Landau Theory of CDT

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Based on [arXiv:1410.0845] (with J. Henson), [arXiv:1612.09533] (with J. Ryan)
and chapter of Handbook of Quantum Gravity (to appear)

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Why Landau theory?

Textbook lattice field theory:

- Identify phases by means of order parameters (observables)
- From within a good phase, take continuum limit by tuning to 2^{nd} order phase transition

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- different minima of L $\Leftrightarrow$ different phases

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Two approaches:

- bottom-up: write L as an expansion in the order parameters, constrained by the symmetries ($\sim$ EFT)
- top-down (difficult): derive L by explicitly coarse-graining the microscopic model, e.g.:

$$Z_{\mathrm{spin}} = \sum_{\{s\}} e^{-\beta H(\{s\})} = \sum_{\{m\}} \sum_{\{s_i | i \in \mathcal{L},\ \frac{1}{N(r)} \sum_{j \in \mathcal{B}_r} s_j = m(r)\}} e^{-\beta H(\{s\})} \equiv \sum_{\{m\}} e^{-L(\{m\})}$$

Observables in quantum gravity

Order parameters in a gauge theory must be gauge-invariant (observables)

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Quantum Gravity: Observables are a long-standing challenge

Diffeomorphism invariance $\Rightarrow$ non-local observables $\int d^dx\sqrt{g}\,\mathcal{O}(x)$

Perhaps not a problem for construction of phase diagram: order parameters are often nonlocal (e.g. average magnetization in the Ising model, Hausdorff dimension in dynamical triangulations)

However, they have limitations:

- Not good for distinguishing non-homogeneous (e.g. spatially modulated) phases
- They do not help in reconstructing a possible continuum local QFT

Observables in quantum gravity

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Wishful question:

Can we infer something about the continuum limit of our pet theory from the continuum limit of its effective description?

Effective dynamics of observables

Causal Dynamical Triangulations (CDT) have a built-in foliation
$\Rightarrow$ time-dependent observables $\mathcal{O}(t)$ are possible

e.g. spatial volume at time $t \Rightarrow$ volume profiles that characterize different phases

Questions:

1. Is it possible to describe dynamics of $\mathcal{O}(t)$ with an effective theory $(S_{\text{eff}}[\mathcal{O}] \sim L[\mathcal{O}])$?

$$\langle \mathcal{O}(t_1) \ldots \mathcal{O}(t_n) \rangle = \int \mathcal{D}g_{\mu\nu}\, e^{-S[g]} \mathcal{O}(t_1) \ldots \mathcal{O}(t_n) \overset{?}{\sim} \int \mathcal{D}\mathcal{O}\, e^{-S_{\text{eff}}[\mathcal{O}]} \mathcal{O}(t_1) \ldots \mathcal{O}(t_n)$$

2. If so, what can we learn from it?

This talk:

S_{eff} for the spatial volume as a Landau free energy of CDT

- Top-down approach in $1+1$ dimensions
- Bottom-up approach in $2+1$ (and $3+1$) dimensions

$\Rightarrow$ strengthen connection to Hořava-Lifshitz gravity

Causal Dynamical Triangulations in a nutshell

- A lattice approach to the nonperturbative quantization of gravity
  $\Rightarrow$ discretization of spacetime with *lattice cutoff* $\equiv a$

- Dynamical spacetime $\Rightarrow$ dynamical lattice:
  random d-dimensional triangulations
  (in Euclidean signature, e^{-S} weigth $\Rightarrow$ Monte Carlo simulations)

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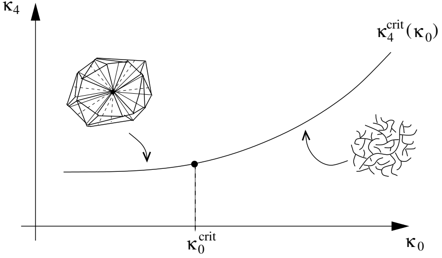
- Experience from the past (DT): no classical geometry and no $2^{\rm nd}$ order phase transition for most general class of geometries



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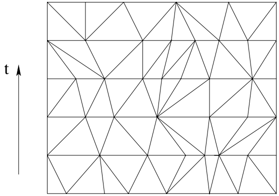
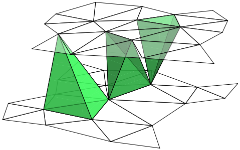
- Restricting the ensemble of geometries to those with a regular foliation both features are obtained $\Rightarrow$ Causal Dynamical Triangulations
  [Ambjørn, Loll - 1998 ($d = 2$); Ambjørn, Jurkiewicz, Loll - 2000,... ($d > 2$)]





