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Self-organization and pattern formation
Sheet 1

Exercise 4 – Lyapunov functions

This exercise is a continuation of Exercise 2. This time, we will analyse two variations of the one-dimensional harmonic oscillator and study the changes in the phase space. We will then introduce the concept of Lyapunov function, useful to study the global stability of a fixed point. The one-dimensional harmonic oscillator (aka the Hookian spring) is characterized by the following equations of motion:

$$m\,\partial_t^2 x = -kx,$$
$$\partial_t \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\frac{k}{m}x \end{pmatrix}.$$

a) Hookian springs are nice to calculate but a real spring will not have a perfectly constant spring constant k. Assume that the spring has a small nonlinear contribution $a > 0$ such that the force exerted follows $F_{\text{spring}} = -(kx + ax^3)$. How does this change the trajectories from the linear case? What stays qualitatively the same?

b) Back to our Hookian spring. In this exercise part, consider that the mass will move in some fluid and experience Stokes drag. Write down the modified equation of motion and plot the trajectory of the damped oscillator in phase space with initial condition $v(0) = 0$ and $x(0) = 1$. Does the behaviour look qualitatively different for different initial conditions?

You can prove that a fixed point is globally stable if you find a *strict Lyapunov function*: Given a dynamical system $\partial_t \vec{u} = \vec{f}(\vec{u})$ with equilibrium solution $\vec{u} = 0$, the scalar and differentiable function $V(\vec{u})$ is called a Lyapunov function, if the following holds

1. $V(\vec{u}) > 0$ for all $\vec{u} \neq 0$ and $V(0) = 0$,
2. $\text{grad}_u\,(V(\vec{u})) \cdot f(\vec{u}) \leq 0$ for all points $\vec{u}$ in phase space; it is called a *strict* Lyapunov function if $\text{grad}_u\,(V(\vec{u})) \cdot f(\vec{u}) < 0$ or all $\vec{u} \neq 0$.

c) Explain why the existence of a strict Lyapunov function ensures that $\vec{u} = 0$ is a globally stable fixed point, i.e. starting from any initial condition, the system will end up at $\vec{u} = 0$. Find a (strict) Lyapunov function for the damped harmonic oscillator from the previous exercise part. What is its physical interpretation?

Identifying a Lyapunov function is a powerful method in nonlinear dynamics, since it characterizes the long-term behavior of the system. However, there is no recipe for finding Lyapunov functions.
To see this, let's have a look at a system from a different context. The Lotka-Volterra model describes predator-prey interactions between a predator species a and a prey species b. The extended Lotka-Volterra equations are

$$\partial_t \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ab - \delta a - \gamma_a a^2 \\ -ab + \beta b - \gamma_b b^2 \end{pmatrix}, \tag{1}$$

where we assume $\beta, \delta, \gamma_a, \gamma_b \geq 0$.

d) Interpret the different terms in the evolution equations.

e) One might have come up with an ansatz of the form $V(a, b) = p_1 [a - a^* \log(a)] + p_2 [b - b^* \log(b)]$ as Lyapunov function for this system. Here, $a^*, b^* > 0$ denote the non-zero equilibrium solution of the system. Determine the parameters p_1, p_2 such that this function is indeed a Lyapunov function for the above system. In which case is it a strict Lyapunov function? Plot trajectories in phase-space when $V(a, b)$ is a (strict) Lyapunov function. How do the two cases differ?
Hint: Separate terms with and without γ_a, γ_b. Rewrite the terms that are left using a^*, b^*.

Exercise 5 – Population of fish

Assume a population of n fish in a pond can be described by the logistic growth equation:

$$\partial_t n = \mu_b n \left(1 - \frac{n}{K_b}\right), \tag{2}$$

where K_b and μ_b are constant parameters.

a) Interpret the meaning of each term.

b) :Now we want to consider a population of these fish, following equation 2.] Consider a population of n fish that is assumed to grow according to this logistic growth law. Extend the logistic growth equation by an additional term which accounts for a constant harvesting denoted by H which is independent of the population size n (this provides a simple model of a fishery.) Show that the system can be rewritten in dimensionless form as

$$\partial_\tau u = u(1 - u) - h \tag{3}$$

for suitably defined dimensionless quantities u, τ, and h.

