Finite-Size Bosonization of 2-Channel Kondo Model: A Bridge between Numerical Renormalization Group and Conformal Field Theory

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We generalize Emery and Kivelson's (EK) bosonization-refermionization treatment of the 2-channel Kondo model to *finite system size* and on the EK line analytically construct its exact eigenstates and finite-size spectrum. The latter crosses over to conformal field theory's (CFT) universal non-Fermi-liquid spectrum (and yields the most-relevant operators' dimensions), and further to a Fermi-liquid spectrum in a finite magnetic field. Our approach elucidates the relation between bosonization, scaling techniques, the numerical renormalization group (NRG), and CFT. All CFT Green's functions are recovered with remarkable ease from the model's scattering states. [S0031-9007(98)06260-7]

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A dynamical quantum impurity interacting with metallic electrons can cause strong correlations and sometimes lead to non-Fermi-liquid (NFL) physics. A prototypical example is the 2-channel Kondo (2CK) model, in which a spin-1/2 impurity is "overscreened" by conduction electrons, leaving a nontrivial residual spin object even in the strong-coupling limit. Many theoretical treatments of this model have been developed [1], including Wilson's numerical renormalization group (NRG) [2,3] for the crossover from the free to the NFL regime, Affleck and Ludwig's (AL) conformal field theory (CFT) [3,4] for exact thermodynamic and transport quantities, valid only near the NFL fixed point, and Emery and Kivelson's (EK) bosonization-refermionization mapping onto a resonant-level model [5], valid on a line in parameter space that connects [6] the free and NFL fixed points. In this Letter we elucidate the well-known yet remarkable fact that these three approaches, despite tremendous differences in style and technical detail, yield mutually consistent results: We show that EK bosonization *in a system of finite size L* yields NRG-like finite-size spectra, and reproduces all known CFT results.

Our method requires *no* knowledge of CFT, only that we bosonize and refermionize with care: Firstly, we construct the boson fields ϕ and Klein factors F in the bosonization relation $\psi \sim F e^{-i\phi}$ explicitly in terms of the model's original fermion operators $\{c_{k\alpha j}\}$. Secondly, we clarify how the Klein factors for EK's refermionized operators act on the original Fock space. Thirdly, we keep track of the gluing conditions on all allowed states. This enables us (i) to explicitly contruct the model's finite-size eigenstates; (ii) to analytically obtain NRG-like finite-size spectra that cross over from free to CFT universal NFL spectra; (iii) to describe magnetic-field-induced cross overs exactly; (iv) to recover with remarkable ease all AL CFT results [4] for $L \to \infty$ [7].

The model.—We consider the standard anisotropic 2CK model with a linearized energy spectrum [3–5], defined by $H = H_0 + H_z + H_h$ ($\hbar = v_F = 1$):

$$H_0 = \sum_{k\alpha j} k :c^\dagger_{k\alpha j} c_{k\alpha j}:, \qquad H_h = h_i S_z + h_e \hat{\mathcal{N}}_s,$$
$$H_z + H_\perp = \Delta_L \sum_{kk'\alpha\alpha' ja} \lambda_a :c^\dagger_{k\alpha j} \frac{1}{2} \sigma^a_{\alpha\alpha'} S_a c_{k'\alpha' j}:.$$

Here $c^\dagger_{k\alpha j}$ creates a free-electron state $|k\alpha j\rangle$ with spin $\alpha = (\uparrow,\downarrow)$, flavor $j = (1,2) = (+,-)$, radial momentum $k \equiv |\vec{p}| - p_F$, and normalization $\{c_{k\alpha j}, c^\dagger_{k'\alpha' j'}\} = \delta_{kk'}\delta_{\alpha\alpha'}\delta_{jj'}$. We let the large-$|k|$ cutoff go to infinity, and quantize k by defining 1D fields with, for simplicity, antiperiodic boundary conditions at $x = \pm L/2$ [4],

$$\psi_{\alpha j}(x) \equiv \sqrt{\Delta_L} \sum_k e^{-ikx} c_{k\alpha j}, \qquad (1)$$

where $k = \Delta_L(n_k - 1/2)$ and $\Delta_L \equiv 2\pi/L$ is the mean level spacing. By : : we denote normal ordering relative to the Fermi ground state $|\vec{0}\rangle_0$. $H_z + H_\perp$ is the Kondo coupling (with dimensionless $\lambda_z \neq \lambda_\perp \equiv \lambda_x \equiv \lambda_y$) to a local spin-1/2 impurity S_a (with S_z eigenstates $|\Uparrow\rangle$, $|\Downarrow\rangle$), and H_h describes magnetic fields h_i and h_e coupled to the impurity spin and the total electron spin $\hat{\mathcal{N}}_s$.

