Defect TQFT and orbifolds

Nils Carqueville

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spacetime $\xrightarrow{\texttt{QFT}}$ **algebra**

$$\textbf{spacetime} \supset \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathrm{Vect}_{\mathbb{C}} \subset \textbf{algebra}$$

$$\textbf{spacetime} \supset \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \xrightarrow{\quad \textbf{defect TQFT} \quad} \mathrm{Vect}_{\mathbb{C}} \subset \textbf{algebra}$$

Goal. Unify and generalise **orbifold** and **state sum constructions**

Method. defects and higher algebra

Slogans.

- "State sum models = orbifolds of the trivial theory"
- "General orbifolds = state sum constructions internal to some QFT"

Result. Worked out for any n-dimensional **defect TQFT**

Applications.

- "generalised symmetry"
- new dualities
- surface defects in Chern-Simons theory
- improved topological quantum computation via orbifolds

2-dimensional closed TQFT

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Atiyah 1988

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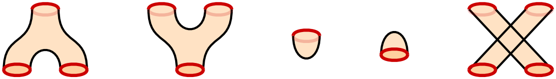

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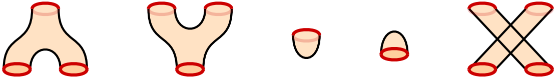

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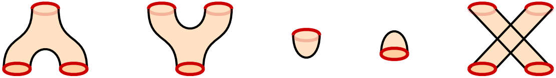

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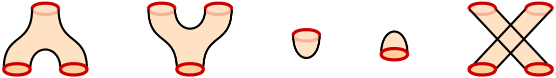

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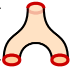
multiplication $\mathcal{Z}\Big($  $\Big)\colon \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$

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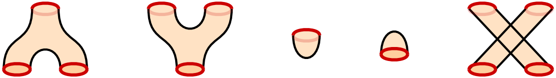

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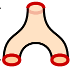
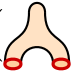
multiplication $\mathcal{Z}\Big($  $\Big)\colon \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$, pairing $\mathcal{Z}\Big($  $\Big)\colon \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathbb{C}$

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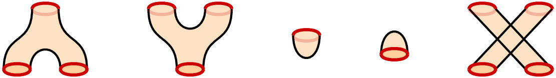

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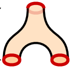
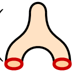
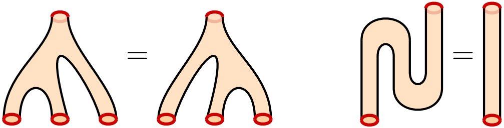
multiplication $\mathcal{Z}\Big($  $\Big) \colon \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$, pairing $\mathcal{Z}\Big($  $\Big) \colon \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathbb{C}$



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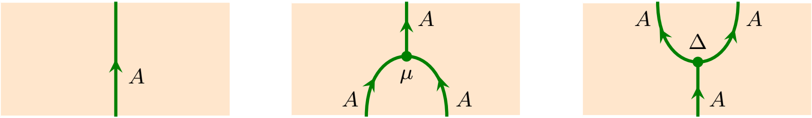
State sum models: ($\supset$ Dijkgraaf-Witten models)

- **input**: separable symmetric Frobenius $\mathbb{C}$-algebra (A, μ, Δ) = matrix algebra
- choose oriented **triangulation** for every bordism(=worldsheet) Σ
- on **Poincaré dual** graph, associate A to edges, (co)multiplication μ, Δ to vertices:



Fukuma/Hosono/Kawai 1992, Lauda/Pfeiffer 2006

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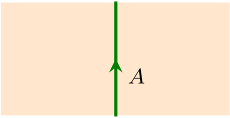

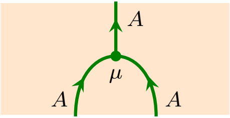

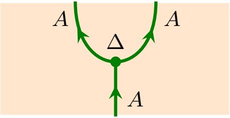

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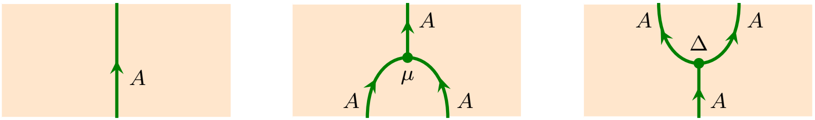

