# ARITHMETIC OF A- AND B-BRANES

Johannes Walcher (Heidelberg)

Geometry and Strings
Schloss Ringberg, July 25, 2018

Work in progress with **Hans Jockers** (Bonn),
and **David Morrison** (Santa Barbara)

# INTRODUCTION AND MOTIVATION

Mirror symmetry is an early example of a physical duality whose mathematical consequences underscore the usefulness of supersymmetry and string theory for applications in the ‘real world’.

Over the years, tremendous progress has been made in understanding both the physical derivation of phenomena related to mirror symmetry and in providing rigorous proofs of its mathematical statements.

In a way, mirror symmetry can be viewed as one of those “*pieces of 21st century mathematics that was discovered by chance in 20th century physics*”

Epistemology notwithstanding, it is of interest to continue exploring the geometric and physical interpretation of even the most abstract mathematical statements.

# INTRODUCTION

A while ago (JW 2012), it was noticed that a full understanding of the predictions of mirror symmetry with D-branes requires an appreciation of arithmetic (number theoretic) properties of Calabi-Yau manifolds and associated mathematics.

The number theory that arises is distinct, though related, to other connections that have been observed in various contexts.

Black holes attractors — Moore 1998
Zeta-Functions — Candelas-de la Ossa-Rodriguez-Villegas 1999
Mathieu Moonshine — Eguchi-Ooguri-Tachikawa 2010
Modular forms — …, Klemm-Scheidegger-Zagier 201?
⋮ ⋮

(in addition to numerous connections discovered in field theory/ amplitudes, etc., … Stieberger, Kleinschmidt…)

# INTRODUCTION

In ongoing work with Hans Jockers and David Morrison, we are elucidating the peculiar predictions that arise in the context of mirror symmetry with D-branes, and this talk is a report on recent progress on this line of research.

In a nutshell, we propose (with evidence) that the arithmetic properties of co-dimension-2 algebraic cycles in Calabi-Yau three-folds (B-branes) are related, as a consequence of mirror symmetry, to the arithmetic properties of hyperbolic three-manifolds embedded as Lagrangian submanifolds (A-branes) in the mirror manifold.

# MIRROR SYMMETRY

The ambiguity in identifying the R-symmetry of N=(2,2) SCFT attached to Calabi-Yau sigma model with the Hodge decomposition of de Rham cohomology led (Dixon, LVW) to the prediction that Calabi-Yau manifolds come in pairs with

$$h^{i,j}(X) = h^{i,n-j}(Y)$$

And such that in string compactification

$$\text{Type IIA}/X \equiv \text{Type IIB}/Y$$

Famous example (Greene-Plesser)

$$X = \text{quintic in } \mathbb{CP}^4$$

$$Y = \left\{ x_1 x_2 x_3 x_4 x_5 - \frac{z^{1/5}}{5}(x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5) = 0 \right\} / \Gamma$$

"mirror quintic"

(Toric geometry)

# MIRROR SYMMETRY

The study of invariant physical observables in 4d N=2 SUGRA+Matter leads to a mathematical relationship between enumerative geometry of X (A-model) and variation of Hodge structure of Y (B-model) (CdGP, 1990; Morrison, Kontsevich).

In the A-model, the prepotential for vector multiplets

$$\mathcal{F}_A(q) = \frac{5}{6}\log q^3 + \cdots + \frac{\zeta(3)}{(2\pi i)^3}\chi(X) + \sum_{d>0}\tilde{N}_d q^d$$

receives non-perturbative worldsheet corrections. In the B-model, it is given by a classical period calculation

$$\mathcal{F}_B = \frac{\varpi_3\varpi_0 - \varpi_1\varpi_2}{2\varpi_0^2} \qquad \varpi_i = \int_{\gamma_i}\Omega(z)$$

"Special geometry" "Yukawa couplings"

# RATIONALITY v. INTEGRALITY

Important under-appreciated fact: In the B-model, there is no a priori mathematical reason why the coefficients of q-expansion should be rational numbers.

More precisely, it is easy to see that the mirror of large volume point in stringy Kähler moduli space has to be a degeneration of “maximal unipotent monodromy” (due to D0/D2/D4/D6-brane mass hierarchy).

All known MUM points in Calabi-Yau moduli are rational, in the (strong) sense that the degeneration is defined over Q. In particular, the coefficients of prepotential are also rational and can be viewed as Gromov-Witten invariants (counting worldsheet instantons) of mirror A-model.

This covers cases in which MS, or even B-model geometry are not known. (Calabi-Yau diff. eq. Almkvist et al.)

