Spontaneous Symmetry Breaking and Goldstone Modes

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Outline

1. Spontaneous Symmetry Breaking
   - Symmetries in Physics
   - The phenomenon of SSB
2. Implications of SSB
   - Goldstone Modes
   - Goldstone Theorem
   - Mermin-Wagner Theorem
3. Conclusion

What is a symmetry in Physics?

Invariance of a physical law under transformations

- Symmetries can arise from physics ...
  - e.g. Galilei-invariance in Newtonian Mechanics
- ... or from mathematics
  - e.g. gauge-invariance in Electrodynamics: $A^{\mu\prime} \rightarrow A^{\mu} + \frac{\partial f}{\partial x^{\mu}}$
- Symmetries form a group

Different kinds of symmetries

we differentiate between:

- global symm.: acts simultaneously on all variables
- local symm.: acts independently on each variable

furthermore:

- continuous symm., e.g. rotations ($\mathbb{SO}(n)$)
- discrete symm., e.g. spin group ($\mathbb{Z}_2$)

Lie group: differentiable manifold that is also a group respecting the continuum properties of the manifold

The concept of SSB

Observation: our world is (mostly) not symmetric!

$\Rightarrow$ General concept in modern physics

- original law is symmetric
- but the solutions are not!
- i.e. the symmetry is broken (by some mechanism realized in our world)

Examples

- Unification of fundamental forces in particle physics
  - early universe: only one force
  - universe cooled down $\Rightarrow$ separation to 4 fundamental forces
- origin: superconductivity (Anderson 1958)
- today: applications in condensed matter physics (superconductivity, superfluidity, BEC) and QFT (particle physics, Standard Model)

Definition of SSB

example: Ising model

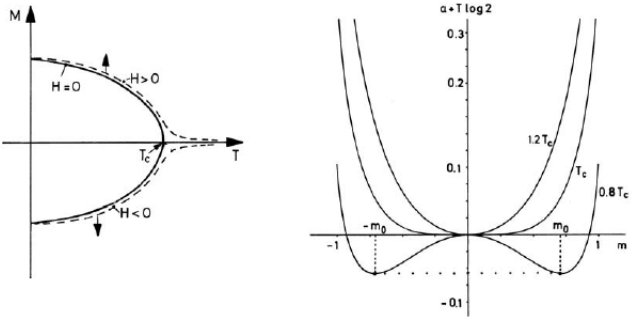
$$\mathcal{H}_0 = -\frac{1}{2}\sum_{ij} \sigma_i J_{ij} \sigma_j \tag{1}$$

$\sigma_i = \pm 1, i \in \mathbb{Z}^d$, d: dimension, no external field here

- $\mathcal{H}_0$ is $\mathbb{Z}_2$-invariant ($\mathbb{Z}_2$: global discrete symm.)
- if $J_{ij} = J(|R_i - R_j|) \Rightarrow$ lattice symm.: $\mathbb{Z}^d$ (Bravais)
- we know: phase transitions for $d \geq 2$
- order parameter: magnetization m

Definition of SSB

- $\mathcal{H}_0$ is $\mathbb{Z}_2$-inv., but solution for m not: it changes sign!
- mean-field: $m \sim |\tau|^\beta$, $\beta_{mf} = \frac{1}{2}$ for $T \leq T_c$
- Helmholtz free energy $a(m, T)$



$\Rightarrow$ former symm. is broken!

Definition of SSB

The general concept:

Definition (SSB)

$\mathbb{G}$ a global symm. group of $\mathcal{H}$
Then SSB occurs if in stable TD equilibrium state:

$$m =:< M >\neq 0 \tag{2}$$

- M: observable not $\mathbb{G}$-inv.
- m: order parameter (of the new phase)

How can SSB occur?

- Problem: There schould not be any SSB!
- Why? Averages w.r.t. ρ and $\rho = \rho(\mathcal{H})$

$$\Rightarrow m =< M >= Tr(M\rho(\mathcal{H})) = 0 \tag{3}$$

- example: Ising model:

$$m =< \frac{1}{N}\sum_i \sigma_i >= \frac{1}{N \cdot Z_N}\sum_i \sum_{\sigma_i=\pm 1} \sigma_i \cdot exp^{-\beta\mathcal{H}(\{\sigma_i\})} = 0 \tag{4}$$

How can SSB occur?

- Solution: take TD limit, but this isn't enough!
- one needs: symm. breaking field h
  $\longrightarrow$ extra term: $-h \cdot M$

Definition (SSB (Bogolyubov))

$$\lim_{h \to 0} \lim_{N \to \infty} < M >_{N,h} = m \neq 0 \tag{5}$$

- Limits cannot be interchanged!

