Open-string T-duality and applications to non-geometric backgrounds

Erik Plauschinn

LMU Munich

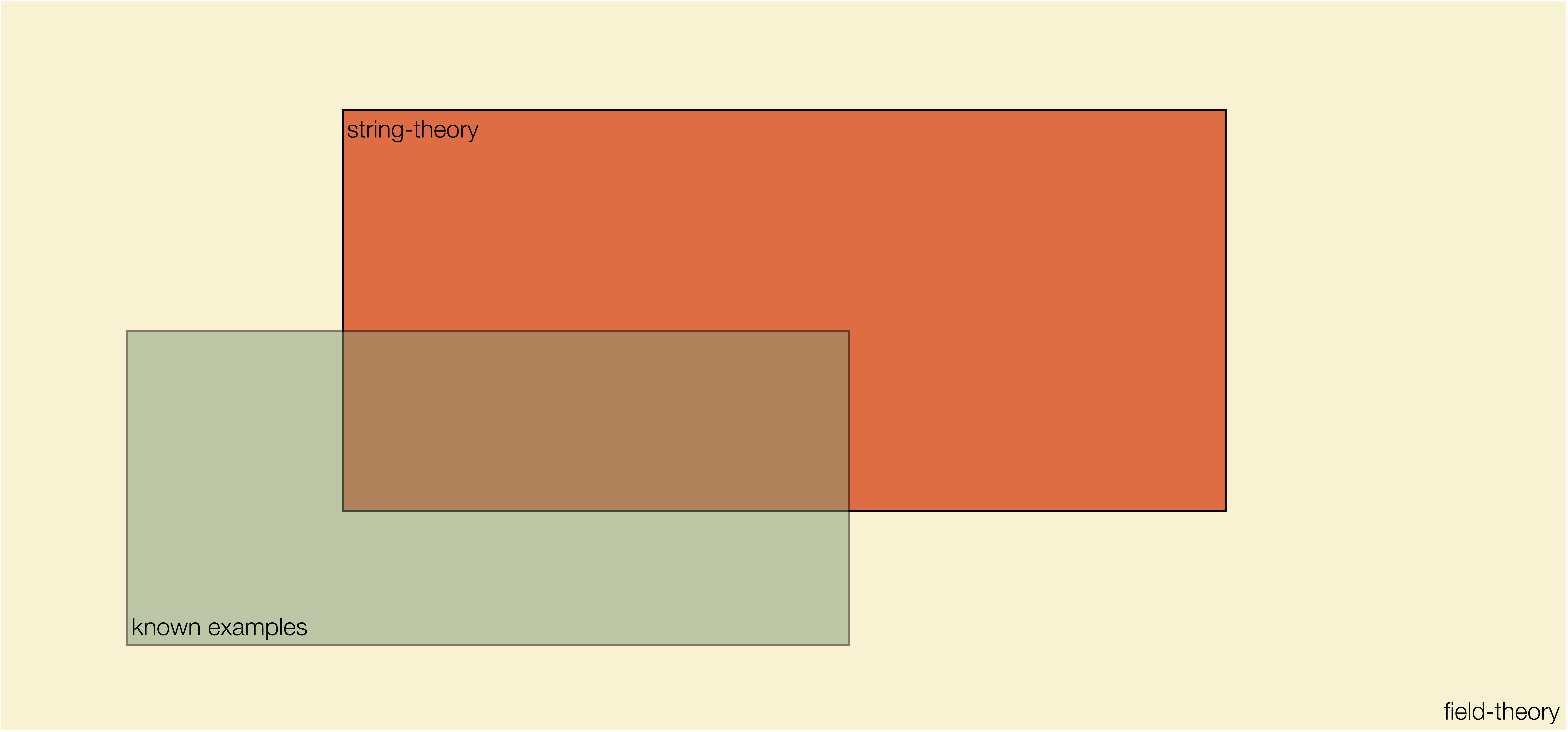
Geometry and Strings — Ringberg Castle

27.07.2018

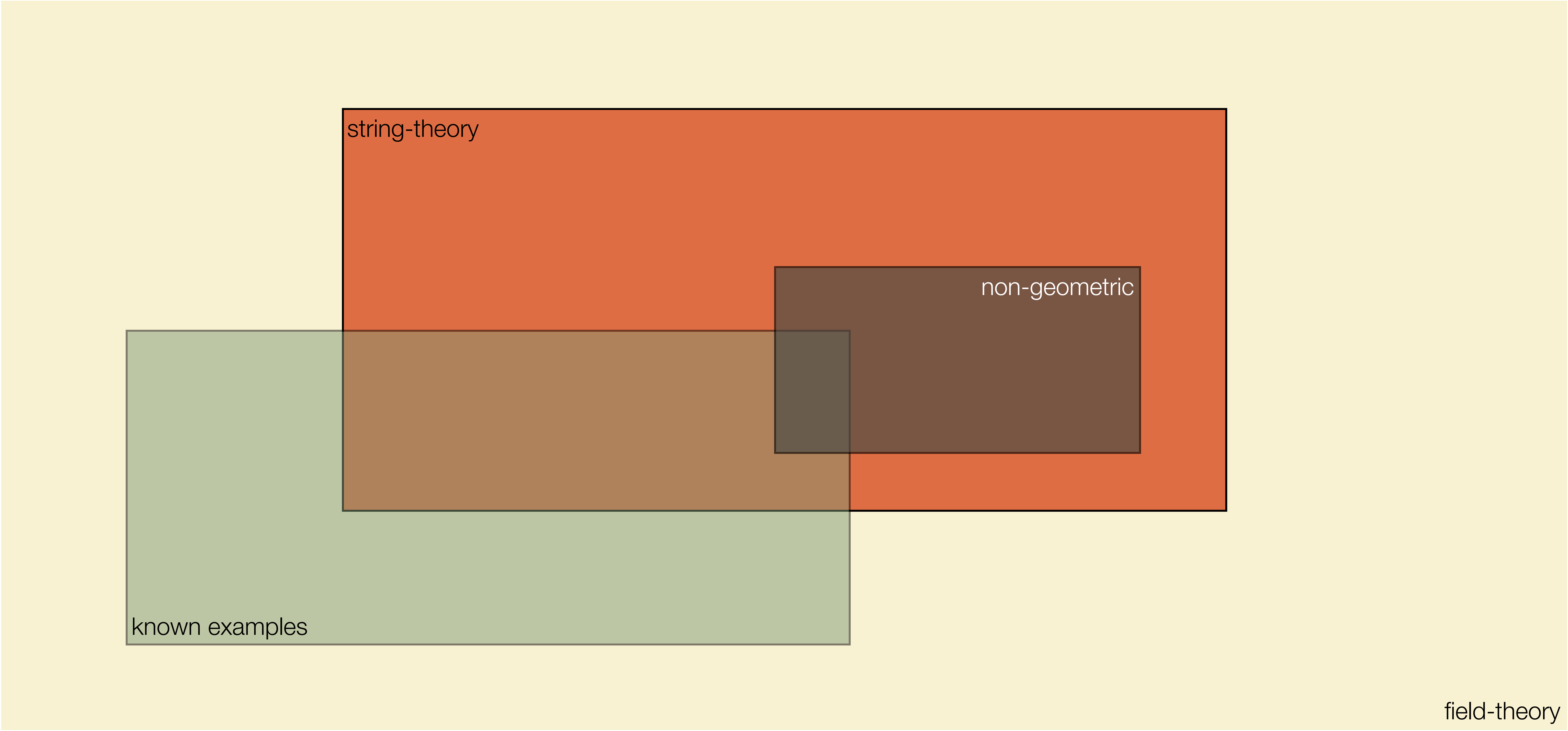
This talk is based on work together with F. Cordonier-Tello and D. Lüst ::

