Non-associative Deformations of Geometry in Double Field Theory

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based on JHEP 04(2014)141 or arxiv:1312.0719 by R. Blumenhagen, MF, F. Haßler, D. Lüst, R. Sun

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Motivation

The Jacobi identity of three QM operators reads

$$\mathrm{Jac}_{[,]}(F,G,H) = [F,[G,H]] + [H,[F,G]] + [G,[H,F]]$$

$$= [F(GH)-(FG)H] - [F(HG)-(FH)G] + \ldots .$$

$\Rightarrow$ Algebraically zero for associative operators!

The Jacobi identity is directly connected to associativity

Canonical quantization:

$$\{\ ,\ \} \quad \longrightarrow \quad \frac{1}{i\hbar}[\ ,\ ]$$

Look at the Poisson bracket in classical mechanics!

$$\{f, g\} := \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$$

The Poisson bracket defined in this way obeys the Jacobi identity by construction

$$\mathrm{Jac}_{\{,\}}(f, g, h) := \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

$\Rightarrow$ QM operators associate/obey the Jacobi identity!

But there are hints for non-associative target spaces in ST!

[Blumenhagen, Lüst, Plauschinn, . . .]

This talk: Resolve this contradiction!

Outline

- Conditions for non-associativity in the Hamiltonian formalism
- Open string
  1. Review of the known deformation
  2. open string deformation in DFT
- Closed string
  1. Review of the known deformation
  2. Closed string deformation in DFT
  3. Possible origin of this deformation

Mathematics of the Hamiltonian Formalism

The Hamiltonian formalism describes dynamics on an even dimensional symplectic manifold equipped with a closed degenerate two form

$$\omega = \omega_{ij}\, dx^i \wedge dx^j \qquad , \det \omega_{ij} \neq 0 \quad \text{and} \quad \mathrm{d}\omega = 0.$$

Define the Poisson bracket as

$$\{f, g\} = \omega^{ij}\, \partial_i f\, \partial_j g \qquad \text{with } \omega^{ij}\omega_{jk} = \delta^i{}_j \text{ and } i, j, k \ \in 1, \ldots, 2D$$

and introduce an evolution parameter t "time" and a real energy function H "Hamiltonian". Postulate the time evolution by

$$\frac{df}{dt} = \{f, H\}.$$

Jacobi identity of this bracket is

$$\mathrm{Jac}_{\{,\}}(f,g,h) = \omega^{[\underline{k}l}\partial_l\omega^{\underline{ij}]}\,\partial_i f\,\partial_j g\,\partial_k h.$$

Zero by assumption $\mathrm{d}\omega = 0$ and

$$\omega^{[\underline{k}l}\partial_l\omega^{\underline{ij}]} = \omega^{ii'}\omega^{jj'}\omega^{kk'}\,(\mathrm{d}\omega)_{i'j'k'}.$$

Or clear from Darboux's theorem: It is possible to choose local coordinates (q,p) such that

$$\omega = dq^i \wedge dp_i \qquad \text{or} \qquad \omega = \begin{pmatrix} 0 & 1_D \\ -1_D & 0 \end{pmatrix}.$$

Why $\mathrm{d}\omega = 0$?

Hamiltonian mechanics is usually defined on the cotangent bundle T^*M which defines a $2D$-dim manifold

$$(\underbrace{q_1, \dots, q_n}_{\in M}\ ,\ \underbrace{p_1, \dots p_n}_{\in T^*_q M})$$

The "tautological one-form" connects the coordinates and their conjugate as

$$\theta = p_i dq^i .$$

Use this to define the symplectic structure

$$\omega = \mathrm{d}\theta = dq^i \wedge dp_i .$$

The symplectic structure of T^*M is exact $\Rightarrow \mathrm{d}\omega = \mathrm{d}^2\theta = 0$

Conclusion

A non-vanishing Jacobi identity is possible if

$$\mathrm{d}\omega \neq 0.$$

Beyond the scope of a Hamiltonian defined on T^*M?

