Real-Space Renormalisation

Bond Shifting

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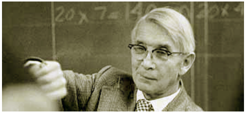
Introduction

Real-Space Renormalisation of the One-Dimensional Ising Model



Ernst Ising

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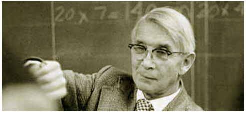

Ernst Ising

- decimation can be carried out exactly in the one-dimensional Ising model

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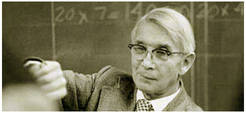

Ernst Ising

- decimation can be carried out exactly in the one-dimensional Ising model
- renormalisation of coupling constant K:
  $K \rightarrow K' = \frac{1}{2} ln(cosh2K)$

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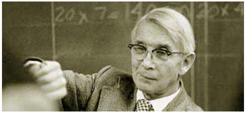

Ernst Ising

- decimation can be carried out exactly in the one-dimensional Ising model
- renormalisation of coupling constant K:
  $K \rightarrow K' = \frac{1}{2} ln(cosh 2K)$
- for arbitrary renormalisation-factor b:
  $K \rightarrow tanh(K') = [tanh(K)]^b$

Introduction

Real-Space Renormalisation of the Two-Dimensional Ising Model

- naive decimation of the two-dimensional Ising model is problematic
- starting from nearest-neighbour Ising model, longer ranged couplings are generated

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Real-Space Renormalisation of the Two-Dimensional Ising Model

- naive decimation of the two-dimensional Ising model is problematic
- starting from nearest-neighbour Ising model, longer ranged couplings are generated
- "approximations" have to be made

The Migdal-Kadanoff Method

Motivation

Leo Kadanoff

- idea: altering the Hamiltonian by an additive factor: $\mathcal{H}' = \mathcal{H} + \Delta$

The Migdal-Kadanoff Method

Motivation

Leo Kadanoff

- idea: altering the Hamiltonian by an additive factor: $\mathcal{H}' = \mathcal{H} + \Delta$
- resulting partition function Z': $Z' = Tr(e^{\mathcal{H}'}) \geq Z + \langle \Delta \rangle_{\mathcal{H}}$

The Migdal-Kadanoff Method

How to Apply the MK Method for Renormalisation

- choose Δ in a way that
  - $\mathcal{H}' = \mathcal{H} + \Delta$ is easily renormalisable
  - $\langle \Delta \rangle_{\mathcal{H}} = 0 \Rightarrow Z' \geq Z + \langle \Delta \rangle_{\mathcal{H}} = Z$
- the original free energy F is than an upper bound for the new F':
  $F' = -ln(Z') \leq -ln(Z) = F$

The Migdal-Kadanoff Method

Iteration

$$
\begin{array}{ccccccc}
\mathcal{H} & & \overline{\mathcal{H}}_1 & & \overline{\mathcal{H}}_2 & & \\
MK \downarrow & R \nearrow & MK \downarrow & R \nearrow & MK \downarrow & R \nearrow & \dots \\
\mathcal{H} + \Delta_1 & & \overline{\mathcal{H}}_1 + \Delta_2 & & \overline{\mathcal{H}}_2 + \Delta_3 & &
\end{array}
$$

with $\langle \Delta_i \rangle_{\overline{\mathcal{H}}_i} = 0$

Bond Shifting in the Two Dimensional Ising Model

How to Choose Δ?

- idea: constructing a quasi one dimensional sub-lattice

Bond Shifting in the Two Dimensional Ising Model

How to Choose Δ?