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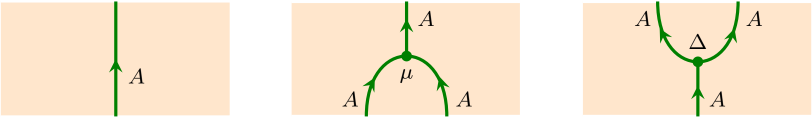

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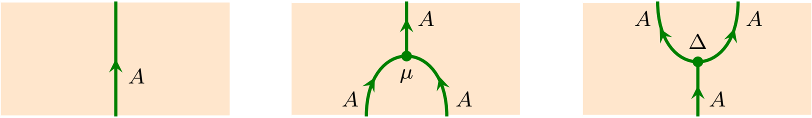

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- define **state sum model**

$$\mathcal{Z}_A^{\mathrm{ss}} \colon \mathrm{Bord}_2 \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

$$S^1 \longmapsto \mathrm{Im}\Big(\pi_k \colon A^{\otimes k} \longrightarrow A^{\otimes k}\Big) \cong Z(A) \quad \text{for all } k$$

$$\Big(\Sigma \colon (S^1)^{\sqcup m} \longrightarrow (S^1)^{\sqcup n}\Big) \longmapsto \Big(\text{induced linear map } Z(A)^{\otimes m} \longrightarrow Z(A)^{\otimes n}\Big)$$

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State sum models in 2d

Theorem.
State sum model for A is independent of choice of triangulation, and $\mathcal{Z}_A^{\mathrm{ss}}(S^1) \cong Z(A)$.

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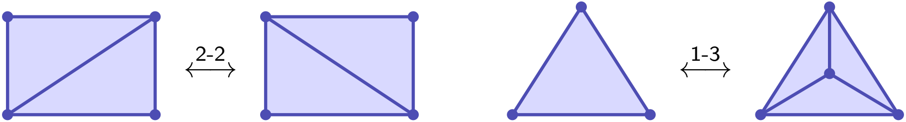

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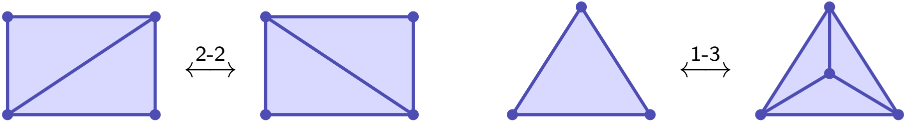

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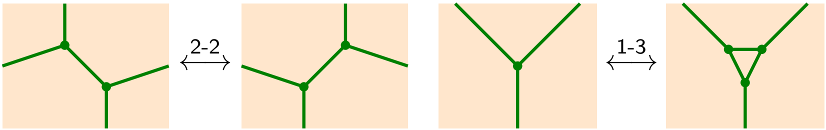

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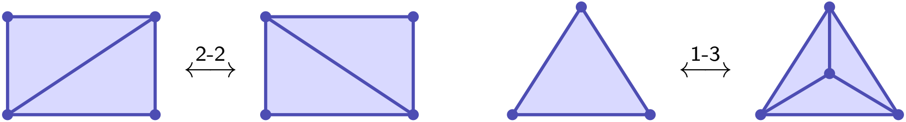

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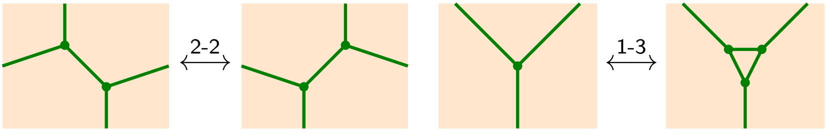

Satisfied for separable symmetric Frobenius $\mathbb{C}$-algebras A!