Open-string T-duality and applications to non-geometric backgrounds [arXiv:1806.01308]

Non-geometric backgrounds in string-theory ::



Non-geometric backgrounds in string-theory ::



Non-geometric backgrounds ::

1) Cannot be described using Riemannian geometry (CFT description).

2) Are globally-defined using (T-)duality transformations.

Non-geometric backgrounds ::

1) Cannot be described using Riemannian geometry (CFT description).
2) Are globally-defined using (T-)duality transformations.

Properties ::

- Give rise to non-commutative & non-associative structures.

  Blumenhagen, Plauschinn - 2010
  Lüst - 2010
  Mylonas, Schupp, Szabo - 2012

- Used for moduli stabilization and inflation.

  Shelton, Taylor, Wecht - 2006
  ...

- Provide origin for gauged supergravities.

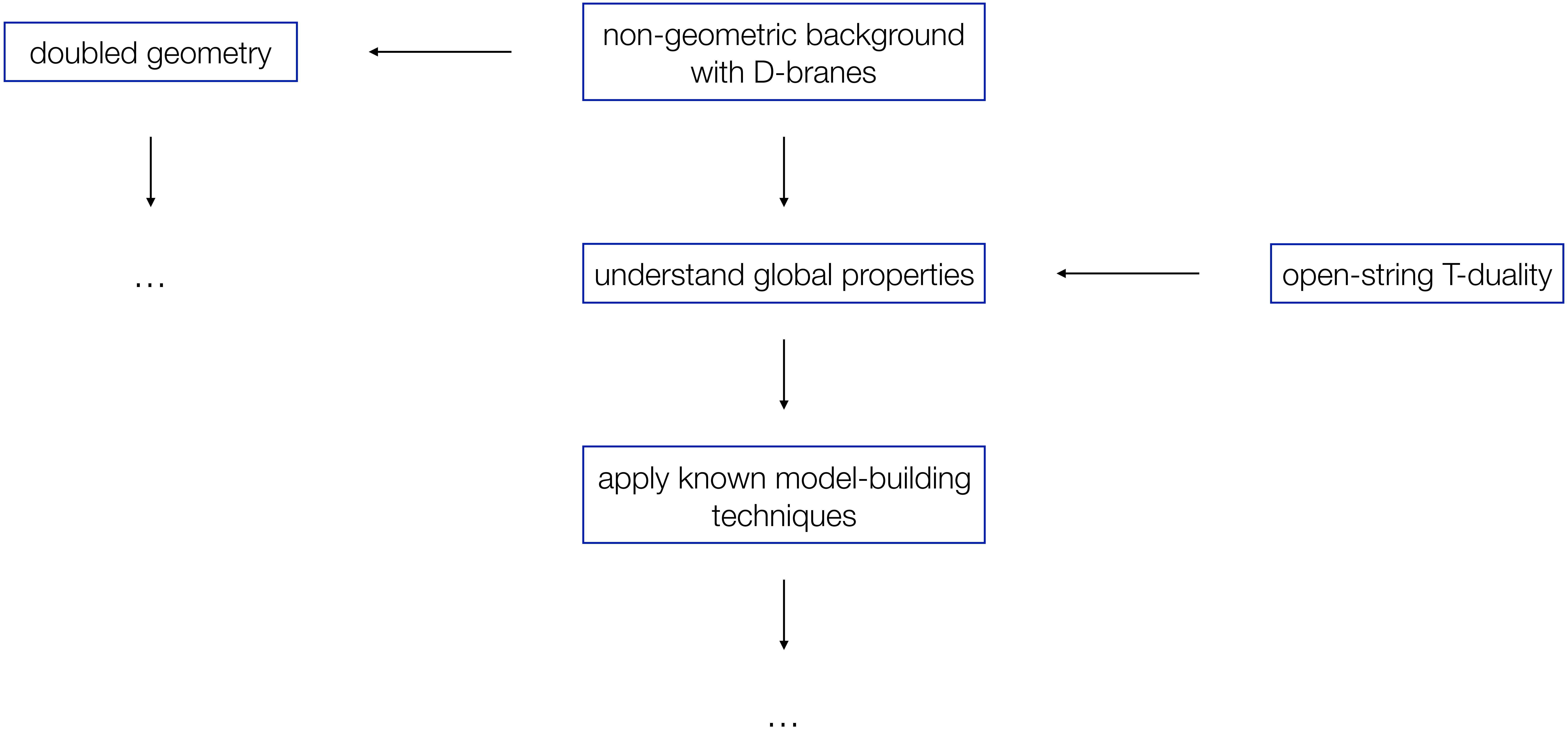
  Grana, Louis, Waldram - 2005
  Cassani - 2008
  Blumenhagen, Font, Plauschinn - 2015

- Needed for mirror symmetry and heterotic/F-theory duality.

  Grana, Louis, Waldram - 2005
  Malmedier, Morrison - 2014
  Gu, Jockers - 2014
  Font, Garcia-Etxebarria, Lüst, Massai, Mayrhofer - 2016

motivation :: d-branes

Objective :: investigate non-geometric backgrounds from an open-string world-sheet perspective.



This talk ::

1) Analyse global properties of D-branes in non-geometric T-fold backgrounds.

2) Discuss Buscher's procedure for open strings (including technical details).