Conserved quantum numbers.—Diagonalizing H requires choosing a suitable basis. Let any (nonunique) simultaneous eigenstate of $\hat{N}_{\alpha j} \equiv \sum_k :c^\dagger_{k\alpha j} c_{k\alpha j}:$, counting the number of (αj) electrons relative to $|\vec{0}\rangle_0$, be denoted by $|\vec{N}\rangle \equiv |N_{\uparrow 1}\rangle \otimes |N_{\downarrow 1}\rangle \otimes |N_{\uparrow 2}\rangle \otimes |N_{\downarrow 2}\rangle$, with $\vec{N} \in \mathbb{Z}^4$. Since H conserves charge, flavor, and total spin, it is natural to define new counting operators, $\hat{\mathcal{N}}_y$ ($y = c, s, f, x$),

$$\begin{pmatrix} \hat{\mathcal{N}}_c \\ \hat{\mathcal{N}}_s \\ \hat{\mathcal{N}}_f \\ \hat{\mathcal{N}}_x \end{pmatrix} \equiv \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{N}_{\uparrow 1} \\ \hat{N}_{\downarrow 1} \\ \hat{N}_{\uparrow 2} \\ \hat{N}_{\downarrow 2} \end{pmatrix}, \qquad (2)$$

which give half the total electron number, the electron spin, flavor, and spin difference between channels, respectively. Equation (2) implies that the eigenvalues $\vec{\mathcal{N}}$

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are either all integers or all half-integers (i.e., $\vec{\mathcal{N}} \in (\mathbb{Z} + P/2)^4$, with $P = (0,1)$ for even/odd total electron number), and that they obey the *free gluing condition*

$$\mathcal{N}_c \pm \mathcal{N}_f = (\mathcal{N}_s \pm \mathcal{N}_x) \bmod 2 . \qquad (3)$$

All nonzero matrix elements of $H_\perp$ have the form $\langle \mathcal{N}_c, S_T - \frac{1}{2}, \mathcal{N}_f, \mathcal{N}_x; \Uparrow | H_\perp | \mathcal{N}_c, S_T + \frac{1}{2}, \mathcal{N}_f, \mathcal{N}_x \pm 1; \Downarrow \rangle$, and since the total spin $S_T = \hat{\mathcal{N}}_S + S_z$, is conserved, the $\hat{\mathcal{N}}_s$ eigenvalue flips only between $S_T \mp \frac{1}{2}$, i.e., it fluctuates only "mildly." In contrast, *the $\hat{\mathcal{N}}_x$ eigenvalue fluctuates* "wildly" [an appropriate succession of spin flips can produce *any* $\mathcal{N}_x$ satisfying (3)]; this will be seen below to be at the root of the 2CK model's NFL behavior (in revealing contrast to the 1CK model, which has no wildly fluctuating quantum number, and lacks NFL behavior). For a given $(\mathcal{N}_c, S_T, \mathcal{N}_f)$ it thus suffices to solve the problem in the corresponding invariant subspace $\sum_{\oplus' \mathcal{N}_x} |\mathcal{N}_c, S_T - \frac{1}{2}, \mathcal{N}_f, \mathcal{N}_x; \Uparrow\rangle \oplus |\mathcal{N}_c, S_T + \frac{1}{2}, \mathcal{N}_f, \mathcal{N}_x + 1; \Downarrow\rangle$, to be denoted by S, where the prime on the sum indicates its restriction to $\mathcal{N}_x$ values respecting (3).

