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T II: Elektrodynamik

(Prof. E. Frey)

Problem set 7

Tutorial 7.1 *Wine cellar*

A good wine cellar should be isolated from atmospheric temperature fluctuations and therefore is often located in a cave. Temperature gradients generate a heat flux according to Fourier's law,

$$\vec{j}(\vec{x},t) = -\kappa\,\vec{\nabla}T(\vec{x},t),$$

where κ denotes the heat conductivity of the rock.

a) Relying on energy conservation, derive the heat diffusion equation

$$\partial_t T(\vec{x},t) = D\,\nabla^2 T(\vec{x},t)$$

with the thermal diffusion coefficient $D = \kappa/\rho c$. The energy density is given by $u(\vec{x},t) = \rho c T(\vec{x},t)$ with mass density ρ and specific heat c of the rock.

b) For simplicity, restrict to the one-dimensional problem across a homogeneous layer of rock, with the atmosphere/rock interface at $x = 0$. Find the steady-state solution of the diffusion equation for a harmonic temperature oscillation of the atmosphere, i.e., boundary conditions

$$T(x=0,t) = T_0 + \Delta T\cos(\omega t).$$

Give a sketch of the solution and discuss its features.

Hint: Use the Ansatz $T(x,t) = T_0 + \mathrm{Re}[\vartheta(x)\,\mathrm{e}^{-\mathrm{i}\omega t}]$ with a time-dependent part that follows the external stimulus with a phase shift, i.e., the amplitude $\vartheta(x)$ is complex in general.

Tutorial 7.2 *Uncertainty relation*

An important consequence of the Fourier transform is the uncertainty relation which states for the widths of a general wave packet that

$$\Delta x\,\Delta k \geq \frac{1}{2}\,;$$

here the squared width Δx^2 of a wave packet $\Phi(x)$ centered at x_c and the squared width Δk^2 of its Fourier transform are defined as

$$(\Delta x)^2 = \int \mathrm{d}x\,(x - x_c)^2 |\Phi(x)|^2 \Big/ \int \mathrm{d}x\,|\Phi(x)|^2$$

and

$$(\Delta k)^2 = \int \frac{\mathrm{d}k}{2\pi}\,(k - k_c)^2 |\widehat{\Phi}(k)|^2 \Big/ \int \frac{\mathrm{d}k}{2\pi}\,|\widehat{\Phi}(k)|^2 .$$

Show that a Gaussian wave packet has minimal uncertainty, i.e. $\Delta x\,\Delta k = 1/2$, and it is the only wave form with this property.

Note: The proof of the uncertainty relation is based on the Cauchy-Schwarz inequality

$$\|f\| \cdot \|g\| \geq |(f, g)|$$

for vectors f, g from a Hilbert space with scalar product $(.,.)$ and derived norm $\|f\| = (f, f)^{1/2}$. Equality holds if and only if f and g are linearly dependent.

Tutorial 7.3 *Gaussian wave packet*

Consider a light pulse propagating in a dispersive medium. The wave numbers of the pulse are concentrated around a value k_c according to a Gaussian distribution. Then the electric field of the pulse is given by its Fourier representation

$$E(x, t) = \int \frac{\mathrm{d}k}{2\pi} \mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega(k)t} \exp\left[-\frac{(k - k_c)^2\sigma^2}{2}\right] .$$

In the vicinity of k_c, the dispersion relation of the medium may be expanded in a Taylor series,

$$\omega(k) = \omega_c + (k - k_c)\, v_g - \frac{1}{2}(k - k_c)^2 \beta,$$

with coefficients $\omega_c = \omega(k_c)$, $v_g = \omega'(k_c)$, and $\beta = -\omega''(k_c)$. Calculate the intensity $I(t) \propto |E(x, t)|^2$ and interpret the result.

Problem 7.4 *Reflection of an electromagnetic wave at a conducting mirror*

A plane polarized electromagnetic wave of frequency ω in free space is incident normally on the flat surface of a nonpermeable medium of conductivity $\sigma \geq 0$ and a constant background susceptibility $\chi_m > 0$.

