Arnold Sommerfeld Center for Theoretical Physics
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Self-organization and pattern formation
Sheet 0

Exercise 1 – Diffusion and Fourier transform

In this course we will often look at dynamical systems that evolve both in space and time. These systems are described by partial differential equations (PDE). To solve PDE's it is often useful to use a Fourier transformation. In this exercise we will remind ourselves of this concept by solving the one-dimensional diffusion equation

$$\frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2 c(x,t)}{\partial x^2}, \tag{1}$$

which describes the temporal evolution of a density $c(x,t)$ on a line with a diffusion coefficient D.

a) Take the Fourier transform of the diffusion equation and solve for $\hat{c}(k,t)$ (use the convention $\hat{c}(k,t) = \int_{-\infty}^{\infty} c(x,t)e^{-ikx}dx$). The solution should read as follows,

$$\hat{c}(k,t) = \hat{c}(k,0)e^{-k^2Dt}. \tag{2}$$

Does the ensuing equation depend on the convention? What assumptions is made for the boundary terms $c(x,t)$ and $\partial_x c(x,t)$ at infinity?

b) Assume that at $t = 0$, the density is concentrated at a single point, i.e. $c(x,t=0) = N\delta(x-x_0)$. Find the Fourier transform of the initial condition. The result should reads $\hat{c}(k,0) = Ne^{-ikx_0}$.

c) Use the inverse Fourier transform $c(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{c}(k,t)e^{ikx}dk$ to get an expression for $c(x,t)$. Does $c(x,t)$ depends on the convention of Fourier transform? *Hint: use Gaussian integral* $\int_{-\infty}^{\infty} e^{-a(x+b)^2}dx = \sqrt{\pi/a}$.

d) Draw a sketch of your solution for $t = 0$, $t \to \infty$ and some intermediate time points $0 < t < \infty$.

Exercise 2 – The harmonic oscillator

One dynamical system you all know well is the harmonic oscillator. Here, we will analyse the dynamics of the one-dimensional harmonic oscillator in phase space. *To plot the flow*

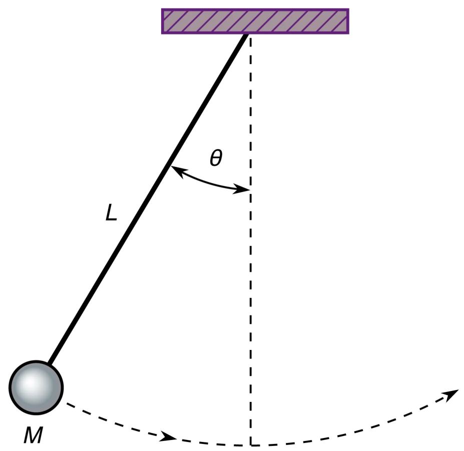

Figure 1: Illustration of pendulum

and trajectories in phase space, you can use the Mathematica template provided along with this exercise sheet. You can get Mathematica through your university account (see `https://collab.dvb.bayern/display/LMULMPHGST/Software`*). For a quick intro, have a look at* `https://www.wolfram.com/language/fast-introduction-for-programmers/en/`*.*

Consider a mass m suspended on a Hookian spring with spring constant k (no friction).

a) Use Newton's 2^{nd} law to write down the equation of motion for the vertical displacement x from the rest position. Then use the displacement x and the velocity v to write the equation of motion as a system of two first-order differential equations: $\partial_t \vec{u} = \vec{f}(\vec{u})$.

b) Sketch the vector field $\vec{f}$ in (x, v)-phase space. What does it describe?

c) Sketch the trajectory of the dynamics of the harmonic oscillator in phase space for initial conditions $v(0) = 0$ and $x(0) = 1, 2, 3$. What stays constant along each single trajectory?

d) Last, we consider the dynamics of a pendulum, which has a fixed length of L and mass M. Describe the inclination of the pendulum with the angle $\theta(t)$ and derive the equation of motions from Newton's 2^{nd} law. As we did in part a), rewrite the second-order equation into a two-component first order differential equations by introducing the conjugate momentum variable $\omega = \partial_t \theta(t)$.

e) Plot the two-variables first order differential equation obtained in e) using Mathematica. Find all the fixed points and argue heuristically for which physical configuration they correspond to. Based on this, do you expect these fixed points to be stable or unstable? And what would you expect, when including friction?