Alvarez, Barbon, Borlaf - 1996
Dorn, Otto - 1996
Förste, Kehagias, Schwager - 1996
Albertsson, Lindström, Zabzine - 2004

outline

1. motivation
2. d-branes & non-geometry
3. open-string t-duality
4. summary

outline

outline

outline

1. motivation
2. d-branes & non-geometry
    a) closed-string sector
    b) boundary conditions
3. open-string t-duality
4. summary

The T-duality group for toroidal compactifications is $O(D,D;\mathbb{Z})$ — which contains ::

- A-transformations ($\mathsf{A} \in GL(D,\mathbb{Z})$)

$$\mathcal{O}_{\mathsf{A}} = \begin{pmatrix} \mathsf{A}^{-1} & 0 \\ 0 & \mathsf{A}^T \end{pmatrix} \longrightarrow \text{diffeomorphisms}$$

- B-transformations (B_{ij} an anti-symmetric matrix)

$$\mathcal{O}_{\mathsf{B}} = \begin{pmatrix} \mathbb{1} & 0 \\ \mathsf{B} & \mathbb{1} \end{pmatrix} \longrightarrow \text{gauge transformations } b \to b + \alpha' \mathsf{B}$$

- β-transformations (β^{ij} an anti-symmetric matrix)

$$\mathcal{O}_{\beta} = \begin{pmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{pmatrix}$$

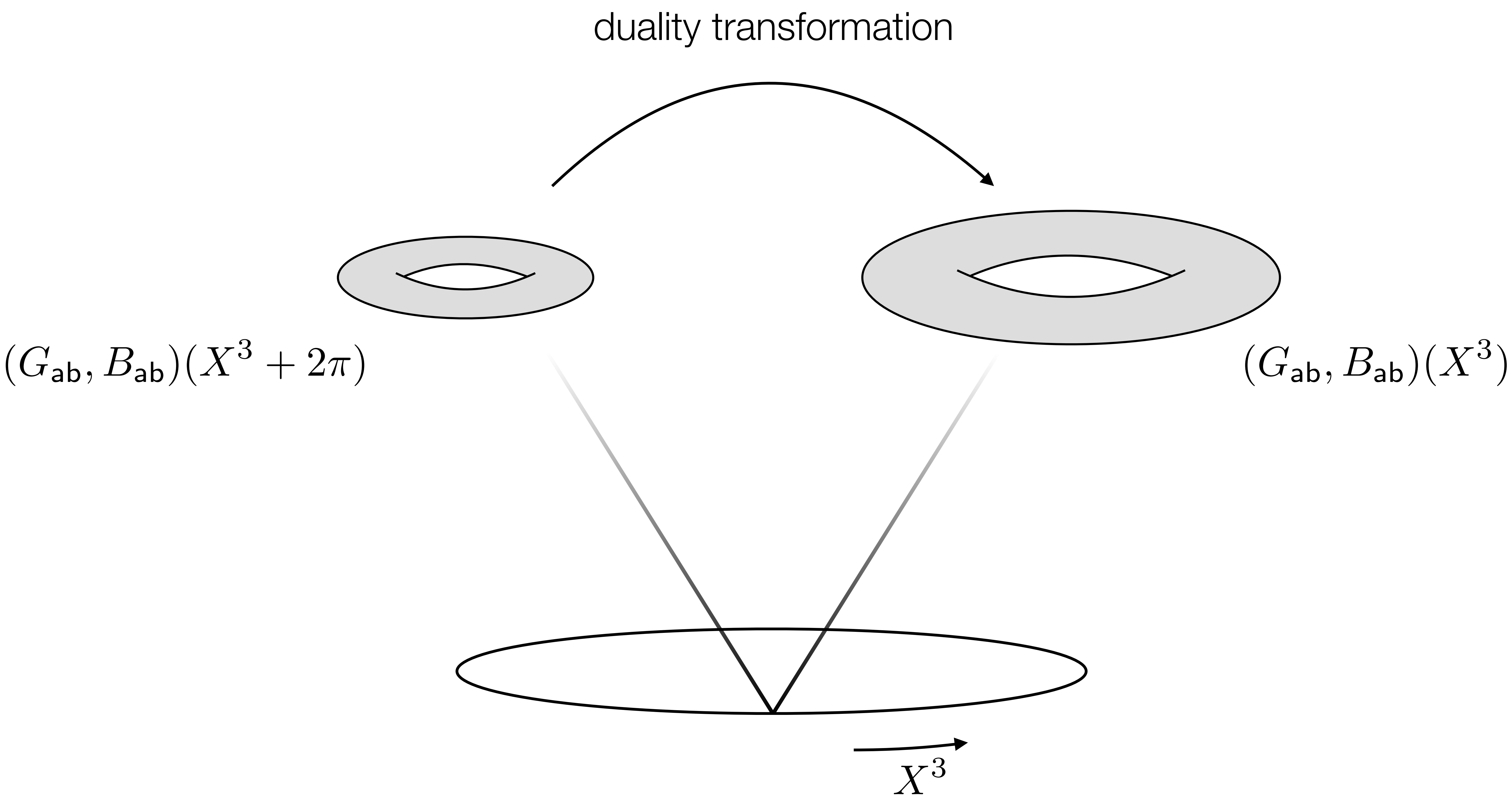
- factorized duality (E_{i} with only non-zero $E_{\mathsf{ii}} = 1$)

$$\mathcal{O}_{\pm\mathsf{i}} = \begin{pmatrix} \mathbb{1} - E_{\mathsf{i}} & \pm E_{\mathsf{i}} \\ \pm E_{\mathsf{i}} & \mathbb{1} - E_{\mathsf{i}} \end{pmatrix} \longrightarrow \text{T-duality transformations } g_{\mathsf{ii}} \to \frac{\alpha'^2}{g_{\mathsf{ii}}}$$

The standard example for a non-geometric background is a $\mathbb{T}^2$-fibration over a circle.

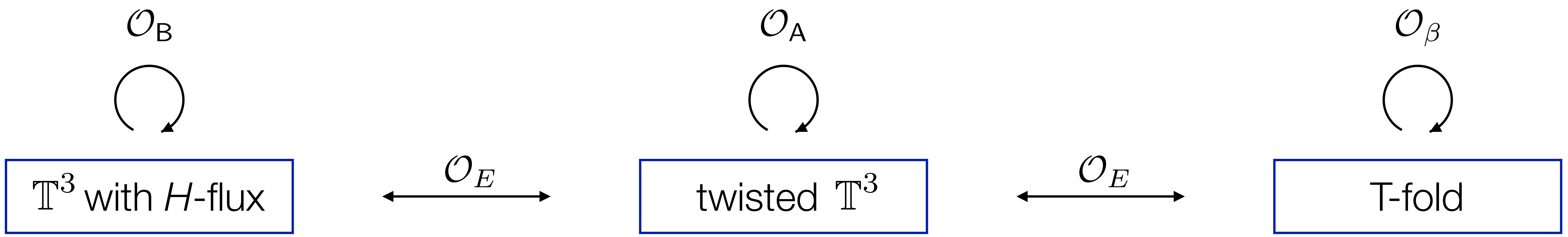
$$G_{ij} = \begin{pmatrix} G_{\mathsf{ab}}(X^3) & 0 \\ 0 & R_3^2 \end{pmatrix}$$

$$B_{ij} = \begin{pmatrix} B_{\mathsf{ab}}(X^3) & 0 \\ 0 & 0 \end{pmatrix}$$



non-geometry :: fibrations II

The non-geometric background is part of a family of $\mathbb{T}^2$-fibrations ::



