Ising Model: Transfer Matrix (H=0)

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Seminar: Renormalization Group Theory and Critical Phenomena, Munich
May 12, 2009

Introduction - A review

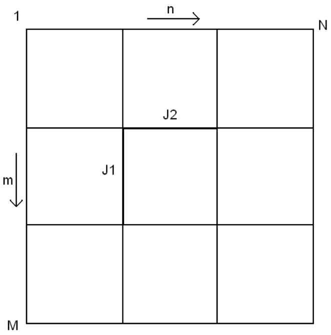

Figure: The square lattice

- We consider a set of spin-1/2's arranged on a square lattice of size M×N, interacting only with nearest neighbours, without any external magnetic field.

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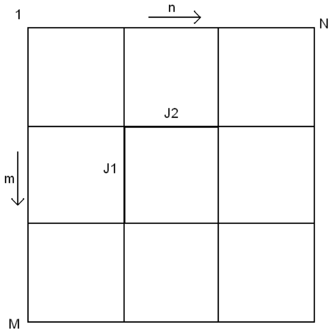

Figure: The square lattice

- We consider a set of spin-1/2's arranged on a square lattice of size M×N, interacting only with nearest neighbours, without any external magnetic field.
- J_1 and J_2 are the coupling constants along vertical and horizontal axes, respectively.

- The Hamiltonian takes the form:

$$-\beta E(\sigma) = \beta J_1 \sum_{n,m=1}^{N,M} \sigma_{n,m}\sigma_{n,m+1} + \beta J_2 \sum_{n,m=1}^{N,M} \sigma_{n,m}\sigma_{n+1,m}$$

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- Note: each $\sigma_{n,m}$ is a classical variable taking value ± 1

Transfer Matrix

- Definition of the Transfer Matrix:

$$V(\vec{\sigma}_n, \vec{\sigma}_{n+1}) = \exp\left(\frac{1}{2}\mathbb{K}_1(\vec{\sigma}_n) + \mathbb{K}_2(\vec{\sigma}_n, \vec{\sigma}_{n+1}) + \frac{1}{2}\mathbb{K}_1(\vec{\sigma}_{n+1})\right)$$

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- $V(\vec{\sigma}_n, \vec{\sigma}_{n+1})$ is a $2^M \times 2^M$ symmetric matrix
- Therefore, we can calculate the partition function in terms of the transfer matrix:

$$\begin{aligned} Z &= \sum_{\vec{\sigma}_1 \ldots \vec{\sigma}_N} \exp(-\beta E(\vec{\sigma}_1, \ldots \vec{\sigma}_N)) \\ &= \sum_{\vec{\sigma}_1 \ldots \vec{\sigma}_N} V(\vec{\sigma}_1, \vec{\sigma}_2) \ldots V(\vec{\sigma}_N, \vec{\sigma}_1) = \sum_{\vec{\sigma}_1} V^N(\vec{\sigma}_1, \vec{\sigma}_1) = TrV^N \end{aligned}$$

New form for the Transfer Matrix

- Eigenvalue of the symmetric transfer matrix:

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- We rewrite this equation in a new form:

matrix $V \implies$ operator $\hat{V}$ on $\bigotimes_1^M H_{(\frac{1}{2})}$

vector $\psi \implies \bar{\psi} \in \bigotimes_1^M H_{(\frac{1}{2})}$

$H_{(\frac{1}{2})}$ is the hilbert space of a spin-$\frac{1}{2}$.

- We get:

$$\bar{\psi} = \sum_{\vec{\sigma}} \psi(\vec{\sigma}) \chi_{\sigma_1}(1) \otimes \dots \otimes \chi_{\sigma_M}(M)$$

with $\chi_{\sigma=+1} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{\sigma=-1} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

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- We define $\hat{\sigma}_m^z = \mathbb{1} \otimes \ldots \otimes \hat{\sigma}^z \otimes \ldots \otimes \mathbb{1}$ where $\hat{\sigma}^z$ is a Pauli Matrix at the m position and we define $\hat{\sigma}_m^x$ and $\hat{\sigma}_m^y$ in the same way. For convenience, we write σ_m^x, σ_m^y and σ_m^z.

$$exp(\frac{1}{2}\mathbb{K}_1), exp(\mathbb{K}_2) \implies \prod_{m=1}^{M} exp(\frac{K_1}{2}\sigma_m^z\sigma_{m+1}^z), \prod_{m=1}^{M} exp(K_2\sigma_m^{z'}\sigma_m^z)$$

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- For each m, we have this matrix form:

$$\begin{aligned} e^{\sigma^{z'}\sigma^z} &= \begin{pmatrix} e^{K_2} & e^{-K_2} \\ e^{-K_2} & e^{K_2} \end{pmatrix} = e^{K_2}(\mathbb{1} + e^{-2K_2}\sigma^x) = \dots \\ &= \sqrt{2\sinh(2K_2)}e^{K_2^*\sigma^x}, \text{ with } \tanh(K_2^*) = e^{-2K_2} \end{aligned}$$

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- Final result:

$$\hat{V} = (2\sinh(2K_2))^{\frac{M}{2}} \times$$
$$exp\left(\frac{K_1}{2}\sum_1^M \sigma_m^z\sigma_{m+1}^z\right) exp\left(K_2^*\sum_1^M \sigma_m^x\right) exp\left(\frac{K_1}{2}\sum_1^M \sigma_m^z\sigma_{m+1}^z\right)$$

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- $\hat{V}$ represents V in the Fock space $\bigotimes_1^M \chi$ (i.e. matrix elements of $\hat{V}$ in Fock space are $V(\vec{\sigma}, \vec{\sigma}'))$

Transfer Matrix - Consequences

- If $K_1, K_2^* << 1$, we can neglect noncommutative terms and we get:

$$\hat{V} \approx (2\sinh(2K_2))^{\frac{M}{2}} e^{-\hat{\mathbb{H}}}$$

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- Now, we want to connect our $\hat{\mathbb{H}}$ to the quantum Hamiltonian of the second talk about the scaling approximation:

$$\frac{\hat{\mathbb{H}}}{K_2^*} = \hat{H} = -\sum_m \sigma_m^x - \Lambda \sum_m \sigma_m^z \sigma_{m+1}^z$$

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- Last remark:

$$K_2^* << 1 \implies \Lambda = \frac{K_1}{K_2^*} \approx \frac{K_1}{tanh(K_2^*)} = \frac{K_1}{e^{-2K_2}} =: \frac{K}{e^{-2K_\tau}} = \lambda$$

Questions

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