Functional Renormalization: Exact Renormalization Flow

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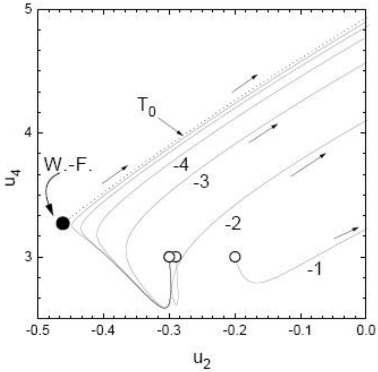
Motivation

- idea: introduce the cutoff in the propagator
- renormalization flow in the vicinity of the Wilson-Fisher fixed point



1. Exact renormalization group equations (ERGE)

2. Local potential approximation (LPA)

Introduction

- What does „exact“ mean?
  - „continuous“ (not discrete) realization of Wilson RG transformation of action
  - no approximations or expansions with respect to some small parameter are made
- formulation: differential form known since 1970´s
- complexity: integro-differential equation

Representations of ERGE

- Representation
  - functional equation
  - infinite set of partial differential equations for couplings
  - infinite hierachy of ordinary differential equations for scaling fields
- no unique form: ERGE characterized by introduction of momentum cutoff Λ
- physical content: embody same physics at large distances and in continuums limit also at small distances

Notations

Notation and language of field theory

- momentum cutoff Λ
- scalar field $\phi(x)$, x coordinate vector in euclidian space of dimension d
- Fourier transformation $\phi(x) = \int_p \phi_p e^{ipx}$ with $\int_p = \int \frac{d^d p}{(2\pi)^d}$
- action $S[\phi] = \frac{\mathcal{H}[\phi]}{k_B T}$

Functionals

- generating functional $Z[J]$ of Green´s function

$$Z[J] = \mathcal{Z}^{-1} \int \mathcal{D}\phi \exp\{-S[\phi] + J \cdot \phi\}$$

with $J \cdot \phi = \int d^d x J(x)\phi(x)$,

- correlation functions

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{\delta^{(n)} Z[J]}{\delta J(x_1) \cdots \delta J(x_2)}$$

- generating functional $W[J]$ of connected Green´s function

$$W[J] = \ln Z[J]$$

$e^{\mathcal{W}} = \mathcal{Z}$, $\mathcal{W}$ minus free energy

Principle of derivation of ERGE

- two step Wilson procedure
  1. decimation: integration of fluctuations $\phi(p)$ over a range $e^{-t}\Lambda < |p| \leq \Lambda$ which leaves $\mathcal{Z}$ invariant
  2. rescaling: change of length scale by $e^{-t}$to restore original scale Λ of the system $p \rightarrow e^t p$

$$\mathcal{Z} = \prod_{p \leq \Lambda} \int \mathcal{D}\phi_p \exp\{-S[\phi]\} \xrightarrow{\text{step 1}} \prod_{p \leq e^{-t}\Lambda} \int \mathcal{D}\phi_p \exp\{-S'[\phi]\}$$

with

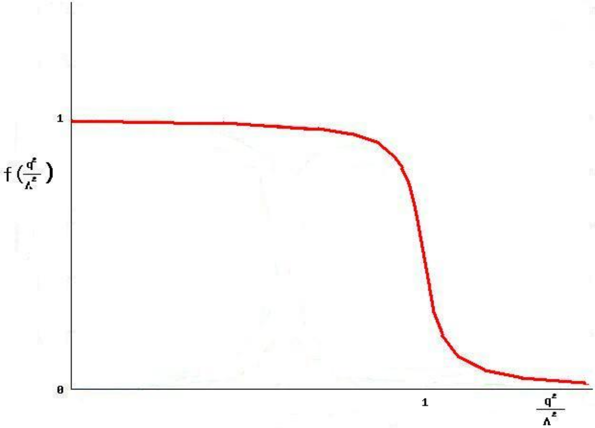
$$\exp\{-S'[\phi]\} = \prod_{e^{-t}\Lambda < p \leq \Lambda} \int \mathcal{D}\phi_p \exp\{-S[\phi]\}$$

- condsidering infinitesimal change of t provides evolution equation for S

$$\dot{S} = \frac{\partial S}{\partial t} = \mathcal{G}_{Dil} S + \mathcal{G}_{Tra} S$$