A three-torus with *H*-flux is characterized as follows ::

1. Metric and *B*-field

$$G_{ij} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \qquad B_{ij} = \begin{pmatrix} 0 & +\frac{\alpha'}{2\pi} h X^3 & 0 \\ -\frac{\alpha'}{2\pi} h X^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h \in \mathbb{Z}\,.$$

2. The background is well-defined under $X^3 \to X^3 + 2\pi$ using a gauge transformation.

3. The *H*-flux $H = dB$ can be expressed in a vielbein basis as

$$H = \frac{\alpha'}{2\pi} \frac{h}{R_1 R_2 R_3}\, e^1 \wedge e^2 \wedge e^3\,, \qquad H_{123} = \frac{\alpha'}{2\pi} \frac{h}{R_1 R_2 R_3}\,.$$

After a T-duality along X^1 one obtains a twisted three-torus ::

1. Metric and *B*-field

$$G_{ij} = \begin{pmatrix} \frac{\alpha'^2}{R_1^2} & -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & 0 \\ -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & R_2^2 + \frac{\alpha'^2}{R_1^2}\left[\frac{h}{2\pi}X^3\right]^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \qquad B_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h \in \mathbb{Z}\,.$$

2. The background is well-defined under $X^3 \to X^3 + 2\pi$ using a diffeomorphism.

3. A geometric *f*-flux is defined via a vielbein basis as

$$de^a = \frac{1}{2} f_{bc}{}^a\, e^b \wedge e^c\,, \qquad f_{23}{}^1 = \frac{\alpha'}{2\pi}\,\frac{h}{R_1\,R_2\,R_3}\,.$$

Kachru, Schulz, Tripathy, Trivedi - 2002

A second T-duality along X^2 gives the T-fold background ::

1. Metric and *B*-field

$$G_{ij} = \begin{pmatrix} \frac{R_2^2}{\rho} & 0 & 0 \\ 0 & \frac{R_1^2}{\rho} & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \qquad B_{ij} = \frac{1}{\rho}\begin{pmatrix} 0 & -\frac{\alpha'}{2\pi}h\,X^3 & 0 \\ +\frac{\alpha'}{2\pi}h\,X^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{aligned} &\rho = \tfrac{R_1^2 R_2^2}{\alpha'^2} + \left[\tfrac{h}{2\pi}X^3\right]^2, \\ &h \in \mathbb{Z}\,. \end{aligned}$$

2. The background is well-defined under $X^3 \to X^3 + 2\pi$ using a β-transformation.

3. A non-geometric *Q*-flux is defined via a vielbein basis and $(G-B)^{-1} = g - \beta$ as

$$Q_i{}^{jk} = \partial_i\,\beta^{jk}\,, \qquad Q_3{}^{12} = \frac{\alpha'}{2\pi}\,\frac{h}{R_1\,R_2\,R_3}\,.$$

Hull - 2004

outline

1. motivation
2. d-branes & non-geometry
   a) closed-string sector
   b) boundary conditions
3. open-string t-duality
4. summary

The world-sheet action for the open string takes the form (Σ is a 2d manifold with $\partial\Sigma \neq \varnothing$)

$$\mathcal{S} = -\frac{1}{2\pi\alpha'}\int_\Sigma \left[\frac{1}{2}G_{ij}\,dX^i\wedge\star dX^j + \frac{i}{2}B_{ij}\,dX^i\wedge dX^j\right] - \frac{1}{2\pi\alpha'}\int_{\partial\Sigma}\left[2\pi i\,\alpha'\,a_i\,dX^i\right].$$

The possible boundary conditions for X^i are

Dirichlet $\quad 0 = \left(dX^{\hat{i}}\right)_{\mathsf{tan}}\,,$

Neumann $\quad 0 = G_{ai}\left(dX^i\right)_{\mathsf{norm}} + 2\pi\alpha' i\,\mathcal{F}_{ab}\left(dX^b\right)_{\mathsf{tan}}\,,$

$$\left(dX^i\right)_{\mathsf{tan}} \equiv t^{\mathsf{a}}\,\partial_{\mathsf{a}}X^i\,ds\big|_{\partial\Sigma}\,,$$

$$\left(dX^i\right)_{\mathsf{norm}} \equiv n^{\mathsf{a}}\,\partial_{\mathsf{a}}X^i\,ds\big|_{\partial\Sigma}\,,$$

$$2\pi\alpha'\mathcal{F} = 2\pi\alpha' F + B\,,$$

$$F = da\,.$$

The open-string boundary conditions can be expressed using (restriction to $\partial\Sigma$ is understood)

$$\begin{pmatrix} \mathsf{D} \\ \mathsf{N} \end{pmatrix} = \begin{pmatrix} \alpha' & 0 \\ 2\pi\alpha'\mathcal{F} & G \end{pmatrix} \begin{pmatrix} i\,(dX)_{\mathsf{tan}} \\ (dX)_{\mathsf{norm}} \end{pmatrix} .$$

A particular type of D-brane is selected using a projection operator

$$\Pi = \begin{pmatrix} \Delta & 0 \\ 0 & \mathbb{1} - \Delta \end{pmatrix}, \qquad \Delta^2 = \Delta\,.$$

Question :: are D-branes globally well-defined on non-geometric backgrounds?

The coordinate differentials behave under transformations $\mathcal{O} \in O(D,D;\mathbb{Z})$ as

$$\begin{pmatrix} i\,(dX)_{\mathsf{tan}} \\ (dX)_{\mathsf{norm}} \end{pmatrix} \xrightarrow{\quad \mathcal{O} \quad} \begin{pmatrix} i\,(d\tilde{X})_{\mathsf{tan}} \\ (d\tilde{X})_{\mathsf{norm}} \end{pmatrix} = \Omega \begin{pmatrix} i\,(dX)_{\mathsf{tan}} \\ (dX)_{\mathsf{norm}} \end{pmatrix},$$

where

$\mathbb{T}^3$ with H-flux
$$\Omega_{\mathsf{B}} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix},$$

twisted $\mathbb{T}^3$
$$\Omega_{\mathsf{A}} = \begin{pmatrix} \mathsf{A}^{-1} & 0 \\ 0 & \mathsf{A}^{-1} \end{pmatrix},$$

