Dynamical scaling for critical states: is Chalker's ansatz valid for strong fractality?

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The dynamical scaling for statistics of critical multifractal eigenstates proposed by Chalker is analytically verified for the critical random matrix ensemble in the limit of strong multifractality controlled by the small parameter $b \ll 1$. The power law behavior of the quantum return probability $P_N(\tau)$ as a function of the matrix size N or time τ is confirmed in the limits $\tau/N \to \infty$ and $N/\tau \to \infty$, respectively, and it is shown that the exponents characterizing these power laws are equal to each other up to the order b^2. The corresponding analytical expression for the fractal dimension d_2 is found.

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Fifty years after the seminal paper by P.W.Anderson [1] which predicted localization it is becoming a common wisdom that the *multi-fractal* scaling properties of critical [2] and near critical [3] states are not just an exotic theory exercise but an important physics reality [4, 5]. Since the pioneer's work by F.Wegner [6], it is known that at the Anderson transition point the local moments of the wave-functions $\psi_n(\mathbf{r})$ scale with the system size L as (summation over n runs over the critical window near the mobility edge)

$$I_q = \sum_{\mathbf{r}} \sum_{n} \left\langle |\psi_n(\mathbf{r})|^{2q} \right\rangle \propto L^{-d_q(q-1)}, \tag{1}$$

where d_q is the fractal dimension corresponding to the q-th moment (*multi*-fractality). This implies [2] that all scales of the amplitude $|\psi_n(\mathbf{r})|^2 \sim L^{-\alpha}$ are present with the corresponding number of sites $N_\alpha \sim L^{f(\alpha)}$. The function $f(\alpha)$ known as *spectrum of fractal dimensions* is given by the Legendre transform of $\tau(q) = d_q(q-1)$. One can distinguish between the weak multi-fractality where d_q is close to the system dimensionality d and is almost linear in q for not too large moments $q > 1$, and the strong multi-fractality where $d_q \ll d$ for all $q > 1$.

Recently the interest in Anderson localization has been shifted towards its effect for a many-body system of interacted particles. The simplest problems of this type are the "multi-fractal" superconductivity [7] and the Kondo effect [8]. For such problems the relevant quantity is the matrix element of (local) interaction which involves wave-functions of two different energies E_m and E_n:

$$K(\omega, \mathbf{R}) = \sum_{\mathbf{r}} \sum_{n,m} \langle |\psi_n(\mathbf{r}+\mathbf{R})|^2 \, |\psi_m(\mathbf{r})|^2 \, \delta(E_m - E_n - \omega) \rangle. \tag{2}$$

A promising playground to observe such effects at controllable strength of disorder and interaction are systems of cold atoms where one-dimensional Anderson localization has been already observed [9] and the observation of their two- and three- dimensional counterparts is on the way.

It was conjectured long ago by Chalker and Daniel [10, 11] and confirmed by numerous computer simulations [3, 10, 12] that at $E_0 \gg \omega \gg \Delta$ (Δ is the mean level separation):

$$K(\omega, 0) \equiv C(\omega) \propto (E_0/\omega)^{\mu}, \;\; \mu = 1 - d_2/d. \tag{3}$$

The scaling relationship Eq.(3) can be viewed as a result of application of the dynamical scaling

$$L \to L_\omega \propto \omega^{-\frac{1}{d}}, \tag{4}$$

to the earlier Wegner's result [13]

$$K(0, R) \propto (L/R)^{d-d_2}, \tag{5}$$

with the simultaneous assumption that the R-dependence is saturated for $R < \ell$; ℓ being the minimum scale of multi-fractality of the order of the elastic scattering length and $E_0 \propto \ell^{-d}$ is the corresponding high-energy cutoff. This property is of great importance for electron interaction in the vicinity of the Anderson transition, as it leads to a dramatic enhancement of matrix elements compared to the case of absence of multi-fractality [3, 7].