Polchinski´s version of ERGE

- derived his own smooth cutoff version of ERGE
- introduced general ultraviolet cutoff function $f(p^2/\Lambda^2)$



Polchinski´s version of ERGE

- derived his own smooth cutoff version of ERGE
- introduced general ultraviolet cutoff function $f(p^2/\Lambda^2)$
- Polchinski´s ERGE obtained from requirement that coarsening step leaves

$$Z_\Delta = \int \frac{\mathcal{D}\phi}{Z_\Delta^\circ} \exp\{-\frac{1}{2}(\phi, \Delta^{-1}\phi) + V[\phi]\}$$

invariant

- definition of the propagator with cutoff function

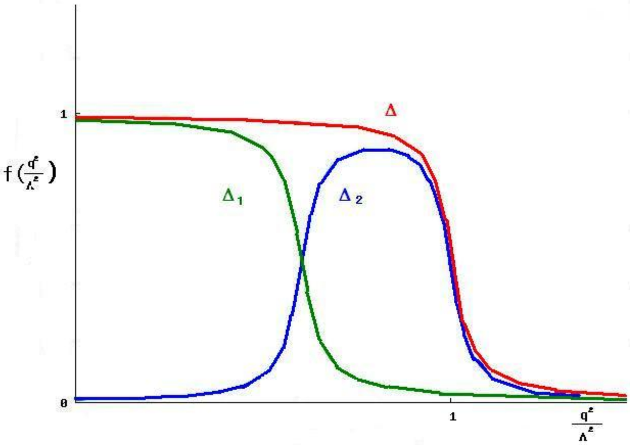
$$\Delta(q;\Lambda) = \frac{1}{q^2} f(q^2/\Lambda^2)$$

- notation

$$(\phi, \psi) = \int d^d x \phi(x)\psi(x) = \int_p \phi_p \psi_{-p}, \ (\phi, \Delta^{-1}\psi) = (\Delta^{-1}\phi, \psi)$$

Splitting theorem

- idea: $\Delta = \Delta_1 + \Delta_2$



Splitting theorem

- idea: $\Delta = \Delta_1 + \Delta_2$
- changing Λ to $\lambda = \frac{\Lambda}{\ell}$ with $\ell \geq 1$ leads to

$$\Delta_1 = \Delta(q;\lambda) = \ell^2 \Delta(q\ell;\Lambda)$$

$$\Delta_2 = \Delta - \Delta_1$$

- goal: decimation by integrating parts with Δ_2
- new effective action

$$Z_\Delta = \int \frac{\mathcal{D}\phi}{Z^\circ_{\Delta_1}} e^{-\frac{1}{2}(\phi,\Delta_1^{-1}\phi)} e^{-V_\ell[\phi]}$$

$$e^{-V_\ell[\phi]} = e^{\frac{1}{2}(\frac{\delta}{\delta\phi},\Delta_2(l)\frac{\delta}{\delta\phi})} e^{-V[\phi]}$$

Repetition of functional methods

- ordinary linearization $\delta f = \frac{df}{dx}\delta x$
- functional case $\delta F = (\frac{\delta F}{\delta \phi}, \delta\phi),\ \delta F = F[\phi + \delta\phi] - F[\phi]$
- „functional" taylor-expansion

$$F[\phi + \psi] = F[\phi] + \int_x \frac{\delta F[\phi]}{\delta \phi(x)}\psi(x)dx$$

$$+\frac{1}{2}\int_x\int_y \frac{\delta^2 F[\phi]}{\delta\phi(x)\delta\phi(y)}\psi(x)\psi(y)dxdy + \cdots$$

- short notation

$$F[\phi + \psi] = e^{(\psi, \frac{\delta}{\delta\phi})}F[\phi]$$

Hint

$$\int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{2\pi(a+b)}} \exp\{-\frac{\alpha^2}{2(a+b)}\} \cdot F(\alpha)$$

$$= \int_{-\infty}^{\infty} \int_{\infty}^{\infty} \frac{d\alpha}{\sqrt{2\pi a)}} \frac{d\beta}{\sqrt{2\pi b)}} \exp\{-\frac{(\alpha-\beta)^2}{2a} - \frac{\beta^2}{2b}\} \cdot F(\alpha)$$