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Davydov/Kong/Runkel 2011

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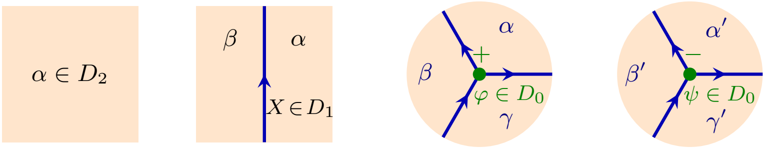

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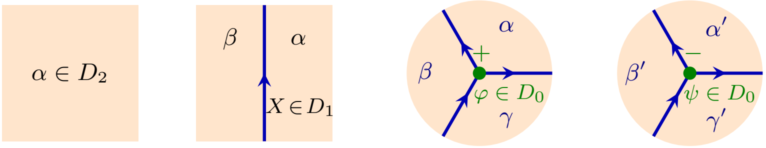

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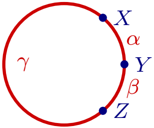

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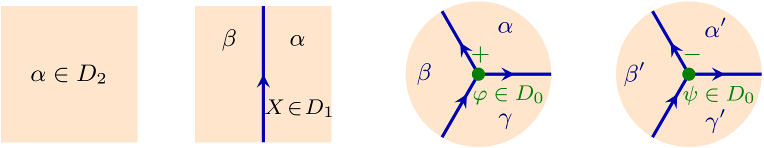
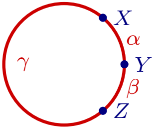
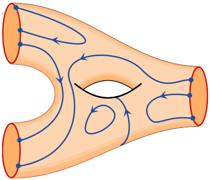


objects: 

morphisms: 

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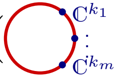
  - $\mathcal{Z}^{\text{triv}}\Bigg($ 

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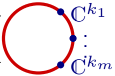
  - $\mathcal{Z}^{\text{triv}}\Big($ 

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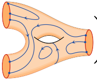
  - $\mathcal{Z}^{\text{triv}}\Big($  $\Big) \overset{\text{def}}{=}$ (evaluate line and point defects in $\operatorname{Vect}_{\mathbb{C}}$)

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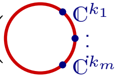
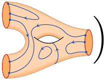
  - $\mathcal{Z}^{\mathrm{triv}}\Big($ 

$\Big) \stackrel{\mathrm{def}}{=} \mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_m}$
  - $\mathcal{Z}^{\mathrm{triv}}\Big($  $\Big) \stackrel{\mathrm{def}}{=}$ (evaluate line and point defects in $\mathrm{Vect}_{\mathbb{C}}$)
- **state sum models 2.0** $\quad \mathcal{Z}^{\mathrm{ss}}\colon \mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}^{\mathrm{ss}}) \longrightarrow \mathrm{Vect}_{\mathbb{C}}$
  - $D_2^{\mathrm{ss}} = \{\text{separable symmetric Frobenius } \mathbb{C}\text{-algebras } A, B, \dots\}$
  - $D_1^{\mathrm{ss}} = \{B\text{-}A\text{-bimodules}\}$
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Davydov/Kong/Runkel 2011

Orbifolds from groups actions

orbifoldable action of finite group G on $\mathcal{Z}\colon \operatorname{Bord}_2^{\mathrm{def}}(\mathbb{D}) \longrightarrow \operatorname{Vect}_{\mathbb{C}}$

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Fröhlich/Fuchs/Runkel/Schweigert 2009, Carqueville/Runkel 2012, Brunner/Carqueville/Plencner 2014

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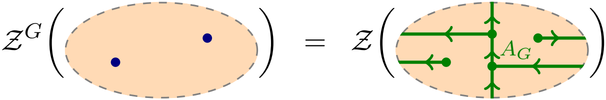

Fröhlich/Fuchs/Runkel/Schweigert 2009, Carqueville/Runkel 2012, Brunner/Carqueville/Plencner 2014

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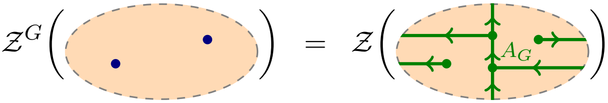

consistent if A_G is **separable symmetric Frobenius algebra** internal to 2-category associated to $\mathcal{Z}$

$\Longrightarrow$ **group orbifolds** from special types of **algebras**

Fröhlich/Fuchs/Runkel/Schweigert 2009, Carqueville/Runkel 2012, Brunner/Carqueville/Plencner 2014

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Let $\mathcal{Z}\colon \mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathrm{Vect}_{\mathbb{C}}$ be any defect TQFT.

