Differential Equations and Associators for Periods

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based on:

St.St., G. Puhlfürst:

- **Differential equations, Associators and Recurrences for Amplitudes,** Nucl. Phys. B902 (2016) 186-245, [arXiv:1507.01582]
- **A Feynman Integral and its Recurrences and Associators,** Nucl.Phys. B906 (2016) 168-193 [arXiv:1511.03630]

Geometry and Physics

numbers or functions

physics: physical observables, couplings

geometry: geometrical data

$\downarrow$

Periods

What is a period ?

Kontsevich, Zagier:

*A period is a **complex number** whose real and imaginary parts are values of absolutely convergent **integrals of rational functions** with rational coefficients over **domains** in R given in **polynomial inequalities with rational coefficients***

$$\sqrt{2} = \int_{2x^2 \leq 1} dx$$

$$\pi = \int_{x^2+y^2 \leq 1} dx\ dy$$

$$\zeta_3 = \int_{0<x<y<z} \frac{dx\ dy\ dz}{(1-x)\ y\ z}$$

Periods and differential equations

consider family of periods (depending on parameter):

e.g. hypergeometric functions

$$ {}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix};x\right] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}\,(1-t)^{c-a-1}\,(1-xt)^{-b} $$

satisfy linear differential equations with algebraic coefficients:
(generalized) Picard-Fuchs equations
(Gauss-Manin systems)

e.g. GKZ systems for periods of CY manifolds systematize for generating a *subclass* of periods

e.g.: construct non—commutative words $w \in \{e_0, e_1\}^*$

$$w = w_1 w_2 \ldots$$

in the letters $w_i \in \{e_0, e_1\}$

generate *series* of periods (iterated integrals)

$$L_{e_0^n}(x) := \frac{1}{n!} \ln^n x \ , \ n \in \mathbf{N} \ ,$$

$$L_{e_1 w}(x) := \int_0^x \frac{dt}{1-t} L_w(t) \ ,$$

$$L_{e_0 w}(x) := \int_0^x \frac{dt}{t} L_w(t) \quad , \quad L_1(x) = 1 \ .$$

$\Longrightarrow$ $L_{e_1} = -\ln(1-x)$ $\qquad L_{e_0^{m-1} e_1}(x) = \mathcal{L}i_m(x)$

multiple polylogarithms (MPLs) in one variable

alphabet specifies underlying MPLs

define generating series of multiple polylogarithms (MPLs) in one variable

$$\Phi(x) = \sum_{w \in \{e_0, e_1\}^*} L_w(x)\ w$$

unique solution to KZ equation:

$$\frac{d}{dx}\Phi = \left(\frac{e_0}{x} + \frac{e_1}{1-x} \right) \Phi$$

Actually, at x=1 we obtain generating series of MZVs:

$$Z(e_0, e_1) = \sum_{w \in \{e_0, e_1\}^*} \zeta(w)\ w = 1 + \zeta_2\ [e_0, e_1] + \zeta_3\ (\ [e_0, [e_0, e_1]] - [e_1, [e_0, e_1]\ )$$

$$+ \zeta_4 \left(\ [e_0, [e_0, [e_0, e_1]]] - \frac{1}{4}\ [e_1, [e_0, [e_0, e_1]]] + [e_1, [e_1, [e_0, e_1]]] + \frac{5}{4}\ [e_0, e_1]^2\ \right) + \ldots$$

Drinfeld Associator

$$Z(e_0, e_1) = \Phi_1(x)^{-1}\ \Phi_0(x)$$

E.g:

$$\int_{z_1<\ldots<z_5} \left(\prod_{l=2}^{3} dz_l\right) \frac{\prod_{i<j}^{4} |z_{ij}|^{\alpha' s_{ij}}}{z_{12}\ z_{23}\ z_{41}}$$

$$= \alpha'^{-2} \left(\frac{1}{s_{12}s_{45}} + \frac{1}{s_{23}s_{45}}\right) + \zeta_2 \left(1 - \frac{s_{34}}{s_{12}} - \frac{s_{12}}{s_{45}} - \frac{s_{23}}{s_{45}} - \frac{s_{51}}{s_{23}}\right) + \mathcal{O}(\alpha')$$