Gauss integration

$$\int_{-\infty}^{\infty} e^{-ax^2} e^{-2bx} = \sqrt{\frac{\pi}{a}} e^{b^2/a}, \ a > 0$$

Polchinski´s ERGE

- ...we finally ended up with an integro-differential equation for the exact renormalization flow of $V_\Lambda[\phi]$

$$\Lambda \frac{d}{d\Lambda} V_\Lambda[\phi] = \frac{1}{2} \int_{xy} \Lambda \frac{d}{d\Lambda} \Delta_2 \left(\frac{\delta}{\delta\phi(x)} - \frac{\delta V_\Lambda[\phi]}{\delta\phi(x)} \right) \frac{\delta V_\Lambda[\phi]}{\delta\phi(y)}$$

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- solution $V_\Lambda[\phi]$ is functional represented by

$$\{ \frac{\delta^n V_\Lambda}{\delta\phi(x_1) \cdots \delta\phi(x_n)} |_{\phi=0} = V_\Lambda^{(n)}(x_1, \ldots, x_n) \}_{n=1}^{\infty}$$

for $V[\phi] = \sum_n \frac{1}{n!} \int_{x_1,\ldots,x_n} V_\Lambda^{(n)}(x_1, \ldots, x_n) \phi(x_1) \cdots \phi(x_n)$

Sharp and smooth cutoffs

- sharp cutoffs
  - sharp/hard cutoffs introduce nonlocal interactions in position space
  - difficulties induced by sharp cutoff circumvented by considering Legendre transform
- smooth cutoffs
  - „incomplete“ integration in which large momenta are more completely integrated than small momenta
- differences between sharp and smooth cutoffs disappear under local potential approximation

History

- Wegner-Houghton‘s sharp cutoff version of ERGE in 1973
- Wilson & Kogut, smooth cutoff version of ERGE in 1974
- Nicoll & Chang, ERGE for Legendre effective action in 1977
- Polchinski‘s smooth cutoff version of ERGE in 1984

Introduction

- idea: consider constant field and neglect all non-trivial momentum dependencies
- derivative expansion of non-pertubative flow equation in 0th order
- still involves infinite number of degrees of freedom
- nonlinear differential equation for (local) potential V_Λ

Local potential approximation for Polchinski´s ERGE

- LPA-Ansatz

$$V_\Lambda[\phi] = \int_x v_\Lambda(\phi(x)) = \int_x \sum_{i=0}^{\infty} v_\Lambda^{(i)} \phi^i(x)$$

- notation

$$\frac{\delta V_\Lambda}{\delta\phi(x)} = v'_\Lambda(\phi(x)), \quad \frac{\delta^2 V_\Lambda}{\delta\phi(x)\delta\phi(y)} = v''_\Lambda(\phi(x))\delta(x-y)$$

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- inserting in Polchinski´s ERGE

$$\int_x \Lambda \partial_\Lambda v_\Lambda(\phi(x)) = \frac{1}{2} \int_{xy} \Lambda \partial_\Lambda \Delta v_\Lambda^{'}(\phi(x)) v_\Lambda^{'}(\phi(y))$$

$$-\frac{1}{2} \int_x \Lambda \partial_\Lambda \Delta(0; \Lambda) v_\Lambda^{''}(\phi(x))$$

Approximations for propagators

- first propagator

$$\Lambda\partial_\Lambda \Delta(x-y;\Lambda) = \Lambda\partial_\Lambda \int_q \frac{1}{q^2} f(\frac{q^2}{\Lambda^2}) e^{iq(x-y)}$$

$$= -2\Lambda^{-2} \int_q f^{'}(\frac{q^2}{\Lambda^2}) e^{iq(x-y)}$$

$$\approx -2\Lambda^{-2} f^{'}(1)\delta(x-y) = \Lambda^{-2} B\delta(x-y); \;\; -2f^{'}(1) = B > 0$$

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- second propagator

$$\Lambda\partial_\Lambda \Delta(0;\Lambda) = -2\Lambda^{-2} \int_q f'(\frac{q^2}{\Lambda^2}) = \Lambda^{d-2} A > 0$$