Bosonization.—To bosonize [5] the model in terms of the original $c_{k\alpha j}$'s [8,9], we define bosonic fields through

$$b^\dagger_{q\alpha j} \equiv \frac{i}{\sqrt{n_q}} \sum_{n_k \in \mathbb{Z}} c^\dagger_{k+q\alpha j} c_{k\alpha j}, \qquad (q \equiv \Delta_L n_q > 0),$$

$$\phi_{\alpha j}(x) \equiv \sum_{0<n_q \in \mathbb{Z}^+} \frac{-1}{\sqrt{n_q}} (e^{-iqx} b_{q\alpha j} + e^{iqx} b^\dagger_{q\alpha j}) e^{-aq/2},$$

which account for particle-hole excitations (the b's by construction satisfy $[b_{q\alpha j}, b^\dagger_{q'\alpha' j'}] = \delta_{qq'}\delta_{\alpha\alpha'}\delta_{jj'}$ and $[b_{q\alpha j}, \hat{N}_{\alpha' j'}] = 0$). Then the usual *bosonization relation,*

$$\psi_{\alpha j}(x) = F_{\alpha j} e^{-i(\hat{N}_{\alpha j} - 1/2)2\pi x/L} e^{-i\phi_{\alpha j}(x)}, \qquad (4)$$

holds as operator identity, where the *Klein factors* [8] $F_{\alpha j} \equiv \sqrt{a}\, \psi_{\alpha j}(0) e^{i\phi_{\alpha j}(0)}$ (see [9]) satisfy $[F_{\alpha j}, \hat{N}_{\alpha' j'}] = \delta_{\alpha\alpha'}\delta_{jj'} F_{\alpha j}$, $[F, \phi] = 0$, and $\{F_{\alpha j}, F^\dagger_{\alpha' j'}\} = 2\delta_{\alpha\alpha'}\delta_{jj'}$. Thus $F_{\alpha j}$, $F^\dagger_{\alpha j}$ ladder between the $N_{\alpha j}$, $N_{\alpha j} \mp 1$ Hilbert spaces without creating particle-hole excitations, and ensure proper ψ, $\psi^\dagger$ anticommutation relations.

To exploit the conserved quantities in the $\mathcal{N}_y$ basis, we now use the transformation (2) to define new Bose fields $b_{q\alpha j} \to b_{qy}$ and $\phi_{\alpha j} \to \varphi_y$. Writing H in terms of these [via (4)], only φ_x and φ_s couple to the impurity [5]:

$$H_0 = \Delta_L \sum_y \frac{1}{2} \hat{\mathcal{N}}_y^2 + \sum_{y, n_q > 0} q b^\dagger_{qy} b_{qy}, \qquad (5)$$

$$H_z = \lambda_z \Delta_L S_z \hat{\mathcal{N}}_s + \lambda_z \Delta_L S_x \sum_{n_q > 0} \sqrt{n_q}\, i(b_{qs} - b^\dagger_{qs}), \qquad (6)$$

$$H_\perp = \frac{\lambda_\perp}{2a} e^{i\varphi_s(0)} S_- \sum_{j=\pm} F^\dagger_{\uparrow j} F_{\downarrow j} e^{ij\varphi_x(0)} + \text{H.c.} \qquad (7)$$

To eliminate H_z, make the EK [5] unitary transformation $H' = UHU^\dagger$, with $U(\lambda_z) \equiv e^{i\lambda_z S_z \varphi_s(0)}$. This yields $H'_h = H_h$, $(H_0 + H_z)' = H_0 + \lambda_z \Delta_L \hat{\mathcal{N}}_s S_z +$ const, $S'_\pm = e^{\pm i\lambda_z \varphi_s(0)} S_\pm$, and φ_s incurs a phase shift:

$$U\varphi_s(x)U^\dagger = \varphi_s(x) - \lambda_z \pi S_z \operatorname{sgn}(x) \equiv \tilde{\varphi}_s(x) . \qquad (8)$$

We henceforth focus on the EK line of fixed $\lambda_z = 1$. Here φ_s decouples from $S_\pm$, and by (4) and (8) the $\psi_{\alpha j}$'s have phase shifts $\pm\pi/4$. Since this is just the value known for the NFL fixed point [3,10], the $\lambda_\perp$-induced crossover between the free and NFL fixed points can be studied on the EK line [6] by solving H' by refermionizing.

