F-Theory Models with Torsion Mordell-Weil Group and Massless Matter via Chow Groups

joint work with D. Morrison, O. Till and T. Weigand: arXiv:1405.3656

&

M. Bies, C. Pehle and T. Weigand: arXiv:1402.5144

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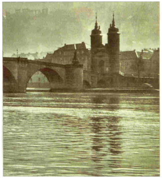


Motivation

- PART I
  - Torsion part of Mordell-Weil group gives info beyond the Lie-algebra of non-abelian gauge symmetries;
- PART II
  - Chiral matter spectrum requires G_4-flux;
  - However, need to know full massless matter spectrum;
    - Not obvious that all (unwanted) vector like pairs obtain mass and lift;
    - Even if, mass might be small (if SUSY breaking scale is low) such that they will contribute to all kind of threshold corrections;
  - To obtain full massless matter spectrum, need gauge data beyond four-form flux;
  - Chow groups will give us handle on them;

Part I

Global Structure of Gauge Groups

Mordell-Weil group

- Points on elliptic curve $E = \mathbb{C}/\Lambda$, with $\Lambda = \langle 1, \tau \rangle$, are additive as complex numbers;
- Points (x, y) with rational coordinates on E,

$$y^2 = x^3 + f\,x + g \quad \text{with} \quad j(\tau) \sim \frac{f^3}{4\,f^3 + 27\,g^2}\,,$$

  over field K form abelian group under addition, $E(K)$;
- Mordell-Weil theorem for elliptic curves states that $E(K)$ is finitely generated;

$$\Rightarrow \; E(K) = \underbrace{\mathbb{Z}^r}_{\text{free part}} \oplus \underbrace{\mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_i}}_{\text{torsion subgroup}}$$

- Can be extended to elliptic fibrations $Y \to \mathcal{B}$; Field K rational functions; Hence, $(x, y) \in$ rational functions over $\mathcal{B}$; Mordell-Weil group becomes group of sections; Group law fibrewise over each point of $\mathcal{B}$;

Torsion subgroup

- Possible torsion subgroups for elliptic K3 surface are:

$$\mathbb{Z}_k \,(k = 2, \ldots, 8), \quad \mathbb{Z}_2 \oplus \mathbb{Z}_{2k} \,(k = 1, 2, 3), \quad \mathbb{Z}_3 \oplus \mathbb{Z}_3, \quad \mathbb{Z}_4 \oplus \mathbb{Z}_4$$

- No classification for higher dimensional Calabi-Yau varieties;
- Among 16 reflexive polygons, 3 admit torsion points/sections as restriction of ambient toric divisors to hypersurface;
  - For these Mordell-Weil groups are: $\mathbb{Z}_2$, $\mathbb{Z} \oplus \mathbb{Z}_2$ and $\mathbb{Z}_3$;
- Torsion of elliptic fibre does not induce torsion in homology of fibration;
- Torsion sections lead (generically) to singularities in co-dim. 1 (order of sing. matches order of tor. section);
  - Can take $\mathbb{Z}_k$-section mode resolution divisors;

$\Rightarrow$ Gives torsion relation;

Shioda map

- Shioda map is homomorphism from group of sections $E(K)$ to group of divisors $NS(Y)$;
- Shioda map of $\mathbb{Z}_k$-torsion section $\mathcal{T}$ gives trivial divisor class on Y;

$$\mathcal{T} \mapsto T - Z - \bar{K} + (\delta) + \frac{1}{k} \sum_{i}^{k} a_i \, F_i \in \mathrm{Pic}_0(Y)$$

with $a_i \in \mathbb{Z}$ and F_i resolution divisors; Not related to $U(1)$;

- Can be used to define:

$$\Xi_k = T - Z - \bar{K} = -\frac{1}{k} \sum_{i}^{k} a_i \, F_i$$

Note fractional coefficients on right-hand side;

Implications for gauge theories

- Intersection pairing of Ξ with split curves over matter loci is integer;
  - $-\frac{1}{k}\sum_i^k a_i F_i$ adds generator for coweight lattice $\Lambda^\vee$ (finer);
- Restricted matter spectrum; Only allowed representations integer charged under Ξ_k; Hence, coarser weight lattice Λ;
- Root and coroot lattices Q and $Q^\vee$ sublattices of weight and coweight lattices Λ and $\Lambda^\vee$, respectively;
- Center Z_G and fundamental group of gauge group G:

$$Z_G = \Lambda/Q \qquad \pi_1(G) = \Lambda^\vee/Q^\vee;$$

- Torsion section refines coweight lattice;
  - Enhances π_1 of G, or equivalently reduces center of G;
- E.g.: A_2 sing. for fibration w/o torsion sec. gives $SU(3)$; If there is $\mathbb{Z}_3$-section, gauge group becomes $SU(3)/\mathbb{Z}_3$;
  - Constrains matter spectrum to representations invariant under action of center $\mathbb{Z}_3$;