$$z_{ij} := z_i - z_j$$

alpha'-expansion of tree-level N-point string scattering:
iterated integrals on sphere -> MZVs

periods on Riemann surface of genus 0 with N-3 marked points

alpha'-expansion of one-level N-point string scattering:
elliptic iterated integrals -> elliptic MZVs

$$\int_0^1 dz_4 \int_0^{z_4} dz_3 \int_0^{z_3} dz_2 \left[\frac{\theta_1(z_{21},\tau)}{\theta_1'(0,\tau)}\right]^{\alpha' s_{12}}$$

$$= \frac{1}{6} + \alpha'\ s_{12} \left\{-\frac{1}{6}\ln(2\pi) - \frac{\zeta_3}{4\pi^2} + \sum_{n\geq 1}\left(\frac{1}{3}\mathcal{L}i_1(q^n) - \frac{1}{2\pi^2}\mathcal{L}i_3(q^n)\right)\right\} + \mathcal{O}(\alpha'^2)$$

periods on Riemann surface of genus 1 with N-1 marked points

Task:

- find efficient and systematic procedure to gain their α'- expansions
- analytic and explicit expressions at each order
- closed and compact expressions
- systemize dependence on parameters

$\Rightarrow$ New techniques for computing α'-expansions for amplitudes

Generalized hypergeometric functions

relevant to both string theory and field-theory

$$ {}_pF_{p-1}(\vec{a};\vec{b};z) \equiv {}_pF_{p-1}\left[\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_{p-1} \end{matrix};z\right] = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)^m}{\prod_{j=1}^{p-1}(b_j)^m}\,\frac{z^m}{m!} \quad , \quad p \geq 1 $$

$a_i, b_j \in \mathbf{R}$

Pochhammer symbol
(raising factorial)

$$(a)^n = \frac{\Gamma(a+n)}{\Gamma(a)} = a\ (a+1)\ldots(a+n-1)$$

<u>Let's first discuss:</u>

$$a_i, b_j \in \mathbf{R} \quad \begin{cases} a_i = \epsilon\ \tilde{a}_i \\ b_j = 1 + \epsilon\ \tilde{b}_j \end{cases}$$

Indeed any other pFq hypergeometric function
with integer shifts can be expressed in terms of the above
by algebraic relations (contiguous relations)

$n = p = q + 1$

$$y(x) = {}_pF_q \left[\begin{matrix} a_1, \ldots, a_p \\ 1 + b_1, \ldots, 1 + b_q \end{matrix} ; x \right]$$

$$\boxed{\begin{aligned} &\theta(\theta + b_1 - 1)(\theta + b_2 - 1) \ldots (\theta + b_{p-1} - 1)y \\ &- z\ (\theta + a_1)(\theta + a_2) \ldots (\theta + a_p)y = 0 \end{aligned}}$$

$$\theta = x \frac{d}{dx}$$

Fuchsian equation with
three regular singularities

$x = 0, 1, \infty$

$$\boxed{\frac{d\boldsymbol{g}}{dx} = \left(\frac{e_0}{x} + \frac{e_1}{1 - x} \right) \boldsymbol{g}}$$

$$\boldsymbol{g} = \begin{pmatrix} y \\ \theta y \\ \vdots \\ \theta^{(n-1)} y \end{pmatrix}$$

representation $e_0, \ e_1$ in terms of n x n matrices
are given in terms of elementary symmetric functions of
$a_1, \ldots, a_p$ and $b_1, \ldots, b_q$

$$e_0 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & -Q^p_{n-1} & -Q^p_{n-2} & \dots & -Q^p_2 & -Q^p_1 \end{pmatrix} \quad , \quad e_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ \Delta^p_n & \Delta^p_{n-1} & \dots & \Delta^p_2 & \Delta^p_1 \end{pmatrix}$$

elementary symmetric functions:

$$P^p_\alpha = \sum_{\substack{i_1,\dots,i_\alpha=1 \\ i_1<i_2<\dots<i_\alpha}}^{p} a_{i_1}\cdot\ldots\cdot a_{i_\alpha} \quad , \quad \alpha = 1,\dots,p \ ,$$

$$Q^p_\beta = \sum_{\substack{i_1,\dots,i_\beta=1 \\ i_1<i_2<\dots<i_\beta}}^{p-1} b_{i_1}\cdot\ldots\cdot b_{i_\beta} \quad , \quad \beta = 1,\dots,p-1 \quad , \quad Q^p_p = 0 \ .$$

$$\Delta^p_\alpha = P^p_\alpha - Q^p_\alpha \quad , \quad \alpha = 1,\dots,p-1 \quad , \quad \Delta^p_p = P^p_p$$

generalized hypergeometric functions $\longleftrightarrow$ linear differential equations

KZ - type equations

$$\frac{d}{dx}\Phi = \left(\frac{e_0}{x} + \frac{e_1}{1-x}\right)\Phi$$

The unique solution $\Phi(x) \in \mathbf{C}\langle\{e_0, e_1\}\rangle$

is known as the generating series of multiple polylogarithms (MPLs) in one variable

$$\Phi(x) = \sum_{w\in\{e_0,e_1\}^\star} L_w(x)\ w$$

non-commutative word w

alphabet specifies underlying MPLs:

$$\begin{aligned}\Phi_0[e_0,e_1](x) = {} & 1 + e_0\ \ln x - e_1\ \ln(1-x) + \frac{1}{2}\ \ln^2 x\ e_0^2 + \mathcal{L}i_2(x)\ e_0 e_1 \\ & - [\ \mathcal{L}i_2(x) + \ln x\ \ln(1-x)\ ]\, e_1 e_0 + \frac{1}{2}\ \ln^2(1-x)\ e_1^2 + \frac{1}{6}\ \ln^3 x\ e_0^3 \\ & + \mathcal{L}i_3(x)\ e_0^2 e_1 - [\ 2\ \mathcal{L}i_3(x) + \ln x\ \mathcal{L}i_2(x)\ ]\, e_0 e_1 e_0 + \mathcal{L}i_{2,1}(x)\ e_0 e_1^2 \\ & + \left[\ \mathcal{L}i_3(x) - \ln x\ \mathcal{L}i_2(x) - \frac{1}{2}\ln^2 x\ \ln(1-x)\ \right] e_1 e_0^2 + \mathcal{L}i_{1,2}(x)\ e_1 e_0 e_1 \\ & - \left[\ \mathcal{L}i_{1,2}(x) + \mathcal{L}i_{2,1}(x) - \frac{1}{2}\ \ln x\ \ln^2(1-x)\ \right] e_1^2 e_0 - \frac{1}{6}\ \ln^3(1-x)\ e_1^3 + \ldots\end{aligned}$$

$$\boxed{{}_pF_{p-1}\left[\begin{matrix} a_1,\ldots,a_p \\ 1+b_1\ldots,1+b_{p-1}\end{matrix};x\right] = \Phi_0[e_0,e_1](x)\ |_{1,1}}$$

in fact:

$${}_pF_{p-1}\left[\begin{matrix} a_1,\ldots,a_2 \\ 1+b_1,\ldots,1+b_{p-1}\end{matrix};1\right] = Z(e_0,e_1)|_{1,1}$$

e.g. ${}_3F_2$:

$${}_3F_2\left[\begin{matrix} \alpha' a_1,\alpha' a_2,\alpha' a_3 \\ 1+\alpha' b_1, 1+\alpha' b_2\end{matrix};x\right] = \Phi_0[B_0,B_1](x)|_{1,1}$$

$$= 1 + \Delta_3\ \mathcal{L}i_3(x) + \Delta_1\ \Delta_3\ \mathcal{L}i_{2,2}(x) - Q_1\ \Delta_3\ \mathcal{L}i_4(x) + \ldots$$

$$e_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -{\alpha'}^2 Q_2 & -\alpha' Q_1 \end{pmatrix} \quad , \quad e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ {\alpha'}^3\Delta_3 & {\alpha'}^2\Delta_2 & \alpha'\Delta_1 \end{pmatrix} \, ,$$