The model

The statistical model of CDT is defined by the partition function

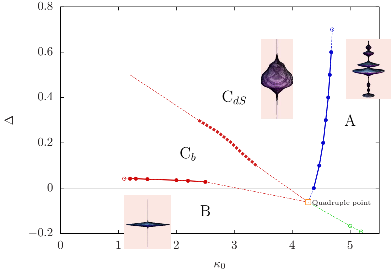
$$Z(\kappa_d, \kappa_{d-2}) = \sum_N e^{-\kappa_d N} \sum_{\mathcal{T}|N_d=N} \frac{1}{C(\mathcal{T})} e^{\kappa_{d-2} N_{d-2}} \equiv \sum_N e^{-\kappa_d N} \tilde{Z}(N, \kappa_{d-2})$$

- $d =$ (space + time) dimensions (but Euclidean)
- $N_n =$ number of n-dimensional simplices ($n = 0, \dots, d$) in simplicial manifold $\mathcal{T}$
- $N_n =$ are constrained by topological relations
  $\Rightarrow$ only 1 independent variable in $1+1$ dimensions,
  and only 2 independent variables in $2+1$ and $3+1$ dimensions
- In CDT we distinguish time-like objects (connecting leaves) and space-like objects (on a single leaf)

  $\Rightarrow$ one more free variable in $3+1$ dimensions (with coupling Δ)
- Monte Carlo simulations:
  (1) $\kappa_0 N_0$ instead of $\kappa_{d-2} N_{d-2}$;
  (2) constant volume (canonical ensemble);
  (3) increase N_d and look for scaling

Emergence of a macroscopic universe

Phase diagram of CDT in $3+1$ dimensions: [Ambjørn, Jurkiewicz, Loll, Görlich, Jordan]



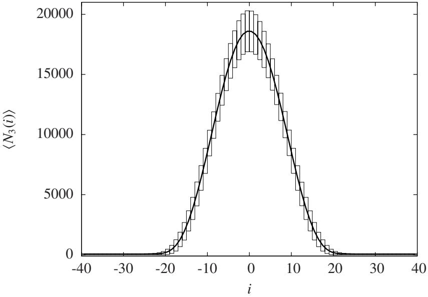
Phase C_{dS}:

- $d_H \sim 4$
- from $S^1 \times S^3$ to an effective S^4
- spontaneous breaking of time translation

$\Leftrightarrow$ "condensation of spacetime"

Volume profile in 3+1 dimensions

Characteristic features of the condensate:
macroscopic blob/droplet surrounded by microscopic stalk



In the bulk of the macroscopic universe:

$$\langle N_3(i) \rangle = \frac{3N_4^{3/4}}{4s_0} \cos^3 \left(\frac{i}{s_0 N_4^{1/4}} \right) \Rightarrow \text{emergence of a classical evolution}$$

(volume profile of a 4-sphere) [Ambjørn, Görlich, Jurkiewicz, Loll - '07]

GR minisuperspace

- The $\cos^3(t)$ profile is obtained also as a solutionn of a GR-inspired minisuperspace model ($g_{ij} = \phi(t)^2\,\hat{g}_{ij} \Rightarrow V_3(t) = \int d^3x\sqrt{g} \propto \phi(t)^3$)

$$S = \frac{1}{2G}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt\,\left(c_1\frac{\dot{V}_3^2(t)}{V_3(t)} + c_2 V_3^{1/3}(t)\right)$$

+ constraint: $V_4 = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt V_3(t)$

$$\Rightarrow \int \mathcal{D}V_3\,\delta\left(V_4 - \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt V_3(t)\right)\,e^{-S}$$

- $c_1 > 0$: unlike in GR!
Nonperturbative cure of the unboundedness, or deviation from diffeomorphism invariance?