Refermionization.—We first have to define Klein factors for the $\mathcal{N}_y$ basis. Since an "off-diagonal" product $F^\dagger_{\alpha j} F_{\alpha' j'}$ acting on any state $|\vec{N}\rangle$ *just changes some of its $N_{\alpha j}$ (and hence $\mathcal{N}_y$) quantum numbers,* we write

$$\mathcal{F}^\dagger_x \mathcal{F}^\dagger_s \equiv F^\dagger_{\uparrow 1} F_{\downarrow 1}, \qquad \mathcal{F}_x \mathcal{F}^\dagger_s \equiv F^\dagger_{\uparrow 2} F_{\downarrow 2},$$
$$\mathcal{F}^\dagger_x \mathcal{F}^\dagger_f \equiv F^\dagger_{\uparrow 1} F_{\uparrow 2}, \qquad (9)$$

thereby defining new Klein factors $\mathcal{F}_y$, $\mathcal{F}^\dagger_y$ satisfying $[\mathcal{F}_y, \mathcal{N}_{y'}] = \delta_{yy'} \mathcal{F}_y$, $[\mathcal{F}, \varphi] = 0$, and $\{\mathcal{F}_y, \mathcal{F}^\dagger_{y'}\} = 2\delta_{yy'}$. Formally, these operators act on an extended Fock space [11] of states with arbitrary $\vec{\mathcal{N}} \in (\mathbb{Z} + P/2)^4$. Its physical subspace contains only those states that obey (3), and by (9) it is closed under the *pairwise* action of $\mathcal{F}_y$'s. This simple construction for keeping track of $\mathcal{N}_y$ quantum numbers is the main innovation of this Letter.

Next we define a *pseudofermion* field $\psi_x(x)$ [5] by

$$\psi_x(x) \equiv a^{-1/2} \mathcal{F}_z e^{-i(\tilde{\mathcal{N}}_s - 1/2)2\pi x/L} e^{-i\varphi_x(x)}, \qquad (10)$$

and expand it as $\sqrt{\Delta_L} \sum_{\bar{k}} e^{-i\bar{k}x} c_{\bar{k}x}$, by analogy with (4) and (1), which imply $\{c_{\bar{k}x}, c^\dagger_{\bar{k}'x}\} = \delta_{\bar{k}\bar{k}'}$. In the $c_{k\alpha j}$ basis, the $c_{\bar{k}x}$'s create highly nonlinear combinations of electron-hole excitations, as in clear from their *explicit* definition, via φ_x and $\mathcal{F}_x$, in terms of the $c_{k\alpha j}$'s. Since $\mathcal{N}_x \in \mathbb{Z} + \frac{P}{2}$, we note that ψ_x has a P-dependent boundary condition, implying $\bar{k} = \Delta_L(n_{\bar{k}} - \frac{1-P}{2})$, and further that $\Delta_L\,(\hat{\mathcal{N}}_x^2/2 + \sum_{q>0} n_q b^\dagger_{qx} b_{qx}) = H_{0x} + P/8$, where $H_{0x} \equiv \sum_{\bar{k}} \bar{k} : c^\dagger_{\bar{k}x} c_{\bar{k}x} :$ and $: :$ means normal ordering of $c_{\bar{k}x}$'s, with $\sum_{\bar{k}} : c^\dagger_{\bar{k}x} c_{\bar{k}x} : \equiv \hat{\mathcal{N}}_x - P/2$. We further define the "local pseudofermion" $c_d \equiv \mathcal{F}^\dagger_s S_-$, implying $c^\dagger_d c_d = S_z + \frac{1}{2}$. Eliminating $\hat{\mathcal{N}}_s$ in the subspace S using $\hat{\mathcal{N}}_s = S_T + \frac{1}{2} - c^\dagger_d c_d$, we can rewrite H' as $H_{csf}(b_c, b_f, b_s, \mathcal{N}_c, \mathcal{N}_f) + H_x + E_G$, where H_{csf} has a trivial spectrum and H_x is quadratic:

$$H_x = \varepsilon_d c^\dagger_d c_d + H_{0x} + \sqrt{\Delta_L \Gamma} \sum_{\bar{k}} (c^\dagger_{\bar{k}x} + c_{\bar{k}x})(c_d - c^\dagger_d),$$

$$E_G = \Delta_L[\tfrac{1}{2}(S_T^2 - \tfrac{1}{4}) + P/8] - \tfrac{1}{2} h_i + h_e(S_T - \tfrac{1}{2}).$$

Here $\Gamma \equiv \lambda_\perp^2/4a$ and $\varepsilon_d \equiv h_i - h_e$ is the spin flip energy cost. As first noted by EK [5], who derived H' for $L \to \infty$, *impurity* properties show NFL behavior since "half the pseudofermion," $(c_d + c^\dagger_d)$, decouples.