T-fold
$$\Omega_{\beta} = \begin{pmatrix} \mathbb{1} + 2\pi\,\beta\,\mathcal{F} & \frac{1}{\alpha'}\,\beta\,G \\ \frac{1}{\alpha'}\,\beta\,G & \mathbb{1} + 2\pi\,\beta\,\mathcal{F} \end{pmatrix}.$$

Boundary conditions for the previous examples are well-defined using $O(D,D;\mathbb{Z})$ transformations

$$\begin{aligned}\begin{pmatrix}\mathsf{D}\\ \mathsf{N}\end{pmatrix}_{X^3+2\pi} &= \begin{pmatrix}\alpha' & 0\\ 2\pi\alpha'\mathcal{F} & G\end{pmatrix}_{X^3+2\pi}\begin{pmatrix} i\,(d\tilde{X})_{\mathsf{tan}}\\ (d\tilde{X})_{\mathsf{norm}}\end{pmatrix}\\ &= \mathcal{O}_\star\begin{pmatrix}\alpha' & 0\\ 2\pi\alpha'\mathcal{F} & G\end{pmatrix}_{X^3}\Omega_\star^{-1}\begin{pmatrix} i\,(d\tilde{X})_{\mathsf{tan}}\\ (d\tilde{X})_{\mathsf{norm}}\end{pmatrix}\\ &= \mathcal{O}_\star\begin{pmatrix}\mathsf{D}\\ \mathsf{N}\end{pmatrix}_{X^3}, \qquad\qquad \star = (\mathsf{B},\mathsf{A},\beta)\,.\end{aligned}$$

The projection onto a particular D-brane has to be performed after the transformation

$$\Pi\left[\begin{pmatrix}\mathsf{D}\\ \mathsf{N}\end{pmatrix}_{X^3+2\pi}\right] = \Pi\left[\mathcal{O}_\star\begin{pmatrix}\mathsf{D}\\ \mathsf{N}\end{pmatrix}_{X^3}\right].$$

Summary ::

- When applying T-duality transformations to geometric $\mathbb{T}^D$-fibrations,
- non-geometric backgrounds can be obtained.

- Open-string boundary conditions are well-defined for such fibrations.

outline

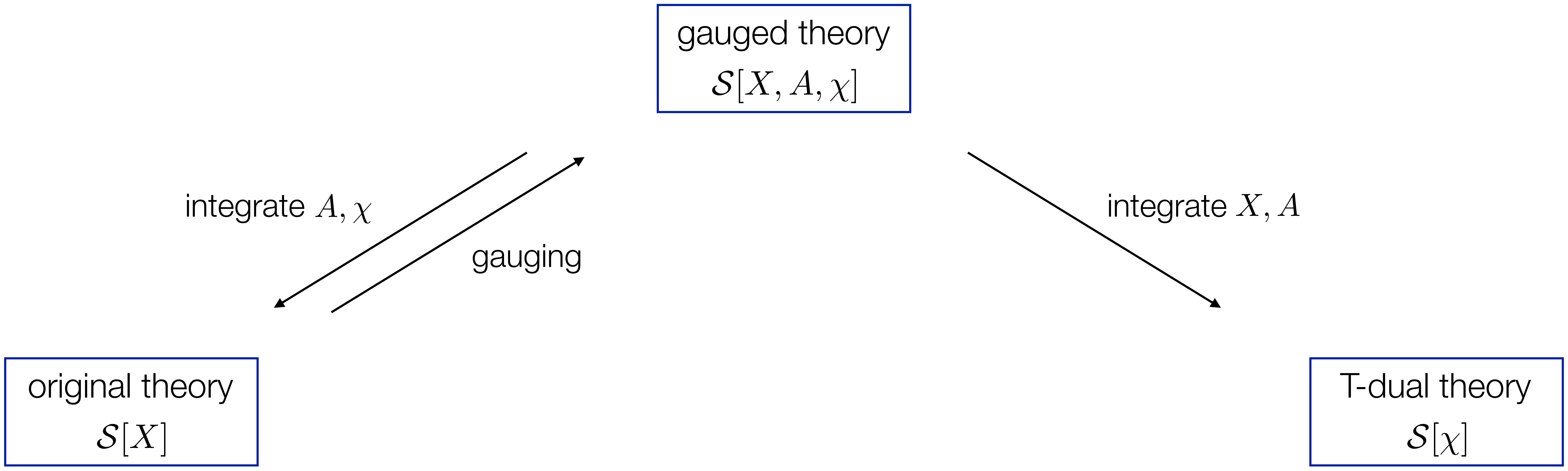
1. motivation
2. d-branes & non-geometry
3. open-string t-duality
4. summary

outline

1. motivation
2. d-branes & non-geometry
3. open-string t-duality
    a) generalities
    b) neumann
    c) dirichlet
    d) summary
4. summary

T-duality transformations for curved backgrounds are obtained following **Buscher's procedure** ::

1) Identify a **global symmetry** (isometry) of the world-sheet action.
2) **Gauge** the global symmetry by introducing a gauge field.
3) **Integrate-out** the gauge field.



The world-sheet action for the open string takes the form (Σ is a 2d manifold with $\partial\Sigma \neq \varnothing$)

$$\mathcal{S} = -\frac{1}{2\pi\alpha'}\int_{\Sigma}\left[\frac{1}{2}G_{ij}\,dX^i \wedge \star dX^j + \frac{i}{2}B_{ij}\,dX^i \wedge dX^j\right] - \frac{1}{2\pi\alpha'}\int_{\partial\Sigma}\left[2\pi i\,\alpha'\,a_i\,dX^i\right].$$

The possible boundary conditions for X^i are

Dirichlet $\quad 0 = \left(dX^{\hat{i}}\right)_{\mathsf{tan}}\,,$

Neumann $\quad 0 = G_{ai}\left(dX^i\right)_{\mathsf{norm}} + 2\pi\alpha' i\,\mathcal{F}_{ab}\left(dX^b\right)_{\mathsf{tan}}\,,$

$$\left(dX^i\right)_{\mathsf{tan}} \equiv t^{\mathsf{a}}\,\partial_{\mathsf{a}}X^i\,ds\big|_{\partial\Sigma}\,,$$

$$\left(dX^i\right)_{\mathsf{norm}} \equiv n^{\mathsf{a}}\,\partial_{\mathsf{a}}X^i\,ds\big|_{\partial\Sigma}\,,$$

$$2\pi\alpha'\mathcal{F} = 2\pi\alpha' F + B\,,$$

$$F = da\,.$$

The Hodge decomposition theorem for manifolds with boundaries can be expressed using

- closed forms $$C^p = \{\omega \in \Omega^p : d\omega = 0\}\,,$$
- exact forms $$E^p = \{\omega \in \Omega^p : \omega = d\eta,\, \eta \in \Omega^{p-1}\}\,,$$
- closed & co-closed, vanishing normal part $$CcC^p_N = \{\omega \in \Omega^p : d\omega = 0, d^\dagger\omega = 0, \omega_{\text{norm}} = 0\}\,.$$

For closed forms on then finds $C^p = E^p \oplus CcC^p_N$.

e.g. Capell, DeTurck, Gluck, Miller - 2005

This implies for Dirichlet directions $X^{\hat{i}}$ that $dX^{\hat{i}}$ is exact.

