BCS Hamiltonian

$$H=\sum_{j,\sigma=\pm}\varepsilon_j c^\dagger_{j\sigma}c_{j\sigma}-g\sum_{jj'}c^\dagger_{j+}c^\dagger_{j-}c_{j'-}c_{j'+}, \tag{1}$$

for electrons in a set of pairs of time-reversed single-particle states $|j,\pm\rangle$ with energies ε_j, which are scattered pairwise from level j' to j, with interaction strength g. Richardson managed to solve this model exactly, for an arbitrary set of levels ε_j (although his solution includes the case of multiply degenerate levels, we shall here consider only the case where $\varepsilon_i\neq\varepsilon_j$ for $i\neq j$): Since any level occupied by only a single electron does not participate in and remains ''blocked'' to the pairscattering described by H, the labels of all such single-occupied levels are good quantum numbers. The eigenstates $|\alpha\rangle$ and corresponding eigenenergies $\mathcal{E}_\alpha$ of H thus have the following general form:

$$|\alpha\rangle=\prod_{i\in B}c^\dagger_{i\sigma_i}|\Psi_P\rangle,\quad \mathcal{E}_\alpha=\mathcal{E}_P+\sum_{i\in B}\varepsilon_i. \tag{2}$$

Here B is the set of singly-occupied, blocked levels, and $|\psi_P\rangle$ is an eigenstate, with eigenvalue $\mathcal{E}_P$ and containing precisely P pairs of electrons, of a Hamiltonian H_U which has precisely the same form as the H of Eq. (1), except that now the j-sums are restricted to run only over the set U of all *unblocked* or nonsingly-occupied levels. It is now convenient to introduce the pseudospin variables

$$S^z_j=\tfrac{1}{2}(1-c^\dagger_{j+}c_{j+}-c^\dagger_{j-}c_{j-}),\quad S^+_j=c_{j-}c_{j+},$$
$$S^-_j=c^\dagger_{j+}c^\dagger_{j-}, \tag{3}$$

which satisfy the standard SU(2) relations

$$[S^+_i,S^-_j]=\delta_{ij}2S^z_j,\quad [S^z_i,S^\pm_j]=\pm\delta_{ij}S^\pm_j, \tag{4}$$

and in terms of which H_U takes the form

$$H_U=\sum_j^U 2\varepsilon_j(1/2-S^z_j)-g\sum_{ij}S^-_iS^+_j. \tag{5}$$

Our choice (3) for the pseudospin variables differs from that used in many other publications[5,6,9] by the replacement

$$S_i^+ \leftrightarrow S_i^-, \quad S_i^z \rightarrow -S_i^z, \tag{6}$$

which preserves the SU(2) relations (4). With our choice, the physical vacuum state $|0\rangle$, containing no pairs, has the maximum possible S_z eigenvalue and hence is a ''highest-weight'' state. This is convenient for our present purpose, namely to establish contact with the ABA, because in the Bethe-ansatz literature it is standard practice to use highest-weight states as reference states [see Eq. (36) below].

Now, Richardson showed that the sought-after P-pair eigenstates (unnormalized) and eigenenergies have the general form[16]

$$|\Psi_P\rangle = \prod_{l=1}^{P} S^-(\mu_l)|0\rangle \ \text{ with } \ S^-(\mu_l) = \sum_i^U \frac{S_i^-}{2\varepsilon_i - \mu_l}, \tag{7}$$

$$\mathcal{E}_P = \sum_{l=1}^{P} \mu_l . \tag{8}$$

Here the P parameters μ_l $(l=1,\ldots,P)$ are a solution of a set of P coupled algebraic equations, which we shall call the ''Richardson equations,''

$$\frac{1}{g} - \sum_i^U \frac{1}{2\varepsilon_i - \mu_l} + \sum_{\substack{l'=1 \\ l' \neq l}}^{P} \frac{2}{\mu_{l'} - \mu_l} = 0 \ \text{ for } \ l = 1, \ldots, P. \tag{9}$$

These are to be solved (numerically, see Appendix B of Ref. 3) subject to the restrictions $\mu_{l'} \neq \mu_l$ if $l' \neq l$. A simple proof of this result may be found in Appendix B of Ref. 3; its strategy is to verify that $(H_U - \mathcal{E}_P)|\Psi_P\rangle = 0$ by simply commuting H_U to the very right past all of the S_l^- operators in Eq. (7).

Moreover, Cambiaggio, Rivas and Saraceno[5] showed that the constants of the motion of H_U have the form[17]

$$\mathbf{H}_i = S_i^z + g \sum_{\substack{j=1 \\ j \neq i}}^{U} \frac{S_i^z S_j^z + \frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+)}{\varepsilon_i - \varepsilon_j}. \tag{10}$$

The operators $\mathbf{H}_i$, $i=1,2,\ldots,N$ commute with each other as well as with the Hamiltonian (5). In the limit $g \rightarrow \infty$, the operators

$$\mathbf{H}_i^{\text{Gaudin}} = \lim_{g\to\infty} \mathbf{H}_i / g = \sum_{\substack{j=1 \\ j \neq i}}^{U} \frac{S_i^z S_j^z + \frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+)}{\varepsilon_i - \varepsilon_j} \tag{11}$$

coincide with the Hamiltonians of the Gaudin chain (see Chap. 13 of Ref. 7). The common eigenstates of the Gaudin Hamiltonians are given by the same Eqs. (7), but with the parameters μ_l satisfying the so-called Gaudin equations, which are simply the $g \rightarrow \infty$ limiting case[18] of Richardsons Eqs. (9). The corresponding eigenvalues of the Hamiltonians $\mathbf{H}_i^{\text{Gaudin}}$ are given by

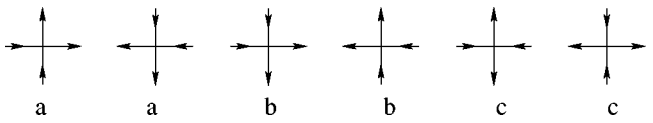

FIG. 1. The six allowed configurations of arrows at a vertex of the six-vertex model, with their corresponding Boltzmann weigths a, b, and c.

$$h_i^{\text{Gaudin}} = -\sum_{l=1}^{P} \frac{1}{2\varepsilon_i - \mu_l} + \frac{1}{2} \sum_{\substack{i'=1 \\ i' \neq i}}^{U} \frac{2}{\varepsilon_i - \varepsilon_{i'}}. \tag{12}$$

III. ALGEBRAIC BETHE ANSATZ FOR THE INHOMOGENEOUS SIX-VERTEX MODEL

A. Definition of model

It is well known that the Gaudin model can be obtained by taking the quasiclassical limit of the IXXX model.[7] The main result of this paper will be to show that a similar construction can be used to obtain the DBCS model from the TIXXX model, as well as generalized DBCS models from the TIXXZ model. To set the scene for these developments, the next two sections give a pedagogical review of the ABA as applied to IXXX and IXXZ models. Since both are special cases of the so-called 6-vertex model, we begin by discussing the latter in full generality. Bethe-ansatz experts may want to skip directly to Sec. V.

The six-vertex model is a classical statistical mechanics model on a two-dimensional regular quadratic lattice, whose dynamical variables are arrows living along the horizontal and vertical edges of the lattice, labeled by $m=1,\ldots,M$ and $i=1,\ldots,N$, respectively. At each vertex, only those six configuration of arrows are ''allowed,'' i.e., have nonzero local Boltzman weights (BW's), for which the total flux into the vertex is zero (see Fig. 1).

Thus, every allowed configurations has exactly two incoming and two out-coming arrows. Furthermore, we take the local BW's to be invariant with respect to the simultaneous reflection of all four arrows. This leaves only three independent BW's per vertex, to be denoted by a_{mi}, b_{mi}, and c_{mi}, where the subscripts give the location (m,i) of the vertex (intersection of row m and column i). Since the BW's are allowed to depend on the location of the vertex, we are considering an ''inhomogeneous'' model. As usual, the total statistical weight of any given configuration is defined as the product of the BW's of all vertices, and the partition function is defined as the sum of these statistical weights over all qallowed configurations.