- idea: constructing a quasi one dimensional sub-lattice
- increase the value of the "good" couplings $\leftrightarrow$ decrease the value of the "bad" ones

Bond Shifting in the Two Dimensional Ising Model

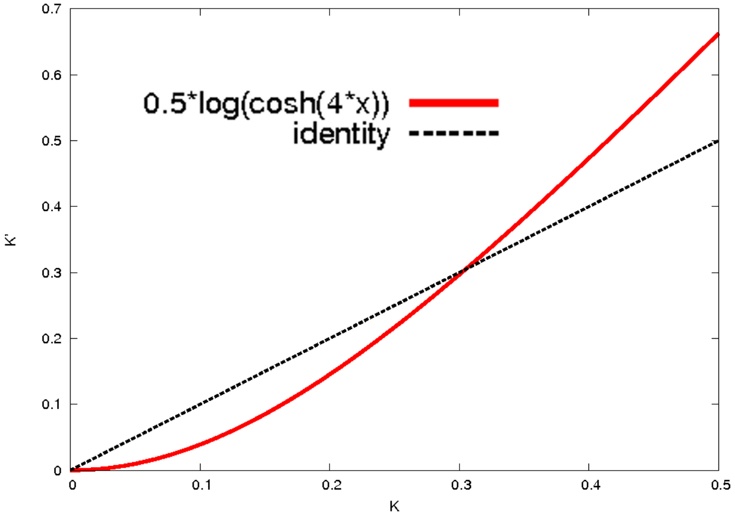
Procedure of Renormalisation

- first step: shifting the bonds
- second step: renormalising the quasi one dimensional rows and columns

Bond Shifting in the Two Dimensional Ising Model

Results

- new coupling constant after renormalisation: $K' = \frac{1}{2} ln(cosh 4K)$
- a non-trivial fixed point exists


0.5*log(cosh(4*x))
identity
K'
K
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7

Generalisation

to Arbitrary Dimensions and Rescaling-Factors

- new coupling constant after renormalisation with rescaling-factor b of a d-dimensional Ising-lattice:

$$\begin{aligned} tanh(K') &= [tanh(b^{d-1}K)]^b \\ \Rightarrow K' &= artan[tanh^b(b^{d-1}K)] \end{aligned}$$

$$tanh(K') = [tanh(b^{d-1}K)]^b$$

$$\begin{array}{c} K \\ \downarrow \\ bK \\ \downarrow \\ \vdots \\ \downarrow \\ b^{d-1}K \end{array}$$

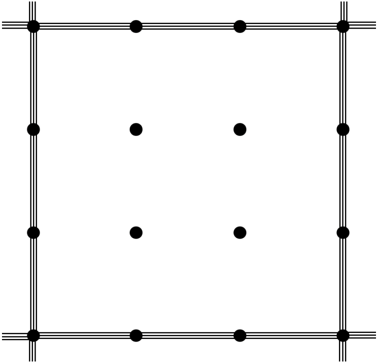


Figure: b=3, d=2

renormalisation in one dimension: $tanh(K') = [tanh(K)]^b$

Generalisation

to Arbitrary Dimensions and Rescaling-Factors

- $K' = artan[tanh^b(b^{d-1}K)]$
- b turns up explicitly $\Rightarrow$ renormalisation can be extended to arbitrary, real numbers: $b \in \mathbb{R}$
- best results expected for $b \approx 1$ $\Rightarrow$ choose $b = 1 + \delta\ell$ to get an continuous flux for K

Generalisation

to Arbitrary Dimensions and Rescaling-Factors

- expanding K' to first order in $\delta\ell$

Generalisation

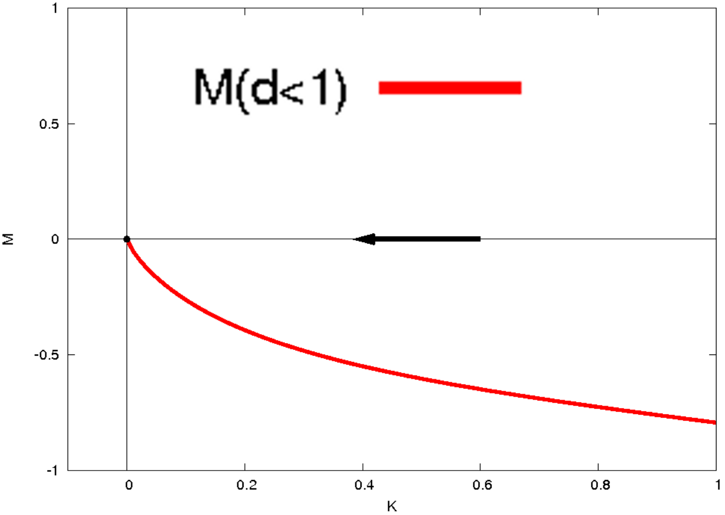
to Arbitrary Dimensions and Rescaling-Factors