Diagonalizing H_x.—To study the NFL behavior of *electron* properties, caused by the nonconservation of $\mathcal{N}_x$, we diagonalize H_x. First, define further pseudofermions having all non-negative energies: $\alpha_{\bar{k}} \equiv \frac{1}{\sqrt{2}}(c_{\bar{k}x} + c^\dagger_{-\bar{k}x})$ and $\beta_{\bar{k}} \equiv \frac{1}{i\sqrt{2}}(c_{\bar{k}x} + c^\dagger_{-\bar{k}x})$ for $\bar{k} > 0$; if $P = 1$ then $\alpha_0 \equiv c^\dagger_{0x}$, and $\alpha_d \equiv c_d$ (or $c_d^\dagger$) for $\varepsilon_d > 0$ (or ≤ 0). Then the $\beta_{\bar{k}}$'s decouple in H_x, and a Bogoliubov transformation $\tilde{\alpha}^\dagger_\varepsilon = \sum_{n=d,\bar{k}} \sum_{\nu=\pm} B_{\varepsilon n \nu}(\alpha_n^\dagger + \nu \alpha_n)$ yields [11]

$$H_x = \frac{\varepsilon_d}{2} + \sum_{\varepsilon \geq 0} \varepsilon\left(\tilde{\alpha}^\dagger_\varepsilon \tilde{\alpha}_\varepsilon - \frac{1}{2}\right) + \sum_{\bar{k}>0} \bar{k}\left(\beta^\dagger_{\bar{k}}\beta_{\bar{k}} + \frac{1}{2}\right),$$

$$4\pi\Gamma\varepsilon/(\varepsilon^2 - \varepsilon_d^2) = -\cot\pi(\varepsilon/\Delta_L - P/2). \quad (11)$$

Equation (11) for the pseudofermion eigenenergies ε implies that each $\bar{k}$ smoothly evolves into a corresponding $\varepsilon(\bar{k})$ as Γ is turned on. Since $\varepsilon(\bar{k}) \simeq \bar{k} + \frac{\Delta_L}{2}$ (or $\simeq \bar{k}$) for $\bar{k} \ll$ (or $\gg$) Γ, we see very nicely that the spectrum's low- and high-energy parts are strongly and weakly perturbed, respectively, with crossover scale $T_K \simeq \Gamma$ [5].

As mentioned above, the pseudofermions act on an extended Fock space. To identify which eigenstates $|\tilde{E}\rangle$ of H' are physical, note that each has to adiabatically develop, as Γ increases from 0, from some state obeying the free gluing condition (3). The latter can be shown [11] to develop into the general gluing condition (GGC) [12] that $\langle \tilde{E}| [\sum_{\varepsilon\geq 0} \tilde{\alpha}^\dagger_\varepsilon \tilde{\alpha}_\varepsilon + \sum_{\bar{k}>0} \beta^\dagger_{\bar{k}}\beta_{\bar{k}}] \bmod 2 |\tilde{E}\rangle$ must be equal to $[\mathcal{N}_c + \mathcal{N}_f - (S_T + \frac{1}{2} + \frac{P}{2} - P_d)] \bmod 2$, where $P_d = 0(1)$ for $\varepsilon_d > 0$ (≤ 0). The GGC and Eqs. (11) together constitute an exact analytical solution of the 2CK model at the EK line for arbitrary $\lambda_\perp$, h_i, and h_e.

Relation to RG methods.—Our exact solution allows us to implement Anderson "poor man's scaling" and Wilson's NRG treatments of the Kondo problem analytically, thus illustrating the main idea behind both, *namely, to try to uncover the low-energy physics via an RG transformation.* In the first, the RG is generated by reducing (at fixed L, usually $= \infty$) the bandwidth while adjusting the couplings to keep the dynamical properties invariant. Since the cutoff used when bosonizing is $1/a(\sim p_F)$ and a occurs in H' *only* through Γ, the scaling equations [6] $\frac{d\ln\lambda_z}{d\ln a} = 0$, $\frac{d\ln\lambda_\perp}{d\ln a} = 1/2$, which imply that $\lambda_\perp$ grows under rescaling [13], are exact along the EK line. Renormalizing the spin flip vertex, possible only approximately in the original $c_{k\alpha j}$ basis by summing selected diagrams, thus becomes trivial after bosonizing and refermionizing, which in effect resums *all* diagrams into a quadratic form.