It is convenient to associate a two-dimensional vector space $\mathbf{C}^2$ with each row, say U_m for row m, and another with each column, say V_i for column i, in such a way that the basis vectors

$$e_1^{(m)} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \rightarrow, \quad e_2^{(m)} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \leftarrow, \tag{13a}$$

$$e_1^{(i)} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \uparrow, \quad e_2^{(i)} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \downarrow, \tag{13b}$$

represent right- and left-pointing arrows along row m, or upward and downward arrows along column i, respectively. Then the local BW's at vertex (m,i) may be viewed as the matrix elements of a linear operator $\mathbf{R}_{mi}$ that acts as follows on the tensor product of the mth "horizontal" and ith "vertical" space $U_m \otimes V_i$:

$$\mathbf{R}_{mi} e_{\bar{l}}^{(m)} \otimes e_{\bar{k}}^{(i)} = (\mathbf{R}_{mi})^{\bar{l}\bar{k}}{}_{\bar{l}\bar{k}} e_{\bar{l}}^{(m)} \otimes e_{\bar{k}}^{(i)}, \tag{14a}$$

where the usual covention of summation over repeated indices $\bar{l},\bar{k} \in \{1,2\}$ is implied. (As a rule, we shall put bars over all repeated indices, and tilde's or nothing over nonrepeated upper or lower indices, respectively.) It follows that the action of $\mathbf{R}_{mi}$ on the coordinates $(\mathbf{w}_{mi})^{lk}$ of a general vector $\mathbf{w}_{mi} \equiv (\mathbf{w}_{mi})^{\bar{l}\bar{k}} e_{\bar{l}}^{(m)} \otimes e_{\bar{k}}^{(i)} \in U_m \otimes V_i$ takes the form

$$(\mathbf{R}_{mi}\mathbf{w}_{mi})^{\tilde{l}\bar{k}} = (\mathbf{R}_{mi})^{\tilde{l}\bar{k}}{}_{\bar{l}\bar{k}} (\mathbf{w}_{mi})^{\bar{l}\bar{k}}. \tag{14b}$$

The only nonzero matrix elements of the operator $\mathbf{R}_{mi}$ are

$$\begin{aligned} (\mathbf{R}_{mi})^{1}{}_{1}{}^{1}{}_{1} &= (\mathbf{R}_{mi})^{2}{}_{2}{}^{2}{}_{2} = a_{mi}, \\ (\mathbf{R}_{mi})^{1}{}_{1}{}^{2}{}_{2} &= (\mathbf{R}_{mi})^{2}{}_{2}{}^{1}{}_{1} = b_{mi}, \\ (\mathbf{R}_{mi})^{1}{}_{2}{}^{2}{}_{1} &= (\mathbf{R}_{mi})^{2}{}_{1}{}^{1}{}_{2} = c_{mi}. \end{aligned} \tag{15}$$

A convenient matrix representation for $\mathbf{R}_{mi}$ is

$$(\mathbf{R}_{mi})^{\tilde{l}}{}_{\bar{l}}{}^{\tilde{k}}{}_{\bar{k}} = \begin{pmatrix} (\boldsymbol{\alpha}_{mi})^{\tilde{k}}{}_{\bar{k}} & (\boldsymbol{\beta}_{mi})^{\tilde{k}}{}_{\bar{k}} \\ (\boldsymbol{\gamma}_{mi})^{\tilde{k}}{}_{\bar{k}} & (\boldsymbol{\delta}_{mi})^{\tilde{k}}{}_{\bar{k}} \end{pmatrix}^{\tilde{l}}{}_{\bar{l}}, \tag{16a}$$

where $\boldsymbol{\alpha}_{mi}$, $\boldsymbol{\beta}_{mi}$, $\boldsymbol{\beta}_{mi}$, $\boldsymbol{\gamma}_{mi}$ are operators acting on the two-dimensional vertical space V_i:

$$\begin{aligned} \boldsymbol{\alpha}_{mi} &= \begin{pmatrix} a_{mi} & 0 \\ 0 & b_{mi} \end{pmatrix}, \quad \boldsymbol{\beta}_{mi} = \begin{pmatrix} 0 & 0 \\ c_{mi} & 0 \end{pmatrix}, \\ \boldsymbol{\gamma}_{mi} &= \begin{pmatrix} 0 & c_{mi} \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{\delta}_{mi} = \begin{pmatrix} b_{mi} & 0 \\ 0 & a_{mi} \end{pmatrix}. \end{aligned} \tag{16b}$$

Even more explicitly, $\mathbf{R}_{mi}$ can be expressed as follows in terms of the unit operator $\mathbf{I}$ and the Pauli $\boldsymbol{\sigma}$-matrices $\boldsymbol{\sigma}^z$, $\boldsymbol{\sigma}^{\pm} = (\boldsymbol{\sigma}^x \pm i\boldsymbol{\sigma}^y)/2$,

$$\begin{aligned} \mathbf{R}_{mi} = \frac{a_{mi}+b_{mi}}{2}(\mathbf{I}_m \otimes \mathbf{I}_i) + \frac{a_{mi}-b_{mi}}{2}(\boldsymbol{\sigma}_m^z \otimes \boldsymbol{\sigma}_i^z) \\ + c_{mi}(\boldsymbol{\sigma}_m^+ \otimes \boldsymbol{\sigma}_i^- + \boldsymbol{\sigma}_m^- \otimes \boldsymbol{\sigma}_i^+), \end{aligned} \tag{17}$$

where the lower indices of the operators indicate the space (U_m or V_i) on which they act.

B. Monodromy matrix

One of the most important objects in the ABA method is the Monodromy matrix $\mathcal{T}_m$. It is defined to be the operator

$$\mathcal{T}_m = \mathbf{R}_{mN}\mathbf{R}_{mN-1}\cdots\mathbf{R}_{m1}, \tag{18a}$$

which acts on the space $U_m \otimes V_1 \cdots \otimes V_N$, with each factor $\mathbf{R}_{mi}$ acting nontrivially only on the "horizontal" space U_m and the "vertical" space V_i. To illustrate this action explicitly, we note that the matrix elements of $\mathcal{T}_m$ are constructed as follows from those of $\mathbf{R}_{mi}$:

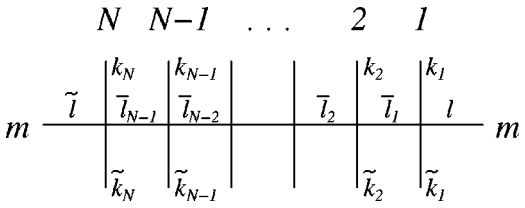

FIG. 2. Construction of the Monodromy matrix: the matrix element $(\mathcal{T}_m)^{\tilde{l},\ \tilde{k}_N \cdots \tilde{k}_1}{}_{l,\ k_N \cdots k_1}$ is equal to the total Boltzmann weight of the mth row, for a fixed configuration of external arrows (specified by the indices of $\mathcal{T}_m$), obtained by summing over all allowed configurations of arrows on internal lines (whose indices carry bars here).

$$\begin{aligned} &(\mathcal{T}_m)^{\tilde{l},\ \tilde{k}_N \ldots \tilde{k}_1}{}_{l,\ k_N \ldots k_1} \\ &\quad = (\mathbf{R}_{mN})^{\tilde{l}}{}_{\bar{l}_{N-1}}{}^{\tilde{k}_N}{}_{k_N} (\mathbf{R}_{mN-1})^{\bar{l}_{N-1}\ \tilde{k}_{N-1}}{}_{\bar{l}_{N-2}\ k_{N-1}} \ldots (\mathbf{R}_{m1})^{\bar{l}_1 \tilde{k}_1}{}_{l\ k_1}. \end{aligned} \tag{18b}$$

This equation has a simple physical interpretation: each such matrix element of $\mathcal{T}_m$ gives the total Boltzmann weight of the mth row, depicted in Fig. 2, for a fixed configuration of external arrows [specified by the indices of $\mathcal{T}_m$ on the left-hand side of Eq. (18b)], obtained by summing over all allowed configurations of arrows on internal horizontal edges [the $\bar{l}_i$ sums, for $i=2,\ldots,N$, on the right-hand side of Eq. (18b)]. Likewise, using this one row construction as building block, the partition function of an M-row lattice can be expressed via the matrix elements of a suitable product of the M Monodromy matrices, as will be seen below.

Because of the different roles played by the horizontal space U_m (usually reffered as the auxiliary space) and the tensor product of remaining vertical spaces $V_1 \cdots V_N$ (the so called quantum space), it is convenient to arrange the matrix elements of $\mathcal{T}_m$ that correspond to the horizontal space U_m, i.e., $(\mathcal{T}_m)^{\tilde{l}}{}_{l}$ in the notation of Eq. (18b), into a 2×2 matrix

$$\mathcal{T}_m \equiv \begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix}. \tag{19}$$

Its entries A, B, C, and D are, of course, operators acting on the quantum space, which implicitly carry the k indices that are displayed in Eq. (18b) (for brevity, we suppressed these above). Each of these four matrix elements corresponds to one of four possible kinds of rows in Fig. 2, depending on how the arrows on the first and last (i.e., external) horizontal edges are fixed:

$$A{:}(\rightarrow,\rightarrow), \quad B{:}(\rightarrow,\leftarrow), \quad C{:}(\leftarrow,\rightarrow), \quad D{:}(\leftarrow,\leftarrow). \tag{20}$$

Note that in this matrix representation, the product of $\mathbf{R}_{mi}$-matrices on the right-hand side of Eq. (18a) may be viewed as conventional multiplication of 2×2 matrices [of the form (16a)], whose entries are, however, operators on the quantum space (and hence carry suppressed k indices).

