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Prof. Dr. E. Frey
Dr. T. Franosch

Lehrstuhl für Statistische Physik
Biologische Physik & Weiche Materie
Arnold-Sommerfeld-Zentrum für Theoretische Physik
Department für Physik

Ludwig Maximilians Universität

T VI: Soft Matter and Biological Physics

(Prof. E. Frey)

Problem set 7

Problem 7.1 *scaling hypothesis*

Close to a continuous phase transitions, various observables depend singularly on thermodynamic control parameters. Using the language of magnets, one defines critical exponents for the magentization (order parameter) m in the low-temperature phase, $m \sim (-\tau)^\beta$ with the reduced temperature $\tau = (T - T_c)/T_c$, the divergence of the zero-field susceptibility, $\chi(T) \sim |\tau|^{-\gamma}$, and the anomalous behavior of the specific heat $C \sim |\tau|^{-\alpha}$. Furthermore, directly at the critical temperature $T = T_c$, the equation of state becomes singular, $m \sim h^{1/\delta}$, where h denotes the magnetic field (field conjugate to the order parameter).

1. The expectation that there should be relations between these exponents is formulated in terms of the scaling hypothesis. The key assumptions is that the free energy density acquires a singular contribution that can be written as

$$f_{\text{sing}} = |\tau|^{2-\alpha} \mathrm{Y}_\pm \left(h|\tau|^{-\Delta} \right) .$$

In particular, f_{sing} does not depend separately on temperature and magnetic field but only on a single scaling variable $x = h|\tau|^{-\Delta}$. The scaling functions $\mathrm{Y}_\pm(\cdot)$ are different for temperatures above ($\tau > 0$) and below ($\tau < 0$) the transition point. They are assumed to be analytic, in particular they may be Taylor expanded for small arguments

$$\mathrm{Y}_\pm(x) \approx \mathrm{Y}_\pm(0) + \mathrm{Y}'_\pm(0)x + \ldots .$$

Using thermodynamic arguments, show that this hypothesis implies

(a) the specific heat acquires the expected behavior.

(b) the exponent for the order parameter is given by $\beta = 2 - \alpha - \Delta$.

(c) the susceptibility exponent fulfills the relation $\gamma = \Delta - \beta$.

(d) the equation of state is singular with $\delta = \Delta/\beta$.

2. In a similar spirit, the order parameter correlation function, $G(\vec{r}; \tau, h) = \langle m(\vec{r}) m(\vec{0}) \rangle$, assumes the scaling form

$$G(\vec{r}; \tau, h) = \frac{1}{r^{d-2+\eta}} \mathcal{G}_\pm \left(r/\xi, h|\tau|^{-\Delta} \right) ,$$

in terms of the divergent correlation length, $\xi \sim |\tau|^{-\nu}$. In particular, distances are measured only relative to ξ. By the fluctuation-response theorem, the susceptibility is connected to the correlation function by

$$\chi = \int \mathrm{d}^d\vec{r}\, G(\vec{r}; \tau, h) \, ,$$

Demonstrate that the scaling hypothesis implies the exponent relation $\gamma = (2 - \eta)\nu$. Compare all exponent relations obtained so far with the mean-field prediction.

3. A *hyperscaling* relation assumes that the singular contribution to the free energy density arises due to *fluctuations.* Then up to factor of order unity, one expects in zero field

$$f_{\text{sing}} \simeq k_B T \frac{\xi^d}{V}\,, \qquad \text{volume of the system: } V\,.$$

i.e. the effective degrees of freedom are correlated regions of linear dimension ξ, and each freedom contributes $k_B T$ to the free energy. Show that this implies $d\nu = 2 - \alpha$. Since this relation involves the dimension of the system, whereas the mean-field exponent are universal, the hyperscaling relation is valid within mean-field only for a special value of $d = d_c$, where d_c is referred to as critical dimension. Determine d_c.

Problem 7.2 *One-dimensional Ising model*

The partition sum of the one-dimensional Ising model can be written in a symmetric form

$$Z = \sum_{\{\sigma_l = \pm 1\}} \exp\left(-\bar{H}[\{\sigma_l\}]\right)\,, \qquad \bar{H}[\{\sigma_l\}] = \sum_l \left(-K\sigma_l\sigma_{l+1} - \frac{L}{2}(\sigma_l + \sigma_{l+1}) + C\right)$$

Here the temperature has been absorbed in the parameters $K = \beta J$ and $L = \beta H$. Furthermore a constant C has been added which will be useful in the following analysis.

1. Trace out every second spin degree of freedom, and show that the *effective* hamiltonian for the remaining spins can be written in the same form as above, provided the parameters K, L, C are replaced by new ones. Find the mappings

$$K' = K'(K, L, C)\,, \qquad L' = L'(K, L, C),\,, \qquad C' = C'(K, L, C)\,.$$

These relations constitute the renormalization procedure.

2. Introduce suitable variables that capture possible strong coupling behavior

$$x = \mathrm{e}^{-4K}\,, \qquad y = \mathrm{e}^{-2L}\,, \qquad w = \mathrm{e}^{-4C}\,,$$

and show that the recursion relation translates to

$$x' = \frac{x(1+y)^2}{(x+y)(1+xy)}\,, \qquad y' = \frac{y(x+y)}{1+xy}\,, \qquad w' = \frac{w^2 x y^2}{(1+y)^2(x+y)(1+xy)}\,.$$

Discuss the behavior of the mapping close to its non-trivial fixed point.