Note that the dynamical scaling hypothesis Eq.(4) implies that the energy scale ω corresponds to the length scale L_ω equal to the size of a sample where the mean level spacing is ω. However simple and natural is this hypothesis, it leads to a somewhat counter-intuitive consequence in the case of strong multi-fractality. Indeed, in the limit $d_2 \to 0$ the exponent μ in Eq.(3) saturates at $\mu = 1$. It signals of a strong overlap of two infinitely sparse fractal wavefunctions, while one would expect such states not to overlap in a typical realization, similar to two localized wavefunctions. At present there is no analytical evidence of validity of Eq.(3) in the limit of strong multi-fractality though numerical simulations seem to be very encouraging to it [3]. The main goal of this paper is to provide such an evidence.

To achieve this objective we consider a model system described by a random matrix Hamiltonian with the multi-fractal eigenstates [14, 15]. The Gaussian ensemble of such random $N\times N$ matrices is defined by the variance of random (complex) entries

$$v_{n-m} \equiv \left\langle |H_{nm}|^2 \right\rangle = \frac{\frac{1}{2}b^2}{b^2+|n-m|^{2(1-\epsilon)}},\ \langle H_{nm}\rangle = 0, \tag{6}$$

and describes a long-range hoppings between sites of the one-dimensional lattice. The parameter b determines the strength of the fractality, with the weak multi-fractality at $b\gg 1$ and the strong multi-fractality at $b \ll 1$. It is the last limit that we will concentrate on in this paper. For convenience of the further calculations we introduced here a regularizing parameter $\epsilon \to +0$.

Numerous computer simulations [2, 3, 16] show that the above model gives an unexpectedly good *quantitative* description of statistics of the critical wave functions in the 3D Anderson model. Inspired by this fact we use the critical RMT, Eq.(6), as the simplest and representative description of multi-fractality at the Anderson transition.

To proceed further on, we note that the moment I_2 defined in Eq.(1) and the dynamic correlation function $C(\omega)$ given by Eq.(3) are related with the physical observable known as the *return probability*:

$$P_N(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} C(\omega). \tag{7}$$

Indeed, plugging $K(\omega, 0)$ defined by Eq.(2) into Eq.(7) and taking into account that the average value of $e^{-i(E_n-E_m)t} \to \delta_{mn}$ in the limit $t\to\infty$ at a fixed mean level separation $\Delta \sim N^{-1}$ we obtain at $N \gg 1$:

$$\lim_{t/N\to\infty} P_N(t) = I_2 \propto N^{-d_2/d}, \tag{8}$$

where in a particular case of the critical RMT the fractal dimension $d_2 = \sum_{n=1}^{\infty} c_m\, b^m$ allows a regular expansion in the parameter $b \ll 1$. The first term of this expansion c_1 was computed in Ref.[17].

Alternatively, if $\Delta \sim N^{-1}$ tends to zero at a fixed large t, the power law in Eq.(3) is unbounded from below and one obtains at $t \gg E_0^{-1} \sim b^{-1}$:

$$\lim_{N/t\to\infty} P_N(t) \propto t^{\mu-1} \propto t^{-d_2/d}. \tag{9}$$

Thus the Chalker's ansatz Eq.(3) is equivalent to the identity:

$$\left.\frac{\partial \ln P_N(t)}{\partial \ln t}\right|_{\substack{N/t\to\infty \\ t\to\infty}} = \left.\frac{\partial \ln P_N(t)}{\partial \ln N}\right|_{\substack{t/N\to\infty \\ N\to\infty}}. \tag{10}$$

It implies that at large t and N the leading dependence of $\ln P_N(t)$ on t and N is logarithmic, and the coefficients in front of $\ln t$ and $\ln N$ are the same in both limits and equal to $-d_2/d$. This is exactly what we are going to demonstrate in the present work.