a) First consider the medium. Show that for harmonically time-varying fields, $\vec{E}(t) = \mathrm{Re}\,\vec{E}_\omega \mathrm{e}^{-\mathrm{i}\omega t}$ etc., the polarization $\vec{P} = \chi_m \vec{E}$ and the current density $\vec{j} = \sigma \vec{E}$ in Ampère's equation can be eliminated in favor of a complex dielectric permittivity,

$$\vec{\nabla} \times \vec{H}_\omega = \frac{-\mathrm{i}\omega}{c} \varepsilon(\omega) \vec{E}_\omega \qquad \text{with} \qquad \varepsilon(\omega) = \varepsilon_m + \frac{4\pi\mathrm{i}\sigma}{\omega}\,, \quad \varepsilon_m = 1 + 4\pi\chi_m\,.$$

b) The incident wave is partially reflected and absorbed by the medium. Choosing the z-axis perpendicularly to the flat surface, a suitable Ansatz for the electric field is given by

$$E_\omega(z) = E_\mathrm{i} \begin{cases} \mathrm{e}^{\mathrm{i}kz} + r\,\mathrm{e}^{-\mathrm{i}kz} & \text{for } z < 0 \ \text{(empty space)}, \\ t\,\mathrm{e}^{\mathrm{i}qz}\,\mathrm{e}^{-\kappa z} & \text{for } z > 0 \ \text{(medium)}. \end{cases}$$

Determine the wave numbers q, k as well as the decay rate κ by solving the corresponding wave equations.

c) Formulate appropriate matching conditions for the electromagnetic fields at the interface ($z = 0$) and determine the *reflection amplitude* r and the *transmission amplitude* t. Calculate the reflection coefficient $R = |r|^2$ and the transmission coefficient $T = 1 - R$.

d) Evaluate the time averaged Poynting vector

$$\langle S \rangle = \frac{1}{2} \operatorname{Re} \left(\frac{c}{4\pi} \vec{E}_\omega \times \vec{H}^*_\omega \right) .$$

in both half spaces and interpret your result.

e) Specialize your results for the case of good conductor $\sigma \gg \omega \varepsilon_m$, i.e., ε_m can be neglected, and discuss the decay rate κ and the reflection coefficient R. Argue that the displacement current is small compared the current density $\vec{j}$ in this case and show that the electromagnetic fields in the medium fulfill diffusion equations rather than wave equations.

f) In the opposite limit of a poor conductor, $\sigma \ll \omega \varepsilon_m$, the decay rate becomes large to the wavelength of the incident wave. Determine the absorption length κ^{-1} and the reflection coefficient R in this case.

Hints: The parts a–d can be solved independently of each other. The calculation of b) may be done for general complex $\varepsilon(\omega)$; the results of c) should be expressed in terms of q, k and κ, the Poynting vector in d) in terms of R and κ; part e) is similar to the problem of the wine cellar and was discussed in parts in the lecture.

Problem 7.5 *Paraxial beams*

Consider a monochromatic beam of angular frequency $\omega = kc$ propagating essentially along the positive z-direction.

a) Argue that the components of the electric field allow for a representation as

$$E(\vec{x}_\perp, z; t) = \mathrm{e}^{-\mathrm{i}\omega t} \int \frac{\mathrm{d}^2 \vec{k}_\perp}{(2\pi)^2} a(\vec{k}_\perp) \exp(\mathrm{i}\vec{k}_\perp \vec{x}_\perp + \mathrm{i}k_\parallel z) ,$$

where $k_\parallel = (k^2 - \vec{k}_\perp^2)^{1/2}$ is to be eliminated in favor of $\vec{k}_\perp$.

b) The complex amplitudes $a(\vec{k}_\perp)$ are assumed to contribute only for $|\vec{k}_\perp| \ll k$. Expanding the square root, $k_\parallel \simeq k - \vec{k}_\perp^2/2k$ to leading order in $k_\perp/k$, show that the field assumes the following form

$$E(\vec{x}_\perp, z; t) = \mathrm{e}^{\mathrm{i}kz - \mathrm{i}\omega t} \mathcal{E}(\vec{x}_\perp, z) ,$$

where the *envelope function* $\mathcal{E}$ is slowly varying along z on the scale of a wavelength, $\partial_z \mathcal{E} \ll k\mathcal{E}$. Relate the envelope to the amplitudes $a(\vec{k}_\perp)$. Show that the envelope satisfies a field equation of the Schrödinger type,

$$\mathrm{i}\partial_z \mathcal{E}(\vec{x}_\perp, z) = -\frac{1}{2k} \nabla^2_\perp \mathcal{E}(\vec{x}_\perp, z).$$

In particular, the field equation is first order in the z-direction.

c) Evaluate the electric field $E(\vec{x}_\perp, z; t)$ for a Gaussian amplitude function

$$a(\vec{k}_\perp) \propto \exp \left(-\frac{1}{4} w_0^2 \vec{k}_\perp^2 \right) , \qquad w_0 > 0 ,$$

and show that the intensity $I \propto |E|^2$ exhibits a Gaussian profile in the perpendicular direction $\vec{x}_\perp$ and a width that depends on z. Where is the width minimal?

Due date: Tuesday, 6/12/2007, at 9 a.m.