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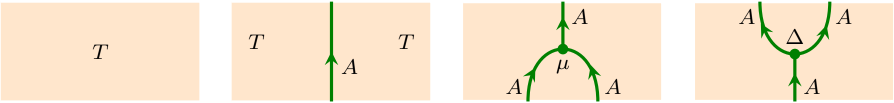

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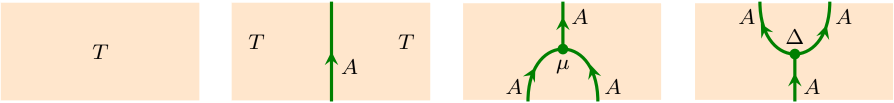

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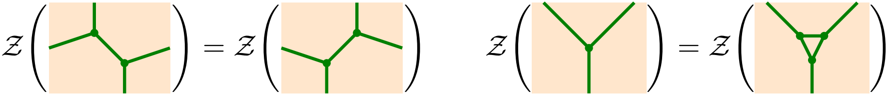

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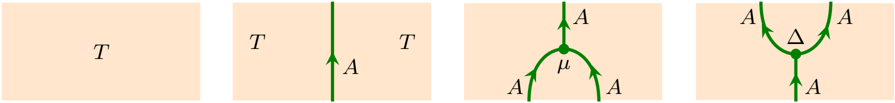

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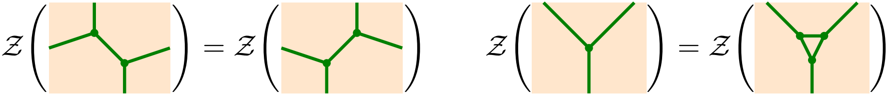

Definition & Theorem.
Applying $\mathcal{Z}$ to $\mathcal{A}$-decorated dual triangulations gives **$\mathcal{A}$-orbifold TQFT**

$$\mathcal{Z}_{\mathcal{A}}\colon \mathrm{Bord}_2 \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

Carqueville/Runkel 2012

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Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013

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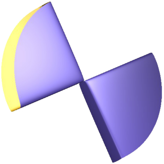
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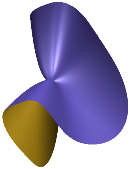

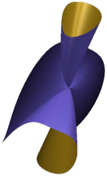
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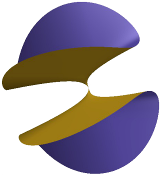

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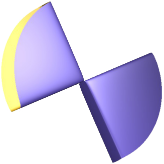
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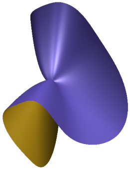

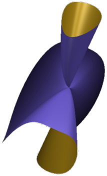
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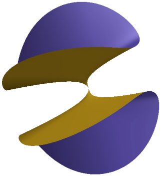

- $\Big($LG model with potential $W_{\mathrm{S}_{11}} = x^2 z + y z^3 + y^4\Big)$
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Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013, Recknagel/Weinreb 2017

In any dimension $n \geqslant 1$, the **generalised orbifold construction** *works for any n-dimensional defect TQFT*

$$\mathcal{Z}\colon\ \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathrm{Vect}_{\mathbb{C}}\,.$$

n-dimensional defect TQFT

An ***n*-dimensional defect TQFT** is a symmetric monoidal functor

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Carqueville/Runkel/Schaumann 2017

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Carqueville/Runkel/Schaumann 2017

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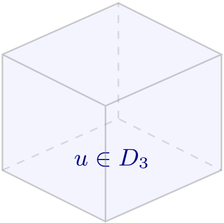

Carqueville/Runkel/Schaumann 2017

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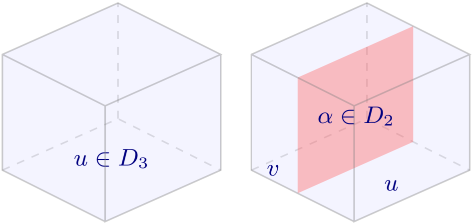