c) Find fixed points ($\partial_\tau u^* = 0$) of Eq.(3) in terms of h.

d) Sketch $y = -u^2$ and $y = h - u$ on the same graph for different values of h and mark the fixed points.

e) If an arbitrary point ($u \neq u^*$) is taken as the initial condition, the system will evolve according to the sign of $\partial_\tau u$. If $\partial_\tau u$ is positive, u increases over time and vice versa. This trend of temporal evolution is called flow and it can be marked by arrows on the u-axis. In Fig. 1 you can find an example of velocity flow for $h = 3$. Sketch the velocity flow for different values of h. The system bifurcates (qualitative changes its flow behavior) at a certain value h_c. Find h_c. Sketch the bifurcation diagram, identify the bifurcation and determine the stability of the fixed points in each plot.

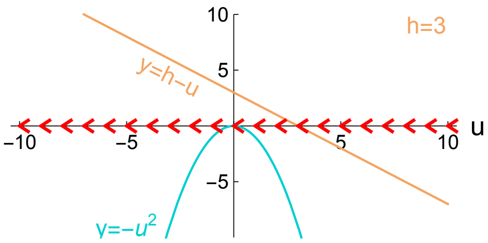

Figure 1: An example of a flow plot for $h = 3$, $-u^2 < h - u$. Here, $\partial_\tau u$ is always negative: The flow (red arrows) is always pointing to the negative direction, i.e. the population size is always decreasing.

f) Discuss the long-term behavior of the fish population for $h < h_c$ and $h > h_c$, and give the biological interpretation in each case.

Exercise 6 – Outbreak of an insect in a forest

In this exercise, we will consider a simple model developed by Ludwig et al.[1] for an outbreak of an insect, called spruce budworm, in a forest. Budworms eat the foliage of a balsam fir forest and, during an outbreak, they can defoliate most trees in about five years. We will only consider the time evolution of the budworm population $n(t)$ and assume the foliage of the forest to be constant. The time evolution of $n(t)$ is given by logistic growth and a phenomenological death term that accounts for predation $p(n)$ by birds.

$$\partial_t n = \mu_b n \left(1 - \frac{n}{K_b}\right) - p(n)\,, \tag{4}$$

where

$$p(n) = \frac{an^2}{b^2 + n^2} \tag{5}$$

and $\mu_b, K_b, a, b > 0$ are parameters.

a) Why might the assumption of constant foliage be a valid simplification? What is the functional role of each of the model's parameters?

[1] Ludwig, Donald, Dixon D. Jones, and Crawford S. Holling. "Qualitative analysis of insect outbreak systems: the spruce budworm and forest." The Journal of Animal Ecology (1978): 315-332.

b) Rescale the system to the following nondimensional form:

$$\partial_\tau u = \mu u \left(1 - \frac{u}{K}\right) - \frac{u^2}{1 + u^2} \,. \tag{6}$$

What are u, μ, K and τ in terms of n, t, μ_b, K_b, a, b?

c) Show by a graphical analysis that there are at most four fixed points u_0, u_1, u_2, u_3. Determine the number of fixed points and their stability for all possible choices of μ and K. Argue which of those fixed points could be associated with the *refuge*, *outbreak*, and *threshold* state of the system. Discuss the possibility of bistability for different parameter configurations.

d) Next we want to compute the curves (K, μ) that parametrize the critical points where the *refuge* state ceases to exist (by undergoing a saddle-node bifurcation with the *threshold* of the system). Show that the critical points are given by the parametric form

$$\mu_u = \frac{2u^3}{(1 + u^2)^2}, \quad K_u = \frac{2u^3}{u^2 - 1}. \tag{7}$$

e) Use MATHEMATICA to plot the bifurcation curves in (K, μ)-space and the surfaces of all stationary states $u(K, \mu)$ in (K, μ, u)-space. How would you expect forest growth to affect the parameters? In this context, discuss the possibility of hysteresis effects.

Your solutions should be handed in in moodle by **Wednesday, October 29$^{\text{nd}}$ 2025, 10 am.**