C. Transfer matrix

In order to investigate our model in the case of periodic boundary conditions in the horizontal direction, it is natural to consider the operator $\mathbf{T}_m$ (called transfer matrix), defined as the trace of $\mathcal{T}_m$ in the horizontal space U_m:

$$\mathbf{T}_m \equiv \mathrm{Tr}_m\{\mathcal{T}_m\} \equiv \sum_{l=1,2} (\mathcal{T}_m)^l_l = A_m + D_m . \tag{21}$$

Its matrix elements $(\mathbf{T}_m)^{\tilde{k}_N \cdots \tilde{k}_1}_{k_N \cdots k_1}$ give the total Boltzmann weight of the mth row depicted in Fig. 2 for a fixed configuration of arrows on the vertical edges [specified by the indices of $\mathbf{T}_m$], obtained by summing over all allowed configurations of arrows on horizontal edges, with the boundary condition that the arrows of the first and last horizontal edges are equal.

It follows that the full partition function for a lattice of M rows and N columns can readily be constructed by a suitable product of M transfer matrices: for the case of double periodic boundary conditions, it is equal to

$$Z_{M,N} \equiv (\mathbf{T}_M)^{\bar{k}_N^M \cdots \bar{k}_1^M}_{\bar{k}_N^{M-1} \cdots \bar{k}_1^{M-1}} (\mathbf{T}_{M-1})^{\bar{k}_N^{M-1} \cdots \bar{k}_1^{M-1}}_{\bar{k}_N^{M-2} \cdots \bar{k}_1^{M-2}} \cdots \times (\mathbf{T}_1)^{\bar{k}_N^1 \cdots \bar{k}_1^1}_{\bar{k}_N^M \cdots \bar{k}_1^M} \tag{22}$$

$$\equiv \mathrm{Tr}\{\mathbf{T}_M \mathbf{T}_{M-1} \cdots \mathbf{T}_1\}, \tag{23}$$

i.e., the trace is over the entire quantum space $V_1 \otimes \cdots V_N$.

The distinguishing feature of *integrable* models is that the transfer matrices for different rows commute, $\mathbf{T}_m \mathbf{T}_{m'} = \mathbf{T}_{m'} \mathbf{T}_m$. In this case, all transfer matrices have common eigenstates, with eigenvalues $\Lambda_m^{(\alpha)}$, say, so that the partition function takes the form

$$Z_{M,N} = \sum_\alpha \prod_{m=1}^{M} \Lambda_m^{(\alpha)} . \tag{24}$$

Thus, the calculation of the partition function reduces to the problem of finding the eigenvalues of the transfer matrices.

D. Yang-Baxter relations

It turns out (and will be shown below) that the transfer matrices commute if the local BW's a_{mi}, b_{mi}, c_{mi} are parametrized as follows:

$$a_{mi} = 1, \quad b_{mi} = b(\lambda_m, \xi_i), \quad c_{mi} = c(\lambda_m, \xi_i), \tag{25a}$$

$$b(\lambda,\xi) = \frac{\phi(\lambda - \xi)}{\phi(\lambda - \xi + 2\eta)}, \quad c(\lambda,\xi) = \frac{\phi(2\eta)}{\phi(\lambda - \xi + 2\eta)}, \tag{25b}$$

where form of the function $\phi(x)$ can be either $\phi(x) = x$ or $\phi(x) = \sinh x$. The parameter λ_m (called "spectral parameter") is associated with the mth horizontal line, and ξ_i (the inhomogeneity parameter) with the ith vertical line. Note that the dependence of the local BW's a_{mi}, b_{mi}, c_{mi} on their indices thus enters only via their spectral parameters, which is why we could introduce functions $b(\lambda,\xi)$ and $c(\lambda,\xi)$ that do not carry the indices (m,i) any longer. Note also that the ratio c/b is antisymmetric under an interchange of its arguments

$$\frac{c(\lambda,\xi)}{b(\lambda,\xi)} = -\frac{c(\xi,\lambda)}{b(\xi,\lambda)}, \tag{26}$$

a property that will be useful later.

Usually the rational case $\phi(x) = x$ is referred to as the XXX model and the trigonometric case $\phi(x) = \sinh x$ as the XXZ model, since the Hamiltonians of the XXX and XXZ Heisenberg magnetic chains can be derived from the corresponding (homogeneous) transfer matrices, by taking a logarithmic derivative with respect to the spectral parameter at some specific point (see, e.g., Chap. 10.14 of Ref. 19).

For the choice of BW's of Eqs. (25), the $\mathbf{R}_{mi}$-matrices have the following very important property, which ultimately leads to the solution of the problem: they satisfies the Yang-Baxter (YB) equation

$$\tilde{\mathbf{R}}_{mm'}(\lambda_m,\lambda_{m'}) \mathbf{R}_{mi}(\lambda_m,\xi_i) \mathbf{R}_{m'i}(\lambda_{m'},\xi_i) = \mathbf{R}_{m'i}(\lambda_{m'},\xi_i) \mathbf{R}_{mi}(\lambda_m,\xi_i) \tilde{\mathbf{R}}_{mm'}(\lambda_m,\lambda_{m'}), \tag{27a}$$

where the operator products on both sides act on the space $U_m \otimes U_{m'} \otimes V_i$, and the arguments in brackets indicate explicitly on which parameters the corresponding operators depend. As before, the $\mathbf{R}_{mi}$-operators act on one horizontal and one vertical space, $U_m \otimes V_i$; their nonzero matrix elements are given in Eq. (15), with the parameters a_{mi}, b_{mi}, and c_{mi} as defined in Eq. (25a), with arguments λ_m and ξ_i. In contrast, the operator $\tilde{\mathbf{R}}_{mm'}$ acts on *two horizontal* spaces, U_m and $U_{m'}$; apart from this replacement of vertical space V_i by the horizontal space $U_{m'}$, however, the structure of $\tilde{\mathbf{R}}_{mm'}$ is exactly the same as that of $\mathbf{R}_{mi}$: the nonzero matrix elements of $\tilde{\mathbf{R}}_{mm'}$ are likewise given by Eq. (15), where now the parameters $a_{mm'}$, $b_{mm'}$ and $c_{mm'}$ have arguments λ_m and $\lambda_{m'}$ (i.e., two λ's instead of λ and ξ):

$$a_{mm'} = 1, \quad b_{mm'} = b(\lambda_m,\lambda_{m'}), \quad c_{mm'} = c(\lambda_m,\lambda_{m'}).$$

To be explicit, the Yang-Baxter equation implies the following relations between matrix elements of the transfer matrices:

$$(\tilde{\mathbf{R}}_{mm'})^{\tilde{l}\tilde{l}'}_{\bar{l}\bar{l}'} (\mathbf{R}_{mi})^{\bar{l}\tilde{k}}_{l\bar{k}} (\mathbf{R}_{m'i})^{\bar{l}'\bar{k}}_{l'k} = (\mathbf{R}_{m'i})^{\tilde{l}'\tilde{k}}_{l'\bar{k}} (\mathbf{R}_{mi})^{\tilde{l}\bar{k}}_{l\bar{k}} (\tilde{\mathbf{R}}_{mm'})^{\bar{l}\bar{l}'}_{l l'} . \tag{27b}$$

(As usual, we used bars for repeated indices, which are summed over.) Graphically, this equation can be represented as the equality of the BW's of the configurations depicted in Fig. 3(b), where the left and right diagrams have the *same* configuration of arrows on all external edges, and sums over all possible configurations of internal indices are implied. The verification of the YB equation (27a) is straightforward (though rather tedious) and reduces to some simple rational or trigonometric identities. [Actually, the solvability of these Yang-Baxter equations dictated the choice of parametrization

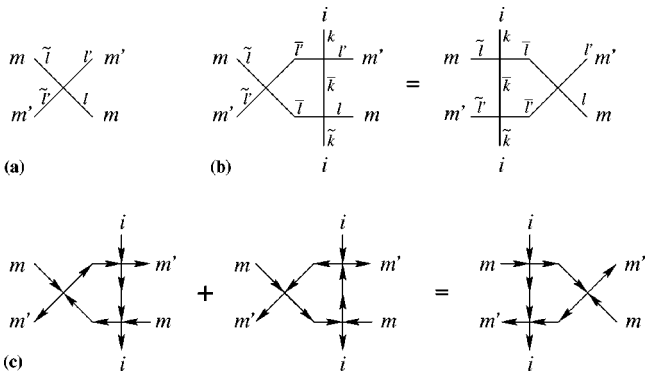

FIG. 3. Graphical depiction of the Yang-Baxter equations. (a) Schematic depiction of the action of $(\tilde{\mathbf{R}}_{\mathbf{mm'}})^{\tilde{l}\tilde{l}'}_{l\,l'}$, which interchanges the order of the rows m' and m. (b) General graph for the Yang-Baxter equation (27b). (c) Specific graph for a particular configuration of external arrows, representing the specific Yang-Baxter equation (28). Summing over all possible configurations of internal arrows consistent with the given choice of external arrows turns out to give two graphs on the left-hand side, but only one on the right-hand side.