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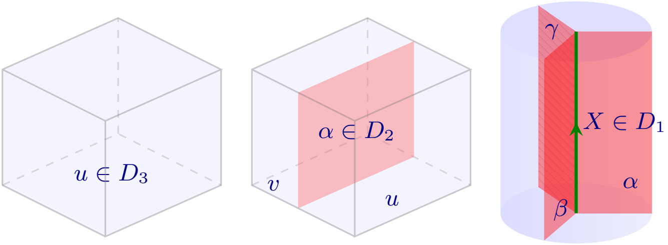

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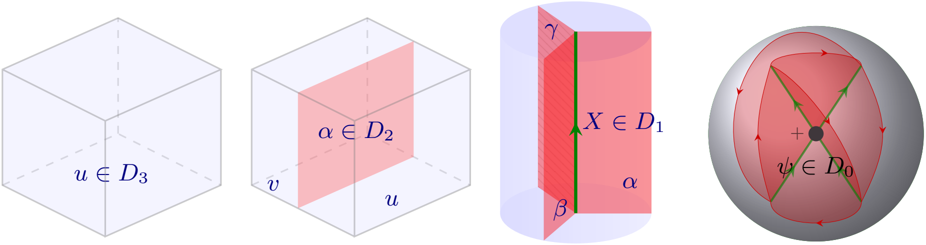

Carqueville/Runkel/Schaumann 2017

Examples of 3d defect TQFTs

- quantum **Chern-Simons theory** (= Reshetikhin-Turaev theory $\mathcal{Z}^{\mathcal{M}}$)
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Kapustin/Saulina 2010, Carqueville/Runkel/Schaumann 2017

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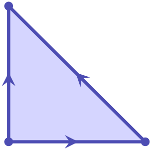
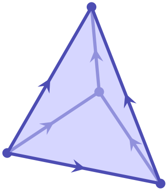
- **Rozansky-Witten theory** (conjecturally)
  - $D_3 = \{\text{holomorphic symplectic manifolds}\}$
  - $D_2 = \{\text{“generalised Landau-Ginzburg models”}\}$ (curved differential graded algebras)
  - $D_1 = \{\text{“fibred matrix factorisations”}\}$ (fibred CDGA bimodules)

Kapustin/Saulina 2010, Carqueville/Runkel/Schaumann 2017, Kapustin/Rozansky/Saulina 2009 + work in progress

Triangulations

standard n-simplex $\Delta^n := \left\{ \sum_{i=1}^{n+1} t_i e_i \;\middle|\; t_i \geqslant 0\,,\ \ \sum_{i=1}^{n+1} t_i = 1 \right\} \subset \mathbb{R}^{n+1}$

$\Delta^2 =$ 

$\Delta^3 =$ 

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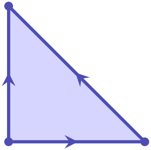
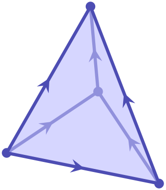
$\Delta^2 =$ 

$\Delta^3 =$ 

A **triangulation** of a manifold M is a decomposition of M into simplices.

Pachner moves

Let $F \subset \partial\Delta^{n+1}$ be collection of n-simplices.

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$$M \longmapsto \left(M \setminus K\right) \cup_{\varphi|_{\partial K}} \left(\partial\Delta^{n+1} \setminus \mathring{F}\right)$$

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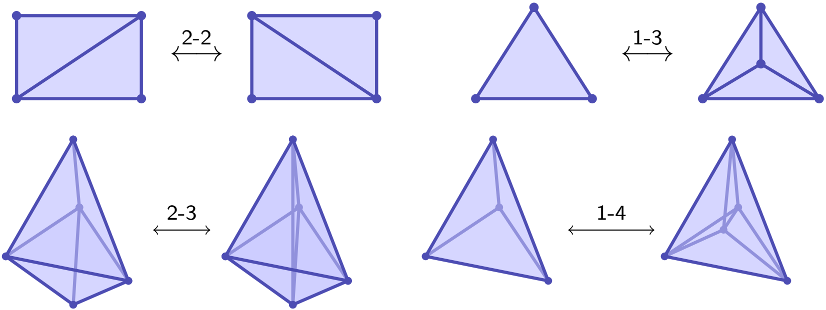

