On the prevalence of elliptic and genus one fibers among Calabi-Yau threefolds

Conference on Geometry and Strings
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Based on:

arXiv: 1805.05907., arXiv: 1808.nnnnn Y-C. Huang, WT

1. Calabi-Yau manifolds and fibrations

Calabi-Yau threefolds:

physically:

- Ricci flat: $R_{\mu\nu} = 0$ (solve vacuum Einstein equations)
- Kähler manifolds (complex structure compatible with SUSY)

mathematically: trivial canonical class $K = 0$ (up to torsion)

Long studied by mathematicians and physicists

— Used in compactification of heterotic, II $\rightarrow$ 4D, F-theory $\rightarrow$ $6D$ (+ M $\rightarrow$ 5D)

Largest class of known Calabi-Yau threefolds:

Kreuzer/Skarke:
Classified 473.8M reflexive 4D polytopes $\rightarrow$ toric hypersurface CY3's

Also: CICY's (80's), gCICY's [Anderson/Apruzzi/Gao/Gray/Lee '15]

Open Question:
Are there a finite number of topological types of Calabi-Yau threefolds?

Elliptic and genus one-fibered CY threefolds

An *elliptic* or *genus one fibered* CY3 X:
$\pi : X \rightarrow B_2$,
$\pi^{-1}(p) \cong T^2$ at a generic point p

Elliptic: $\exists$ section $\sigma : B_2 \rightarrow X, \pi\sigma = \text{Id}$

Elliptic Calabi-Yau threefold has Weierstrass model

$$y^2 = x^3 + fx + g, \quad f \in \Gamma(\mathcal{O}(-4K_B)), g \in \Gamma(\mathcal{O}(-6K_B))$$

Finite number of topological types of elliptic Calabi-Yau threefolds
[Grassi, Gross]

Constructive proof [Kumar/Morrison/WT]: (using principles of F-theory)
Bases blow-ups of $\mathbb{F}_m$ (Grassi);
Finite number of distinct strata in space of B_2 W. models (Hilbert basis thm)

Upshot of recent work:

Can construct elliptic Calabi-Yau threefolds by:

1. Classify allowed bases B
   (Morrison/WT: 65k toric bases; Wang/WT: non-toric bases)

2. "Tune" Weierstrass model [Johnson/WT, . . .]

Tuning gives increased singularities (Kodaira, etc.) and Mordell-Weil group
Physics interpretation via F-theory: gauge groups and matter

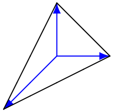
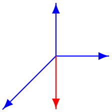
In principle gives all elliptic Calabi-Yau manifolds
Various technical challenges, particularly for CY4's

Growing evidence: most known Calabi-Yau threefolds are elliptic or g1 fibered!
[Candelas/Constantin/Skarke, Gray/Haupt/Lukas, AGGL, . . .]

This talk: explicitly explore KS database
i) directly analyze fiber structure
ii) construct simple fibrations, seive $\rightarrow$ more exotic fibrations

Toric hypersurface construction [Batyrev, Kreuzer/Skarke]

Toric geometry: simple combinatoric version of algebraic geometry

Δ  $\Rightarrow$ (blow up point) 

$\mathbb{P}^2$ $\mathbb{F}_1 = dP_1$

Toric variety: characterized by toric divisors $D_i \leftrightarrow$ rays $v_i \in \mathbb{Z}^d$

Anti-canonical class $-K = \sum_i D_i$ (never compact CY)

Anti-canonical hypersurface $\Rightarrow$ CY by adjunction

Δ polytope: convex hull of v_i

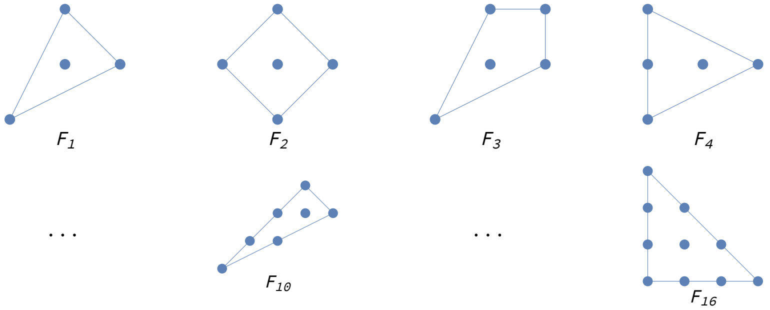
$\{$monomials$\} \leftrightarrow$ lattice points in *dual polytope* $\Delta^* = \{w : w \cdot v \geq -1\}$