$$\Delta_1 = a_1 + a_2 + a_3 - b_1 - b_2,$$

$$\Delta_2 = a_1 a_2 + a_2 a_3 + a_3 a_1 - b_1 b_2,$$

$$\Delta_3 = a_1 a_2 a_3,$$

$$Q_1 = b_1 + b_2,$$

$$Q_2 = b_1 b_2.$$

$$ {}_pF_{p-1}(\vec{a};\vec{b};z) \equiv {}_pF_{p-1}\left[\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_{p-1} \end{matrix};z\right] = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)^m}{\prod_{j=1}^{p-1}(b_j)^m} \frac{z^m}{m!} \quad , \quad p \geq 1 $$

coefficients of ϵ - expansions
for a subclass of hypergeometric functions
give rise to **multiple polylogarithms (MPLs)**

$$ a_i, b_j \in \mathbf{R} \quad \begin{cases} a_i = \epsilon\ \tilde{a}_i \\ b_j = 1 + \epsilon\ \tilde{b}_j \end{cases} $$

$$ \mathcal{L}i_{n_1,\ldots,n_l}(z_1,\ldots,z_l) = \sum_{0<k_l<\ldots<k_1} \frac{z_1^{k_1}\cdot\ldots\cdot z_l^{k_l}}{k_1^{n_1}\cdot\ldots\cdot k_l^{n_l}} $$

Rational values

$$ a_i, b_j \in \mathbf{R} \quad \begin{cases} a_i = \dfrac{p_i}{q_i} + \epsilon\ \tilde{a}_i \\ b_j = \dfrac{p_j}{q_j} + \epsilon\ \tilde{b}_j \end{cases} $$

What is the underlying differential equation ?

At **rational** values
(real parameters shifted by p/q):
Fuchsian differential equation
with q+2 regular singular points
at 0,

$$\exp(2\pi i r/q),\ \infty,\ r = 1, \ldots, q$$

"roots of unity"

$$\frac{d\Phi}{dx} = \left(\sum_{i=0}^{q} \frac{A_i}{x - x_i} \right) \Phi$$

Schlesinger system

→ MPLs of q-th roots of unity

$$\frac{d\Phi}{dx} = \left(\sum_{i=0}^{q} \frac{A_i}{x - x_i} \right) \Phi$$

hyperlogarithms are defined recursively from words w built from an alphabet $\{w_0, w_1, \ldots\}$

(with $w_i \simeq A_i$) with $l+1$ letters: $l = q$

$$L_{w_0^n}(x) := \frac{1}{n!} \ln^n(x - x_0) \ , \ n \in \mathbf{N} \ ,$$

$$L_{w_i^n}(x) := \frac{1}{n!} \ln^n \left(\frac{x - x_i}{x_0 - x_i} \right) \ , \ 1 \leq i \leq l \ ,$$

$$L_{w_i w}(x) := \int_0^x \frac{dt}{t - x_i} L_w(t) \quad , \quad L_1(x) = 1 \ .$$

$$L_{w_0^{n_l-1} w_{\sigma_l} \ldots w_0^{n_2-1} w_{\sigma_2} w_0^{n_1-1} w_{\sigma_1}}(x) = (-1)^l \ \mathcal{L}i_{n_l,\ldots,n_1} \left(\frac{x - x_0}{x_{\sigma_l} - x_0}, \frac{x_{\sigma_l} - x_0}{x_{\sigma_{l-1}} - x_0}, \ldots \frac{x_{\sigma_3} - x_0}{x_{\sigma_2} - x_0}, \frac{x_{\sigma_2} - x_0}{x_{\sigma_1} - x_0} \right)$$