Part II

Gauge Data via Chow Groups

Fluxes I

- F-theory has higher form gauge potential

$$C_3 \simeq C_3 + d\Lambda_2;$$

$\Rightarrow$ Four-form flux of F-theory

$$G_4 = dC_3$$

- General condition on flux for 4d Poincaré invariance (from dual M-theory picture): 'one leg in the fibre, three legs in the base'

$$\begin{aligned} &\int_{\tilde{Y}_4} G_4 \wedge \omega_b = 0 \\ &\int_{\tilde{Y}_4} G_4 \wedge [Z] \wedge \omega_a = 0\,; \end{aligned} \qquad \forall \omega_a,\, \omega_b \;\text{ with legs in base}$$

- Supersymmetry: $G_4 \in H^{2,2}(Y_4)$;
- Has to be quantised: $G_4 + \frac{c_2}{2} \in H^4(Y_4, \mathbb{Z})$;

Chirality

- Type IIB: chirality along curve of intersecting branes given by

$$q \int_{\mathcal{C}_{R_q}} F_X \, ;$$

$\mathcal{C}_{R_q}$ denotes curve with states in representation R_q and q $U(1)_X$-charge;

- F-theory: replaced by integral of four-form flux over matter surfaces C_{R_q} in $\hat{Y}_4$,

$$\int_{C_{R_q}} G_4 \, ;$$

- Matter surfaces, C_{R_q}, consist of linear combinations of blow-up $\mathbb{P}^1$'s fibred over enhancement curve $\mathcal{C}_{R_q}$;
- Recall: linear combination such that in dual M-theory picture, M2-brane wrapping this combination is one state of R_q ;

Beyond the Chiral Index I

- To calculate number chiral states in rep. R and $\bar{R}$—not just index—need information about gauge data C_3;
- Encoded in Deligne cohomology $H^4_D(\hat{Y}_4, \mathbb{Z}(2))$ or Cheeger-Simons twisted differential characters;
- Can get intuition from IIB:
  1. Discrete (bundle) data: 2-form field strength $\frac{1}{2\pi}F$
  2. Gauge field adds continues/discrete info: Wilson lines $\oint A$

  1. + 2. form Picard group (Pic), i.e. class of holomorphic line bundles modulo gauge transformations;
- Splitting of Pic encoded via:

$$0 \to \mathcal{J}^1(X) \to \mathrm{Pic}(X) \xrightarrow{c_1} H^{1,1}_{\mathbb{Z}}(X) \to 0$$

  $c_1(L) = \frac{1}{2\pi}F$ linear map onto $H^{1,1}(X, \mathbb{Z})$,
  $\mathcal{J}^1(X) = H^{0,1}(X, \mathbb{C})/H^1(X, \mathbb{Z})$ Jacobian of X (space of flat connections).

Beyond the Chiral Index II

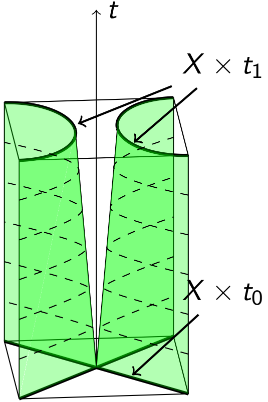
- For A_3 & G_4 in F-theory exists similar decomposition:

$$0 \to \underbrace{\mathcal{J}^2(\hat{Y}_4)}_{\substack{\text{2^{nd} intermediate}\\ \text{Jakobian}}} \to \underbrace{H^4_D(\hat{Y}_4, \mathbb{Z}(2))}_{\substack{\text{4^{th} Deligne}\\ \text{cohomology class}}} \xrightarrow{c_2} H^{2,2}_{\mathbb{Z}}(\hat{Y}_4) \to 0$$

  - $H^4_D(\hat{Y}_4, \mathbb{Z}(2)) \leftrightarrow$ equivalence classes of gauge data
  - $H^{2,2}_{\mathbb{Z}}(\hat{Y}_4) \leftrightarrow$ field strength G_4;
  - $\mathcal{J}^2(\hat{Y}_4) \simeq H^3(\hat{Y}_4, \mathbb{C})/(H^{3,0}(\hat{Y}_4) + H^{2,1}(\hat{Y}_4) + H^3(\hat{Y}_4, \mathbb{Z})) \leftrightarrow$ data beyond flux (flat connections);
- Usually difficult to work directly with Deligne cohomology;
- But can work indirectly by using Chow groups;