Batyrev: $\Delta = \Delta^{**}$ reflexive $\leftrightarrow$ 1 interior point
$\leftrightarrow$ hypersurface CY generically smooth (avoids singularities)

Simple toric fibrations:

$\Delta_2 \subset \Delta$, Δ_2 reflexive

Only 16 reflexive Δ_2's (e.g. F-theory fibers:
[Braun, Braun/Grimm/Keitel, Klevers/Mayorga Pena/Oehlmann/Piragua/Reuter])



-1 curve $C = D_i^{(2)}$: satisfies $-K \cdot C = C \cdot C + 2 = 1$

All but $F_1 = \mathbb{P}^2, F_2 = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1, F_4 = \mathbb{F}_2$ have -1 curves $\Rightarrow$ toric sections

2. Results

First approach: look at KS database, directly identify polytope fibers
(paper w/Huang to appear)

Basic algorithm:

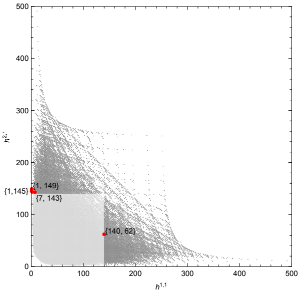
- Identify $v, w \in \Delta$: span $(v, w) \cap \Delta = \Delta_2 = F_i, i = 1, \ldots, 16$
- Some algorithmic efficiency implemented
  (e.g. $v, w \subset S$ w/limited $v \cdot u, u \in \Delta^*$)
- Currently in mathematica (faster implementation possible)

Finding $F_i, i \in \{1, \ldots, 16\} \Rightarrow \exists$ g1/elliptic toric fibration

If only $F_{1,2,4}$: genus one, not necessarily section.

Any other $F_i \Rightarrow$ elliptic

Results I: all KS polytopes giving CY w/ $h^{1,1} \geq 140$ or $h^{2,1} \geq 140$



Only 4 (of 495515) lack genus one fibers:

$$(h^{1,1}, h^{2,1}) = (1, 149), \quad (1, 145), \quad (7, 143), \quad (140, 62)$$

When $h^{1,1} = 1$, clearly no fiber (Shioda-Tate-Wazir)

- Only 384 (of 495515) have only genus one fibers

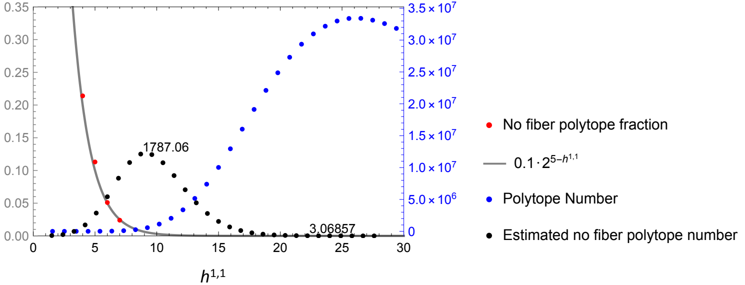
Do the others really have non-toric elliptic/g1 fibers?

$h^{1,1} = 140(194)$ largest known value w/o explicit g1 (elliptic) fiber

Results II: Small $h^{1,1}$

Probability that a CY3 is not g1/elliptic fibered decreases as $2^{-h^{1,1}}$ for $h^{1,1} > 1$

$h^{1,1}$	2	3	4	5	6	7
# without fiber Δ_2	23	91	256	562	872	1202
Total #	36	244	1197	4990	17101	50376
%	0.639	0.373	0.214	0.113	0.051	0.024



Why exponentially unlikely to not have fiber?

Theorem (Oguiso/Wilson):
A Calabi-Yau 3-fold X, X is genus one (or elliptically) fibered iff there exists a divisor $D \in H^2(X, \mathbb{Q})$ that satisfies $D^3 = 0, D^2 \neq 0$, and $D \cdot C \geq 0$ for all algebraic curves $C \subset X$.

Assuming "random" data for triple intersection form C_{ijk},
how likely is this to occur?

Possible obstructions:

A) Number theoretic (no solution to $C_{ijk} d_i d_j d_k = 0$ over integers)

B) Cone obstruction, no solution over reals when $D \subset$ positive cone

Consider each in turn

Number theoretic obstructions

For example:

$$x^3 + x^2y + y^3 + 2z^3 + 4w^3 = 0$$

has no solutions over the integers $\mathbb{Z}$ (or over $\mathbb{Q}$); ($\mathbb{Z}_2$ obstruction)

Mordell (1937) identified homogeneous degree d polynomial in d^2 variables with obstruction

Subsequent conjectures: d^2 is maximum number of variables with obstruction

Proven for $d = 1, 2$

Counterexample: quartic with 17 variables has obstruction!