In order to reach this goal one needs to find the return probability at finite time t as well as its limiting value at $t\to\infty$. In this Letter we calculate $P_N(t)$ using the virial expansion in the number of resonant states, each of them is localized at a certain site n. The virial expansion formalism was developed in Ref.[18, 19] following the initial idea of Ref.[20]. The supersymmetric version of the virial expansion [19] is formulated in terms of integrals over super-matrices. In particular, it allows us to represent $P_N(t)$ as an infinite series of integrals over an increasing number of super-matrices Q_n associated with different sites n. As it was shown in Ref.[19], the terms $P_N^{(i)}(t)$ involving integration over i different super-matrices result in the contribution to $P_N(t) = \sum_i P_N^{(i)}(t)$ of the form:

$$P_N^{(i)}(t) = b^{i-1}\, p_N^{(i)}(bt), \tag{11}$$

with the known [19] explicit expressions for $p_N^{(2,3)}(bt)$ and $p_N^{(1)} = 1$.

It follows from Eq.(11) that

$$\ln P_N(t) \simeq b\, p_N^{(2)}(bt) + b^2 \left[p_N^{(3)}(bt) - \frac{1}{2}\left(p_N^{(2)}(bt)\right)^2 \right]. \tag{12}$$

For Eq.(10) to be valid, one requires that in the limit of large bt, N:

(i) $p_N^{(2)}(bt) = -c_1\, \ln(\min\{bt, N\})$

(ii) $\left[p_N^{(3)}(bt) - \frac{1}{2}\left(p_N^{(2)}(bt)\right)^2\right] = -c_2\, \ln(\min\{bt, N\})$

(iii) terms proportional to $\ln^2(bt)$ and $\ln^2 N$ cancel out in the combination (ii).

This cancelation, as well as the logarithmic asymptotic behavior with equal coefficients in front of $\ln(bt)$ and $\ln N$ in the two different limits, is not trivial. We show below that these properties are hidden in the general structure of the virial expansion and in the power law dependence of the variance of the critical RMT $v_{n\neq 0} \propto 1/n^2$, Eq(6).

We start by analyzing the explicit expression for $P_N^{(2)}(t)$ obtained in Ref.[19]:

$$P_N^{(2)}(t) = -2\sqrt{\pi} \sum_{n=1}^{N} \left\{ v_n |t|\, e^{-v_n t^2} + \frac{\sqrt{\pi v_n}}{2}\, \mathrm{erf}\left(\sqrt{v_n}|t|\right) \right\}, \tag{13}$$

where $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi^2} d\xi$.

In the large t and N limit the sum over n in Eq.(13) is dominated by large n at any $\epsilon > 0$. Replacing the sum by an integral $\sum_n \to \int_0^\infty dn$ one can represent $p_N^{(2)}(bt)$ as a double integral:

$$p_N^{(2)}(bt) = -2\int_0^\tau \frac{dy}{y^2} \int_{N^{-1}}^\infty \frac{dn}{n^2}\, \mathcal{F}_2\left(y\, n^{1-\epsilon}\right), \tag{14}$$

where $\tau=\frac{bt}{\sqrt{2}}$, $\mathcal{F}_2(z)=\sqrt{2\pi}\,z^2(1-z^2)e^{-z^2}$, and $\epsilon>0$ ensures convergence at small y. Now we take the logarithmic derivatives of $p_N^{(2)}(bt)$ as in Eq.(10) and implement the limits $N/t\to\infty$ or $t/N\to\infty$. The results are

$$-2\sqrt{2\pi}\tau^{\frac{\epsilon}{1-\epsilon}}\int_0^\infty \frac{dz}{z^2}\,\mathcal{F}_2(z^{1-\epsilon}),\;\; -2\sqrt{2\pi}N^\epsilon\int_0^\infty \frac{dz}{z^2}\,\mathcal{F}_2(z), \tag{15}$$

respectively. Finally we take the limit $\epsilon\to 0$. One can see that both expressions in Eq.(15) coincide and

$$c_1=2\sqrt{2\pi}\int_0^\infty \frac{dz}{z^2}\,\mathcal{F}_2(z)=\frac{\pi}{\sqrt{2}}, \tag{16}$$

provided that the operations of taking the limit $\epsilon\to 0$ and integrating over z commute.