- expanding K' to first order in $\delta\ell$
- the following equation arises:

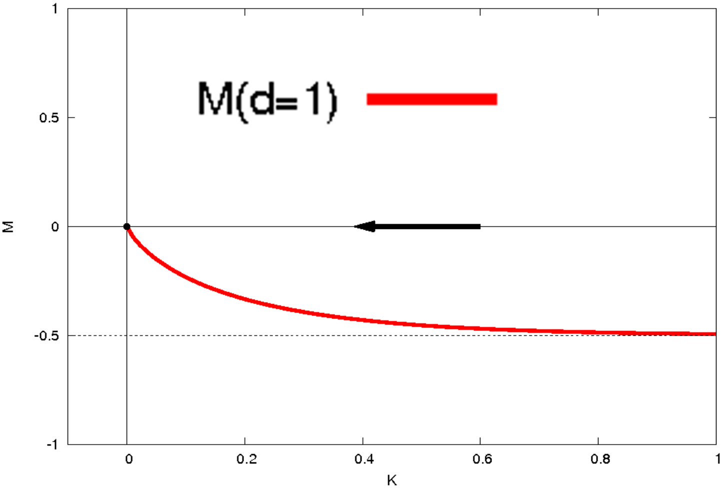
$$M(d,K) := \frac{dK}{d\ell} = \frac{K'-K}{\delta\ell} =$$
$$= (d-1)K + \frac{1}{2} sinh(2K) ln|tanh(K)|$$