Heath-Brown (1983): every non-singular cubic in ≥ 10 variables with rational coefficients has nontrivial rational zero.

Also proven for general cubic in ≥ 16 variables

Upshot: no number-theoretic obstruction when $h^{1,1}(X) > 15$ (likely 9)

Cone obstructions: apparently exponentially suppressed

Simple heuristic argument:

Assume cone has $D = \sum_i d_i D_i$, $d_i \geq 0$

Look for positive solution of cubic $\sum_{i,j,k} C_{ijk} d_i d_j d_k = 0$

Proceed by induction:

First, check $M = 2$, $\sum_{i,j,k}^{M} C_{ijk} d_i d_j d_k = 0$
$\sim$ cubic in two variables, has ≥ 1 real solution; 50% chance in cone

Add one variable: pick random other numbers in cone; probability solution in last variable is positive: 1/2, ...

$\Rightarrow$ suggests probability $\leq\sim 2^{-h^{1,1}}$ that no fiber exists

Very heuristic argument, but matches data!

Strong evidence: almost all known CY3's have elliptic/g1 fibers

Supported by other recent work, particularly Anderson + Gray + collaborators

E.g. all CICY threefolds with $h^{1,1} > 4$ have g1/elliptic fibers
[Anderson, strings 2018 talk]

If most Calabi-Yau threefold are elliptic/g1 fibered
+ finite number of elliptic/g1 fibered CY threefolds
$\Rightarrow$ would prove finite number of CY threefolds!

Classification of elliptic/g1 CY threefolds $\Rightarrow$ CY3's,
non-fibered threefolds $\sim$ special cases

Note: all elliptic CY's connected by extremal transitions $\rightarrow\sim$ Reid's fantasy?

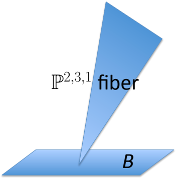
Structure of fibrations: investigate in more detail

Much recent work (see Huang/WT paper), only touch on some points here

Close connection between "Tate form" general Weierstrass model

$$y^2 + a_1 yx + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

and simple "standard stacking" $\mathbb{P}^{2,3,1}$ (F_{10}) fibered polytopes



$$v_x = (0,0,1,0), v_y = (0,0,0,1), v_z = (0,0,-2,-3)$$

$$v_i^{(a)} = (v_{i,1}^{(B)}, v_{i,2}^{(B)}, -2, -3)$$

geometry $\Rightarrow \nabla = \Delta^* = \{$monomials in Tate$\}$ ($\sim$ "top" construction)

(some subtleties with non-Higgsable and tuned gauge groups, $\Delta = \Delta^{**}$)

These constructions dominate at large Hodge numbers
(other fibers problematic when base contains -12 curve)

Systematically implement Tate forms on toric bases [Huang/WT]

$\Rightarrow$ All but 18 (of 1827) Hodge pairs with $h^{1,1}$ or $h^{2,1} \geq 240$ realized directly.

Looked at other examples: all elliptic fibrations with more subtle structure

• Exotic matter: usual SU(6) Tate tuning: ord $(a_1, a_2, a_3, a_4, a_6) = (0, 1, 3, 3, 6)$.
Gives generic (**6**, **15**) SU(6) matter.
Exotic tuning: (0, 2, 2, 4, 6) gives 3-index antisymmetric (**20**) rep

• Large tunings:
e.g. $(h^{1,1}, h^{2,1}) = (135, 15) \rightarrow (261, 9)$, w/ SO(20) on -4 curve

• Automatic U(1)'s on some examples

• Gauge groups on non-toric curves
(examples at lower Hodge numbers w/genus ≥ 1)

Cases with many fibers

Some polytopes give many distinct fibrations (see also AGGL, …)

In some cases from symmetries of polytope

Polytopes with most fibrations: e.g.
$(h^{1,1}, h^{2,1}) = (149, 1)[58], (145, 1)[37], (144, 2)[37]$
occur at small $h^{2,1}$, large $h^{1,1}$

Possibly related observation [WT/Wang]: dominant CY4's after multiple blow-ups are mirrors of known CY's with small $h^{1,1}$ (?)

Further directions

• Genus one/multisection structure at small $h^{1,1}$ for $F_{1,2,4}$ fibers

• Similar fibration analysis for CY4's

• Understand of structure of effective cone, triple intersection
Proof finite number CY3's?

• Reid fantasy extended to non-elliptic CY3's?

• Physics: use understanding of elliptic/g1 fibration structure to better understand heterotic, II, F-theory compactifications