Wilson's NRG [2,3] is, in effect, a finite-size scaling method which increases (at fixed bandwidth and couplings) the system size, thus decreasing the mean level spacing and pushing ever more eigenenergies down into the spectrum's strongly perturbed regime below T_K. Each RG step enlarges the system by order $\Lambda > 1$ by including an extra "onion-skin shell" of electrons, then rescales $H \rightarrow \Lambda H$ to measure energy in units of the new reduced level spacing. We can mimick this by transforming $L \rightarrow L' = \Lambda L$ (thus $\Gamma/\Delta_L \rightarrow \Lambda\Gamma/\Delta_L$) and plotting the spectrum in units of $\Delta_L' = \frac{2\pi}{L'}$.

Figure 1 displays $(\tilde{E} - \tilde{E}_{\min})/\tilde{\Delta}_L$ for the lowest few $|\tilde{E}\rangle$ that satisfy the GGC. Figure 1(a) shows the evolution of the spectrum *toward* the EK line for $\lambda_z \in [0,1]$ at $\Gamma = \varepsilon_d = 0$ [i.e., free fermions, phase shifted by $\pm\lambda_z\pi/2$ in the spin sector, see (8)]. Figure 1(b) shows its further evolution *on* the EK line for $\Gamma/\Delta_L \in [0,\infty]$ at $\lambda_z = 1$, $\varepsilon_d = 0$. Decreasing Δ_L at *fixed* Γ yields an NRG-like crossover spectrum that for $\Delta_L \rightarrow 0$ indeed reproduces the NRG's universal NFL fixed point spectrum [2,3] (irrespective of the specific Γ value, illustrating the irrelevance of spin anisotropy [3]). This NFL spectrum also agrees with that found by AL using a so-called *fusion hypothesis* [4], which our GGC thus proves simply and directly (in contrast to the CFT proof of Ref. [14(b)]). Note that the ground state (with degeneracy 2) has entropy $\ln 2$, as it must for finite L [15] (in contrast, the celebrated result $\frac{1}{2}\ln 2$ requires taking $L \rightarrow \infty$ before $T \rightarrow 0$).

Next we illustrate Wilson's program of extracting the most relevant operator's dimensions from the L dependence of the finite-size corrections, $\delta\tilde{E}(L) \equiv \tilde{E}(\Gamma/\Delta_L) - \tilde{E}(\infty)$, to the universal NFL spectrum: For $\varepsilon_d = 0$, Eq. (11) gives $\frac{\delta\tilde{E}}{\Delta_L} \sim \frac{1}{\Gamma L}$, thus *on* the EK line the least irrelevant operator has dimension 1, but perturbative corrections in $\lambda_z - 1$ yield $\frac{\delta\tilde{E}}{\Delta_L} \sim \frac{\lambda_z - 1}{(\Gamma L)^{1/2}}$, thus the general leading irrelevant operators (absent *on* the EK line) have dimension $\frac{1}{2}$ [4,11,14]. Next, turning on a local field $\varepsilon_d = h_i$, we find from (11) that for $h_i \ll h_c \equiv \sqrt{\Gamma/L}$ the NFL spectrum is only slightly affected, while for $h_c \ll h_i \ll \Gamma$ the spectrum has three distinct regions: It is Fermi-liquid-like [3] (with uniform level spacing) for $\varepsilon \ll h_K \equiv \frac{h_i^2}{\Gamma}$ and $\varepsilon \gg \Gamma$, and NFL-like (nonuniform level spacings) for $h_K \ll \varepsilon \ll \Gamma$. Both the L dependence of h_c and the h_i dependence of the crossover scale h_K show that the local magnetic field is relevant, with dimension $-\frac{1}{2}$; it causes a crossover, shown in Fig. 1(c), to a Fermi-liquid spectrum for all states with $\varepsilon \ll h_K$.

For $\Gamma/\Delta_L \rightarrow \infty$, $h \rightarrow 0$, we find logarithmic divergences for the susceptibility $\chi \approx \frac{1}{4\pi^2\Gamma}\ln(\Gamma L)$ and the $\hat{\mathcal{N}}_x$ fluctuations $\langle \hat{\mathcal{N}}_x^2 \rangle \approx \frac{1}{\pi^2}\ln(\Gamma L)$ (with $\langle \hat{\mathcal{N}}_x \rangle = 0$). Both are clear signs of 2CK NFL physics: The first shows