For Buscher's procedure, one assumes that the action is invariant under a global transformation

$$\delta_\epsilon X^i = \epsilon\, k^i(X)\,, \qquad\qquad \epsilon = \text{const.} \ll 1\,.$$

The variation of the action vanishes provided that

$$\mathcal{L}_k\, G = 0\,,$$

$$\mathcal{L}_k\, B = dv\,,$$

$$2\pi\alpha' \mathcal{L}_k a\big|_{\partial\Sigma} = (-v + d\omega)\big|_{\partial\Sigma}\,.$$

v globally-defined one-form on Σ,

ω globally-defined function on $\partial\Sigma$, The global symmetry can be gauged by introducing a gauge field A (and a Lagrange multiplier χ)

$$\hat{\mathcal{S}} = -\frac{1}{2\pi\alpha'}\int_\Sigma \Big[\tfrac{1}{2}G_{ij}(dX^i + k^i A)\wedge\star(dX^j + k^j A) - \tfrac{i}{2}B_{ij}\,dX^i\wedge dX^j - i\,(v - \iota_k B + d\chi)\wedge A\Big] - \frac{1}{2\pi\alpha'}\int_{\partial\Sigma}\Big[2\pi i\,\alpha'\,a_a\,dX^a - i\,\Omega_{\partial\Sigma}\Big]\,.$$

The local symmetry transformations take the form

$$\hat{\delta}_\epsilon X^i = \epsilon\, k^i\,, \qquad \hat{\delta}_\epsilon A = -d\epsilon\,, \qquad \hat{\delta}_\epsilon \chi = -\epsilon\,\iota_k v\,.$$

The possible boundary conditions for the gauge field are

Dirichlet $$0 = A_{\mathsf{tan}}\big|_{\partial\Sigma}\,,$$

Neumann $$0 = G_{ai}\,k^i A_{\mathsf{norm}} + 2\pi\alpha' i\,\mathcal{F}_{ab}\,k^b A_{\mathsf{tan}}\big|_{\partial\Sigma}\,.$$

Albertsson, Lindström, Zabzine - 2004

The boundary term has the following form ::

- For Dirichlet boundary conditions the variation parameter satisfies $\epsilon|_{\partial\Sigma} = 0$ and hence

$$\Omega_{\partial\Sigma} = 0\,.$$

- For Neumann boundary conditions a second Lagrange multiplier is needed and

$$\Omega_{\partial\Sigma} = \big(\chi + \phi + \omega - 2\pi\,\alpha'\,\iota_k a\big)\,A\,,$$

χ globally-defined function on $\partial\Sigma$,

ϕ constant function on $\partial\Sigma$.

outline

1. motivation
2. d-branes & non-geometry
3. open-string t-duality
    a) generalities
    b) neumann
    c) dirichlet
    d) summary
4. summary

The original action is obtained from the gauged action using the Lagrange multipliers ::

- equation of motion for χ (globally-defined on Σ) $\longrightarrow$ $F = dA = 0\,,$
- equation of motion for ϕ (globally-defined on $\partial\Sigma$) $\longrightarrow$ $A_{\mathsf{tan}}|_{\partial\Sigma} = 0\,.$

Using Hodge decomposition for manifolds with boundary the original action is recovered via ($\omega^{\mathsf{m}} \in CcC^1_N$)

$$A \xrightarrow{\;dA=0\;} A = da_{(0)} + a_{(\mathsf{m})}\,\omega^{\mathsf{m}} \xrightarrow{\;A_{\mathsf{tan}}=0\;} a_{(\mathsf{m})} = 0$$

$$\xrightarrow{\;\hat{\delta}_\epsilon A\;} a_{(0)} = 0$$

Integrating-out the gauge field gives the action (with $k^i = (1,0,\ldots,0)$ and $\tilde{B}_{m1} = B_{m1} - v_m$)

$$\check{\mathcal{S}} = -\frac{1}{2\pi\alpha'}\int_\Sigma \Big[\ \ \frac{1}{2}\left(G_{mn} - \frac{G_{m1}G_{n1} - \tilde{B}_{m1}\tilde{B}_{n1}}{G_{11}}\right) dX^m \wedge \star dX^n + \frac{1}{2}\frac{1}{G_{11}}\,d\chi\wedge\star d\chi + \frac{\tilde{B}_{m1}}{G_{11}}\,d\chi\wedge\star dX^m$$

$$-\frac{i}{2}\left(B_{mn} - \frac{\tilde{B}_{m1}G_{n1} - G_{m1}\tilde{B}_{n1}}{G_{11}}\right) dX^m\wedge dX^n - i\,\frac{G_{m1}}{G_{11}}\,dX^m\wedge d\chi + i\,dX^1\wedge(d\chi + v)\ \ \Big]$$

$$-\frac{1}{2\pi\alpha'}\int_{\partial\Sigma}\Big[2\pi i\,\alpha'\,a_a\,dX^a\Big]\,.$$

Interpreting $d\tilde{X}^1 = \pm\frac{1}{\alpha'}\,d\chi$ as the dual coordinate, the Buscher rules can be read-off

$$\check{G}_{11} = \frac{\alpha'^2}{G_{11}}\,,$$

$$\check{G}_{m1} = \pm\alpha'\,\frac{B_{m1}}{G_{11}}\,, \qquad \check{B}_{m1} = \pm\alpha'\,\frac{G_{m1}}{G_{11}}\,,$$

$$\check{G}_{mn} = G_{mn} - \frac{G_{m1}G_{n1} - B_{m1}B_{n1}}{G_{11}}\,, \qquad \check{B}_{mn} = B_{mn} - \frac{B_{m1}G_{n1} - G_{m1}B_{n1}}{G_{11}}\,.$$

Buscher - 1987, 1988

The variation of A on the boundary introduces a constraint, which is implemented as

$$0 = 2\pi\alpha'\,\iota_k a - (\chi + \phi + \omega)\Big|_{\partial\Sigma} \quad \xrightarrow{\text{path integral}} \quad \delta(\phi - \tilde{\chi})_{\partial\Sigma}\,,$$

$$\tilde{\chi} = \chi + \omega - 2\pi\alpha'\iota_k a\,.$$

The Neumann boundary condition for A becomes as Dirichlet condition for $\tilde{\chi}$

$$0 = d\tilde{\chi}\big|_{\partial\Sigma}\,.$$

After integrating-out the gauge field, the path integral takes the form

$$\mathcal{Z} = \int \frac{[\mathcal{D}X^i]\,[\mathcal{D}\chi]}{\mathcal{V}_{\mathsf{gauge}}} \int [\mathcal{D}\phi]\; \delta\big(\phi - \tilde{\chi}\big)_{\partial\Sigma}\; \exp\Big(\check{\mathcal{S}}[X^i,\chi]\Big)\,.$$