Thus the validity of the Chalker's ansatz, in the form given by Eq.(10), in the first order in $b\ll 1$ is based on the symmetry of the integrand in Eq.(14) with respect to n and y at $\epsilon=0$. However the necessary condition for this argument to work is the *commutativity* of integrating up to infinity and taking the limit $\epsilon\to 0$. Such a commutativity can be checked straightforwardly for Eq.(15).

Now we proceed with b^2 contributions. Here the higher powers of $\ln N$ or $\ln\tau$ may arise and their cancelation in $\ln P_N(\tau)$ is of principal importance. The cancelation, if it happens, excludes in Eq.(8) presence of further *additive* terms such as $N^{-k\,d_2}$ or $\tau^{-k\,d_2}$ with $k>1$. Note that such terms, albeit not changing the leading large-N,τ behavior, could significantly alter the perturbative expansion in $b\ln N$ and $b\ln\tau$. Their absence is a strong argument in favor of a *pure power-law* behavior Eqs.(8,9).

Using the representation for $p_N^{(3)}(bt)$ derived in Ref.[19] one can cast the combination $\mathcal{P}_N^{(3)}(bt)=p_N^{(3)}(\tau)-\frac{1}{2}\left(p_N^{(2)}(\tau)\right)^2$ in the following form:

$$\mathcal{P}_\tau=\tau^{\frac{2\epsilon}{1-\epsilon}}\left\{\left(\frac{1-\epsilon}{2\epsilon}\right)\iint_{-\infty}^{+\infty}dxdy\,\mathcal{F}_3\left(x,y;\epsilon\right)-\frac{1}{2}\left(\frac{1-\epsilon}{\epsilon}\right)^2\left[\int_{-\infty}^{+\infty}dx\,\mathcal{F}_2\left(\frac{1}{|x|^{1-\epsilon}}\right)\right]^2\right\}, \tag{17}$$

$$\mathcal{P}_N=N^{2\epsilon}\left\{\int_0^\infty\frac{d\beta}{\beta^{1-2\epsilon\kappa}}\iint_{\frac{-1}{\beta^\kappa}}^{\frac{1}{\beta^\kappa}}dxdy\mathcal{F}_3\left(x,y;\epsilon\right)\left(1-|x-y|\beta^\kappa\right)-\frac{1}{2}\left[\int_0^\infty\frac{d\beta}{\beta^{1-\epsilon\kappa}}\int_{\frac{-1}{\beta^\kappa}}^{\frac{1}{\beta^\kappa}}dx\mathcal{F}_2\left(\frac{1}{|x|^{1-\epsilon}}\right)\left(1-|x|\beta^\kappa\right)\right]^2\right\} \tag{18}$$

where $\kappa\equiv 1/(1-\epsilon)$, $\mathcal{P}_N=\lim_{\tau/N\to\infty}\mathcal{P}_N^{(3)}(\tau)$ and $\mathcal{P}_\tau=\lim_{N/\tau\to\infty}\mathcal{P}_N^{(3)}(\tau)$, and

$$\mathcal{F}_3\left(x,y;\epsilon\right)\;=\;-\frac{\sqrt{\pi}i}{8}\int_{\mathcal{C}}d\alpha\,\sqrt{\alpha}\,e^\alpha\,\left[f_1(Z_x)f_1(Z_y)\,(f_3-f_2)(Z_{x-y})+f_2(Z_x)f_3(Z_y)\,(4f_1-f_3)(Z_{x-y})\right], \tag{19}$$

$$f_1(Z)\;=\;Z/\sqrt{1+Z},\;f_2(Z)=1/\sqrt{1+Z},\;f_3(Z)=Z/(1+Z)^{\frac{3}{2}},\;Z_x=1/(\alpha|x|^{2(1-\epsilon)}). \tag{20}$$

The contour $\mathcal{C}$ is the Hankel contour encompassing the negative part of the real axis $\Re(\alpha)<0$.

