Arnold Sommerfeld Center for Theoretical Physics
Ludwig-Maximilians-Universität München
Winter Term 2025

Prof. Erwin Frey
A. Winter, D. Toffenetti,
N. Dominko Kobilica,
K. Wharam

Self-organization and pattern formation
Sheet 7

Exercise 19 – Stability of wave solutions of the complex Ginzburg-Landau equation

As introduced in the lecture, the complex Ginzburg-Landau equation can be derived as an amplitude equation, reading in two spatial dimensions

$$\partial_t A = \left[1 + (1 + ic_1)\nabla^2 - (1 - ic_3)|A|^2\right] A\,. \tag{1}$$

Considering the stability of perturbed planar traveling wave solutions (*phase winding solutions*)

$$A(\boldsymbol{x}, t) = \left\{R_q + \delta A_+ e^{i\boldsymbol{k}\cdot\boldsymbol{x}+\sigma(\boldsymbol{k})t} + \delta A_-^* e^{-i\boldsymbol{k}\cdot\boldsymbol{x}+\sigma^*(\boldsymbol{k})t}\right\} e^{i(\boldsymbol{q}\cdot\boldsymbol{x}-\Omega_q T)}\,, \tag{2}$$

one can derive the characteristic equation for the linear growth rates of the perturbations as

$$\sigma^2 + 2\sigma\left[R_q^2 + k^2 + 2ic_1\boldsymbol{q}\cdot\boldsymbol{k}\right] + (1 + c_1^2)\left[k^4 - 4\left(\boldsymbol{q}\cdot\boldsymbol{k}\right)^2\right] + 2R_q^2\left[(1 - c_1c_3)k^2 + 2i\left(c_1 + c_3\right)\boldsymbol{q}\cdot\boldsymbol{k}\right] = 0\,. \tag{3}$$

a) Accounting for longitudinal perturbations only, namely $\boldsymbol{k} \parallel \boldsymbol{q}$, show that in the long wavelength limit, the dispersion relation can be expressed as

$$\sigma = -iv_g k - D_2 k^2 + i\Omega_g k^3 - D_4 k^4 + \mathcal{O}(k^5)\,, \tag{4}$$

and give expressions for the group velocity v_g, phase diffusion D_2, and the remaining coefficients Ω_g and D_4. You may want to use Mathematica to obtain the expressions.

b) Recapitulate the conditions on Eckhaus instability as well as the Benjamin-Feir (BF) limit on the basis of the long wavelength dispersion relation Eq. (4) and discuss the physical significance of the two instabilities.

c) In the following we want to investigate convective as well as absolute instabilities in the system. As localized perturbations arising by instabilities may move away over time due to convection, the system can retain its stability even though the criteria from part b) would already indicate an instability. Derive the criteria for the absolute and convective instabilities,

$$\text{absolute instability:} \quad \Re\left[\sigma(k_0)\right] > 0\,, \quad \text{with} \quad \partial_k\sigma(k)|_{k=k_0} = 0, \tag{5}$$
$$\text{convective instability:} \quad \exists k_0 : \Re\left[\sigma(k_0)\right] > 0 \quad \text{and} \quad \Im\left[\sigma(k_0)\right] \neq 0\,, \tag{6}$$

along the lines of the lecture notes. Start from the relation

$$S(x,t) \sim \int_{-\infty}^{\infty} S_0(k) \exp\left[ikx + \sigma(k)t\right] \mathrm{d}k \tag{7}$$

giving the temporal evolution of a one-dimensional localized perturbation with initial Fourier spectrum $S_0(k)$. Explain the difference between convective and absolute instabilities. What effect could lead to a destabilization even in regions in which the system is only convectively unstable?

d) By numerically evaluating the condition on absolute instability from part c) provided with the dispersion relation obtained from Eq. (3), plot the instability regimes resulting from the criteria on Eckhaus-, BF-, and absolute instability in the $c_1 - c_3$-parameter plane. The value of the wavenumber depends on the solution under consideration. For the purpose of illustration, here you can simply select a value, e.g. $q = 1/3$. Interpret the different regions in parameter space and comment on the physical implications.