Integration over ϕ is trivially performed.

t-duality :: integrating-out the original coordinate

The terms in the action depending on the original coordinate read (with $k^i = (1, 0, \ldots, 0)$)

$$+\frac{i}{2\pi\alpha'}\int_\Sigma (d\chi + v)\wedge dX^1 - \frac{i}{2\pi\alpha'}\int_{\partial\Sigma} 2\pi\alpha' a_1 dX^1 = +\frac{i}{2\pi\alpha'}\int_{\partial\Sigma}\tilde\chi\, dX^1 \,.$$

Expand $dX^1 = dX^1_{(0)} + X^1_{(\mathsf{m})}\,\omega^{\mathsf{m}}$. For X^1 compact and free $X^1_{(\mathsf{m})} \in 2\pi\mathbb{Z}$, and

$$\begin{aligned}
&\int \frac{[\mathcal{D}X^1]}{\mathcal{V}_{\text{gauge}}}\exp\left[\frac{i}{2\pi\alpha'}\int_{\partial\Sigma}\tilde\chi\, dX^1\right] \\
&= \int \frac{\big[\mathcal{D}X^1_{(0)}\big]}{\mathcal{V}_{\text{gauge}}}\sum_{X^1_{(\mathsf{m})}\in 2\pi\mathbb{Z}}\exp\left[\frac{i}{2\pi\alpha'}\int_{\partial\Sigma}\tilde\chi\, X^1_{(\mathsf{m})}\,\omega^{(\mathsf{m})}\right] \\
&= \sum_{m_{(\mathsf{m})}\in\mathbb{Z}}\delta\left[\frac{1}{2\pi\alpha'}\tilde\chi - m_{(\mathsf{m})}\right]_{\partial\Sigma} \qquad \longrightarrow \qquad \tilde\chi\big|_{\partial\Sigma}\in 2\pi\alpha'\,\mathbb{Z}\,.
\end{aligned}$$

The dual coordinate $\tilde X^1 = \pm\frac{1}{\alpha'}\tilde\chi$ is quantized on the boundary and thus compact.

Summary ::

- T-duality along a Neumann direction results in a T-dual Dirichlet direction.
- A Wilson loop along X^1 shifts the dual coordinate as $\tilde{X}^1 = \pm\frac{1}{\alpha'}(\chi + \omega - 2\pi\alpha' a_1)$.
- Momentum modes of X^1 determine winding modes via $\tilde{X}^1|_{\partial\Sigma} \in 2\pi\mathbb{Z}$.
- The dual metric and *B*-field can be identified as (contain open-string gauge flux)

$$\check{G}_{11} = \frac{\alpha'^2}{G_{11}},$$

$$\check{G}_{m1} = \pm\alpha' \frac{\tilde{B}_{m1}}{G_{11}}, \qquad \check{B}_{m1} = \pm\alpha' \frac{G_{m1}}{G_{11}},$$

$$\check{G}_{mn} = G_{mn} - \frac{G_{m1}G_{n1} - \tilde{B}_{m1}\tilde{B}_{n1}}{G_{11}}, \qquad \check{B}_{mn} = B_{mn} - \frac{\tilde{B}_{m1}G_{n1} - G_{m1}\tilde{B}_{n1}}{G_{11}}.$$

- The dual gauge field reads $\check{a} = a_m dX^m$.

outline

1. motivation
2. d-branes & non-geometry
3. open-string t-duality
   a) generalities
   b) neumann
   c) dirichlet
   d) summary
4. summary

The original action is obtained using the Lagrange multiplier ::

- Perform a Hodge decomposition ($\omega^{\mathsf{m}} \in CcC^1_N$) $\quad d\chi = d\chi_{(0)} + \chi_{(\mathsf{m})}\,\omega^{\mathsf{m}}\,,$
- and recall the boundary conditions $\quad A_{\mathsf{tan}}|_{\partial\Sigma} = 0\,.$

Perform then the following steps to recover the original action ::

- The equation of motion for $\chi_{(0)}$ leads to $F = dA = 0$ $\longrightarrow$ $A = da_{(0)} + a_{(\mathsf{m})}\,\omega^{\mathsf{m}}\,.$
- The boundary conditions imply $\quad a_{(\mathsf{m})} = 0\,.$
- The equation of motion for $\chi_{(\mathsf{m})}$ gives $\quad a_{(0)}|_{\partial\Sigma} = 0\,.$
- The gauge symmetry can be used to set $\quad a_{(0)} = 0\,.$

Integrating-out the gauge field leads to the Buscher rules similarly as before ::

$$\check{\mathcal{S}} = -\frac{1}{2\pi\alpha'}\int_\Sigma \Bigg[\ \ \frac{1}{2}\left(G_{mn} - \frac{G_{m1}G_{n1} - \tilde{B}_{m1}\tilde{B}_{n1}}{G_{11}}\right) dX^m \wedge \star dX^n + \frac{1}{2}\frac{1}{G_{11}}\,d\chi\wedge\star d\chi + \frac{\tilde{B}_{m1}}{G_{11}}\,d\chi\wedge\star dX^m$$
$$- \frac{i}{2}\left(B_{mn} - \frac{\tilde{B}_{m1}G_{n1} - G_{m1}\tilde{B}_{n1}}{G_{11}}\right) dX^m\wedge dX^n - i\,\frac{G_{m1}}{G_{11}}\,dX^m\wedge d\chi + i\,dX^1\wedge(d\chi + v)\ \ \Bigg]$$
$$-\frac{1}{2\pi\alpha'}\int_{\partial\Sigma}\Big[2\pi i\,\alpha'\,a_a\,dX^a\Big]\,.$$

The Dirichlet boundary condition for A becomes as Neumann condition for $\tilde{X}^1 = \pm\frac{1}{\alpha'}\chi$

$$0 = \check{G}_{1i}\big(d\tilde{X}^i\big)_{\mathsf{norm}} + i\,\check{B}_{1i}\big(d\tilde{X}^i\big)_{\mathsf{tan}}\,.$$

The terms in the action depending on the original coordinate read (with $k^i = (1,0,\ldots,0)$ and $v=0$)

$$-\frac{i}{2\pi\alpha'}\int_\Sigma dX^1 \wedge d\chi = -\frac{i}{2\pi\alpha'}\int_{\partial\Sigma} 2\pi\alpha' \left[\frac{X^1|_{\partial\Sigma}}{2\pi\alpha'}\, d\chi\right].$$