of a, b, and c made in Eq. (25b).] As an illustration, the graph in Fig. 3(b) corresponds to the following explicit realization of Eq. (27b):

$$(\tilde{\mathbf{R}}_{mm'})^{12}_{21}(\mathbf{R}_{mi})^{22}_{22}(\mathbf{R}_{m'i})^{12}_{12} + (\tilde{\mathbf{R}}_{mm'})^{12}_{12}(\mathbf{R}_{mi})^{12}_{21}(\mathbf{R}_{m'i})^{21}_{12} = (\mathbf{R}_{m'i})^{22}_{22}(\mathbf{R}_{mi})^{12}_{12}(\tilde{\mathbf{R}}_{mm'})^{12}_{21} \tag{28a}$$

or, using Eqs. (15),

$$c_{mm'}a_{mi}b_{m'i}+b_{mm'}c_{mi}c_{m'i}=a_{m'i}b_{mi}c_{mm'}, \tag{28b}$$

which can be verified to hold if a_{mi}, b_{mi}, and c_{mi} have the form specified in Eqs. (25).

Now consider two Monodromy matrices $\mathcal{T}_m$ and $\mathcal{T}_{m'}$ with identical sets of inhomogeneity parameters $\xi_1,\ldots,\xi_N$,

$$\mathcal{T}(\lambda_m)\equiv\mathcal{T}_m(\lambda_m;\xi_1,\ldots,\xi_N),$$

$$\mathcal{T}(\lambda_{m'})\equiv\mathcal{T}_{m'}(\lambda_{m'};\xi_1,\ldots,\xi_N), \tag{29}$$

so that it suffices to display only the functional dependence on the (arbitrary) spectral parameters λ_m or $\lambda_{m'}$. For this case, we shall write the components of $\mathcal{T}(\lambda_m)$ as $A(\lambda_m)$, $B(\lambda_m)$, $C(\lambda_m)$, and $D(\lambda_m)$, and the transfer matrix as $\mathbf{T}(\lambda_m)$, while using $\tilde{\mathbf{R}}_{mm'}$ as shorthand for $\tilde{\mathbf{R}}_{mm'}(\lambda_m,\lambda_{m'})$. A direct consequence of the YB equation is that two such Monodromy matrices satisfy the following exchange relations:

$$\tilde{\mathbf{R}}_{mm'}\mathcal{T}(\lambda_m)\mathcal{T}(\lambda_{m'})=\mathcal{T}(\lambda_{m'})\mathcal{T}(\lambda_m)\tilde{\mathbf{R}}_{mm'} \tag{30a}$$

or, in terms of matrix elements,

$$(\tilde{\mathbf{R}}_{mm'})^{\tilde{l}\tilde{l}'}_{\bar{l}\bar{l}'}\mathcal{T}^{\bar{l},\tilde{k}_N\cdots\tilde{k}_1}_{l,\bar{k}_N\cdots\bar{k}_1}(\lambda_m)\mathcal{T}^{\bar{l}',\bar{k}_N\cdots\bar{k}_1}_{l',k_N\cdots k_1,}(\lambda_{m'}) = \mathcal{T}^{\tilde{l}',\tilde{k}_N\cdots\tilde{k}_1}_{\bar{l}',\bar{k}_N\cdots\bar{k}_1}(\lambda_{m'})\mathcal{T}^{\tilde{l},\bar{k}_N\cdots\bar{k}_1}_{\bar{l},k_N\cdots k_1}(\lambda_m)(\tilde{\mathbf{R}}_{mm'})^{\bar{l}\,\bar{l}'}_{l\,l'}. \tag{30b}$$

Figure 4 is a graphical representation of this relation. To prove it, one successively ''pulls'' the crossing of the two horizontal lines across from the right-most edge of the quantum space to the left-most edge, using the graphical representation Fig. 3 of the YB equation (27a). Rewriting Eq. (30a) in the form

$$\tilde{\mathbf{R}}_{mm'}\mathcal{T}(\lambda_m)\mathcal{T}(\lambda_{m'})\tilde{\mathbf{R}}^{-1}_{mm'}=\mathcal{T}(\lambda_{m'})\mathcal{T}(\lambda_m), \tag{31}$$

taking traces over the spaces U_m, $U_{m'}$ and using the cyclic property of the trace operation, one immediately concludes that the corresponding transfer matrices commute:

$$\mathbf{T}(\lambda_m)\mathbf{T}(\lambda_{m'})=\mathbf{T}(\lambda_{m'})\mathbf{T}(\lambda_m). \tag{32}$$

It is the existence of a one-parameter family of commuting transfer matices that makes the exact calculation of their eigenvalues and construction of their common eigenstates possible.

IV. EIGENSTATES AND EIGENVALUES OF THE TRANSFER MATRIX

Equation (30a) represents in a compact form 16 commutation relations among the matrix entries $A(\lambda_m)$, $B(\lambda_m)$, $C(\lambda_m)$, and $D(\lambda_m)$ of the Monodromy matrix $\mathcal{T}(\lambda_m)$ [see Eq. (19)]. Below we wright down three of them, which are essential for solving our eigenvalue eigenstate problem (the full set of relations can be found, e.g., in Chap VII of Ref. 13):

$$[B(\lambda_m),B(\lambda_{m'})]=0, \tag{33a}$$

$$A(\lambda_m)B(\lambda_{m'})=\frac{1}{b(\lambda_{m'},\lambda_m)}B(\lambda_{m'})A(\lambda_m) + \frac{c(\lambda_m,\lambda_{m'})}{b(\lambda_m,\lambda_{m'})}B(\lambda_m)A(\lambda_{m'}), \tag{33b}$$

$$D(\lambda_m)B(\lambda_{m'})=\frac{1}{b(\lambda_m,\lambda_{m'})}B(\lambda_{m'})D(\lambda_m) + \frac{c(\lambda_{m'},\lambda_m)}{b(\lambda_{m'},\lambda_m)}B(\lambda_m)D(\lambda_{m'}). \tag{33c}$$

These three equations correspond to the graphical equations of Figs. 4(b), 4(c), and 4(d) (in that order); for example, Fig. 4(d) represents the following specific realization of Eq. (30b) (whose k-indices we suppress here):