Remarks

- SSB $\rightarrow$ long range order, e.g. in ferro-/antiferro-magnets

- SSB $\Rightarrow$ phase transition
    - But there are phase transitions without SSB, e.g. liquid-vapour: both fully rotational and translational invariant!
    - "order parameter" $|\rho_l - \rho_v|$

Goldstone Modes

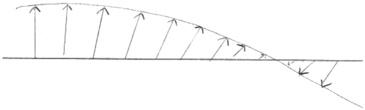
- SSB of **continuous** symm. $\Rightarrow$ new excitations: Goldstone Modes (cost little energy)
- example: Spin waves in Heisenberg model

$$\mathcal{H} = -J \sum_{ij} S_i S_j = -J \sum_{ij} cos(\theta_{ij}), S_i \in \mathbb{R}^3, |S_i| = 1 \tag{6}$$

- ferromagnetic phase (SSB) $\Rightarrow$ Groundstate: all spins in one direction

Goldstone Modes

- energy cost to rotate one spin: $E_g \sim J(1 - cos(\theta))$, (infinitesimal small angle θ)
- energy cost to rotate all spins: Nothing! (due to cont. symm. $\mathbb{O}(3)$)
- $\Rightarrow$ long-wavelength spin-waves



- remark: cannot happen with discrete broken symm.!
- e.g. Ising model: $E_g \sim J$ (every excitation costs finite energy)

Goldstone Theorem

general result proving existence of long-range order

- consider correlation functions $\mathcal{G}_{ij}^{\alpha\beta}$, e.g. spin-spin corr.
- $\vec{m}_i$: order parameter
- $\vec{h}_i$: symm. breaking field
- Gibbs free energy: $G\{h\} = -ln(Z)$

Proof of Goldstone Theorem

- vertex function: $F\{m\} = G\{h\} + \sum_i \vec{h}_i \vec{m}_i$ (Legendre transformation)
- take F $\mathbb{G}$-invariant (in zero external field)
- $\frac{\partial}{\partial m_i^\alpha} F\{m\} = h_i^\alpha$
- corr. function: $\mathcal{G}_{ij}^{\alpha\beta} = -\frac{\partial^2 G}{\partial h_i^\alpha \partial h_j^\beta}\Big|_{\vec{h}_i=0} = \beta^{-1}\chi_{ij}$ (Fluctuation-Dissipation Theorem)

Proof of Goldstone Theorem

- general property of Legendre transf.: $\frac{\partial^2 F}{\partial m^2} = -(\frac{\partial^2 G}{\partial h^2})^{-1}$
  $\Rightarrow \frac{\partial^2 F}{\partial m_i^\alpha \partial m_j^\beta} = (\mathcal{G}^{-1})_{ij}^{\alpha\beta}$
- Fourier transformation:
  $[\mathcal{G}^{-1}(\vec{q})]^{\alpha\beta} = \sum_i e^{-i\vec{q}(\vec{R}_i - \vec{R}_j)} \frac{\partial^2 F}{\partial m_i^\alpha \partial m_j^\beta}$
- now: zero field case, uniform order parameter ($\vec{m}_i = const. = \vec{m}$, i.e. $\vec{q} = 0$) and infinitesimal transformation $g_\beta^\alpha = id + t_\beta^\alpha$ (Lie group)
- t_β^α: non-trivial transformation ($\hat{=}$ transversal modes!)
- acting on $m^\alpha \rightarrow g_\beta^\alpha(m^\beta) = m^\alpha + \underbrace{t_\beta^\alpha m^\beta}_{\delta m^\alpha}$

Proof of Goldstone Theorem

- variation yields: $0 = \delta[\frac{\partial F}{\partial m_i^\alpha}] = \sum_{\beta j}[\frac{\partial^2 F}{\partial m_i^\alpha \partial m_j^\beta}]_{\vec{m}_i=\vec{m}} \delta m^\beta$

$$\Rightarrow \sum_{\beta\gamma} [\mathcal{G}^{-1}(\vec{q}=0)]^{\alpha\beta} t_\gamma^\beta m^\gamma = 0 \qquad (7)$$

- no SSB $\Rightarrow$ trivial, since $\vec{m} = 0$
- "longitudinal" transformation $\Rightarrow$ trivial, since $t_\gamma^\beta m^\gamma = 0$
- but if $\vec{m} \neq 0 \Rightarrow det\mathcal{G}^{-1}(\vec{q}=0) = 0 \Rightarrow det\mathcal{G}(\vec{q}=0) = \infty$

Goldstone Theorem

Theorem (Goldstone)