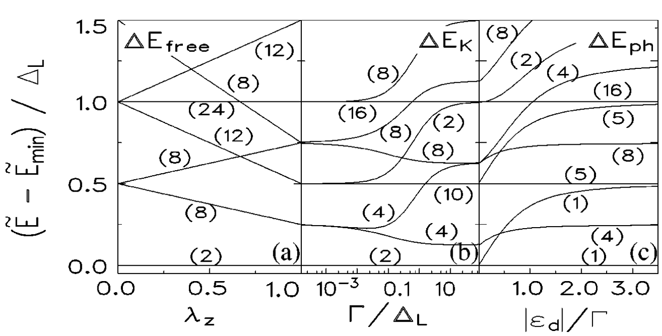

FIG. 1. All eigenenergies $\Delta\tilde{E} = (\tilde{E} - \tilde{E}_{\min})/\Delta_L \leq 1$ (degeneracies in parentheses) of the full H' as functions of (a) $\lambda_z \in [0,1]$ at $\Gamma = \varepsilon_d = 0$; (b) $\Gamma/\Delta_L \in [0,\infty]$ at $\lambda_z = 1$, $\varepsilon_d = 0$; (c) $|\varepsilon_d|/\Gamma \in [0, 3.5]$ at fixed $\Gamma/\Delta_L \gg 1$, $\lambda_z = 1$.

that no spin singlet is formed due to "overscreening," the second how strongly this perturbs the electron sea.

Relation to CFT.—Recent CFT [7] and scaling [6] arguments showed that the NFL regime can be described by *free boson fields.* This can be confirmed very easily by finding the scattering state operators $\tilde{c}^{\dagger}_{\bar{k}x}$ [and field $\tilde{\psi}^{\dagger}_{x}(x)$] into which the free $c^{\dagger}_{\bar{k}x}$'s $[\psi^{\dagger}_{x}(x)]$ develop when Γ is turned on adiabatically as $e^{\eta t}\Gamma$ (at $\varepsilon_d = 0$), and deducing from these the behavior of the $\tilde{\varphi}_y$ fields. In the continuum limit [$L \to \infty$, then ($\Delta_L \ll$) $\eta \to 0^+$], the $\tilde{c}^{\dagger}_{\bar{k}x}$'s obey [16] the Lippmann-Schwinger equation $[H_x, \tilde{c}^{\dagger}_{\bar{k}x}] = \bar{k}\tilde{c}^{\dagger}_{\bar{k}x} + i\eta(\tilde{c}^{\dagger}_{\bar{k}x} - c^{\dagger}_{\bar{k}x})$, which gives [16]

$$\tilde{c}^{\dagger}_{\bar{k}x} = c^{\dagger}_{\bar{k}x} + \int d\bar{k}' \times \frac{2\Gamma\bar{k}(c^{\dagger}_{\bar{k}'x} + c_{-\bar{k}'x})}{[(\bar{k} + i\eta)(\bar{k} + i4\pi\Gamma) - \varepsilon_d^2](\bar{k} - \bar{k}' + i\eta)}.$$

To find the asymptotic behavior ($|x| \to \infty$) of $\tilde{\psi}^{\dagger}_{x}(x) \equiv \sqrt{\Delta_L}\int d\bar{k}e^{i\bar{k}x}\tilde{c}^{\dagger}_{\bar{k}x}$, we may take $\bar{k}/\Gamma \to 0$; this gives

$$\tilde{\psi}^{\dagger}_{x}(x) \sim 1/\sqrt{\Delta_L}\int d\bar{k}' e^{i\bar{k}'x}[c^{\dagger}_{\bar{k}'x}\theta(x) - c_{-\bar{k}'x}\theta(-x)].$$

Adopting AL's notation of L and R movers, $\tilde{\psi}^{\dagger}_{x}(x) \equiv \theta(x)\tilde{\psi}^{\dagger}_{xL}(x) + \theta(-x)\tilde{\psi}^{\dagger}_{xR}(x)$, then gives $-\tilde{\psi}^{\dagger}_{xR} \sim \tilde{\psi}^{\dagger}_{xL} \sim \psi^{\dagger}_{x}$. To translate this into "boundary conditions" on the $\tilde{\varphi}_y$ boson fields, we write $\tilde{\psi}_{xL/R} \equiv \tilde{\mathcal{F}}_{xL/R}a^{-1/2}e^{-i\tilde{\varphi}_{xL/R}}$ and note that $\tilde{\varphi}_c$, $\tilde{\varphi}_f$ decouple and $\tilde{\varphi}_s$ is phase shifted as in (8). Thus the free and scattering boson fields are asymptotically related (with $\eta_c, \eta_s, \eta_f = 1 = -\eta_x$) by