- Discretization:

$$S = \kappa\sum_i\,\left(c_1\frac{(N_3(i+1) - N_3(i))^2}{N_3(i)} + c_2 N_3^{1/3}(i)\right)$$

– Reconstructed directly from the CDT data (inside the droplet) by studying correlators $\langle N_3(i)N_3(j)\rangle$ [Ambjorn et al. '08-'12-'13]

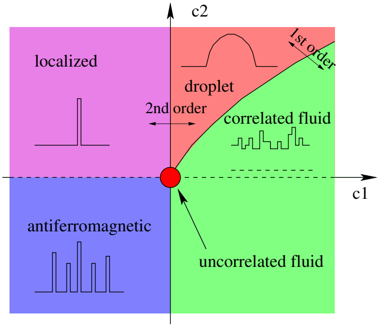
Condensation from Balls-in-Boxes model

GR-inspired minisuperspace model explains not only the bulk evolution, but also the stalk, as well as occurrence of other phases [Bogacz, Burda, Waclaw - '12]

$\Rightarrow$ Balls-in-Boxes model = discrete path integral with a constraint

$$Z_{\rm BIB}(T,M) = \sum_{m_1=m_{\min}}^{M} \dots \sum_{m_T=m_{\min}}^{M} \delta_{M,\sum_i m_i} \prod_{j=1}^{T} g(m_j, m_{j+1})$$
$$= \sum_{\{m_j\}} e^{-S[\{m_j\}]} \delta_{M,\sum_i m_i} \, ,$$

$$g(m,n) = \exp\left\{ -c_1 \frac{2(m-n)^2}{m+n} - c_2 \frac{m^{1/3}+n^{1/3}}{2} \right\} \quad \Rightarrow$$



CDT in 1+1 dimensions

Example of top-down approach to a Landau free energy in a quantum gravity model

$$\begin{aligned}\tilde{Z}_{(1+1)\mathrm{d-CDT}}(T, N_2) &= \sum_{\mathcal{T}(T,N_2)} 1 \\ &= \sum_{l_1=1}^{N_2} \cdots \sum_{m_T=1}^{N_2} \delta_{N_2, 2\sum_i m_i} \prod_{i=1}^{T} g(l_i, l_{i+1}) \\ &\equiv \sum_{\{l\}} e^{-L(\{l\};T,N_2)}\end{aligned}$$

is a *balls-in-boxes* (BIB) model, with l_i giving the length of the spatial slice, and (for open boundary conditions in space) reduced transfer matrix

$$g(l_i, l_{i+1}) = \frac{(l_i + l_{i+1})!}{l_i!\, l_{i+1}!}$$

Summing over N_2 with a Boltzmann weight $e^{-\kappa_2 N_2}$, we obtain:

$$L_{\text{grand can.}}(\{l\}; T, \kappa_2) = 2\kappa_2 \sum_i l_i - \sum_2 \ln\left(g(l_i, l_{i+1})\right)$$

The model is exactly solvable [Ambjørn, Loll - '98] and it has no droplet phase

CDT in 1+1 dimensions – continuum limit

By Stirling's formula:

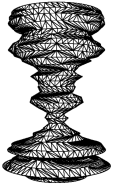
$$g(l_i, l_{i+1}) = \frac{(l_i + l_{i+1})!}{l_i!\, l_{i+1}!} \sim 2^{l_i + l_{i+1}} e^{-\frac{(l_{i+1} - l_i)^2}{l_i + l_{i+1}}}$$

Effective continuum action:

$$S_{\text{eff}}[\ell] = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt\, \frac{\dot{\ell}^2(t)}{4\ell(t)}$$

$\Rightarrow$ Minimized by constant configuration

- No condensation
- Large fluctuations



- The effective action is not a reduction of Einstein-Hilbert (topological in $d = 2$), but of Horava-Lifshitz gravity in $1 + 1$ dimensions [Ambjorn, Glaser, Sato, Watabiki - '13]

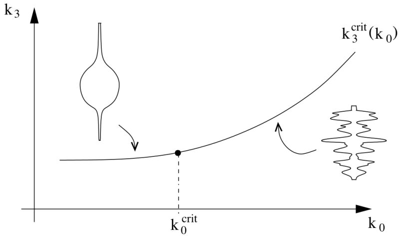
Hořava-Lifshitz gravity (and CDT)

- HL gravity: a dynamical theory of geometries with a preferred foliation
  $\Rightarrow$ Reduced symmetry: foliation-preserving diffeomorphisms $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$