We present the cumbersome Eqs.(17)-(19) not only to give a flavor of real complexity of the calculations but also to uncover the ultimate reason for the Chalker's ansatz to hold. As in Eq.(15), there is a certain similarity in Eqs.(17) and (18) which can be traced back to $(n-m)^{-2}$ behavior of the variance Eq.(6). To exploit this similarity, we use the approximate equality:

$$\int_0^\infty\frac{d\beta}{\beta^{1-\delta}}f(\beta)\simeq\frac{f(0)}{\delta}-\int_0^\infty d\beta\,\ln\beta\,\frac{\partial f}{\partial\beta}-\frac{\delta}{2}\int_0^\infty d\beta\,\ln^2\beta\,\frac{\partial f}{\partial\beta}. \tag{21}$$

Applying this formula to Eq.(18), one can see that the first term on the r.h.s. reproduces immediately Eq.(17) up to the change of the pre-factor $\tau^{\frac{2\epsilon}{1-\epsilon}}$ by $N^{2\epsilon}$. Like in the dimensional regularization calculus [21], after taking the logarithmic derivatives w.r.t. τ or N and taking the subsequent limit $\epsilon\to 0$, the terms $\propto\epsilon^{-1}\,\tau^{\frac{2\epsilon}{1-\epsilon}}$ and $\propto\epsilon^{-1}\,N^{2\epsilon}$ tend to a constant determining the b^2 contribution to d_2.

The full calculation, however, is complicated by the presence of ϵ^{-2} singularity in $\mathcal{F}_3$ leading to the appearance of the additional contributions of the form $\epsilon\,\epsilon^{-2}$ etc. Thus one has to keep not only the first term on the r.h.s. of Eq.(21) but the next two terms as well. An accurate account of all such terms shows that Eq.(10) is valid to the b^2 order, and the fractal dimension d_2 is equal to:

$$d_2=\frac{\pi b}{\sqrt{2}}+\frac{(\pi b)^2}{4}\left[10-\frac{56}{3\sqrt{3}}-\ln 4+\pi\,I\right]+O(b^3), \tag{22}$$

where

$$I=\left(\frac{2}{\pi}\right)^3\int_0^{\frac{\pi}{2}}\frac{d\varphi_1\,d\varphi_2\,d\varphi_3}{(\cos\varphi_1+\cos\varphi_2)\,(\cos\varphi_1+\cos\varphi_2+\cos\varphi_3)}.$$

I can be evaluated numerically

$$I=0.79426047250532455983. \tag{23}$$

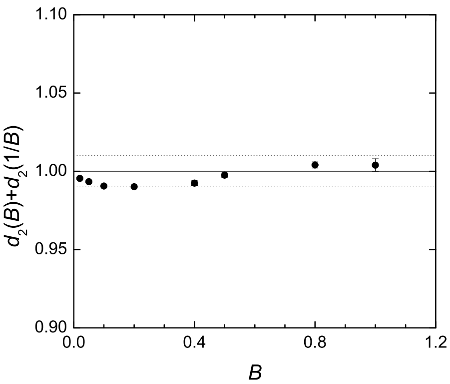

FIG. 1: Duality relation verified by numerical diagonalization of RMT Eq.(6). The deviation from Eq.(24) is less than 1 % for all values of $B = (\pi b)\, 2^{\frac{1}{4}}$.

However, we believe that this integral may have a geometrical meaning which is still evading our comprehension and thus it can be evaluated analytically. It remotely resembles the integrals arising in the problem of resistance of the regular 3-dimensional resistor lattice [22] which are known to have an intimate relation with the number theory.

Note that the leading terms in $d_2(b)$ at small and large b [2] can be represented in the form of the duality relation:

$$d_2(B) + d_2(B^{-1}) = 1, \;\; B \equiv (\pi b) 2^{\frac{1}{4}}. \tag{24}$$

It appears that this relation is well fulfilled *at all* values of B (see Fig.1). Our analytical result Eq.(22) together with the result [23] which gives no $1/b^2$ terms in $d_2(b)$ for large b, implies that the duality relation is not exact. However, its extremely accurate *approximate validity* is only possible because of the *anomalously small* value of the coefficient ≈ 0.083 in front of $(\pi b)^2$ in Eq.(22).