CFT

In general: CFT's are usual QFT's, therefore the CFT operator algebra must be associative ($\Leftrightarrow$ crossing symmetry).
But note: The coordinates are not well defined CFT operators (not even quasi primaries, $h = 0$)!

- The **closed string** worldsheet has an $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ symmetry

Commutativity expected for vertex operators inserted at the bulk.

- The **open string** worldsheet has an $\mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2$ symmetry

Vertex operators inserted at the boundary (D-brane) must be cyclic, but may be non-commutative, for instance

$12 = 21$ and $123 = 231$ $\qquad$ $123 \neq 132$ or $1234 \neq 1243$.

Open Strings

in non-vanishing $\mathcal{F} = B + 2\pi\alpha' \, \mathrm{d}A$ background.

[Chu, Ho, Seiberg, Witten, Cornalba, Schiappa, Schomerus, Herbst, Kling, Kreuzer, ... $\sim$ '98-'01]

For constant $\mathcal{F}$ one gets on the D-brane $\partial\mathbb{H} = \mathbb{R}$

$$\langle X^\mu(\tau) X^\nu(\tau') \rangle_{\mathcal{F}} = -\alpha' \left[G^{\mu\nu} \log |\tau - \tau'|^2 + i\pi \, \Theta^{\mu\nu} \, \epsilon(\tau - \tau') \right]$$

where the open string metric G and the antisymmetric θ are

$$\begin{aligned} G^{\mu\nu} &= \quad \left[(g - \mathcal{F})^{-1} \, g \, (g + \mathcal{F})^{-1} \right]^{\mu\nu} , \\ \theta^{\mu\nu} &= - \left[(g - \mathcal{F})^{-1} \, \mathcal{F} \, (g + \mathcal{F})^{-1} \right]^{\mu\nu} . \end{aligned}$$

Open String Product

$$\Big\langle :e^{ipX(\tau)}::e^{ip'X(\tau')}: \Big\rangle_{\mathcal{F}}$$
$$= e^{-i\pi\alpha'\,\theta^{\mu\nu}\,p_\mu p'_\nu\,\epsilon(\tau-\tau')} \times \langle :e^{ipX(\tau)}::e^{ip'X(\tau')}:\rangle_0.$$
$$= \exp\left[i\pi\alpha'\,\theta^{\mu\nu}\,\frac{\partial}{\partial X_1^\mu}\,\frac{\partial}{\partial X_2^\nu}\right] \times \langle :e^{ipX(\tau)}::e^{ip'X(\tau')}:\rangle_0.$$

The background field can be captured by changing the multiplication law to a Moyal-Weyl star-product

$$f\star g := \exp\left[i\pi\alpha'\,\theta^{\mu\nu}\,\frac{\partial}{\partial x_1^\mu}\,\frac{\partial}{\partial x_2^\nu}\right] f(x_1)\,g(x_2) + \mathcal{O}(\partial\theta).$$

then for instance $\langle V_1 V_2\rangle_{\mathcal{F}} = \langle V_1\star V_2\rangle_{\mathcal{F}=0}$

Higher Orders in $\partial\theta$

$$
\begin{aligned}
f \star g = f \cdot g \, + \, \frac{i}{2}\theta^{ij}\, \partial_i f \, \partial_j g - \frac{1}{8}\theta^{ij}\theta^{kl}\, \partial_i\partial_k f \, \partial_j\partial_l g \\
- \frac{1}{12}\big(\theta^{im}\partial_m\theta^{jk}\big)\big(\partial_i\partial_j f \, \partial_k g - \partial_i\partial_j g \, \partial_k f\big) + \mathcal{O}\big((\partial\theta)^2, \partial^2\theta, \theta^2\big)
\end{aligned}
$$

[Cornalba, Schiappa and Herbst, Kling, Kreuzer '01]

Same as the Kontsevich deformation quantization formula but θ might be a quasi-Poisson $\mathrm{d}\theta \neq 0$ tensor here $\Rightarrow$ Non-associative!

$$
(f \star g) \star h - f \star (g \star h) \; \propto \; \theta^{[\underline{\mu\rho}}\partial_\rho\theta^{\underline{\nu\sigma}]}\, \partial_\mu f \, \partial_\nu g \, \partial_\sigma h \neq 0!
$$

Remember: $Jac \propto \theta^{[\underline{\mu\rho}}\partial_\rho\theta^{\underline{\nu\sigma}]}$ as well.