e) Plot the regions of absolutely unstable, convectively unstable and stable states in the $q - c_3$-plane for values of $c_1 \in \{-3.0, -1.5, -1\}$. Interpret your findings. Against which condition does the range of convective instability converge at $q = 0$?

f) In the provided Mathematica notebook perform one-dimensional numerical simulations of the complex Ginzburg-Landau equation, Eq. (1). Set the parameter $c_1 = -3.0$ and explore the three different regions of behavior (stable, convectively unstable, and absolutely unstable). Therefore initiate the system with a phase winding solution with the respective wavevector q. Describe and illustrate your results and relate them to your finding from exercise part e).

g) Using the provided Mathematica notebook, perform two-dimensional numerical simulations of the complex Ginzburg-Landau equation, Eq. (1). Explore the rich behavior of the equation and discuss your findings.

Exercise 20 – Phase Equations of the CGLE

The complex Ginzburg-Landau equation (CGLE) in polar form reads

$$\partial_t R = \nabla^2 R + \left(1 - (\nabla\phi)^2 - R^2\right) R - c_1 \left(R\nabla^2\phi + 2\nabla R \cdot \nabla\phi\right), \tag{8}$$
$$R\partial_t\phi = R\nabla^2\phi + 2\nabla R \cdot \nabla\phi + c_3 R^3 + c_1 \left(\nabla^2 R - R(\nabla\phi)^2\right). \tag{9}$$

Where $R = R(\boldsymbol{x},t), \phi = \phi(\boldsymbol{x},t)$ are the (real) modulus and phase and the (complex) solution is written as $A(\boldsymbol{x},t) = Re^{i\phi}$.

a) First, convince yourself that the phase-winding ansatz

$$A_{\boldsymbol{q}} = R_q \exp(i\phi_{\boldsymbol{q}}(\boldsymbol{x}, t)) = R_q \exp(i\boldsymbol{q} \cdot \boldsymbol{x} - i\Omega_q t), \tag{10}$$

is a valid solution to the CGLE and derive the expressions given in the lecture

$$R_q^2 = 1 - q^2, \qquad \Omega_q = -c_3 + (c_1 + c_3)q^2. \tag{11}$$

b) Now we perturb such a solution in both the modulus and the phase, i.e. we insert

$$R(\boldsymbol{x}, t) = R_q + \rho(\boldsymbol{x}, t), \qquad \phi(\boldsymbol{x}, t) = \phi_{\boldsymbol{q}}(\boldsymbol{x}, t) + \varphi(\boldsymbol{x}, t), \tag{12}$$

into CGLE (8), (9). Derive the linearized equations for ρ and φ.

c) From the linearized equations you just derived (also given in the lecture), perform a coordinate shift into the frame, moving with $\boldsymbol{v} = 2(c_1 + c_3)\boldsymbol{q}$. This frame is moving with the group velocity you derived in the lectures. Furthermore, we will be interested in long-range diffusive time-scale phenomena. To this end, also rescale space of the linearized equations by ϵ and time by ϵ^2.
In your answer, order the terms in powers of ∇. And argue why we may use the quasi steady state approximation
Hint: In the moving and rescaled frame the time derivative is transformed as $\partial_t \to \epsilon^2 \partial_T - \epsilon \boldsymbol{v} \cdot \nabla$

d) Insert the perturbative ansatz $\rho = \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \cdots$ (and similarly for φ) and write down the equations to second order. Derive how the modulus perturbation always follows the phase perturbation and write down the (closed form) differential equation for φ_0. You should get the equation also given in the lecture notes

$$\partial_T \varphi_0 = (1 - c_1 c_3)\nabla^2 \varphi_0 - 2(1 + c_3^2)R_q^{-2}(\boldsymbol{q} \cdot \nabla)^2 \varphi_0. \tag{13}$$

e) Perform linear stability analysis of the above equation. Discuss the connection to what you have done in the lecture.

Your solutions should be handed in in moodle by **Wednesday, December 10 th 2025, 10 am.**