- inserting approximation for propagators in the Polchinski´s ERGE

$$\int_x \Lambda\partial_\Lambda v_\Lambda(\phi(x)) = \frac{1}{2}\int_x\int_y v_\Lambda'(\phi(x))\Lambda^{-2}B\delta(x-y)v_\Lambda'(\phi(y))$$

$$-\frac{1}{2}\int_x \Lambda^{d-2}Av_\Lambda''(\phi(x))$$

- inserting approximation for propagators in the Polchinski´s ERGE

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$$-\frac{1}{2}\int_x \Lambda^{d-2}Av''_\Lambda(\phi(x))$$

$$= \frac{1}{2}\int_x \Lambda^{-2}Bv'^2_\Lambda(\phi(x)) - \frac{1}{2}\int_x \Lambda^{d-2}Av''_\Lambda(\phi(x))$$

- inserting approximation for propagators in the Polchinski´s ERGE

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$$= \frac{1}{2}\int_x \Lambda^{-2}Bv'^2_\Lambda(\phi(x)) - \frac{1}{2}\int_x \Lambda^{d-2}Av''_\Lambda(\phi(x))$$

- considering the integrand

$$\Lambda\partial_\Lambda v_\Lambda = \frac{1}{2}\{\Lambda^{-2}Bv'^2_\Lambda - \Lambda^{d-2}Av''_\Lambda\}$$

Introducing dimensionless quantities

- first substitution $\phi \to \sqrt{A}\,\phi$ and $v_\Lambda \to \frac{A}{B} v_\Lambda$

$$\Lambda\partial_\Lambda \frac{A}{B} v_\Lambda = \frac{1}{2}\{\Lambda^{-2} B \left(\frac{A}{B}\frac{1}{\sqrt{A}}\right)^2 v_\Lambda'^2 - \Lambda^{d-2} A \left(\frac{1}{\sqrt{A}}\right)^2 v_\Lambda''\}$$

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- yields

$$\Lambda \partial_\Lambda v_\Lambda = \frac{1}{2}\{\Lambda^{-2} v_\Lambda'^2 - \Lambda^{d-2} v_\Lambda''\}$$

Introducing dimensionless quantities

- dimensionless field and potential

$$\phi_x = \Lambda^{(d-2)/2}\zeta_x, \;\; v_\Lambda(\phi) = \Lambda^d u_\Lambda(\zeta(\Lambda))$$

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- dimensionless field and potential

$$\phi_x = \Lambda^{(d-2)/2}\zeta_x, \;\; v_\Lambda(\phi) = \Lambda^d u_\Lambda(\zeta(\Lambda))$$

- left-hand side

$$\Lambda\partial_\Lambda v_\Lambda(\phi) = \Lambda\partial_\Lambda \Lambda^d u_\Lambda(\zeta(\Lambda))$$

$$= \Lambda^d \Lambda \frac{\partial}{\partial\Lambda} u_\Lambda + d\Lambda^d u_\Lambda + \Lambda^{d+1} u'_\Lambda \frac{\partial\zeta(\Lambda)}{\partial\Lambda}$$

$$= \Lambda^d \Lambda \frac{\partial}{\partial\Lambda} u_\Lambda + d\Lambda^d u_\Lambda - \Lambda^d \frac{d-2}{2}\zeta u'_\Lambda$$

Renormalization flow equation of $u_\Lambda(\zeta)$

- right-hand side

$$= \frac{1}{2}\{\Lambda^{-2}\left(\frac{\partial}{\partial\phi}\Lambda^d u_\Lambda\right)^2 - \Lambda^{d-2}\frac{\partial^2}{\partial\phi^2}\Lambda^d u_\Lambda\}$$

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$$= \frac{1}{2}\{\Lambda^{-2}\left(\Lambda^d\right)^2\left(\Lambda^{-(d-2)/2}\right)^2 u_\Lambda'^2 - \Lambda^{d-2}\left(\Lambda^{-(d-2)/2}\right)^2 u_\Lambda''\}$$

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$$= \frac{1}{2}\{\Lambda^d u_\Lambda'^2 - \Lambda^d u_\Lambda''\}$$