Resolution

Integrate the deformation! Captures

- low-energy effective actions and
- correlators [Schomerus, Seiberg, Witten '98: Integration to implement momentum conservation and more general Herbst, Kling, Kreuzer '02]

$$\int d^n x \sqrt{g - \mathcal{F}} \, (f \star g - f \cdot g) \overset{PI}{=} - \int d^n x \, f \underbrace{\partial_\mu \left(\sqrt{g - \mathcal{F}} \theta^{\mu\nu} \right)}_{\text{DBI-eom } = 0} \partial_\nu g$$

- $\int f \star g \overset{eom}{=} \int f \cdot g$
- But $\int f \star g \star h \neq \int f \cdot g \cdot h$!
- Also associative

$$\int d^n x \sqrt{g - \mathcal{F}} (f \star g) \star h - f \star (g \star h) \overset{eom}{=} 0.$$

Summary

The open string product matches the expected properties

- $12 = 21$
- $123 \neq 132$, $1234 \neq 1243$, ...
  $\Rightarrow$ additional terms in low-energy effective action
- cyclic [also in higher orders Herbst, Kling, Kreuzer '03]
- vanishing Jacobi identity

up to boundary terms.

Open String Product in DFT

DFT [Hull, Zwiebach, Hohm, ...] has only closed string degrees of freedom. Therefore

- vertex operators are expected to commute,
- the gauge invariant object is H

We use the flux formulation of DFT [Aldazabal, Geissbuhler, Marques, Nunez, Penas]. There the product reads

$$f \triangle g \triangle h := f\, g\, h + H^{abc}\ \partial_a f\, \partial_b g\, \partial_c h + R_{abc}\ \tilde{\partial}^a f\, \tilde{\partial}^b g\, \tilde{\partial}^c h + \ldots$$

$$\stackrel{\text{DFT}}{=} f\, g\, h + \check{\mathcal{F}}_{ABC}\ \partial^A f\, \partial^B g\, \partial^C h.$$

Write this deformation under an integral

$$\int dXe^{-2d}\ \check{\mathcal{F}}_{ABC}\ \partial^A f\,\partial^B g\,\partial^C h \stackrel{\text{PI}}{=} -\int dXe^{-2d} \underbrace{\mathcal{G}_{AB}}_{eom:\,\mathcal{G}_{AB}=0!} f\,\partial^A g\,\partial^B h.$$

The same mechanism is present here! Holds for product of n-functions as expected in a closed string setting!

Matter (e.g. RR fields) in form of an energy momentum tensor $\mathcal{T}^{AB}$ changes the eom to

$$\mathcal{G}^{AB} = \mathcal{T}^{AB},$$

which breaks the associativity.

Matter Corrections

Associativity can be restored by adding a $\mathcal{T}^{AB}$ term:

$$f \triangle g \triangle h = f\, g\, h + \mathcal{T}^{AB} \left(f\, \partial_A g\, \partial_B h + cycl. \right) + \check{\mathcal{F}}^{ABC} \partial_A f \partial_B g \partial_C h$$

This term arises naturally, if the geometry is also deformed by $\mathcal{T}^{AB}$

$$f \triangle_2 g \;:= f \cdot g + \mathcal{T}^{AB}\, \partial_A f\, \partial_B g$$

which vanishes by continuity equation under an integral!