# RATIONALITY v. INTEGRALITY

True or false?

$$\forall \text{ M.U.M. points, } \tilde{N}_d := [\mathcal{F}]_{q^d} \in \mathbb{Q}??$$

If coefficients are rational, they will satisfy (Kontsevich-Schwarz-Vologodsky) the integrality properties that allow for BPS interpretation in A-model/spacetime (Gopakumar-Vafa, … )

Otherwise, such a BPS interpretation would appear to be rather impossible.

☛ This is precisely what happens in the presence of D-branes!

# D-BRANES

Supersymmetric cycles in Calabi-Yau manifolds come in two types, related by mirror symmetry.

A-branes: special Lagrangian submanifolds with flat connection

B-branes: holomorphic submanifolds with holomorphic vector bundle and suitable stability condition

In the mathematical incarnation of mirror symmetry,

$$\mathrm{Fuk}(X) \cong D^b(Y) \qquad \Big(\text{and } D^b(X) \cong \mathrm{Fuk}(Y)\Big)$$

proven in many important cases, coherent sheaves are easily accessible (e.g., by correspondence with matrix factorizations). In contrast, (special) Lagrangian geometry is much harder, and for compact C-Y, only very few examples are known explicitly. Enumerative geometry remains largely a mystery.

☛ One result of our work is to improve this situation.

# D-BRANES

In string theory, A/B-branes are used to engineer vacua with N=1 SUSY in 4d. The relevant holomorphic observable arises from the space-time superpotential.

(Fukaya-Oh-Ohta-Ono, Witten, … )

$$\mathcal{W}_A = \cdots + \mathrm{CS}(A) + \sum_{\mathrm{hol.disks}} e^{\oint A + \int \omega}$$

(Witten, …, Costello-Li, …)

$$\mathcal{W}_B = \int_Y \mathrm{Tr}\Big(A \wedge \bar{\partial} A + \frac{2}{3} A \wedge A \wedge A\Big) \wedge \Omega$$

By restriction to critical points of ‘open string moduli’

$$\mathcal{W}|_{F_A=0} = \int_{C_0}^{C} \Omega \qquad [C] = \begin{array}{l}\text{algebraic second}\\ \text{Chern class}\end{array}$$

“Abel-Jacobi map/semi-period”

☛ BPS domain walls

# EXAMPLE — The real quintic

Only known construction of sLAGs in compact CY:

$$L = \{x_i = \bar{x}_i\} \subset X \qquad \text{topologically, } L \cong \mathbb{RP}^3\,,\ H_1(L) = \mathbb{Z}_2$$

Originally studied by Brunner et al., the mirror of the real quintic is represented by "Deligne conics" (hol. curves, co-dim 2 algebraic cycle)

$$C = \{x_2 = -x_1, x_4 = -x_3, x_5^2 = \varphi x_1 x_3\} \subset Y$$

$$Y = \left\{\prod x_i - \frac{z^{1/5}}{5}\sum x_i^5\right\}$$

$$\Leftrightarrow\ z^{1/5}\varphi^2 = 5$$

(If desired, HMS gives a holomorphic vector bundle (object in derived category) with algebraic second Chern class C.)

# EXAMPLE — The real quintic

Indeed, under mirror map,

$$\mathcal{W}(q) = \frac{\log q}{2} + \frac{1}{4} + \frac{1}{\pi^2}\Big(30q^{1/2} + \frac{4600}{3}q^{3/2} + \cdots\Big)$$

is generating function for genus 0 open Gromov-Witten invariants of pair (quintic, real quintic), counting BPS domainwalls

$$\sum \tilde{n}_d q^{d/2} = \sum n_d \,\mathrm{Li}_2(q^{d/2}) \qquad \tilde{n}_d \in \mathbb{Q}\,,\ n_d \in \mathbb{Z}$$

In distinction to classical case, there is only a small set of examples of this “extended mirror symmetry” (compact C-Y).

(This is not to say that open string mirror symmetry is not well understood in many cases, non-compact and toric. … Klemm, Mayr, Jockers …)

# EXAMPLE — van Geemen lines

$$x_1x_2x_3x_4x_5 - \frac{z^{1/5}}{5}\left(x_1^5+x_2^5+x_3^5+x_4^5+x_5^5\right)$$