If Lie symm. group is spontaneously broken and order parameter is uniform then the order parameter-order parameter response function $\mathcal{G} =< mm >$ *develops a pole.*

- long range order
- appearence of Goldstone Modes
- gapless excitation spectrum ("zero mass")

Goldstone Theorem: Remark

In the language of field theory:

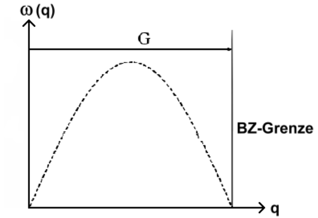
- partition function: $\mathcal{Z} = \int \mathcal{D}\{\phi\} exp[S\{\phi\}]$
  with action $S\{\phi\} = \int d^d x \mathcal{L}[\phi(x)]$,
  order parameter field $\phi(x)$
- Lagrange density: $\mathcal{L} = \frac{1}{2}\sum_i |\nabla \phi^i|^2 + P[C_n(\phi)]$
- propagators: $\Delta^{ij}(\vec{x}, \vec{y}) = -\frac{\delta^2 \Omega\{\lambda\}}{\delta\lambda_i(\vec{x})\delta\lambda_j(\vec{y})}$
  $\Omega = -ln(\mathcal{Z})$, λ_i: source term = external field
- $\Rightarrow det(\Delta^{-1}(p = 0)) = 0$

Goldstone Theorem: Remark

remark: order parameter modulated with wave-vector Q

- $det\mathcal{G}^{-1}(Q) = 0$ (above: special case Q $= 0$)
- e.g. liquid-solid phase transition
- consider $\tilde{\rho}(q) \Rightarrow$ order parameter $\tilde{\rho}(G)$ (G: reciprocal lattice vector)
- Goldstone modes?

Goldstone Theorem: Remark



- $\lim_{q \to 0} \omega(q) = 0$: consequence of short range forces (nothing special, usual sound waves that appear in liquids and solids)
- Goldstone Modes: “Umklapp” phonons at q = G

Goldstone Theorem: Example

Example: superconductors

- Field operators $\hat{\Psi}_\sigma(\vec{r})$
- global $\mathbb{U}(1)$ gauge symmetry: $\hat{\Psi}_\sigma(\vec{r}) \longrightarrow \hat{\Psi}_\sigma(\vec{r})e^{i\theta}$
- order parameter $\Delta_{sc}(\vec{r}) =< \hat{\Psi}_\uparrow(\vec{r})\hat{\Psi}_\downarrow(\vec{r}) >\neq 0$
- short range forces: collective density excitations with $\lim_{q\to 0}\omega(q) = 0$
- long range forces (more realistic, e.g. Coulomb force): Goldstone Modes pushed to the plasma frequency: $\lim_{q\to 0}\omega(q) = \Omega_{pl}$ ($\Rightarrow$ Cooper Pairs!)
- caveat: for long range forces: $\omega(q=0) \neq \lim_{q\to 0}\omega(q)$
- spectrum has a gap $\Rightarrow$ minimum mass ($\longrightarrow$ the same in Higgs mechanism [except that we have our particles!])

Mermin-Wagner Theorem

SSB, phase transitions and the role of dimension ...

Theorem (Mermin-Wagner)

If we have:

- *SSB of a Lie symm. group*
- *short range forces* ($\sum_i |\vec{R}_i|^2 |J_{i0}| < \infty$)
- *poisson bracket structure (classical) or (anti-)commutator structure (quantum mechanics) [fulfilled for all standard Hamiltonians]*

Then there is no phase transition (associated with a long range order!) for dimension $d \leq 2$ *(for* $T > 0$*).*

Mermin-Wagner Theorem

Theorem (Mermin-Wagner)

There is no phase transition (associated with a long range order!) for dimension $d \leq 2$ (for $T > 0$).

- Proof uses Bogolyubov's inequality
- e.g. Heisenberg model:

$$S^2 \geq \int_{BZ} \frac{d^d k}{(2\pi)^d} \frac{2Tm^2}{|\vec{k}|^2 S^2 \sum_i |\vec{R}_i|^2 |J_{i0}| + |h||m|} \qquad (8)$$

$\rightarrow$ diverges for $d \leq 2$ (if there is SSB; for zero field)

- d = 2: there is phase transition associated with "quasi-long range" order $\Rightarrow$ Kosterlitz-Thouless

What can you take home?

- SSB is a general concept in modern physics applicable to a variety of fields
- SSB $\Rightarrow$ new excitations: Goldstone Modes/Bosons (e.g. Higgs mechanism)
- SSB $\Rightarrow$ remarkably general results about phase transitions (Mermin-Wagner)

Thank you for your attention