Chow Groups I

- 'Bundle data' via rational equivalence class of 4-cycles
- Rational equivalence: $C_1 \cong C_2 \in Z_n(X)$ if $C_1 - C_2$ is zero/pole of meromorphic function defined on $(n+1)$-dim. irreducible subvariety of X; Equivalently: two algebraic cycles C_1, $C_2 \in Z_i(X)$ rationally equivalent if $\exists$ rationally parametrised family of cycles interpolating between them;



Chow Groups II

- Chow group $\mathrm{CH}^k(X)$ = group of rational equivalence classes of (complex) codim. k-cycles; $\mathrm{CH}_k(X) = \ldots$ dim. k-cycles;
- Special case: $\mathrm{CH}^1(X) = \mathrm{Pic}(X)$
  - $\Rightarrow$ Rational equivalence is finer than homological equivalence.
- $\exists$ map (homomorphism) from second Chow group into Deligne cohom.: $\hat{\gamma} : \mathrm{CH}_2(X) \to H^4_D(\hat{Y}_4, \mathbb{Z}(2))$ (refined cycle map);

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{CH}^p_{\mathrm{hom}}(X) & \longrightarrow & \mathrm{CH}^p(X) & \xrightarrow{\gamma_p} & H^{p,p}_{alg}(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle AJ} & & \downarrow{\scriptstyle \hat{\gamma}_p} & & \downarrow & & \\
0 & \longrightarrow & J^p(X) & \longrightarrow & H^{2p}_D(X, \mathbb{Z}(p)) & \xrightarrow{\hat{c}_p} & H^{p,p}_{\mathbb{Z}}(X) & \longrightarrow & 0
\end{array}
$$

Chow Groups III

- Can use Chow groups to describe gauge data;
- Has clear advantage if we know Chow groups of $\hat{Y}_4$;
- In case of hypersurface or CICYs in toric varieties, we know at least parts of it;
  - Part which inherited from ambient space

$$\mathrm{CH}^*(X_5) \simeq H^*(X_5, \mathbb{Z});$$

- $\hat{\gamma}_2$ in general not surjective, but every complex 2-cycle class gives gauge data up to gauge equivalence;
- Strategy:
  1. Fix cycle (class) $\alpha_G \in \mathrm{CH}_2(X)$ with $G_4 = [\alpha_G] = \hat{c}_2 \circ \hat{\gamma}_2(\alpha_G)$;
  2. Manipulations modulo rational equivalence preserve C_3 modulo gauge equivalence;

Matter I

- With gauge data given by $\alpha_G \in CH_2(\hat{Y}_4)$, we have natural pairing with matter surfaces;
  - matter surface $C_R \in Z_2(\hat{Y}_4)$ with projection $\pi_R : |C_R| \to \mathcal{C}_R$;
  - $C_R \cdot_r \alpha_G \in \mathrm{CH}^2(|C_R|) =$ Chow class of points on $|C_R|$ where $\cdot$ denotes intersection, i.e. map from $\mathrm{CH}^p \times \mathrm{CH}^q$ to CH^{p+q};
- Projection to base B_3 gives points on matter curve $\mathcal{C}_R$:

$$\pi_{R*}(C_R \cdot_r \alpha_G) \in \mathrm{CH}_0(\mathcal{C}_R) \cong \mathrm{Pic}(\mathcal{C}_R)$$

- This collection of points $A_{R,G} \in Z_0(\mathcal{C}_R)$ can be used to define line bundle $L_{G,R} = \mathcal{O}_{\mathcal{C}_R}(A_{R,G})$ on $\mathcal{C}_R$;

Matter II

- **Proposal**:
massless $\mathcal{N} = 1$ chiral multiplets counted by
$$H^i(C_R, L_{G,R} \otimes \sqrt{K_{\mathcal{C}_R}}), \qquad i = 0, 1$$
with $\sqrt{K_{\mathcal{C}_R}}$ the spin bundle of the matter curve $\mathcal{C}_R$ (induced by embedding);
- Can checked proposal for fluxes/gauge data coming from e.g. $U(1)$-symmetries;

Applied to U(1)-model I

- In F-theory $U(1)$'s in one-to-one relation with rank of Mordell-Weil group (without torsion part);
- Rank of MW corresponds to number of independent sections (minus one);
  - $\Rightarrow$ Call additional section s_A (consider only one $U(1)$);
- Take $\mathrm{w}_A \in \mathrm{CH}^1(\hat{Y}_4)$ such that

$$\gamma(\mathrm{w}_A) \stackrel{!}{=} \varphi(\gamma(s_A))$$

where φ denotes Shioda map;

$\Rightarrow$ $\gamma(\mathrm{w}_A)$ together with $f \in \mathrm{CH}^1(B_3)$ gives four-form flux (class)

$$G_4^A = \pi^* \gamma(f) \cup \gamma(\mathrm{w}_A) \in H^{2,2}(\hat{Y}_4)$$

which satisfies 'one leg ...' condition and leaves all non-abelian sym. untouched.