Closed Strings

in a constant $H = \mathrm{d}B$ background on T^3. Fulfills eom in linear order $\Rightarrow$ still a CFT. [Blumenhagen, Deser, Lüst, Plauschinn, Rennecke '11]

Correlator of the coordinates is corrected as

$$\langle X^\mu(z_1, \bar{z}_1) X^\nu(z_2, \bar{z}_2) X^\sigma(z_3, \bar{z}_3) \rangle_H \ \propto \ H^{\mu\nu\sigma} \left[\mathcal{L}\left(\frac{z_{12}}{z_{13}}\right) - \mathcal{L}\left(\frac{\bar{z}_{12}}{\bar{z}_{13}}\right) \right]$$

Using this the Jacobi identity at equal space and time is zero

$$\mathrm{Jac}\left(X^\mu(z,\bar{z}), X^\nu(z,\bar{z}), X^\sigma(z,\bar{z})\right)_H = 0.$$

T-duality in all directions gives the winding coordinate $\tilde{X}$. Their correlator has a crucial $+$

$$\langle \tilde{X}^\mu(z_1,\bar{z}_1)\tilde{X}^\nu(z_2,\bar{z}_2)\tilde{X}^\sigma(z_3,\bar{z}_3)\rangle_H \;=\; \theta^{\mu\nu\sigma}\left[\mathcal{L}\left(\frac{z_{12}}{z_{13}}\right)+\mathcal{L}\left(\frac{\bar{z}_{12}}{\bar{z}_{13}}\right)\right]$$

The contributions now add up in the Jacobi identity

$$\mathrm{Jac}\,(\tilde{X}^\mu(z,\bar{z}),\tilde{X}^\nu(z,\bar{z}),\tilde{X}^\sigma(z,\bar{z}))_H \propto H^{\mu\nu\sigma}.$$

Dualizing this gives normal coordinates in the T-dual to the H-flux, named R-flux

$$\mathrm{Jac}\left(X^{\mu}(z,\bar{z}), X^{\nu}(z,\bar{z}), X^{\sigma}(z,\bar{z})\right)_R \propto R^{\mu\nu\sigma}.$$

$\Rightarrow$ Non-associative target space for non-vanishing R-flux!

How is this possible?

Normal coordinates in non-vanishing $R^{\mu\nu\sigma} = \tilde{\partial}^{[\mu}\beta^{\nu\sigma]}$ means coordinates and winding at the same time. The description needs

$$TM \oplus T^*M.$$

A restriction to TM or T^*M is not possible. This is beyond usual Hamiltonian formalism on T^*M with $\omega = d\theta$. More concretely later!

Correlator of vertex operators gives $\langle V_1 V_2 V_3 \rangle_H = \langle V_1 V_2 V_3 \rangle_0$ and

$$\langle V_1 V_2 V_3 \rangle_R \, \propto \, (1 + R^{\mu\nu\sigma} p_{1,\mu} p_{2,\nu} p_{3,\sigma}) \, \times \, \langle V_1 V_2 V_3 \rangle_0$$

Capture the R-flux in a deformed tri-product

$$(f \triangle g \triangle h)(x) := f\, g\, h \, + \, R^{\mu\nu\sigma} \; \partial_\mu f \, \partial_\nu g \, \partial_\sigma h + \mathcal{O}(\theta^2).$$

whose totally antisymmetric tri-bracket of the coordinates reproduces the Jacobi identity.

The tri-product trivializes for tachyon vertex operators by momentum conservation.

Closed String Product in DFT

Motivation: Need for simultaneous winding and momentum.
In the flux formulation the product reads

$$
\begin{aligned}
f \triangle g \triangle h =& f\, g\, h + \mathcal{F}_{ABC}\, \partial^A f \partial^B g \partial^C h \\
=& f\, g\, h + R^{abc}\, \partial_a f \partial_b g \partial_c h + H_{abc}\, \tilde{\partial}^a f \tilde{\partial}^b g \tilde{\partial}^c h + \dots
\end{aligned}
$$

Here the flux is $\mathcal{F}_{ABC} = \Omega_{[ABC]}$ with the Weitzenböck connection $\Omega_{ABC} = \partial_A E_B{}^M\, E_{CM}$

Constraints in DFT

The generalized Lie-derivative in DFT:

$$\mathcal{L}_\xi V^M = \xi^N \partial_N V^M + (\partial^M \xi_N - \partial_N \xi^M) V^N$$

The gauge algebra does not close, constraints are needed for the fields and the gauge parameters of theory (not coordinates).