In conclusion, we have shown that the Chalker's dynamical scaling and its drastic consequence for strong correlations of the sparse multi-fractal wavefunctions is valid in the critical random matrix ensemble in the limit of strong multi-fractality $d_2 \ll 1$. We checked its validity in the form of Eq.(10) up to the second order in the small parameter b that controls the strength of the multi-fractality. Specifically (i) we observed the cancelation of the $\ln^2 N$ and $\ln^2 \tau$ terms in $\ln P_N(\tau)$ required by a pure power-law behavior; (ii) we demonstrated that the coefficients in front of the $\ln N$ and $\ln \tau$ terms are the same up to the order b^2, and (iii) we found analytically the b^2 term in the fractal dimension d_2. The validity of the Chalker's ansatz in the form Eq.(10) is encoded in the possibility of symmetric representation of the two different limits which can be traced back to the $(m-n)^{-2}$ dependence of the critical variance.

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[1] P.W. Anderson, Phys. Rev. **109**, 1492 (1958).
[2] F.Evers, A.D.Mirlin, Rew. Mod. Phys. **80**, 1355 (2008).
[3] E.Cuevas, V.E.Kravtsov, Phys.Rev.B **76**, 235119 (2007).
[4] A.Richardella, et al, Science **327**, 665 (2010).
[5] S.Faez, et al, Phys. Rev. Lett. **103**, 155703 (2009).
[6] F. Wegner, Z. Phys. B **36**, 209 (1980).
[7] M.V. Feigel'man, et al, Phys. Rev. Lett. **98**, 027001 (2007); M.V.Feigelman, et al, Annals of Physics **325**, 1368 (2010).
[8] S. Kettemann, E. R. Mucciolo, I. Varga, Phys. Rev. Lett. **103**, 126401, (2009).
[9] J.Billy et al., Nature **453**, 891 (2008); Roati et al., Nature **453**, 895 (2008).
[10] J.T. Chalker, G.J. Daniell, Phys. Rev. Lett. **61**, 593 (1988).
[11] J.T.Chalker, Physica A, **167**, 253 (1990).
[12] B. Huckenstein, L. Schweitzer, Phys. Rev. Lett. **72**, 713 (1994); T. Brandes, et al, Ann. Phys. (Leipzig) **5**, 633 (1996); K. Pracz, et al, J. Phys.: Condens. Matter **8**, 7147 (1996).
[13] F.Wegner, in *Localisation and Metal Insulator transitions*, ed. by H. Fritzsche and D. Adler (Plenum, N.Y. 1985), p.337.
[14] A.D. Mirlin, et al, Phys. Rev. E **54**, 3221 (1996).
[15] V.E.Kravtsov, K.A.Muttalib, Phys. Rev. Lett. **79**, 1913 (1997).
[16] V.E.Kravtsov, in: *Handbook on random matrix theory* (Oxford University Press, 2010), arXiv:0911.0615v1.
[17] A.D. Mirlin, F. Evers, Phys. Rev. B **62**, 7920 (2000).
[18] O. Yevtushenko, V.E. Kravtsov J.Phys.A: Math.Gen. **36**, 8265 (2003); Phys. Rev. E **69**, 026104 (2004). V.E. Kravtsov, O. Yevtushenko, E. Cuevas, J.Phys.A: Math.Gen. **39**, 2021(2006).
[19] O. Yevtushenko, A. Ossipov, J. Phys. A: Math. Theor. **40**, 4691 (2007). S. Kronmüller, O.M. Yevtushenko, E. Cuevas, J. Phys. A: Math. Theor. **43**, 075001 (2010).
[20] L.S. Levitov, Phys.Rev.Lett. **64**, 547 (1990).
[21] C. Itzykson, J.B. Zuber, "Quantum field theory" (New York, NY: McGraw-Hill, 1980).
[22] L.S.Levitov, A.B.Shitov, *Green's functions* (Fizmatlit, Moscow, 2003, in Russian), p.40.
[23] F.Wegner, Nucl.Phys. **B280**, 210 (1987); I. Rushkin, A. Ossipov, Y. V. Fyodorov, in preparation.