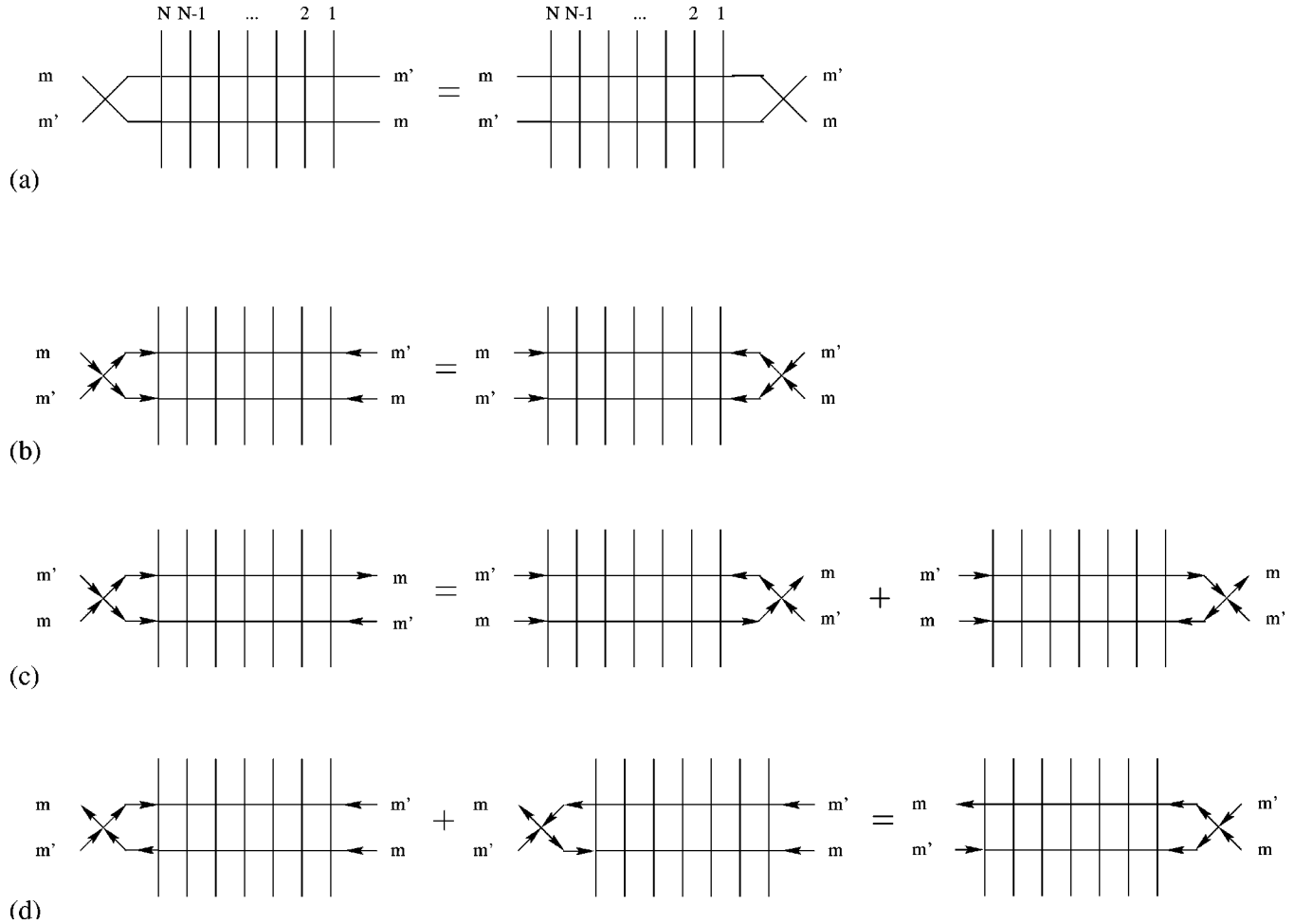

FIG. 4. Illustration of the exchange relations (30a) for the Monodromy matrix. (a) The general relation (30a). (b), (c), (d). Three specific choices of external arrows, leading to the three Eqs. (33a), (33b), and (33c), respectively.

$$(\tilde{\mathbf{R}}_{mm'})^{2\ 1}_{2\ 1}\mathcal{T}^{2}{}_{2}(\lambda_m)\mathcal{T}^{1}{}_{2}(\lambda_{m'}) + (\tilde{\mathbf{R}}_{mm'})^{2\ 1}_{1\ 2}\mathcal{T}^{1}_{2}(\lambda_m)\mathcal{T}^{2}{}_{2}(\lambda_{m'}) \tag{34}$$

$$= \mathcal{T}^{1}{}_{2}(\lambda_{m'})\mathcal{T}^{2}{}_{2}(\lambda_m)(\tilde{\mathbf{R}}_{mm'})^{22}_{22}. \tag{35}$$

Using Eqs. (15) and the antisymmetry property (26), this readily reduces to Eq. (33c).

Now we are ready to construct some eigenstates of the model. As reference state we use the following so-called vacuum state or highest-weight state,

$$|0\rangle \equiv e_1^{(1)} \otimes \cdots \otimes e_1^{(N)} \in V_1 \otimes \cdots \otimes V_N, \tag{36}$$

which can be visualized as a row of vertical edges, each of which carries an upward-pointing arrow. It is easy to verify that

$$C(\lambda_m)|0\rangle = 0, \tag{37a}$$

$$A(\lambda_m)|0\rangle = |0\rangle, \tag{37b}$$

$$D(\lambda_m)|0\rangle = \prod_{i=1}^{N} b(\lambda_m, \xi_i)|0\rangle. \tag{37c}$$

To check, e.g., Eq. (37c), recall that the operator D has the graphical representation shown in Fig. 2, with left-pointing arrows put on the first and last horizontal edges. The action on $|0\rangle$ implies arranging upward-pointing arrows on all the vertical edges above the horizontal line, but then the only allowed arrangement of arrows on the edges below the horizontal line is again a sequence of exclusively upward-pointing arrows. By Eq. (18b), this implies that we must have N successive b-type vertices, as specified in Eq. (37c).

To obtain more general states, one can act on the reference state $|0\rangle$ by an arbitrary number P (with $1 \leqslant P \leqslant M$) of operators B:

$$|\mu_1, \ldots, \mu_P\rangle \equiv B(\mu_1) \ldots B(\mu_P)|0\rangle. \tag{38}$$

Note that the parameters μ_l, $l = 1, \ldots, P$, in the arguments of the B's are arbitrary now, and unrelated to the spectral parameters introduced earlier, for which λ_m is associated with row m. We shall now show that when these μ_l parameters are solutions of a particular system of equations [the famous Bethe-ansatz equations, see Eq. (45) below], the vector (38) becomes an eigenstate of the transfer matrix $\mathbf{T}(\lambda)$ (we shall henceforth usually drop the index m on λ_m, since only the functional dependence is important). To this end, let

us analyze in detail the action by the operator $D(\lambda)$ on the Bethe vector (38) [subsequently, the action of $A(\lambda)$, which is analogous, will be outlined somewhat more briefly]. Our strategy is simple: using the exchange relations (33c), we move the operator D to the right past all B's until it appears next to the vacuum state $|0\rangle$, on which it acts according to Eq. (37c). The result can be represented as

$$D(\lambda)|\mu_1,\ldots,\mu_P\rangle=\prod_{l=1}^{P}\frac{1}{b(\lambda,\mu_l)}\prod_{i=1}^{N}b(\lambda,\xi_i)|\mu_1,\ldots,\mu_P\rangle + \sum_{l=1}^{P} f_l^D|\mu_1,\ldots,\mu_{l-1},\lambda,\mu_{l+1},\ldots,\mu_P\rangle, \tag{39a}$$

$$f_l^D=\frac{c(\mu_l,\lambda)}{b(\mu_l,\lambda)}\prod_{\substack{l'=1\\ l'\neq l}}^{P}\frac{1}{b(\mu_l,\mu_{l'})}\prod_{i=1}^{N}b(\mu_l,\xi_i). \tag{39b}$$

The first (so-called ''wanted'') term on the right-hand side (RHS) of Eq. (39a) arises from the case for which one picks up, at each of the series of commutations, the first term of the exchange rule (33c), which in our case is equal to

$$\frac{1}{b(\lambda,\mu_l)}B(\mu_l)D(\lambda). \tag{40}$$

Below we will refer to this term in exchange relation as a ''regular'' term. All other terms (the so called unwanted terms), which have contributions from the second term of (33c), are combined in the second line of Eq. (39a). The form of the coefficient f_l^D occuring in this term is given in Eq. (39b) and can be derived as follows: the left-hand side (LHS) of the Eq. (39a) is symmetric with respect to the permutations of the parameters $\mu_1,\ldots,\mu_P$, due to the commutativity (33a) of B's. Since the first term on the RHS of Eq. (39a) is symmetric as well, the second should be too. This means that if one succeedes to determine a single coefficient f_l^D for some fixed l, then all other f's can be straightforwardly found using the symmetry. Let us consider the case $l=1$. It is not difficult to see that the only possibility to obtain a term that is proportional to $|\lambda,\mu_2,\ldots,\mu_P\rangle$ and does not contain the operator $B(\mu_1)$, is to choose the ''wrong'' term of (33c),

$$\frac{c(\mu_1,\lambda)}{b(\mu_1,\lambda)}B(\lambda)D(\mu_1), \tag{41}$$

at the very first step when commuting $D(\lambda)$ with $B(\mu_1)$, and then everywhere else to choose ''regular'' ones. Thus, for f_1^D we obtain

$$f_1^D=\frac{c(\mu_1,\lambda)}{b(\mu_1,\lambda)}\prod_{l=2}^{P}\frac{1}{b(\mu_1,\mu_l)}\prod_{i=1}^{N}b(\mu_1,\xi_i). \tag{42}$$

The abovementioned symmetry under the permutations of μ_l's then immediately implies that in general f_l^D must have the form given in Eq. (39b).

A similar consideration of the action by the operator A on the Bethe vector (38) gives

$$A(\lambda)|\mu_1,\ldots,\mu_P\rangle=\prod_{l=1}^{P}\frac{1}{b(\mu_l,\lambda)}|\mu_1,\ldots,\mu_P\rangle + \sum_{l=1}^{p} f_l^A|\mu_1,\ldots,\mu_{l-1},\lambda,\mu_{l+1},\ldots,\mu_P\rangle, \tag{43a}$$

$$f_l^A=\frac{c(\lambda,\mu_l)}{b(\lambda,\mu_l)}\prod_{\substack{l'=1\\ l'\neq l}}^{P}\frac{1}{b(\mu_{l'},\mu_l)}. \tag{43b}$$

Combining Eqs. (39) and (43), we see that the state (38) is an eigenstate of the transfer matrix $\mathbf{T}(\lambda)=A(\lambda)+D(\lambda)$

$$\mathbf{T}(\lambda)|\mu_1,\ldots,\mu_P\rangle=t(\lambda;\mu_1,\ldots,\mu_P)|\mu_1,\ldots,\mu_P\rangle \tag{44a}$$

with the eigenvalue

$$t(\lambda;\mu_1,\ldots,\mu_P)=\prod_{l=1}^{P}\frac{1}{b(\mu_l,\lambda)}+\prod_{l=1}^{P}\frac{1}{b(\lambda,\mu_l)}\prod_{i=1}^{N}b(\lambda,\xi_i), \tag{44b}$$

provided that $f_l^A+f_l^D=0$ for every $l=1,2,\ldots,P$. Using the antisymmetry property of the ratio c/b, Eq. (26), it is easy to see that this condition is satisfied provided that the P parameters $\mu_1,\ldots,\mu_P$ satisfy the following system of P equations:

$$\prod_{\substack{l'=1\\ l'\neq l}}^{P}\frac{b(\mu_l,\mu_{l'})}{b(\mu_{l'},\mu_l)}=\prod_{i=1}^{N}b(\mu_l,\xi_i), \tag{45}$$

which are known as the Bethe equations. Every solution of the system of Bethe equations defines an eigenstate and corresponding eigenvalue of the transfermatrix $\mathbf{T}(\lambda)$ via Eqs. (38) and (44b).