For instance the generalized Lie derivative of a generalized scalar f is not a scalar anymore but must be enforced

$$\Delta_{\xi'} \mathcal{L}_\xi f \ := \ (\delta_{\xi'} - \mathcal{L}_{\xi'}) \mathcal{L}_\xi f = -\xi_M \, \partial_N \xi'^M \, \partial^N f \stackrel{!}{=} 0.$$

Choosing the vielbein as the parameters $\xi = E_B$ and $\xi' = E_A$ gives

$$\Omega_{CAB} \, \partial^C f \stackrel{!}{=} 0 \qquad \text{(note also } \partial_A f \, \partial^A g = 0\text{)}.$$

The deformation is zero by demanding closure since

$$\underbrace{\mathcal{F}_{ABC}}_{\Omega_{[ABC]}} \partial^A f \partial^B g \partial^C h \stackrel{!}{=} 0$$

Summary

As expected vertex operators commute and associate due to

- momentum conservation in CFT
- the consistency constraints and
- the Bianchi identity (after partial integration) in DFT.

We have a non-associative target space in CFT and DFT for a non-vanishing R-flux, thus for description on $TM \oplus T^*M$ (see also Blair '14).

Why?

Hamiltonian Origin of the Non-associativity

The appearing Jacobi identity could also arise from the commutator algebra [Andriot, Larfors, Lüst, Patalong '13 and Blair '14]

$$[x^i, x^j] \propto R^{ijk}\, p_k \qquad \text{and} \qquad [x^i, p_j] = i\delta^i{}_j .$$

Underlying classical symplectic structure reads

[Mylonas, Schupp, Szabo '13,'14 and Bakas, Lüst '13]

$$\omega^{ij} = \begin{pmatrix} R^{ijk} p_k & \delta^i{}_k \\ -\delta_i{}^j & 0 \end{pmatrix} .$$

Interpret this as a special case of the DFT generalization

$$\Omega^{\mathcal{I}\mathcal{J}} = \begin{pmatrix} \mathcal{F}^{IJK} P_K & \delta^I{}_K \\ -\delta_I{}^J & 0 \end{pmatrix} .$$

Speculative Origin of the Symplectic Structure

Similar to the symplectic structure of T^*M we start with the tautological one-form Θ whose exterior derivative is the symplectic structure Ω

$$\Theta = P_I dX^I$$

Inspired by generalized geometry $(TM \oplus T^*M)$ use a twisted derivative $\mathrm{d}_{\mathcal{F}^{(3)}} = \mathrm{d} + \mathcal{F}^{(3)}$!

The symplectic structure

$$\Omega = \mathrm{d}_{\mathcal{F}}\Theta = dP_I \wedge dX^I + \mathcal{F}^{(3)}_{IJK} P^K \, dX^I \wedge dX^J$$

is precisely the non-associative symplectic structure emerging in Hamiltonian formalism.

Conclusion

No contradiction between the non-vanishing Jacobi identity and the non-associative deformations in string theory and DFT

1. Closed string:

Vertex operators commute and associate due to

- momentum conservation in CFT
- consistency constraints and
- Bianchi identity (after partial integration) in DFT.

The **target space** is non-associative for non-zero R-flux due to $TM \oplus T^*M$ (see also talk by Erik: No non-geometry on the sphere)

Conclusion

2. Open string:

Vertex operators do not commute but are associative due to the

- equation of motion
- consistency constraints and
- continuity equation of energy-momentum tensor in DFT.

Although cured, why was there non-associativity at all (No $TM \oplus T^*M$ here)?

Freed-Witten anomaly: A D3 brane wrapping a T^3 with a constant H-flux is anomalous, therefore a non-constant B-field is forbidden $\Rightarrow$ no non-associativity at all.
(Note: T-duality gives D0 brane (point particle) in R-flux)

Thank you!