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Progress on Threshold Corrections in N=2 Superstring Vacua

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$\mathcal{N}$=2 (Heterotic) Superstring Vacua have been largely studied in the late 90's.
The low-energy effective action has been reconstructed,
including (one-loop) radiative corrections to various couplings.

$\mathcal{N}$=2 supersymmetry imposes stringent constraints on the couplings of
the classical and quantum Lagrangian,
leading to a number of strong non-renormalisation theorems.

$\mathcal{N}$=2 vacua have represented a fertile play-ground to study
non-perturbative properties and dualities of
quantum field theories and superstring theories

In this presentation I will reconsider some of the old results under the light of our recent methods in the theory of modular integrals. This is instrumental to fully dissect the low-energy couplings.

OUTLINE:

- *Introduction to the main tools and conventions;*
- *N=2 Heterotic vacua **without** Wilson lines;*
- *N=2 Heterotic vacua **with** Wilson lines;*
- *Conclusions.*

INTRODUCTION TO THE MAIN TOOLS AND CONVENTIONS

We shall be interested in N=2 heterotic vacua obtained as a T^2 reduction of N=1 six-dimensional vacua. The moduli of the two-torus will be denoted by

$$T = -B_{12} + i\sqrt{G} \qquad\qquad U = \frac{G_{12} + i\sqrt{G}}{G_{11}}$$

The moduli space spanned by these fields is

$$\mathscr{M}_{T,U} = \left(\frac{\mathrm{SO}(2,2)}{\mathrm{SO}(2)\times\mathrm{SO}(2)}\right)_{T,U} \sim \left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)_{T} \otimes \left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)_{U}$$

All physical properties of these classes of N=2 vacua are thus covariant with respect to the discrete group:

$$\mathrm{PSL}(2;\mathbb{Z})_T \times \mathrm{PSL}(2;\mathbb{Z})_U \times \sigma_{T\leftrightarrow U}$$

INTRODUCTION TO THE MAIN TOOLS AND CONVENTIONS

When Wilson lines are turned on along the spectator T^2, the moduli space

$$\mathscr{M}_{T,U,W} = \frac{SO(2+k,2)}{SO(2+k)\times SO(2)}$$

is spanned by the complex moduli

$$T = -B_{12} + i\sqrt{G} + \tfrac{1}{2} Y_1^a W^a \qquad U = \frac{G_{12} + i\sqrt{G}}{G_{11}} \qquad W^a = U\,Y_1^a - Y_2^a$$

In these cases, properties of these classes of N=2 vacua are covariant with respect to the discrete group:

$$SO(2+k,2;\mathbb{Z}) \supset PSL(2;\mathbb{Z})_T \times PSL(2;\mathbb{Z})_U$$

INTRODUCTION TO THE MAIN TOOLS AND CONVENTIONS

The resulting geometry of the vector-scalar manifold is *special Kähler*, and is encoded in a holomorphic function $F(X)$ called the *pre-potential*.

The Kähler metric, gauge couplings, Yukawa couplings, … all descend from it.

Non-renormalisation theorems of N=2 supersymmetry imply that the holomorphic pre-potential is

$$F(X) = F_{\text{classical}}(X) + F_{\text{one-loop}}(X) + F_{\text{non-pert}}(X)$$

In the present talk I shall confine the discussion to the perturbative contributions, and shall neglect possible non-perturbative effects.

INTRODUCTION TO THE MAIN TOOLS AND CONVENTIONS

The discrete symmetries – i.e. the T duality of the Narain lattice – are exact symmetries of the heterotic vacuum and of its quantum corrected action. Thus the couplings it contains should still be covariant under the discrete group.

An exception is the pre-potential that is in general not invariant under the symmetry group. Actually, based on general arguments one finds that it transforms in-homogeneously.

[Antoniadis, Ferrara, Gava, Narain, Taylor, 1995]
[de Wit, Kaplunovsky, Louis, Lüst, 1995]

GOAL: ***compute the quantum pre-potential and determine its transformation properties under the action of the symmetry group.***

A WAY TO COMPUTE THE ONE-LOOP PRE-POTENTIAL

The one-loop moduli metric is

$$K_{T\bar{T}} \sim S_2^{-1} G^{(1)}_{T\bar{T}} \sim S_2^{-1}\, T_2^{-1}\, D_T\, D_U\, f(T,U) \qquad \text{where} \qquad F(S,T,U) = STU + f(T,U)$$

From String Theory

$$D_T D_U f(T,U) = \int_{\mathscr{F}} d\mu\, \Phi(\tau)\, \partial_\tau\, \Lambda(T,U)$$

Narain lattice

weight -2 holomorphic modular form with simple pole at infinity

Use properties of the Narain lattice to determine a differential equation for $f(T,U)$.