- Evidence for a CDT-HL relation comes from
  - Presence of a foliation
  - Analogies in phase diagram (CDT in $(3+1)$d) [Ambjorn, Goerlich, Jurkiewicz, Loll - '10]
  - Short-scale spectral dimension in $(2+1)$d [DB, Henson - '09]
  - Large-scale geometry (stretched sphere) in $(2+1)$d [DB, Henson - '09]
  - Minisuperspace action with positive kinetic term: no wrong sign of conformal mode! (in $(2+1)$d, compared to kinetic term of moduli [Budd - '11])
  - Quantum Hamiltonian in $(1+1)$d [Ambjorn, Glaser, Sato, Watabiki - '13]

CDT in 2+1 dimensions

Much easier than $d = 3 + 1$ (also one less coupling in the lattice action),
but richer than $d = 1 + 1$



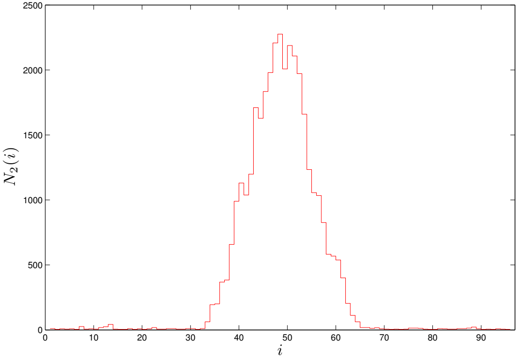
[Ambjørn, Jurkiewicz, Loll - '00]

$\Rightarrow$ again an extended phase ($d_H \sim 3$) with a condensation phenomenon

Note: phase transition is $1^{\rm st}$ order, but there are no propagating degrees of freedom in 3d general relativity with spherical slices, so perhaps not a problem

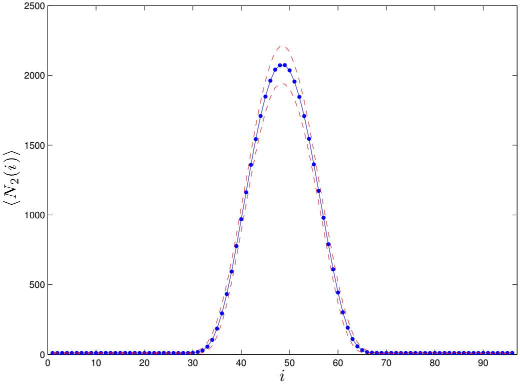
Volume profile in 2+1 dimensions [DB, Henson -'14]

Snapshot of a typical configuration:



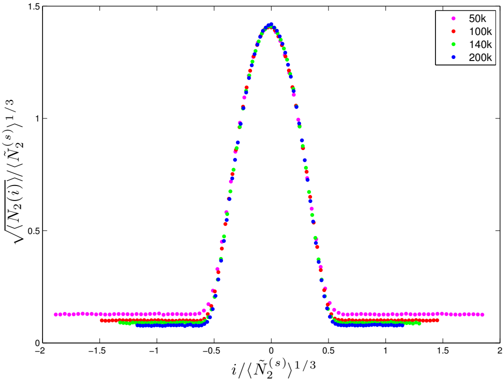
Volume profile in 2+1 dimensions [DB, Henson -'14]

Average:



Volume profile in 2+1 dimensions [DB, Henson -'14]

Scaling:



Failure of the GR-inspired Landau free energy

- No potential for $V_2(t)$ in GR-inspired minisuperspace model
  $(g_{ij} = \phi(t)^2\, \hat{g}_{ij} \Rightarrow V_2(t) = \int d^2x \sqrt{g} = 4\pi\phi^2(t))$

$$S_{(2+1)\mathrm{d-mini}} = \frac{1}{2G} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt\, \frac{\dot{V}_2^2(t)}{V_2(t)}$$

  + constraint: $V_3 = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt V_2(t)$

- Droplet solution:

$$\bar{V}_2(t) = \begin{cases} A\cos^2\left(\frac{2\pi^2 At}{V_3}\right), & \text{for } t \in [-\frac{V_3}{4\pi A}, +\frac{V_3}{4\pi A}], \\ 0, & \text{for } t \in [-\frac{\tau}{2}, -\frac{V_3}{4\pi A}) \cup (+\frac{V_3}{4\pi A}, +\frac{\tau}{2}] \end{cases}$$

- On-shell action: $S_{(2+1)\mathrm{d-mini}}[\bar{V}_2] = \frac{A^2\pi^3}{2GV_3}$
  minimized by $A = 0$, but this violates $\frac{V_3}{4\pi A} \leq \frac{\tau}{2} \Rightarrow \bar{A} = \frac{V_3}{2\pi\tau}$

$$\Rightarrow \quad S_{(2+1)\mathrm{d-mini}}[\bar{V}_2; \bar{A}] = \frac{\pi V_3}{8G\tau^2} > 0$$

- However:

$$S_{(2+1)\mathrm{d-mini}}[V_2(t) = V_3/\tau] = 0$$

$\Rightarrow$ Constant configuration is the absolute minimum!