Exercise 3 – A model of a laser

In this exercise we will look at an example system that can be described by a set of coupled partial differential equations (PDEs). You will learn how to interpret the different terms in the PDE and get introduced to common techniques for analyzing such dynamical systems.
We'll look at a special type of laser called a solid-state laser, as shown in Fig. 2. It's made up of unique type of atoms embedded in a solid-state matrix, bounded by partially reflecting mirrors at either end. Those atoms can be "pumped" with energy from an external source, moving them out of their ground state. Each atom can be pictured as a tiny antenna emitting energy. When the energy input is low, the atoms vibrate independently, emitting light randomly. However, when the energy input is high enough, the atoms start vibrating together. Here, all these tiny antennas work together like a giant antenna, producing a powerful, focused beam of light, i. e., a laser beam.

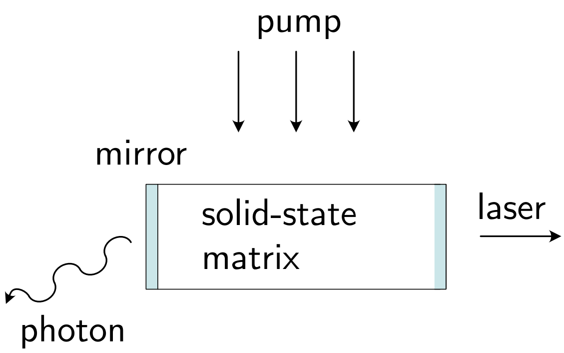

Figure 2: A sketch of the described laser model. Energy in pumped into a solid-state matrix containing the "laser-atoms". Partially reflecting mirrors are shown in light blue. The laser light is emitted at the right, while a photon is escaping the solid-state matrix on the left.

To make a simple model of the phenomena, following Milonni and Eberly [1], we consider two variables: N, the number of excited atoms, and n, the number of laser photons emitted by these atoms. The number of photons increases by one when a photon bumps into an excited atom, that is therefore stimulated to emit another photon. As photons move randomly into the solid-state matrix, the probability of this encounter is proportional to both the number of excited atoms N and of photons in the laser n. Additionally, some photons may escape the laser with a certain rate. On the other hand, whenever an excited atom emits a photon, it drops back to its ground state, effectively reducing the number of excited atoms in the system N by one. Excited atoms can also spontaneously decay to the ground state with some constant rate. Finally, the number of excited atoms is increased at a constant rate, given by the pump strength. The above process can be formalized with the following equations:

$$\dot{n} = GnN - kn,$$
$$\dot{N} = -GnN - fN + p.$$

All parameters are positive, except p, which can have either sign.

a) Describe the meaning of each coefficient G, k, f and p, associating the phenomena described above with each term in the equations.

[1] Milonni, P. W. and Eberly, J. H. (2010). Laser physics. John Wiley and Sons.

b) Suppose that N relaxes much more rapidly than n. Then we may make the quasi-static approximation $\dot{N} \approx 0$. Given this approximation, express $N(t)$ in terms of $n(t)$ and derive a first-order system for n. (This procedure is often called adiabatic elimination, and is possible because of the *time-scale separation* of the two variables' dynamics n and N.)

c) Calculate the fixpoints ($\dot{n}\big|_{n=n^*} = 0$) of the first order system for n.

d) In a next step, consider a small deviation around the steady state $n = n^* + \delta n$ and linearize the first order system for n around the fixed point $n^* = 0$ to obtain a linear equation for $\dot{\delta n}$.

e) Determine the solution of the linear equation for $\dot{\delta n}$ you derived in the previous exercise. Determine the critical parameter p_c that changes the behavior of the solution as a function of the other model's parameters. Describe what happens for $p > p_c$ and $p < p_c$. What does this imply for the stability of the homogeneous steady state $n^* = 0$?

> Your solutions should be handed in in moodle by **Wednesday, October 22$^{\text{nd}}$ 2025, 10 am.** Note that the first sheet is not graded. It is a good exercise for you to get started with the course!