[Antoniadis, Ferrara, Gava, Narain, Taylor, 1995]

ANOTHER WAY TO COMPUTE THE ONE-LOOP PRE-POTENTIAL

Integrate by parts to obtain

$$D_T D_U f(T,U) = \int_{\mathscr{F}} d\mu\, \tilde{\Phi}(\tau)\, \Lambda(T,U)$$

Narain lattice

quasi holomorphic modular function with simple pole at ∞

$$\tilde{\Phi}(\tau) \sim \Delta^{-1}(\alpha\, \hat{E}_2 E_4 E_6 + \beta E_4^3 + \gamma E_6^2)$$

$$\alpha = 1\,, \quad \beta = 3\,, \quad \gamma = 2\,.$$

Actually, for the previous relation to hold the coefficients cannot be arbitrary.

REMARK: Gauge thresholds have a similar integrand but they cannot be written as $D_T\, D_U$ of a holomorphic function.

ANOTHER WAY TO COMPUTE THE ONE-LOOP PRE-POTENTIAL

However, from any integral of the type

$$\int_{\mathscr{F}} d\mu \, \tilde{\Phi}(\tau) \, \Lambda(T, U)$$

we can extract the (generalised) holomorphic pre-potential, up to an overall, model-dependent, constant.

A CONVENIENT BASIS FOR QUASI MEROMORPHIC MODULAR FORMS

It is convenient to introduce a basis of (quasi) holomorphic modular forms with (higher-order) poles at the cusp

$$\mathcal{F}(1-\tfrac{w}{2}+n,\kappa,w) \sim \frac{1}{q^\kappa}\left(\frac{1}{\tau_2^n}+\frac{1}{\tau_2^{n-1}}\dots\right)+O(1)$$

These functions have several interesting properties:

- their Poincaré series is uniformly and absolutely convergent (even for $w<0$);
- they are eigenmodes of the modular Laplacian;
- they are closed under the action of modular derivatives:
  - $D_w\,\mathcal{F}(s,\kappa,w) \propto \mathcal{F}(s,\kappa,w+2)$
  - $\bar{D}_w\,\mathcal{F}(s,\kappa,w) \propto \mathcal{F}(s,\kappa,w-2)$

A CONVENIENT BASIS FOR QUASI MEROMORPHIC MODULAR FORMS

Any modular form whose q-Laurent expansion has principal part

$$\Phi_w^- = \sum_{0<m\leq\kappa} \sum_{\ell=0}^{n} \frac{c_\ell(m)\, \tau_2^{\ell-n}}{q^m}$$

can be decomposed as

$$\Phi_w(\tau) = \sum_{0<m\leq\kappa} \sum_{\ell=0}^{n} d_\ell(m)\, \mathcal{F}(1 - \tfrac{w}{2} + \ell, m, w)$$

with the coefficients $d_l(m)$ uniquely determined in terms of the $c_l(m)$

A CONVENIENT BASIS FOR QUASI MEROMORPHIC MODULAR FORMS

As a result,

$$\int_{\mathscr{F}} d\mu\, \Phi_w(\tau)\, \Lambda(T,U) = \sum_{0<m\leq\kappa} \sum_{\ell=0}^{n} d_\ell(m) \int_{\mathscr{F}} d\mu\, \mathcal{F}(1-\tfrac{w}{2}+\ell, m, w)\, \Lambda(T,U)$$

For String Theory, $\kappa = 1$ and

$$\int_{\mathscr{F}} d\mu\, \tilde{\Phi}_0(\tau)\, \Lambda_{2,2}(T,U) = \sum_{\ell=0}^{n} d_\ell \int_{\mathscr{F}} d\mu\, \mathcal{F}(1+\ell, 1, 0)\, \Lambda_{2,2}(T,U)$$

$$\int_{\mathscr{F}} d\mu\, \tilde{\Phi}_{-k/2}(\tau)\, \Lambda_{2+k,2}(T,U,W) = \sum_{\ell=0}^{n} d_\ell \int_{\mathscr{F}} d\mu\, \mathcal{F}(1+\tfrac{k}{4}+\ell, 1, -\tfrac{k}{2})\, \Lambda_{2+k,2}(T,U,W)$$

A CONVENIENT BASIS FOR (GENERALISED) HOLOMORPHIC PRE-POTENTIAL

Actually, we can say more. If from low-energy supergravity one knows that the one-loop correction to a given coupling can be written as

$$G(S,T,U) = G_0(S,T,U) + (D_T D_U)^\ell f_\ell(T,U)$$

then we can immediately conclude that a stringy calculation would yield

$$f_\ell(T,U) \propto \int_{\mathscr{F}} d\mu\, \Lambda_{2,2}(T,U)\, \mathcal{F}(1+\ell,1,0)$$