An HL-inspired Landau free energy [DB, Henson -'14]

- HL gravity (with constant lapse N):

$$S_{(2+1)-\mathrm{HL}} = \frac{1}{16\pi G} \int dt\, d^2x N\sqrt{g}\, \{\lambda\, K^2 - K_{ij}K^{ij} + b\, R - \gamma\, R^2\}$$

+ volume constraint: $V_3 = \int dt\, d^2x N\sqrt{g}$

- Minisuperspace reduction: $g_{ij} = \phi^2(t)\, \hat{g}_{ij}$ $\quad (V_2(t) = \int d^2x\sqrt{g} = 4\pi\phi^2(t))$

$$\Rightarrow \quad S_{(2+1)-\mathrm{mini}} = \frac{1}{2\kappa^2}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt\, \left\{\dot{\phi}^2 - \frac{\xi}{\phi^2} + b'\right\}$$

+ volume constraint: $\mathcal{V} \equiv V_3 - 4\pi N \int dt\, \phi^2(t) = 0$

+ kinematic constraint: $\phi(t) \geq \epsilon$

Competing effects

$$Z_{(2+1)-\mathrm{mini}} = \int_{\phi(t)>\epsilon} \mathcal{D}\phi(t)\, \delta(\mathcal{V})\, \exp\left\{ -\frac{1}{2\kappa^2} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \left[\dot{\phi}^2 - \frac{\xi}{\phi^2} \right] \right\}$$

In the limit $\kappa \to 0$ we expect the partition function (and the observables) to be dominated by those configurations that minimize the action

- Kinetic term favors constant solutions $\Rightarrow$ for $\xi = 0$, taking into account volume constraint we have

$$\bar{\phi}_0(t) = \sqrt{\frac{V_3}{4\pi\tau}}$$

  as we saw before

- For $\xi > 0$, potential favors configurations saturating the kinematic constraint (i.e. $\phi(t) = \epsilon$)

$\Rightarrow$ dominance of configurations with a stalk saturating the kinematic constraint, and a droplet taking care of the missing volume

(Note: due to unboundedness of the action for $\xi > 0$, dominant configuration is not necessarily a saddle point)

Simulations of the BIB model [DB, Ryan -'16]

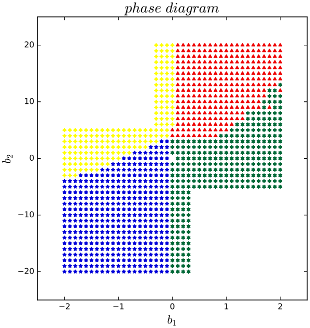
Minimization analysis is far from rigorous, and it relies on several assumptions
⇒ comparison to direct Monte Carlo simulations of the BIB model is important

$$
\begin{aligned}
Z_{\rm BIB}(T,M) &= \sum_{m_1=m_{\rm min}}^{M} \dots \sum_{m_T=m_{\rm min}}^{M} \delta_{M,\sum_i m_i} \prod_{j=1}^{T} g(m_j, m_{j+1}) \\
&= \sum_{\{m_j\}} e^{-S[\{m_j\}]} \delta_{M,\sum_i m_i} ,
\end{aligned}
$$

with

$$
g(m_j, m_{j+1}) = \exp\left\{ -\frac{2(m_{j+1}-m_j)^2}{m_{j+1}+m_j} b_1 + \frac{2}{m_{j+1}+m_j} b_2 \right\}
$$

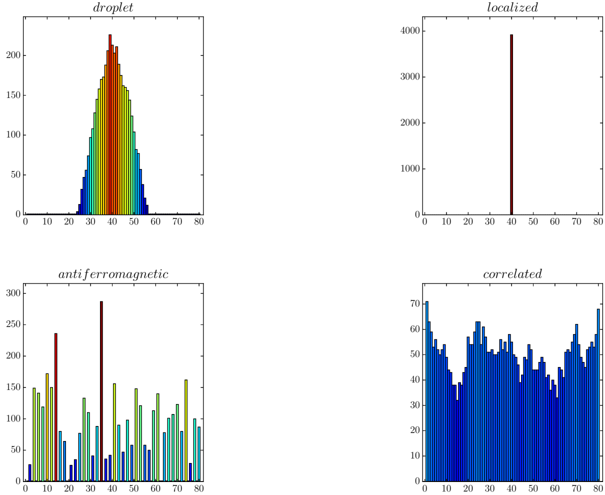
Phases [DB, Ryan -'16]



Phase diagram for system with $T = 80$, $M = 4000$: droplet (red triangles), localized (yellow squares), antiferromagnetic (blue pentagons), correlated fluid (green hexagons).