- ERGE flow for dimensionless potential $u_\Lambda(\zeta)$

$$\Lambda\frac{\partial}{\partial\Lambda}u_\Lambda = \frac{1}{2}u_\Lambda'^2 + \frac{1}{2}u_\Lambda'' - d\cdot u_\Lambda + \frac{d-2}{2}\zeta u_\Lambda'$$

Quest for fixed points

- condition for fixed points

$$\frac{\partial}{\partial\Lambda}u^* = 0$$

- and we get an ordinary differential equation for $u^*(\zeta)$

$$\frac{1}{2}u_\Lambda^{*\prime 2} + \frac{1}{2}u_\Lambda^{*\prime\prime} - d \cdot u_\Lambda + \frac{d-2}{2}\zeta u_\Lambda^{*\prime} = 0$$

- symmetry

$$u^*(\zeta) = u^*(-\zeta)$$

- initial conditions

$$u^*(0) = u_0,\ \ u^{*\prime}(0) = 0$$

List of fixed points

- Gaussian fixed point

$$u_G^*(\zeta) = 0$$

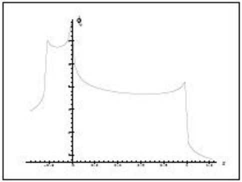
- High temperature fixed point

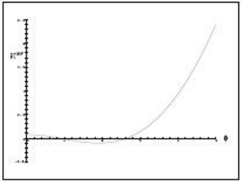
$$u_{HT}^*(\zeta) = \zeta^2 - \frac{1}{d}$$

- Wilson-Fisher fixed point

$$u_{WF}^*(\zeta) \sim \zeta^2,\ \zeta \to \infty$$

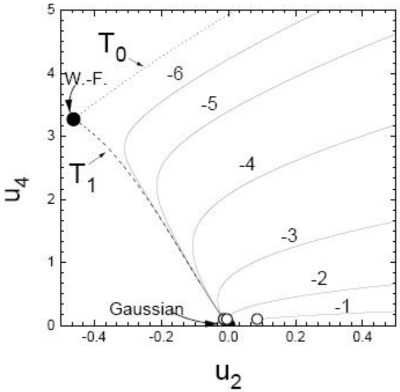
- numerical „shooting“ method





Visualization of renormalization flow trajectories

- truncate higher powers of the field and consider only $u(\zeta) = u_0 + u_2\zeta^2 + u_4\zeta^4$
- one gets an ordinary differential equations for u_2 and u_4



Linearization in vicinity of fixed point

- linearize near fixed point

$$u_\Lambda = u^* + u_1(\Lambda)$$

- inserting in our renormalization flow equation and neglecting higher derivatives

$$\Lambda \frac{\partial}{\partial \Lambda} u_1 = u^{*'} u_1^{'} - \frac{1}{2} u_1^{''} - u_1 \cdot d + \frac{d-2}{2} \zeta u_1^{'}$$

- Ansatz

$$u_1(\zeta; \Lambda) = \Lambda^\omega y(\zeta), \text{ with eigenvalue } \omega$$

- cases

$$\begin{array}{lll} \omega > 0 : & \lim_{\Lambda \to 0} u_1 = 0; & \text{irrelevant pertubation} \\ \omega < 0 : & \lim_{\Lambda \to 0} u_1 = \infty; & \text{relevant pertubation} \\ \omega = 0 : & \lim_{\Lambda \to 0} u_1 = ?; & \text{marginal pertubation} \end{array}$$

Numerical example in $d = 3$

- only one relevant eigenvalue for Wilson-Fisher fixed point in $d = 3$
- we can perform a τ-like pertubation
- calculation of critical exponents

$$\omega = \frac{1}{\nu}, \ \nu_{LPA} \approx 0.65$$

Comparison of different approximation methods

Method	ν	ω
Lattice calculation	0.6305	
ϵ-expansion at $O(\epsilon^5)$	0.6310	0.81
Six loop pertubation series	0.6300	0.79
Local potential approximation (Pol.)	0.6496	0.6557
LPA Variation method	0.6347	0.6093
Local potential approximation (Leg.)	0.6604	0.6285
Momentum expansion at $O(p^2)$	0.620	0.898

Tabelle: Exponents for three dimensional one component Z_2-invariant scalar field theory

The End

Thank you for your attention.