$3 \nmid 2875(\#\text{lines})$

$Y\ S_5\text{-invariant}$ $\Rightarrow \exists\, \mathbb{Z}_3\text{-invariant lines}$

$$C = \Big\{ x_1 + \omega x_2 + \omega^2 x_3 = 0\,, \\ x_4 = \frac{a}{3}(x_1+x_2+x_3)\,, x_5 = \frac{b}{3}(x_1+x_2+x_3) \Big\} \subset Y$$

$$\Leftrightarrow ab = 6z^{1/5}\,, a^5+b^5 = 27\,, 1+\omega+\omega^2 = 0$$

$$\mathcal{W} = i\,\underbrace{13.368560\cdots}_{=:\frac{V}{\pi^2}} + \frac{\sqrt{-3}}{\pi^2}\Big(140000q + \frac{11148100000}{3}q^2 + \cdots\Big)$$

Question: What is A-model geometry, and (enumerative/BPS?) interpretation of coefficients, valued in algebraic number field

$$K = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\omega)$$

# GENERAL RESULT — Algebraic integrality

$$\begin{array}{ccccc} C & \subset & \hat{Y} & = & Y \\ \downarrow & & \downarrow & & \downarrow \\ \hat{D} & = & \hat{D} & \to & D \end{array}$$

(w/ Schwarz-Vologodsky, 2017)

$$\begin{aligned} Y \to D &: \text{defined over } \mathbb{Q}, \\ K &: \text{residue field of extended period ring} \\ K \supset \mathcal{O} &: \text{ring of integers} \end{aligned}$$

$$\exp(\delta\mathcal{W}) \in \mathrm{Ext}^1_{\mathrm{VMHS}}(\mathbb{Z}(-2), \mathcal{H})$$

$$\forall p \text{ rational prime} : \frac{1}{p^2}\mathrm{Frob}_p \mathcal{W}_p - \mathcal{W}_p \in \mathcal{O}_p[[q]]$$

$$x \in \mathcal{O}\,, P(x) = 0 : \mathrm{Frob}_p(x) = x^p - \frac{P(x^p)}{P'(x^p)} + \cdots$$

“Frobenius element” in Gal(K/Q)

# GENERAL RESULT — Algebraic integrality

For van Geemen lines (K abelian extension of Q)

$$\pi^2 \mathcal{W} = iV + \underbrace{\sqrt{-3}\left(140000q + \frac{11148100000}{3}q^2 + \cdots\right)}_{= \sum_d n_d \sum_k \frac{1}{k^2}\left(\frac{-3}{k}\right) q^{kd}}$$

$$\left(\frac{-3}{k}\right) = \begin{cases} 0 & k = 0 \bmod 3 \\ 1 & k = 1 \bmod 3 \\ 2 & k = 2 \bmod 3 \end{cases}$$

"Dirichlet character"

$$n_d \in \sqrt{-3}\,\mathbb{Z}\left[\tfrac{1}{3}\right]$$

Most interesting for non-abelian K: A-model geometry knows about Galois group, Frobenius mod p^2.

# POSSIBLE INTERPRETATIONS

1. The Lagrangian (or more general object of Fukaya category) mirror to van Geemen lines is obstructed, the critical points of superpotential are irrational, but algebraic — WHY?

FOOO: $$\sum_{n=0}^{\infty} m_n(b, \ldots, b) = 0\,, \quad b \in C^1(L, K) \quad \text{"bounding cochain"}$$

2. More conservatively (?), the arithmetic is intrinsic to the Lagrangian submanifold itself — perhaps this is a missing ingredient for their difficult construction.

☛Natural candidate: Hyperbolic three-manifolds
Play a central role in 3-manifold theory (Thurston)
The generic Lagrangian submf of C-Y might be hyperbolic

3. Speculating wildly, the Galois group acts on open string vacua (and thereby on BPS spectra).

# HYPERBOLIC 3-MANIFOLDS

$$L \text{ hyperbolic: constant sectional curvature } -1$$

$$\Rightarrow L = \mathbb{H}^3/\Gamma\,, \quad \Gamma \subset \mathrm{PSL}(2,\mathbb{C})$$

Mostow rigidity: Action of fundamental group through isometries of euclidean AdS is unique up to conjugation.

☛ Geometric invariants (volume etc.) are topological.

Arithmetic invariants (finite volume, compact):

$$\text{trace field: } K = \mathbb{Q}[\mathrm{Tr}\gamma, \gamma \in \Gamma] \text{ (number field)}$$

$$\text{quaternion algebra: } \mathbb{A} = K[\gamma, \gamma \in \Gamma]$$

# HYPERBOLIC 3-MANIFOLDS

$$\text{central simple algebra, } \mathbb{A} = \left(\frac{a,b}{K}\right)$$

$$= K[i,j,k]\,, i^2 = a, j^2 = b, ij = -ji = k$$

$$L = \mathbb{H}^3/\Gamma \text{ compact} \Leftrightarrow \mathbb{A} \not\cong \mathrm{Mat}(2,K)$$

A is determined by "discriminant", places of K where A is ramified (not split).