- fixed point for $M(d,K)=0$

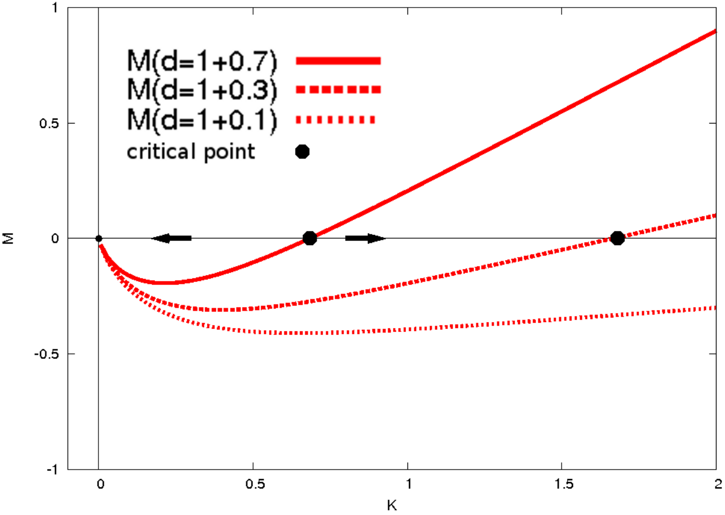
Graph for $d = 0.7$



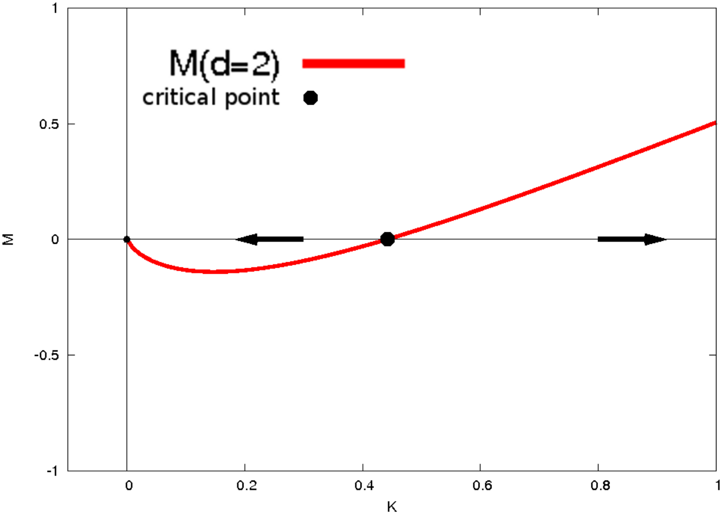
Graph for $d = 1$



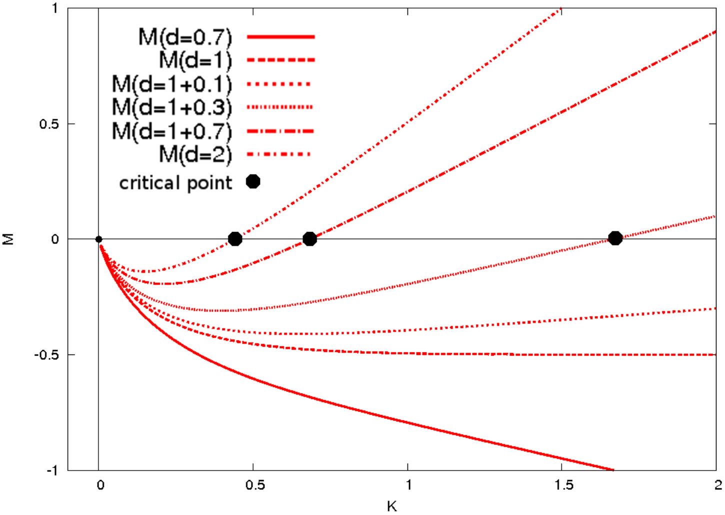
Graph for $d = 1 + 0.1$, 0.3 and 0.7



Graph for $d = 2$



Comparison


M(d=0.7)
M(d=1)
M(d=1+0.1)
M(d=1+0.3)
M(d=1+0.7)
M(d=2)
critical point
1
0.5
0
-0.5
-1
0
0.5
1
1.5
2
M
K

Generalisation

Results

- no fixed points for $d \leq 1$
- for $d = 1 + \epsilon$ with $\epsilon \ll 1$: fixed point at $K_c = \dfrac{1}{2\epsilon}$
- for $d = 2$: fixed point at $K_c = \frac{1}{2} ln(1 + \sqrt{2})$
  $\leftrightarrow$ same fixed point as in Onsager's solution

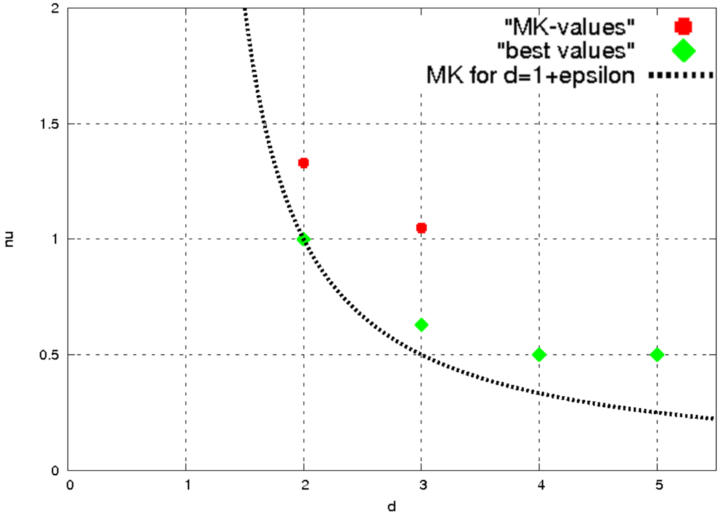
Generalisation

Results

- critical exponent ν: $\nu = (\frac{dM}{dK})^{-1}|_{K=K_c} = \nu(d)$

$$\Rightarrow \nu(d = 1 + \epsilon) = \frac{1}{\epsilon}$$
$$\nu(d = 2) = \frac{1}{0.75} = 1.33 \leftrightarrow exact : \nu = 1$$
$$\nu(d = 3) = \frac{1}{0.94} = 1.05 \leftrightarrow measured : \nu \approx 0.63$$

Critical Exponents



Other Applications

of the Migdal-Kadanoff Method

- variational methods

Other Applications

of the Migdal-Kadanoff Method

- variational methods
- calculation of surface effects

Thank you for your attention.