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Self-organization and pattern formation
Sheet 8

Exercise 21 – Kuramoto models – global synchronization of heterogenous oscillators

Coupled oscillators show interesting and surprising behavior, such as the unexpected wobbling of the Millenium Bridge. A very simple model for the dynamics of people walking on the Millenium Bridge is an ensemble of coupled harmonic oscillators. Each oscillator has it's own natural oscillation frequency ω_i which is drawn from a random distribution (e.g. a Gaussian distribution). Coupling between oscillators is global. Each oscillator is coupled to all other oscillators via the sine of their phase difference $\theta_i(t)-\theta_j(t)$. The dynamics for the phase of the i-th oscillator reads

$$\partial_t \theta_i(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin\left(\theta_j(t) - \theta_i(t)\right),$$

where K/N is the coupling strength normalized to the total number of oscillators.

a) Simulate the dynamics of $N = 100$ such oscillators with natural frequencies ω_i drawn from a Gaussian distribution with mean 0.5 Hz and standard deviation 0.5 Hz and a coupling strength $K = 0.8$. Visualize the dynamics of the oscillators as points rotating on a circle ($x_i = \cos\theta_i, y_i = \sin\theta_i$). What do you observe?
Hint: A template for the setup of the simulation can be found in the Mathematica notebook that was uploaded together with this exercise.

b) Vary the coupling strength K, how does the dynamics change when you go to lower and higher coupling strengths? Based on the observed dynamics, estimate the value for the critical coupling strength K_c seen in the lecture?

c) Implement the order parameter

$$Z(t) = r(t)e^{i\psi(t)} = N^{-1}\sum_{i=1}^{N} e^{i\theta_i(t)} \tag{1}$$

from the lecture, where r the degree of synchronization of the oscillators. Plot the time evolution of $r(t)$ for different values of K. Average $r(t)$ over time (excluding the initial transient, so only for fluctuations around a steady state). Plot the average synchronization $\langle r \rangle_t$ versus the coupling strength K. What kind of relation do you expect for $N \to \infty$?

d) We now want to analyze a simple example of two coupled Kuramoto oscillators. Use this to show one can only have complete phase synchronaization (meaning $r = 1$) between Kuramoto oscillators for identical natural frequencies

e) In the following we will assume identical Kuramoto oscillators, with one natural frequency ω. We additionally want to introduce the phase lag α, as

$$\partial_t \theta_i = \omega + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i + \alpha). \tag{2}$$

This effectively accounts for a constant time delay of the interaction.
Analyze the conditions needed for complete phase synchronization. You can assume that the natural frequency is given by $\omega = 0$.

Exercise 22 – Kuramoto models – spatially coupled identical oscillators

In the previous exercise, the oscillators were globally coupled. In most physical systems, interactions have a range though such that the (physical) distance of the oscillators plays a role. Such systems can exhibit very counterintuitive behavior as we will learn in the following.
Consider a system of identical oscillators arranged on a ring with unit circumference identical distances d/N, oscillating at a natural frequency ω.
Each oscillator is coupled to its neighbors with a distance dependent coupling kernel $G(x, x') = G(|x - x'|)$, so the dynamics read

$$\partial_t \theta_i = \omega - N^{-1} \sum_{j=\lceil i-N/2 \rceil}^{\lfloor i+N/2 \rfloor} G\big(d/N(i-j)\big) \sin(\theta_i - \theta_j + \alpha)\,,$$

where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denotes the ceiling and the floor of the respective number and further we use the ring arrangement, meaning $\theta_i = \theta_{i+N}$. Note that this sum actually counts the case $j = i - N/2$ twice, for an even number of oscillators, but since this oscillator is furthest away and therefore contributes the least to the sum, we assume that this is justified, especially for large N.
We will use the coupling kernel

$$G(x) = \mathcal{N} \exp(-\kappa |x|).$$

a) Sketch the kernel and find the constant $\mathcal{N}$ such that the total coupling is normalized to unity, i.e.

$$\int_{-1/2}^{1/2} \mathrm{d}x\, G(x) = 1.$$

b) Use the provided Mathematica notebook (you have to enter the kernel function yourself!) to simulate and visualize the dynamics of the oscillators. Start with the parameters $\kappa = 4$ and $\alpha = 1.45$. What do you observe? Repeat the simulation for different values of κ and α and (shortly) describe the three types of behavior the system exhibits. Compare your observation to the case of globally coupled heterogenous oscillators.

Exercise 23 – Slow Manifold of Predator-Prey Dynamics

The Rosenzweig–MacArthur model is a classical predator–prey model where we assume a maximal capacity K for the prey x and a bounded kill rate by the predators y. We now want to consider a fast–slow version of this model given by

$$x' = x\left(1 - \frac{x}{K}\right) - a\frac{x}{x+d}y, \tag{3}$$

$$y' = \varepsilon\left(a\frac{x}{x+d}y - by\right), \tag{4}$$

where $a, b, d, K > 0$. For simplicity we set $K = 1$ and $b = 1$. We assume $0 < \varepsilon \ll 1$.

a) Explain, why such a separation of timescales could be important, if we want to model predator–prey dynamics? And how can we interpret ε

b) Using the small parameter ε find the critical manifold C_0 as well as the fast and slow subsystems.

c) Analyze the fast and slow subsystem separately: find all equilibria, their stability and sketch the corresponding phase portraits.

d) Combine the information from the two subsystems to a global phase portrait. Would you expect the system to have a periodic orbit?

Your solutions should be handed in in moodle by **Wednesday, December 17 th 2025, 10 am.**