$$(\eta_y\tilde{\varphi}_{yR} - \pi S_z\delta_{ys}) \sim (\tilde{\varphi}_{yL} + \pi S_z\delta_{ys}) \sim \varphi_y, \quad (12)$$

while $\eta_y(\tilde{\mathcal{F}}_{yR})^{\eta_y} = \tilde{\mathcal{F}}_{yL} = \mathcal{F}_y$ for $y = s, f, x$. This central result, first found in Ref. [7] (with different phases since Klein factors were neglected), shows *that the NFL regime can be described by boson fields $\tilde{\varphi}_{yL/R}$ that are, asymptotically, free* (with only a trivial S_z dependence).

Next we consider the 16 bilinear fermion currents $\tilde{J}^{aA}_{y} \equiv :\tilde{\psi}^{\dagger}_{\alpha j}(T^{aA}_{y})_{\alpha\alpha',jj'}\tilde{\psi}_{\alpha'j'}:$ (with $T^{00}_{c} = \frac{1}{2}\delta\delta$, $T^{a0}_{s} = \frac{1}{2}\sigma^a\delta$, $T^{0A}_{f} = \frac{1}{2}\delta\sigma^A$, $T^{aA}_{x} = \frac{1}{2}\sigma^a\sigma^A$), for which (12) yields [11] the boundary conditions $\tilde{J}^{aA}_{yR} \sim \eta_y\tilde{J}^{aA}_{yL}$. For $y = c, s, f$, these express the reemergence at the NFL fixed point of the full $U(1) \times SU(2)_2 \times SU(2)_2$ Kac-Moody symmetry assumed by AL; for $y = x$ they are just what AL derived using their fusion hypothesis. Since these boundary conditions fully determine all AL's CFT Green's functions [4], the boson approach will *identically* reproduce them also, if one proceeds as follows: To evaluate $\langle\tilde{\psi}_{\alpha j}(1)\ldots\tilde{\psi}^{\dagger}_{\alpha'j'}(1')\rangle$, simply insert (4), rewrite the result in terms of $\tilde{\varphi}_{yL/R}$ and $\tilde{\mathcal{F}}_{yL/R}$, and combine (12) with standard free-boson results such as

$$\frac{\langle e^{-i\lambda\tilde{\varphi}_{yR}(x)}e^{i\lambda'\tilde{\varphi}_{y'L}(x')}\rangle}{a^{(\lambda^2+\lambda'^2)/2}} \sim \frac{\delta_{yy'}L^{-1/2(\eta_y\lambda-\lambda')^2}}{(ix - ix')^{\eta_y\lambda\lambda'}}. \quad (13)$$

All asymptotic NFL behavior of electron Green's functions arises from the fact that $\eta_x = -1$, combined with relations such as (13); it directly yields, e.g., the so-called "unitarity paradox" [7] $\langle\tilde{\psi}_{\alpha jR}(x)\tilde{\psi}^{\dagger}_{\alpha'j'L}(x')\rangle \sim 0$ (for $L \to \infty$, then $|x' - x| \to \infty$). Note, though, that probability is not lost during scattering: $\tilde{\psi}^{\dagger}_{x}(x)$ shows that each *pseudoparticle* $c^{\dagger}_{\bar{k}'x}$ incident from $x > 0$ is "Andreev-scattered," emerging at $x < 0$ as *pseudohole* $c_{-\bar{k}'x}$, *orthogonal to what was incident;* this very NFL-like behavior dramatically illustrates the effects of $\hat{\mathcal{N}}_x$ nonconservation.

To find AL's *boundary operators* in terms of the $\tilde{\varphi}_y$'s [6,11], one calculates the operator product expansion of $\tilde{\psi}_{R\alpha j}\tilde{\psi}^{\dagger}_{L\alpha'j'}$. Since $\eta_x = -1$, *all* terms contain a factor $e^{\pm i\varphi_y}$ ($y = s, f$ or x) with dimension $\frac{1}{2}$; this ultimately causes the famous $T^{1/2}$ in the resistivity [4,6,7].

In conclusion, finite-size bosonization allows one (i) to mimick, in an *exact* way, the strategy of standard RG approaches and (ii) to recover with remarkable ease all exact results known from CFT for the NFL fixed point. It thus constitutes a bridge between these theories.

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