Expand $d\chi = d\chi_{(0)} + \chi_{(\mathsf{m})}\,\omega^{\mathsf{m}}$, and for X^1 compact perform the path-integral

$$\begin{aligned}
&\int \frac{[\mathcal{D}X^1]}{\mathcal{V}_{\text{gauge}}}\,\exp\left[-\frac{i}{2\pi\alpha'}\int_{\partial\Sigma} X^1|_{\partial\Sigma}\, d\chi\right] \\
&= \int \frac{[\mathcal{D}X^1_0]}{\mathcal{V}_{\text{gauge}}} \sum_{n_{\partial\Sigma}\in\mathbb{Z}} \exp\left[-\frac{i}{2\pi\alpha'}\int_{\partial\Sigma}\left(X^1_0|_{\partial\Sigma} + 2\pi\, n_{\partial\Sigma}\right) d\chi\right] \\
&= \sum_{m_{(\mathsf{m})}\in\mathbb{Z}} \delta\left[\frac{1}{2\pi\alpha'}\,\chi_{(\mathsf{m})} - m_{(\mathsf{m})}\right] \exp\left[-\frac{i}{2\pi\alpha'}\int_{\partial\Sigma} 2\pi\alpha'\,\frac{X^1_0|_{\partial\Sigma}}{2\pi}\,\frac{d\chi}{\alpha'}\right] \qquad \longrightarrow \qquad \chi_{(\mathsf{m})} \in 2\pi\alpha'\mathbb{Z}\,.
\end{aligned}$$

This gives Wilson loop and quantized momenta for the dual coordinate $\tilde{X}^1 = \pm\frac{1}{\alpha'}\chi$.

Summary ::

- T-duality along a Dirichlet direction results in a T-dual Neumann direction.
- The position of $X^1|_{\partial\Sigma}$ determines a Wilson loop for $\tilde{X}^1$.
- Winding modes of X^1 determine momentum modes of $\tilde{X}^1$.

- The dual metric and *B*-field can be identified as

$$\check{G}_{11} = \frac{\alpha'^2}{G_{11}} ,$$

$$\check{G}_{m1} = \pm\alpha' \frac{B_{m1}}{G_{11}} , \qquad \check{B}_{m1} = \pm\alpha' \frac{G_{m1}}{G_{11}} ,$$

$$\check{G}_{mn} = G_{mn} - \frac{G_{m1}G_{n1} - B_{m1}B_{n1}}{G_{11}} , \qquad \check{B}_{mn} = B_{mn} - \frac{B_{m1}G_{n1} - G_{m1}B_{n1}}{G_{11}} .$$

- The dual gauge field reads $\check{a} = \frac{X_0^1|_{\partial\Sigma}}{2\pi} d\tilde{X}^1 + a_m dX^m$.

outline

1. motivation
2. d-branes & non-geometry
3. open-string t-duality
   a) generalities
   b) neumann
   c) dirichlet
   d) summary
4. summary

Summary ::

Neumann boundary conditions	T-duality ⟷	Dirichlet boundary conditions
▪ momentum modes		▪ winding modes
▪ Wilson loop		▪ D-brane position

Here ::

- CFT results are reproduced for curved backgrounds.
- T-duality along Dirichlet directions.
- Inclusion of non-trivial world-sheet topologies.

Paper :: includes the generalization to collective T-duality along multiple directions.

$$
\begin{aligned}
\check{\mathsf{G}}_{mn} = G_{mn} &- \mathsf{k}_{\alpha m} \left[(\mathcal{G}+\mathcal{D})^{-1}\,\mathcal{G}\,(\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \mathsf{k}_{\beta n} \\
&- \mathsf{k}_{\alpha m} \left[(\mathcal{G}+\mathcal{D})^{-1}\,\mathcal{D}\,(\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \tilde{v}_{\beta n} \\
&+ \tilde{v}_{\alpha m} \left[(\mathcal{G}+\mathcal{D})^{-1}\,\mathcal{D}\,(\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \mathsf{k}_{\beta n} \\
&+ \tilde{v}_{\alpha m} \left[(\mathcal{G}+\mathcal{D})^{-1}\,\mathcal{G}\,(\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \tilde{v}_{\beta n}
\end{aligned}
$$

$$
\begin{aligned}
\check{\mathsf{B}}_{mn} = B_{mn} &+ \mathsf{k}_{\alpha m} \left[(\mathcal{G}+\mathcal{D})^{-1}\,\mathcal{D}\,(\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \mathsf{k}_{\beta n} \\
&+ \mathsf{k}_{\alpha m} \left[(\mathcal{G}+\mathcal{D})^{-1}\,\mathcal{G}\,(\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \tilde{v}_{\beta n} \\
&- \tilde{v}_{\alpha m} \left[(\mathcal{G}+\mathcal{D})^{-1}\,\mathcal{G}\,(\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \mathsf{k}_{\beta n} \\
&- \tilde{v}_{\alpha m} \left[(\mathcal{G}+\mathcal{D})^{-1}\,\mathcal{D}\,(\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \tilde{v}_{\beta n}
\end{aligned}
$$

$$
\begin{aligned}
\check{\mathsf{G}}^{\alpha}{}_{n} = &+ \left[(\mathcal{G}+\mathcal{D})^{-1}\,\mathcal{D}\,(\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \mathsf{k}_{\beta n} \\
&+ \left[(\mathcal{G}+\mathcal{D})^{-1}\,\mathcal{G}\,(\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \tilde{v}_{\beta n}
\end{aligned}
$$

$$
\begin{aligned}
\check{\mathsf{B}}^{\alpha}{}_{n} = &- \left[(\mathcal{G}+\mathcal{D})^{-1}\,\mathcal{G}\,(\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \mathsf{k}_{\beta n} \\
&- \left[(\mathcal{G}+\mathcal{D})^{-1}\,\mathcal{D}\,(\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \tilde{v}_{\beta n}
\end{aligned}
$$

$$
\check{\mathsf{G}}^{\alpha\beta} = + \left[(\mathcal{G}+\mathcal{D})^{-1}\,\mathcal{G}\,(\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta}
$$

$$
\check{\mathsf{B}}^{\alpha\beta} = - \left[(\mathcal{G}+\mathcal{D})^{-1}\,\mathcal{D}\,(\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta}
$$

outline

1. motivation
2. d-branes & non-geometry
3. open-string t-duality
4. summary

Summary ::

- Boundary conditions for D-branes on certain flux-backgrounds
- are well-defined using $O(D,D;\mathbb{Z})$ transformations.

- Open-string T-duality via Buscher's procedure has been discussed,
- taking into account non-trivial world-sheet topologies.