V. SKLYANIN'S K-MATRIX

In this section, we shall generalize, following Sklyanin,[10–12] the formalism described above to the case when the boundary conditions in the horizontal direction is not strictly periodic: instead, the first and $N+$ first horizontal bonds are to be identified only up to a ''twist,'' implemented using a (fixed) linear transformation. We will show later on that the DBCS model (and also some of its possible generalizations) is some special limiting case of the inhomogeneus XXX(XXZ) model with such a twisted boundary condition. Consider a diagonal 2×2 matrix $\mathbf{K}_m$, first introduced by Sklyanin,[10,11] acting on the horizontal space U_m:

$$\mathbf{K}_m=\begin{pmatrix}(K_m)_{11} & 0\\ 0 & (K_m)_{22}\end{pmatrix}. \tag{46}$$

It is easy to check that the following relation holds[20] (illustrated in Fig. 5):

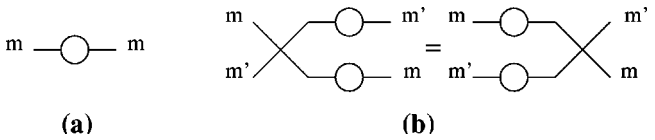

FIG. 5. (a) Sklyanin's **K**-matrix and (b) a graphical depiction of Eq. (47).

$$[\tilde{\mathbf{R}}_{mm'},\mathbf{K}_m\mathbf{K}_{m'}]=0, \tag{47}$$

where, in accord with our earlier conventions, the subscripts (m,m') specify the horizontal spaces on which the operators act non trivially. Let us define a modified monodromy matrix as

$$\tilde{\mathcal{T}}_m\equiv\tilde{\mathcal{T}}(\lambda_m)\equiv\mathbf{K}_m\mathcal{T}(\lambda_m), \tag{48}$$

or, in the 2×2 block form of Eq. (19),

$$\tilde{\mathcal{T}}_m\equiv\begin{pmatrix}\tilde{A}_m & \tilde{B}_m\\ \tilde{C}_m & \tilde{D}_m\end{pmatrix}=\begin{pmatrix}(K_m)_{11} & 0\\ 0 & (K_m)_{22}\end{pmatrix}\begin{pmatrix}A_m & B_m\\ C_m & D_m\end{pmatrix}. \tag{49}$$

Equation (47) ensures that the new monodromy matrix $\tilde{\mathcal{T}}$ obeys exactly the same exchange relation (30a) as $\mathcal{T}$, so that in particular Eqs. (32) and (33) remain valid also after the substitution $\mathbf{T}\to\tilde{\mathbf{T}}$, $A\to\tilde{A}$, $B\to\tilde{B}$, $C\to\tilde{C}$, $D\to\tilde{D}$. Furthermore, the analogs of Eqs. (37) take the form

$$\tilde{C}(\lambda)|0\rangle=0,$$

$$\tilde{A}(\lambda)|0\rangle=K_{11}|0\rangle,$$

$$\tilde{D}(\lambda)|0\rangle=K_{22}\prod_{i=1}^{N}b(\lambda,\xi_i)|0\rangle. \tag{50}$$

It follows that the Bethe vector [see Eq. (38)]

$$|\mu_1,\ldots,\mu_P\rangle_{\mathbf{K}}\equiv\tilde{B}(\mu_1)\cdots\tilde{B}(\mu_P)|0\rangle \tag{51}$$

is an eigenstate of $\tilde{\mathbf{T}}(\lambda)$,

$$\tilde{\mathbf{T}}(\lambda)|\mu_1,\ldots,\mu_P\rangle_{\mathbf{K}}=\tilde{t}(\lambda;\mu_1,\ldots,\mu_P)|\mu_1,\ldots,\mu_P\rangle_{\mathbf{K}}, \tag{52}$$

with eigenvalue

$$\tilde{t}(\lambda;\mu_1,\ldots,\mu_P)=K_{11}\prod_{l=1}^{P}\frac{1}{b(\mu_l,\lambda)}+K_{22}\prod_{l=1}^{P}\frac{1}{b(\lambda,\mu_l)}\prod_{i=1}^{N}b(\lambda,\xi_i), \tag{53}$$

provided that the parameters μ_j satisfy the following Bethe equations, for $l=1,\ldots,P$:

$$K_{11}\prod_{\substack{l'=1\\ l'\neq l}}^{P}\frac{b(\mu_l,\mu_{l'})}{b(\mu_{l'},\mu_l)}=K_{22}\prod_{i=1}^{N}b(\mu_l,\xi_i). \tag{54}$$

VI. THE "QUASICLASSICAL" LIMIT

In this section we show how the DBCS pairing model, or a generalisation thereof, can be recovered[21] by taking the so-called "quasiclassical" limit ($\eta\to0$) of the TIXXX or TIXXZ model, respectively. We shall present explicitly calculations for the TIXXZ case, i.e., for $\phi(x)=\sinh(x)$; to recover the corresponding results for the TIXXX case, one simply has to replace all hyperbolic functions by the corresponding rational ones.

A. Generator for conserved operators

Before taking the limit $\eta\to0$, it is convenient to write the inhomogeneity parameters as

$$\xi_i=2\varepsilon_i+\eta, \tag{55}$$

where the new parameters ε_i's are taken to be independent of η, and rescale the $\mathbf{R}_{mi}$ operators by a scalar factor, as follows:

$$\mathbf{R}_{mi}\to\frac{2\mathbf{R}_{mi}}{b(\lambda_m,\xi_i)+1}=\frac{\sinh(\lambda_m-2\varepsilon_i+\eta)}{\sinh(\lambda_m-2\varepsilon_i)\cosh(\eta)}\mathbf{R}_{mi}. \tag{56}$$

These transformations are convenient because, first, then the leading term in $\mathbf{R}_{mi}$ is simply a direct product of unit matrices [see Eq. (57) below]; and second, as we will see later, then many equations transform simply under $\eta\to-\eta$ (being either symmetric or antisymmetric), which considerably simplifies all expansions in powers of η. When written in terms of a direct product of 2×2 Pauli matrices as in Eq. (17), the rescaled $\mathbf{R}_{mi}$ of [Eq. (56)] takes the form

$$\mathbf{R}_{mi}=\mathbf{I}_m\otimes\mathbf{I}_i+\frac{\tanh\eta}{\tanh(\lambda_m-2\varepsilon_i)}\boldsymbol{\sigma}_m^z\otimes\boldsymbol{\sigma}_i^z+\frac{2\sinh\eta}{\sinh(\lambda_m-2\varepsilon_i)}(\boldsymbol{\sigma}_m^+\otimes\boldsymbol{\sigma}_i^-+\boldsymbol{\sigma}_m^-\otimes\boldsymbol{\sigma}_i^+). \tag{57}$$

Now choose the following form for the **K** matrix:

$$\mathbf{K}=\mathbf{I}+\frac{\eta}{g}\boldsymbol{\sigma}^z, \tag{58}$$

and expand the transfer matrix in powers of the parameter η, using Eqs. (57) and (58). This readily yields

$$\tilde{\mathbf{T}}(\lambda_m)\equiv\mathrm{Tr}_m\{\mathbf{K}_m\mathbf{R}_{mN}\cdots\mathbf{R}_{m1}\}=2\mathbf{I}+\frac{4}{g}\eta^2\mathbf{P}(\lambda_m)+O(\eta^3), \tag{59}$$

where

$$\mathbf{P}(\lambda)=\frac{1}{2}\sum_{i=1}^{N}\frac{\boldsymbol{\sigma}_i^z}{\tanh(\lambda-2\varepsilon_i)}+g\sum_{\substack{i,j\\ i<j}}^{N}\left(\frac{\boldsymbol{\sigma}_i^z\boldsymbol{\sigma}_j^z}{2\tanh(\lambda-2\varepsilon_i)\tanh(\lambda-2\varepsilon_j)}+\frac{\boldsymbol{\sigma}_i^+\boldsymbol{\sigma}_j^-+\boldsymbol{\sigma}_i^-\boldsymbol{\sigma}_j^+}{\sinh(\lambda-2\varepsilon_i)\sinh(\lambda-2\varepsilon_j)}\right), \tag{60}$$

and to the commutativity (32) of transfer matrices for different spectral parameters guarantees that

$$[\mathbf{P}(\lambda),\mathbf{P}(\lambda')]=0. \tag{61}$$

$\mathbf{P}(\lambda)$ can be viewed as the generating operator for all possible conserved operators of the model.