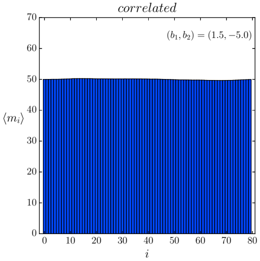
Phases [DB, Ryan -'16]

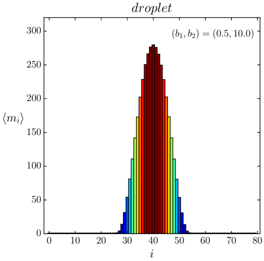
Typical configurations for the various phases:



Phases [DB, Ryan -’16]

Mean value $\langle m_i \rangle$ as a function of i, for samples in the correlated phase and in the droplet phase:





Conclusions

- CDT is a nonperturbative lattice approach to quantum geometry, and a rather unique case in which a minisuperspace model can be derived as effective (Landau-type) description, not as approximation
- In $(1+1)d$ continuum limit of CDT is HL gravity
- In $(2+1)d$ the GR-inspired minisuperspace model has no potential term for the spatial volume
  $\Rightarrow$ the droplet phase is never favorable
- HL-inspired model succeeds very well in reproducing the spacetime condensation of (2+1)d CDT!
- It would be interesting to study volume fluctuations in CDT and directly extract the effective action from there
- In naive continuum limit (no tuning), the coupling of R^2 goes to zero, but a nontrivial limit might be reached if a Lifshitz point exists
- Relation to Lifshitz-type theories does not prevent the possibility of recovering full diffeo-invariance by taking continuum limit in a submanifold of the theory space (compare to recovering diffeo-invariance in functional RG)

The End

Backup slides

Minimization of the action – 1

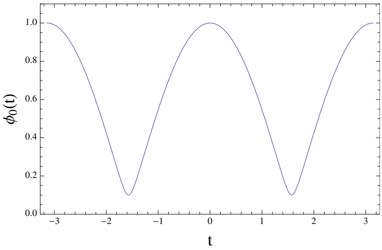
Local minima

e.o.m.: $$\ddot{\phi} + \omega^2 \phi - \frac{\xi}{\phi^3} = 0$$

(ω is a Lagrange multiplier to enforce volume constraint)

It is exactly solvable (isotonic oscillator):

$$\phi_0(t) = \frac{1}{\omega A} \sqrt{(\omega^2 A^4 - \xi) \cos^2(\omega t + \psi) + \xi}$$



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For $\frac{\pi}{\omega} = \frac{\tau}{n}$, $n \in \mathbb{N}$ (and solving the volume constraint):

$$\Rightarrow \quad S_{(2+1)-\mathrm{mini}}[\phi_0(t)] = \frac{n\,\pi}{8\kappa^2}\left(\frac{n\,V_3}{N\tau^2} - 8\sqrt{\xi}\right)$$

However, for $\phi(t) = \xi^{1/4}/\sqrt{\omega} \equiv \bar{\phi}_0$ and $\omega = 4\pi N\tau\sqrt{\xi}/V_3$:

$$S_{(2+1)-\mathrm{mini}}[\bar{\phi}_0] = -\frac{2\pi N\tau^2\xi}{\kappa^2 V_3} \leq S_{(2+1)-\mathrm{mini}}[\phi_0(t)]$$