Similarly, in the presence of Wilson lines

$$f_\ell(T,U,W) \propto \int_{\mathscr{F}} d\mu\, \Lambda_{2+k,2}(T,U,W)\, \mathcal{F}\left(1+\tfrac{k}{4}+\ell,1,-\tfrac{k}{2}\right)$$

N=2 HETEROTIC VACUA WITHOUT WILSON LINES

$$f_\ell(T,U) \propto \int_{\mathscr{F}} d\mu\, \Lambda_{2,2}(T,U)\, \mathcal{F}(1+\ell,1,0)$$

The integral can be evaluated using your favourite method

and the T-Fourier expansion of result can be cast in the form

Of course, I will choose mine!?!

$$2^{2(1+\ell)}\,\sqrt{4\pi}\,\Gamma(\ell+\tfrac{1}{2})\,T_2^{-\ell}\,E(1+\ell,U) + 8\,\mathrm{Re}\sum_{M>1}\sqrt{\frac{T_2}{M}}\,e^{2i\pi M T_1}\,K_{\ell+\frac{1}{2}}(2\pi M T_2)\,\mathcal{F}(1+\ell,M,0;U)$$

(this expression is valid in the chamber $T_2>U_2$)

N=2 HETEROTIC VACUA WITHOUT WILSON LINES

Remember the property

$$D\,\mathcal{F}(1+\ell,1,-2) \propto \mathcal{F}(1+\ell,1,0)$$

and the fact that $\mathcal{F}(1+\ell,1,w)$ is holomorphic when $w=-2$

Also

$$\sqrt{y}\,K_{\ell+\frac{1}{2}}(2\pi y)\,e^{2i\pi x} = D^{\ell}(e^{2i\pi z})$$

$$8\,\mathrm{Re}\sum_{M>1}\sqrt{\frac{T_2}{M}}\,e^{2i\pi M T_1}\,K_{\ell+\frac{1}{2}}(2\pi M T_2)\,\mathcal{F}(1+\ell,M,0;U) \propto \mathrm{Re}\sum_{M>1}(-D_T D_U)^{\ell}\,f_{\ell}(T,U;M)$$

with

$$f_{\ell}(T,U;M) \propto \frac{q_T^M}{M^{2\ell+1}}\,\mathcal{F}(1+\ell,M,-2\ell;U)$$

N=2 HETEROTIC VACUA WITHOUT WILSON LINES

For the zero Fourier mode

$$T_2^{-\ell}\, E(1+\ell; U)$$

$$\propto \mathrm{Re}\,(-D_T D_U)^{\ell}\, E(1+\ell, -2\ell; U)$$

Non holomorphic, and

$$U^{2\ell}\, E(1+\ell, -2\ell; -1/U) = E(1+\ell, -2\ell; U)$$

$$\propto \mathrm{Re}\,(-D_T D_U)^{\ell}\, \tilde{E}_{-2\ell}(U)$$

Holomorphic, but

$$U^{2\ell}\, \tilde{E}_{-2\ell}(-1/U) = \tilde{E}_{-2\ell}(U) + P_{\gamma,2n}(U)$$

What is its interpretation in SUGRA?

$$2^{2(1+\ell)}\sqrt{4\pi}\,\Gamma(\ell+\tfrac{1}{2})\, T_2^{-\ell}\, E(1+\ell; U) \propto \mathrm{Re}\,(-D_T D_U)^{\ell}\, f_{\ell}(T, U; 0)$$

N=2 HETEROTIC VACUA WITHOUT WILSON LINES

The full one-loop correction to the holomorphic pre-potential is (l=1)

$$f_1(T,U) = \frac{4\pi^4}{\zeta(4)}\,\tilde{E}_{-2}(U) - 2\sum_{M>1}\frac{q_T^M}{(2M)^3}\,\mathcal{F}(2,M,-2;U)$$

Under the generators of PSL(2,Z)$_U$

$$f_1(T,U+1) = f_1(T,U) + \frac{(2\pi i)^3}{240}\left[\tfrac{1}{6} + \tfrac{1}{2}U + \tfrac{1}{2}U^2\right]$$

$$U^2 f_1(T,-1/U) = f_1(T,U) + \frac{(-2\pi i)^3}{288}\,U$$

[*c.f.* Harvey, Moore, 1996]

(Similar expression exist also for higher l)

N=2 HETEROTIC VACUA WITH WILSON LINES

$$f_\ell(T,U,W) \propto \int_{\mathcal{F}} d\mu\, \Lambda_{2+k,2}(T,U,W)\, \mathcal{F}\left(1+\tfrac{k}{4}+\ell, 1, -\tfrac{k}{2}\right)$$