A convenient way of obtaining a complete set of commuting conserved operators, is to take the residues of $\mathbf{P}(\lambda)$ at the points $\lambda=2\varepsilon_i$ for $i=1\cdots N$,

$$\mathbf{H}_i=\mathrm{Res}[\mathbf{P}(\lambda);\lambda\to 2\varepsilon_i]=\oint_{\mathcal{C}_i}\frac{d\lambda}{2\pi i}\mathbf{P}(\lambda), \tag{62}$$

where $\mathcal{C}_i$ is a small contour in the complex λ-plane, encircling the point $2\varepsilon_i$. Explicitly evaluating the residues for the present model, one obtains

$$\mathbf{H}_i=\frac{\boldsymbol{\sigma}_i^z}{2}+g\sum_{\substack{j=1\\ j\neq i}}^{N}\left(\frac{\boldsymbol{\sigma}_i^z\boldsymbol{\sigma}_j^z}{2\tanh(2\varepsilon_i-2\varepsilon_j)}+\frac{\boldsymbol{\sigma}_i^+\boldsymbol{\sigma}_j^-+\boldsymbol{\sigma}_i^-\boldsymbol{\sigma}_j^+}{\sinh(2\varepsilon_i-2\varepsilon_j)}\right). \tag{63}$$

Equation (61) immediately implies that all of these operators commute: $[\mathbf{H}_i,\mathbf{H}_j]=0$. Furthermore, it is not difficult to show that the set of all $\mathbf{H}_i$ is complete, in the sense that $\mathbf{P}(\lambda)$ can be expressed purely in terms of these operators. Indeed, $\mathbf{P}(\lambda)$ is a rational (matrix-valued) function of the variable $z\equiv\exp(2\lambda)$, which is regular at $z\to\infty$ and has simple poles at $z\to\exp(4\varepsilon_i)$; it is thus completely determined in terms of the corresponding residues, which are equal to $2\exp(4\varepsilon_i)\mathbf{H}_i$, so that we have

$$\mathbf{P}(\lambda)=\mathbf{P}(\infty)+\sum_{i=1}^{N}\frac{2e^{4\varepsilon_i}\mathbf{H}_i}{e^{2\lambda}-e^{4\varepsilon_i}}. \tag{64}$$

The term $\mathbf{P}(\infty)$ itself also can be expressed via $\mathbf{H}_i$:

$$\mathbf{P}(\infty)=g\left(\sum_{i=1}^{N}\mathbf{H}_i\right)^2+\sum_{i=1}^{N}\mathbf{H}_i-\frac{Ng}{4}, \tag{65}$$

where we have used the fact that

$$\sum_{i=1}^{N}\mathbf{H}_i=\frac{1}{2}\sum_{i=1}^{N}\boldsymbol{\sigma}_i^z. \tag{66}$$

The commuting operators $\mathbf{H}_i$ are in fact just the so-called generalized Gaudin Hamiltonians. Moroever, in the limit $g\to\infty$, in which $\mathbf{K}=\mathbf{I}$ so that one recovers periodic boundary conditions, the $\mathbf{H}_i/g$ reduce to the standard Gaudin Hamiltonians $\mathbf{H}_i^{\mathrm{Gaudin}}$ of Eq. (11).

B. Eigenvectors

To obtain the quasiclassical limit of the Bethe eigenvectors $|\mu_1,\ldots,\mu_P\rangle_{\mathbf{K}}$ of Eq. (51), we have to investigate the $\eta\to 0$ limit of the operators $\tilde{B}$ which enter in its definition of Eq. (51). Now, recall that the operator $\tilde{B}_m=\tilde{B}(\lambda_m)$ is the (1,2) component (in auxiliary space U_m) of the monodromy matrix $\tilde{\mathcal{T}}_m=\mathbf{K}_m\mathbf{R}_{mN}\cdots\mathbf{R}_{m1}$, which has the following expansion [using Eqs. (57) and (58)]:

$$\tilde{\mathcal{T}}(\lambda_m)=\mathbf{I}+\frac{\eta}{g}\boldsymbol{\sigma}_m^z+\sum_{i=1}^{N}\left[\frac{\eta}{\tanh(\lambda_m-2\varepsilon_i)}\boldsymbol{\sigma}_m^z\boldsymbol{\sigma}_i^z+\frac{2\eta}{\sinh(\lambda_m-2\varepsilon_i)}(\boldsymbol{\sigma}_m^+\boldsymbol{\sigma}_i^-+\boldsymbol{\sigma}_m^-\boldsymbol{\sigma}_i^+)\right]+O(\eta^2). \tag{67}$$

In this equation, the only terms having non-zero (1,2) components and hence contributing to $\tilde{B}(\lambda_m)$ are those proportional to $\boldsymbol{\sigma}_m^+$. Thus we have

$$\tilde{B}(\lambda)=2\eta S^-(\lambda)+O(\eta^2), \tag{68}$$

where

$$S^-(\lambda)=\sum_{i=1}^{N}\frac{\boldsymbol{\sigma}_i^-}{\sinh(\lambda-2\varepsilon_i)}. \tag{69}$$

We see that the quasiclassical (unnormalized) Bethe vector of Eq. (51) takes the form

$$|\mu_1\cdots\mu_P\rangle_{\mathbf{K}}=S^-(\mu_1)\cdots S^-(\mu_P)|0\rangle, \tag{70}$$

where, as before, the reference state $|0\rangle$ is defined by Eq. (36).

C. Eigenvalues

The eigenvalues h_i of the conserved operators $\mathbf{H}_i$ can be found from the eigenvalue $p(\lambda)$ of their generator $\mathbf{P}(\lambda)$. Since $(4/g)\mathbf{P}(\lambda)$ is the order-η^2 coefficient of the transfer matrix $\tilde{\mathbf{T}}(\lambda)$ [Eq. (59)], its eigenvalue $(4/g)p(\lambda)$ is given by the order-η^2 coefficient of the corresponding eigenvalue $\tilde{t}(\lambda)$, which can be found by multiplying Eq. (53) by the factor

$$\prod_{i=1}^{N}\frac{\sinh(\lambda-2\varepsilon_i+\eta)}{\sinh(\lambda-2\varepsilon_i)\cosh(\eta)} \tag{71}$$

[see Eq. (56)] and setting $\xi_i=2\varepsilon_i+\eta$ [see Eq. (55)]:

$$\tilde{t}(\lambda)=\left[\left(1+\frac{\eta}{g}\right)\prod_{i=1}^{N}\frac{\sinh(\lambda-2\varepsilon_i+\eta)}{\sinh(\lambda-2\varepsilon_i)\cosh(\eta)}\times\prod_{l=1}^{P}\frac{\sinh(\lambda-\mu_l-2\eta)}{\sinh(\lambda-\mu_l)}+\left(1-\frac{\eta}{g}\right)\prod_{i=1}^{N}\frac{\sinh(\lambda-2\varepsilon_i-\eta)}{\sinh(\lambda-2\varepsilon_i)\cosh(\eta)}\times\prod_{l=1}^{P}\frac{\sinh(\lambda-\mu_l+2\eta)}{\sinh(\lambda-\mu_l)}\right]. \tag{72}$$

Expanding this expression in η, the coefficient of η^2, multiplied by $g/4$, is found to be

$$p(\lambda)=Pg+\frac{1}{2}\sum_{i=1}^{N}\frac{1}{\tanh(\lambda-2\varepsilon_i)}-\sum_{l=1}^{P}\frac{1}{\tanh(\lambda-\mu_l)}+\frac{1}{2}\sum_{\substack{i,i'\\ i<i'}}^{N}\frac{g}{\tanh(\lambda-2\varepsilon_i)\tanh(\lambda-2\varepsilon_{i'})}+\sum_{\substack{l,l'\\ l<l'}}^{P}\frac{2g}{\tanh(\lambda-\mu_l)\tanh(\lambda-\mu_{l'})}-\sum_{i}^{N}\sum_{l=1}^{P}\frac{g}{\tanh(\lambda-2\varepsilon_i)\tanh(\lambda-\mu_l)}. \tag{73}$$

The eigenvalues of the generalized Gaudin Hamiltonians $\mathbf{H}_i$, say h_i, can be obtained by taking the residues at the points $\lambda=2\varepsilon_i$:

$$h_i=\frac{1}{2}-\sum_{l=1}^{P}\frac{g}{\tanh(2\varepsilon_i-\mu_l)}+\frac{1}{2}\sum_{\substack{i'=1\\ i'\neq i}}^{N}\frac{g}{\tanh(2\varepsilon_i-2\varepsilon_{i'})}. \tag{74}$$

Finally, since all the $\mathbf{H}_i$ commute, we can immediately write down the eigenvalue of any function of these operators. In particular, the general Hamiltonian

$$\mathbf{H}=\mathcal{P}(\mathbf{H}_1,\ldots,\mathbf{H}_N), \tag{75}$$

where $\mathcal{P}$ is some arbitrary polinomial of its arguments, has eigenvalues $\mathcal{P}(h_1,\ldots h_N)$. For example, the general class of models recently discussed by Amico, Di Lorenzo, and Osterloh[9] in the context of superconductivity in small grains, is obtained by considering certain second order polynomials (i.e., quadratic combinations of $\mathbf{H}_i$'s).