Applied to U(1)-model II

- Have now more then flux, because

$$\alpha_{F,A} = \mathrm{w}_A \cdot_\pi f \ \in \ \mathrm{CH}^2(\hat{Y}_4)$$

specifies "$U(1)$ bundle" with associated flux G_4^A;

- Via projection can extract actual line bundle on C_R, matter curve on base

$$\pi|_{C_R *}(\alpha_R \cdot_{\iota_R} \alpha_{F,A}) = \pi|_{C_R *}(\alpha_R \cdot_{\iota_R} (\mathrm{w}_A \cdot_\pi f)) = \pi|_{C_R *}(\alpha_R \cdot_{\iota_R} \mathrm{w}_A) \cdot_{\iota_R|_{B_3}} f;$$

- In many cases, CY four-fold embedded in toric ambient space and w_A is pullback ($\mathrm{w}_A = j^* \tilde{\mathrm{w}}_A$) then

$$\pi|_{C_R *}(\alpha_R \cdot_{\iota_R} \mathrm{w}_A) = \pi|_{C_R *}(\alpha_R \cdot_{j\iota_R} \tilde{\mathrm{w}}_A);$$

Applied to U(1)-model III

- In such cases, can use intersections of toric ambient variety to calculate $\alpha_R \cdot_{j\iota_R} \tilde{\mathrm{w}}_A$ and find:

$$\pi|_{C_R *}(\alpha_R \cdot_{\iota_R} \mathrm{w}_A) = \pi|_{C_R *}(\alpha_R \cdot_{j\iota_R} \tilde{\mathrm{w}}_A) = q_A(R)\,[C_R], \in \mathrm{CH}_1(C_R)$$

  with $q_A(R)$ number of intersections over C_R (is interpreted as $U(1)$-charge);
- Finally from $[C_R] \cdot_{\iota_R|_{B_3}} f$ obtain collection of points $A_{R,A} \in Z_0(C_R)$ on C_R;
  - $\Rightarrow$ Defines line bundle $\mathcal{O}_{C_R}(A_{R,A})$; Massless matter states correspond to

$$H^i(C_R, L_{R,A} \otimes \sqrt{K_{C_R}}), \qquad L_{R,A} = [\mathcal{O}_{C_R}(A_{R,A})]^{\otimes q_A(R)};$$

- In geometries with well-defined Sen limit, agrees with massless matter states in Type IIB limit;

Explicit example I

- Can work out explicit examples;
- As starting point took hypersurface ($D_{B_3} = H_1 + 2\,H_2 + H_3$) in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$;
- GUT surface placed at $D_{\mathrm{GUT}} = (H_1 + H_3)|_{B_3}$ and $f = \tilde{f}|_{B_3} = (n\,H_1 + m\,H_2 + o\,H_3)|_{B_3}$
- By means of *cohomCalg* and self-write Mathematica code, could work out spectral sequences to obtain cohomologies of line bundles on curves;

Explicit example II

- Consider $U(1)$-restricted case;
- With flux choice $\gamma(\tilde{f}) = \frac{1}{2}(-7, 0, 9)$, in agreement with quantisation cond.;

curve	$h^0(C, \mathcal{L}\|_C)$	representation	$h^1(C, \mathcal{L}\|_C)$	representation
$C_{\mathbf{10}}$	4	$\mathbf{10}_{-\mathbf{1}}$	1	$\overline{\mathbf{10}}_{+\mathbf{1}}$
$C_{\overline{\mathbf{5}}_{\mathbf{m}}}$	6	$\overline{\mathbf{5}}_{\mathbf{3}}$	3	$\mathbf{5}_{-\mathbf{3}}$
$C_{\mathbf{5}_{\mathbf{H}}}$	9	$\mathbf{5}_{\mathbf{2}}$	9	$\overline{\mathbf{5}}_{-\mathbf{2}}$
$C_{\mathbf{1}}$	585	$\mathbf{1}_{\mathbf{5}}$	0	$\overline{\mathbf{1}}_{-\mathbf{5}}$

Summary & Outlook

- Showed implications of torsion sections on gauge group;
- To go beyond $\mathbb{Z}_2$ and $\mathbb{Z}_3$ need complete intersections or non-toric methods;
  - MSSM with gauge group $SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6$?
- General procedure to compute all light matter states; Not only chiral index;
- Applied it already to the $U(1)$-restricted case;
- See whether there is projection formulae (overall factor) also in other cases;
  - Will not be the general case; Immediate counter example universal spectral cover flux which appears in $SU(5)$-models even without abelian symmetry;
- Apply our methods to cases where intersections have to be done on CY itself;

Thank you for your attention!