Following similar steps, for $l>0$ one can express the integrals in terms of holomorphic pre-potentials. For $l=1$,

$$\int_{\mathcal{F}} d\mu\, \mathcal{F}(1+2+1,1,-4)\, \Lambda_{10,2}(T,U,W) \propto \mathrm{Re}\, \Box\, f_1(T,U,W)$$

with now

[*c.f.* Kiritsis, Obers, 1997]

$$\Box = -\frac{1}{2\pi^2}\left[\eta^{ab}\partial_a\partial_b + \frac{k-2w}{(y_2,y_2)}\left(iy_2^a\partial_a + \tfrac{1}{2}w\right)\right]$$

N=2 HETEROTIC VACUA WITH WILSON LINES

The holomorphic pre-potential admits a T-Fourier series expansion

$$f_1(T,U,W) = f_1(T,U,W;0) + \sum_{M>1} f_1(T,U,W;M)$$

with positive frequencies given by

$$f_1(T,U,W;M) \propto \frac{q_T^M}{M^3}\,\mathcal{F}(2,M,-2;W|U)$$

a Jacobi form generated by the same seed

c.f. with the symmetric lattice case

$$f_1(T,U;M) \propto \frac{q_T^M}{M^3}\,\mathcal{F}(2,M,-2;U)$$

N=2 HETEROTIC VACUA WITH WILSON LINES

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with positive frequencies given by

$$f_1(T,U,W;M) \propto \frac{q_T^M}{M^3}\,\mathcal{F}(2,M,-2;W|U)$$

They are "trivially" covariant with respect to the Jacobi group

N=2 HETEROTIC VACUA WITH WILSON LINES

The zero-frequency mode is fairly more complicated

$$f_1(T,U,W;0) \propto \sum_{Q^2=2} \left[\tfrac{1}{2}\,\mathrm{Li}_3\left(e^{2i\pi W\cdot Q}\right) + \sum_{M>0} \mathrm{Li}_3\left(e^{2i\pi(MU+W\cdot Q)}\right)\right]$$

$$+\tilde{E}_{-2n} + \left[-\tfrac{1}{3}\,U^3 + 3T\,W^2 + \tfrac{5}{2}\,U\,W^2\right]$$

From general arguments, it is expected to be a Jacobi integral, *i.e.* to transform in-homogeneously under the Jacobi group.

N=2 HETEROTIC VACUA WITH WILSON LINES

Any Jacobi form $\varphi(z|\tau)$ admits a Taylor expansion

$$\varphi(z|\tau) = \sum_{n>0} \varphi_n(\tau)\, z^n$$

whose coefficients are modular forms of given weight.

One could try to compute the Taylor expansion of the zero-frequency mode of the holomorphic pre-potential to find

N=2 HETEROTIC VACUA WITH WILSON LINES

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$$f_1(T,U,W;0) \propto \sum_{\substack{Q^2=2\\ Q>0}} \Bigg[\tilde{E}_{-2}(U) - U^3$$

$$+(Q\cdot W)^2 \; \left[H_2 + U - \log(2\pi W\cdot Q\, \eta^2(U)\right]$$

$$+ \sum_{\ell>1} (Q\cdot W)^{2\ell} E_{2\ell-2} \Bigg]$$

The coefficients are now Eichler integrals and modular forms whose transformation properties are fully under control.

N=2 HETEROTIC VACUA WITH WILSON LINES

Putting things together, one finds

$$U^2 f_1(T - W^2/2U, -1/U, W) = f_1(T,U,W) + 37800\, i\, \pi^3\, W^2$$

$$f_1(T, U+1, W) = f_1(T,U,W) + 12600\, i\, \pi^3\, W^2$$

$$f_1(T + \lambda \cdot W + \tfrac{1}{2}\lambda^2 U, U, W + U\lambda) = f_1(T,U,W) + S_\lambda(T,U,W)$$

with

$$S_\lambda(T,U,W) = \sum_{\substack{Q^2=2 \\ Q\cdot\lambda>0}} \left[\tfrac{1}{2} B_3((W+U\lambda)\cdot Q) + \tfrac{1}{2} B_3(W\cdot Q) - \sum_{M=0}^{\lambda\cdot Q} B_3(MU + W\cdot Q) \right]$$

$$+(\lambda\cdot W)(6TU + 5U^2 + 3W^2) + \lambda^2(3TU^2 + \tfrac{5}{2}U^3 + \tfrac{3}{2}UW^2) + 6U(\lambda\cdot W)^2 + 6U^2(\lambda\cdot W)\lambda^2 + \tfrac{3}{2}U^3(\lambda^2)^2$$

CONCLUSIONS

- *I have revisited the computation of the radiative corrections to the holomorphic pre-potential.*
- *In the absence of Wilson lines its T-Fourier zero mode can be identified with an Eichler integral, thus fixing its transformation properties under the duality group.*
- *In the presence of Wilson lines the in-homogeneous transformation under the action of the Jacobi group also come entirely from the T-Fourier zero mode, that is now a Jacobi integral whose behaviour under the duality group is fully under control.*

THANK YOU!