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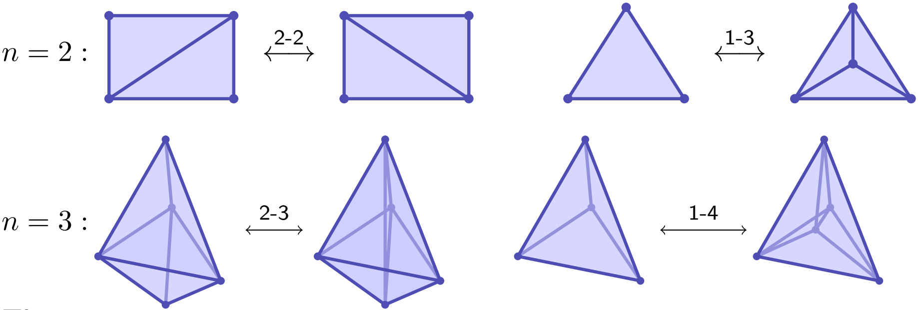

Theorem.
If triangulated PL manifolds are PL isomorphic, then there exists a finite sequence of Pachner moves between them.
Pachner 1991

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Carqueville/Runkel/Schaumann 2017

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- **compatibility**:
  $\mathcal{A}_j$ can consistently label j-strata dual to $(n-j)$-simplices in Δ^n

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- two elements $\mathcal{A}_0^+, \mathcal{A}_0^- \in D_0$, such that
- **compatibility**:
  $\mathcal{A}_j$ can consistently label j-strata dual to $(n-j)$-simplices in Δ^n;
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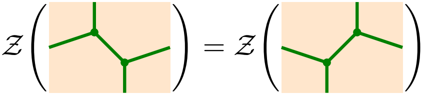
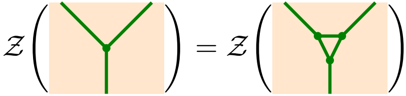
Recovers case $n = 2$:





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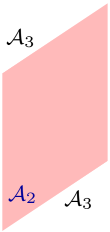

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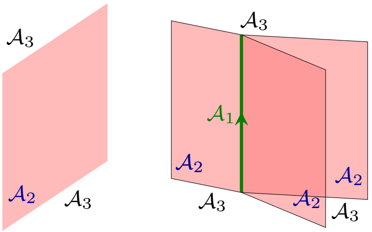

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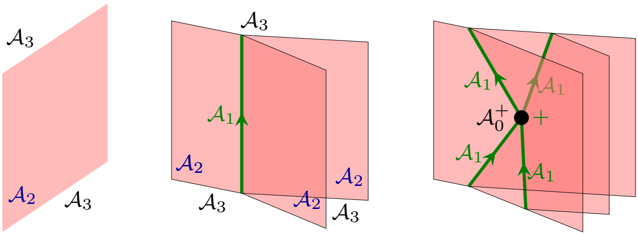

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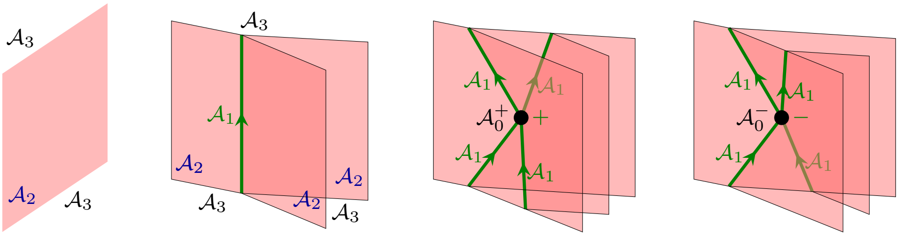

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Results: 3 classes of examples of 3d orbifolds

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- *commutative* separable symmetric Frobenius algebras in $\mathcal{M}$
- **topological quantum computation**: $\mathcal{M} = \mathcal{C}^{\boxtimes n}$ and $G \subseteq S_n$

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