In a putative correspondence with algebraic cycles, the (invariant) trace field is identified with field of definition of representative cycle (residue field of extended period ring).

We do not have any interpretation of quaternion algebra, or other invariants such as length spectrum of geodesics.

# HYPERBOLIC VOLUMES

By triangulating hyperbolic three-manifolds into ideal tetrahedra (Thurston, … Neumann-Zagier,…)

$$\mathrm{vol}(L) \in D(\mathcal{B}_{\overline{\mathbb{Q}}})$$ "volume spectrum"

$$\mathcal{B}_{\overline{\mathbb{Q}}} = \mathrm{Ker}\big(\mathbb{Z}[\mathbb{C}^\times] \to \mathbb{Z}[\mathbb{C}^\times \times \mathbb{C}^\times\big)/\mathrm{relations}$$
$$z \mapsto z \wedge 1 - z$$

"Bloch group"

$$D(z) = \mathrm{Im} Li_2(z) + \arg(1-z)\log|z|$$ "Bloch-Wigner function"

In physics,

$$\mathrm{vol}(L) = \mathrm{Im}\big(CS_{SL(2,\mathbb{C})}(L)\big)$$

is classical contribution to FOOO superpotential for A-branes on Calabi-Yau.

# HIGHER CHOW GROUPS

On the B-side (Bloch, Beilinson, Goncharov,…,Green-Griffiths-Kerr)

$$D(\mathcal{B}_{\overline{\mathbb{Q}}}) \ni \mathrm{Im}\left( \lim_{z\to 0} \int_{C_0}^{C} \Omega \right)$$

More precisely, in intermediate Jacobian fibration,

$$J^2(Y_z) = F^2H^3(Y)^\vee / H_3(\mathbb{Z}) \underset{z\to 0}{\longrightarrow} \text{disc.gp.} \ltimes \mathbb{C}/\mathbb{Z}(2)$$

And, given co-dim 2 algebraic cycle as before,

$$\begin{array}{ccc} C(z) \in \mathrm{CH}^2(Y) & \xrightarrow{z\to 0} & C(0) \in \mathrm{CH}^2(\mathbb{C},3) \\ \big\downarrow \mathrm{AJ} & & \big\downarrow \text{"regulator"} \\ J^2(Y) & \xrightarrow{z\to 0} & \mathbb{C}/\mathbb{Z}(2) \end{array}$$

# CALCULATION — van Geemen lines

$$\mathcal{W} = i\frac{V}{\pi^2} + \frac{\sqrt{-3}}{\pi^2}\Big(140000q + \frac{11148100000}{3}q^2 + \cdots\Big)$$

$$V = 131.9424088\ldots$$

Following Green-Griffiths-Kerr, we obtain:

$$V = \lim_{z\to 0}\pi^2\mathcal{W} = \frac{12\cdot 65\cdot |\Delta|^{3/2}\zeta_K(2)}{4\pi^2} = 195D(\omega)$$

$$\Delta = \mathrm{disc}(K/\mathbb{Q}) = -3 \qquad \zeta_K(s) = \sum_{I\subset\mathcal{O}_K}\frac{1}{(N_{K/\mathbb{Q}}(I))^s}$$

☛Does there exist a hyperbolic three-manifold with this volume?

# CONSTRUCTIONS

Arithmetic hyperbolic 3-manifolds: Γ commensurable with $SL(2,O_K)$

Humbert 1919: $$\mathrm{vol}\big(\mathbb{H}^3/SL(2,\mathcal{O}_K\big) = \frac{|\Delta|^{3/2}\zeta_K(2)}{4\pi^2}$$

Rational factor 12*5*13 from discrete group, choice of (i) quaternion algebra and (ii) an "order" E in A, characterized by discriminant and level.

Fact: There exists a quaternion algebra $\mathbb{A}/K$ and an "Eichler" order $\mathcal{E}$ (intersection of two maximal orders) such that

$$\mathrm{vol}\big(\mathbb{H}^3/\mathcal{E}^\times\big) = V_{\text{van Geemen}}$$

☛Can this 3-manifold be embedded as Lagrangian in quintic?

# OPEN PROBLEMS

- Substantiate volume calculation: 3-loop beta-function of sigma-model w/ Lagrangian b.c., higher-derivative corrections to DBI action (Bachas et al., Sevrin et al.)
- Construct embedding of L into Y using Strominger-Yau-Zaslow description of mirror symmetry, or otherwise
- Study its open Gromov-Witten theory
- Off-shell version? (Jockers et al.)
- Interpret quaternion algebra and other arithmetic data in physics / B-model