Minimization of the action – 2

Absolute minima

$$\bar{\phi}(t) = \begin{cases} \sqrt{\left(\frac{\tilde{V}_3 \bar{\omega}}{2\pi^2 N} - \epsilon^2\right) \cos^2(\bar{\omega} t) + \epsilon^2}\,, & \text{for } t \in [-\frac{\pi}{2\bar{\omega}}, +\frac{\pi}{2\bar{\omega}}]\,, \\ \epsilon\,, & \text{for } t \in [-\frac{\tau}{2}, -\frac{\pi}{2\bar{\omega}}) \cup (+\frac{\pi}{2\bar{\omega}}, +\frac{\tau}{2}] \end{cases}$$

$$\bar{\omega} \equiv \omega(A_\epsilon) = \left(\frac{2\pi^2 N \sigma^2}{\tilde{V}_3}\right)^{\frac{1}{3}}\,, \quad \sigma^2 = \frac{\xi}{\epsilon^2}$$

$$\tilde{V}_3 = V_3 - 4\pi N \epsilon^2 \tau + \left(2\pi^2\right)^{\frac{2}{3}} \left(\frac{V_3}{\sigma^2}\right)^{\frac{1}{3}} \epsilon^2 + O(\epsilon^4)$$

The action evaluates to

$$S_{(2+1)-\mathrm{mini}}[\bar{\phi}(t)] = \frac{1}{\kappa^2}\left(-\frac{\xi\tau}{2\epsilon^2} + \frac{3}{4}\left(\frac{\pi V_3 \xi^2}{2N\epsilon^4}\right)^{\frac{1}{3}} - \pi\sqrt{\xi} + \frac{1}{2}\left(\frac{\pi^5 N \epsilon^4 \xi}{4V_3}\right)^{\frac{1}{3}}\right)$$

which is smaller than for other configurations, for $\epsilon^2 \ll V_3/\tau$.

Minimization of the action – 3

The droplet/condensate is stable in a finite interval: $\tau_- < \tau < \tau_+$

- $\frac{\pi}{\bar{\omega}} < \tau \Rightarrow$ there is a minimal value of τ below which the droplet is unstable:

$$\tau_- \simeq \left(\frac{\pi V_3 \epsilon^2}{2N\xi} \right)^{\frac{1}{3}}$$

$\Rightarrow$ constant configurationn dominates for $\tau < \tau_-$
(consistent with [Ambjørn, Jurkiewicz, Loll - '00; Cooperman, Miller - '13])

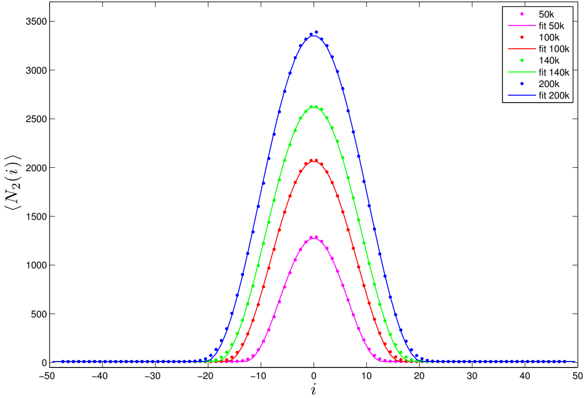
- $S_{(2+1)-\mathrm{mini}}[\bar{\phi}_0] \sim -\tau^2$ vs. $S_{(2+1)-\mathrm{mini}}[\bar{\phi}(t)] \sim -\tau$

$\Rightarrow$ there is a maximal value of τ above which the constant solution is favourable

and $\tau_+ < \tau_{\max} \equiv V_3/(4\pi N\epsilon^2)$

(for $\tau > \tau_{\max}$ the constraint $\phi(t) > \epsilon$ is incompatible with the volume constraint)

Fitting the data



Further hints for the unbounded potential?

In [Benedetti, Loll, Zamponi - '07] we obtained the following continuum Hamiltonian from a very special model of $(2+1)d$ CDT:

$$\hat{H} = -\frac{\partial}{\partial V_2} V_2^{3/2} \frac{\partial}{\partial V_2} - \frac{1}{16} \frac{1}{V_2^{1/2}} + \Lambda V_2$$

to be compared with the Hamiltonian of our HL minisuperspace model:

$$\hat{H} = -G \left(\frac{\partial}{\partial V_2} V_2 \frac{\partial}{\partial V_2} + \gamma \frac{1}{V_2} \right) + \Lambda V_2$$

Notice: roughly the same for $G \to V_2^{1/2}$

Maybe possible to obtain missing G from the more realistic model? (from ABAB matrix model)
Or maybe just a problem with scaling G canonically? ($\Rightarrow$ Lifshitz scaling?)

Of course just a speculation, but presence of a term singular at $V_2 = 0$ and seemingly unbounded from below is very suggestive!