D. Bethe equations

The quasiclassical Bethe state $|\mu_1,\ldots,\mu_P\rangle_{\mathbf{K}}$ is an eigenstate of the generator $\mathbf{P}(\lambda)$, and consequently also of each of the generalized Gaudin Hamiltonians $\mathbf{H}_i$, only if the parameters μ_l satisfy the limit $\eta\to 0$ of the Bethe equations (54). The latter are of course not affected by the rescaling transformation (56), and take the following form upon inserting $\xi_i=2\varepsilon_i+\eta$ of Eq. (55):

$$\left(1+\frac{\eta}{g}\right)\prod_{\substack{l'=1\\ l'\neq l}}^{P}\frac{\sinh(\mu_{l'}-\mu_l+2\eta)}{\sinh(\mu_{l'}-\mu_l-2\eta)}=\left(1-\frac{\eta}{g}\right)\prod_{i=1}^{N}\frac{\sinh(\mu_l-2\varepsilon_i-\eta)}{\sinh(\mu_l-2\varepsilon_i+\eta)}.$$

In the "quasiclassical" limit $\eta\to 0$ we obtain the following set of equations, for $l=1,\ldots,P$, which may be viewed as generalized Gaudin equations:

$$\frac{1}{g}-\sum_{i=1}^{N}\frac{1}{\tanh(2\varepsilon_i-\mu_l)}+\sum_{\substack{l'=1\\ l'\neq l}}^{P}\frac{2}{\tanh(\mu_{l'}-\mu_l)}=0. \tag{76}$$

We would like to emphasize that these are the *on-shell* Bethe equations of quasiclassical limit of the TIXXZ model. In contrast, in Refs. 6 and 9, who did not consider twisted boundary conditions as we do here, these equations are *off-shell* Bethe-ansatz equations.

Of course, Eqs. (76) can be derived, if desired, without reference to the ABA, by pursuing the following strategy (described in detail in Appendix B of Ref. 3): in order to show that the state $|\mu_1\cdots\mu_P\rangle_{\mathbf{K}}$ of Eq. (70) is an eigenstate of any $\mathbf{H}_i$, one would commute $\mathbf{H}_i$ past all the operators $S^-(\mu_l)$ in Eq. (70) [whose form (69) is reminiscent of the operators B_μ defined Ref. 3 if we identify $\boldsymbol{\sigma}_i^-$ with $b_i^\dagger$]; this would generate "unwanted" terms that only vanish if Eqs. (76) are satisfied.

E. Specialization to Richardson's equations

It is straightforward to recover the DBCS model and Richardson's solution thereof, as summarized in Sec. II, by considering the case $\phi(x)=x$ appropriate for the XXX model (instead of the XXZ case $\phi=\sinh x$), and replacing everywhere

$$\tanh x\to x,\quad \sinh x\to x. \tag{77}$$

First, we note that the generalized Gaudin equations (76) then reduce to Richardson's equations (9). Furthermore, the generalized Gaudin Hamiltonians $\mathbf{H}_i$ of Eq. (63) reduce to the form given in Eq. (10) for the conserved operators of the DBCS model. This fact was noted by Sklyanin himself in a side remark in Ref. 11, and first derived by him already in 1989 in Ref. 12. However, he was at the time unaware of the fact that the resulting $\mathbf{H}_i$ were useful in the context of the DBCS model, and in particular, that they be used to construct the Hamiltonian H_U of Eq. (5) of the DBCS model. It is straightforward to check that this can be done through the following construction:

$$H_U(\mathbf{H}_i)=\sum_{i=1}^{N}\left[(g-2\varepsilon_i)\mathbf{H}_i+(\varepsilon_i-3g/4)\right]+g\left(\sum_{i=1}^{N}\mathbf{H}_i\right)^2. \tag{78}$$

To calculate its eigenvalues $\mathcal{E}_P=H_U(h_i)$ explicitly, the following identities [derived by repeated use of Eqs. (74) and (76)] are useful:

$$\sum_{i=1}^{N} h_i = \frac{N}{2} - \sum_{l=1}^{P} \sum_{i=1}^{N} \frac{g}{2\varepsilon_i - \mu_l}$$

$$= \frac{N}{2} - \sum_{l=1}^{P} \left(1 + \sum_{\substack{l'=1 \\ l' \neq l}}^{P} \frac{2g}{\mu_{l'} - \mu_l} \right) = \frac{N}{2} - P, \quad (79a)$$

$$\sum_{i=1}^{N} 2\varepsilon_i h_i = \sum_{i=1}^{N} \varepsilon_i + \sum_{\substack{i,i' \\ i \neq i'}}^{N} \frac{g\varepsilon_i}{2(\varepsilon_i - \varepsilon_{i'})} - \sum_{i=1}^{N} \sum_{l=1}^{P} \frac{2g\varepsilon_i}{2\varepsilon_i - \mu_l} \quad (79b)$$

$$= \sum_{i=1}^{N} \varepsilon_i + gN(N-1)/4$$

$$- \left[\sum_{l=1}^{P} \mu_l + gPN - gP(P-1) \right], \quad (79c)$$

where the last term of Eq. (79c) was obtained from the last term of Eq. (79b) by rewriting the latter as follows:

$$\sum_{i=1}^{N} \sum_{l=1}^{P} \frac{g(2\varepsilon_i - \mu_l + \mu_l)}{2\varepsilon_i - \mu_l}$$

$$= gPN + \sum_{l=1}^{P} \mu_l \left(1 + \sum_{\substack{l'=1 \\ l' \neq l}}^{P} \frac{2g}{\mu_{l'} - \mu_l} \right). \quad (79d)$$

Using Eqs. (79a) and (79c), it is straightforward to check that $\mathcal{E}_P$ reduces to the simple form $\mathcal{E}_P = \Sigma_{l=1}^{P} \mu_l$ of Eq. (8).

VII. CONCLUSIONS

The TIXXZ results of the previous section for the conserved operators $\mathbf{H}_i$, their eigenvalues h_i and eigenvectors $|\mu_1, \ldots, \mu_P\rangle_{\mathbf{K}}$, and the corresponding consistency condition (76), have been found independently before by Amico, Di Lorenzo and Osterloh.[9] They managed to construct the $\mathbf{H}_i$ apparently by inspection, without presenting a systematic approach for their derivation, and in their approach the consistency condition (76) appears as a set of off-shell Bethe-ansatz equations. In our work, we presented a systematic derivation of these results from a vertex model with twisted boundary conditions, and the consistency condition (76) corresponds directly to the *on-shell* Bethe-ansatz equations of this model. Thus, we hope to have shed some additional light on the reasons why the DBCS model and its generalizations are integrable and Bethe-ansatz solvable, and on the underlying algebraic structure of the solutions. We hope that our work shows the way towards further progress in applying the powerful formalism of the ABA to the DBCS and related models, e.g., for the calculation of correlation functions[15] such as $\langle S_i^z S_j^z \rangle$ or $\langle S_i^- S_j^+ \rangle$, which are of importance for understanding the nature of pairing correlations in nanoscale superconducting grains.[2,3]

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integrability, using more general choices of **K**: for the XXX model, Eq. (47) holds for an arbitrary 2×2 **K**-matrix. For the XXZ model, Eq. (47) holds for any diagonal **K**-matrix, or any purely-off-diagonal **K**-matrix (i.e., vanishing diagonal elements). For both the XXX and XXZ models, though, only the case of a diagonal **K**-matrix is solvable by conventional Bethe-ansatz methods. The off-diagonal cases are solvable, however, using Sklyanin's method of separation of variables.

[21] Note that our derivation below of the mutually commuting operators in the quasiclassical limit does not need to introduce the notion of a ''quantum determinant,'' and thus is slightly more direct than the method used by Sklyanin in Ref. 12.