A group-like solution Φ taking values in $\mathbf{C}\langle A\rangle$
with the alphabet $A = \{A_0, A_1, \ldots, A_q\}$
can be given as formal weighted sum over iterated integrals
(with the weight given by the number of iterated integrations)

$$\Phi(x) = \sum_{w \in A^*} L_w(x)\ w$$

Actually:

$$\Phi_i(x) \longrightarrow (x - x_i)^{A_i} \quad , \quad i = 0, 1, \ldots, q$$

$$\Phi_i(x) = \sum_{w \in A^*} L^i_w(x)\ w\ ,$$

$$L^b_{w_i w}(x) = \int_b^x \frac{dt}{t - x_i}\ L^b_w(t)$$

q (independent) associators:

$$Z^{(x_k)}(A_0, A_1, \ldots, A_q) = \Phi_k(x)^{-1}\ \Phi_0(x) \quad , \quad k = 1, \ldots, q$$

<u>Again:</u> $\Phi_0(x_k)_{reg.} = Z^{(x_k)}(A_0, \ldots, A_q) \quad k = 1, \ldots, q$

q regularized zeta series:

$$Z^{(x_k)}(A_0, A_1, \ldots, A_q) = \sum_{w \in A^*} \zeta^{x_k}(w)\ w \quad , \quad k = 1, \ldots, q$$

$$Z^{(x_k)}(A_0, \ldots, A_q) = 1 + \ln(x_k - x_0)\ A_0 + \sum_{1 \leq i \neq k} \ln\left(\frac{x_k - x_i}{x_0 - x_i}\right)\ A_i - \ln(x_0 - x_k)\ A_k + \ldots\ .$$

Brown: map: $w \mapsto \zeta^{x_k}(w)$

homomorphism for the shuffle product:

$$\zeta^{x_k}(w \sqcup\!\sqcup \tilde{w}) = \zeta^{x_k}(w\tilde{w}) + \zeta^{x_k}(\tilde{w}w) = \zeta^{x_k}(w)\ \zeta^{x_k}(\tilde{w})$$

$$\zeta^{x_k}(w_k) = -\ln(x_0 - x_k)\ ,$$
$$\zeta^{x_k}(w_0) = \ln(x_k - x_0)\ ,$$
$$\zeta^{x_k}(w) = L_w(x_k)\ ,\ \ w \notin A^* w_0 \cup w_k A^*\ ,$$

with w neither beginning in w_k nor ending in w_0

enough to compute the map ζ^{x_k} for any word $w \in A^*$
because every word w can be written uniquely as a linear combination
of shuffle products of the words w_0 and w_k and words which neither begin in w_k nor end in w_0.

$$Z^{(x_k)}(A_0, \ldots, A_q) = 1 + \ln(x_k - x_0)\ A_0 + \sum_{1 \le i \ne k} \ln\left(\frac{x_k - x_i}{x_0 - x_i}\right)\ A_i - \ln(x_0 - x_k)\ A_k + \ldots\ .$$

(i) q=2 (parameters shifted by half-integer):

$$\underbrace{{}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}+c \end{matrix}; z\right],\ {}_2F_1\left[\begin{matrix} \frac{1}{2}+a, \frac{1}{2}+b \\ \frac{1}{2}+c \end{matrix}; z\right],\ {}_2F_1\left[\begin{matrix} \frac{1}{2}+a, b \\ \frac{1}{2}+c \end{matrix}; z\right],\ {}_2F_1\left[\begin{matrix} \frac{1}{2}+a, b \\ 1+c \end{matrix}; z\right]}_{\text{can algebraically be expressed in terms of e.g.:}}$$

$${}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}+c \end{matrix}; z\right]$$

$$\underbrace{{}_2F_1\left[\begin{matrix} \frac{1}{2}+a, \frac{1}{2}+b \\ 1+c \end{matrix}; z\right]}_{\text{elliptic function}}$$

elliptic function K
of first kind

$${}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; z\right] = \frac{2}{\pi}\ K(\sqrt{z})$$

we want to determine:

$${}_2F_1\left[\begin{matrix} a\epsilon, b\epsilon \\ \frac{1}{2}+c\epsilon \end{matrix}; z\right] = \sum_k \epsilon^k\ u_k(z)$$

coefficients functions are given in terms of harmonic polylogarithsm (HPLs)

$$ {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}+c \end{matrix}; z\right] $$

With:

$$ \Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} $$

$$ \varphi_1(y) := {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}+c \end{matrix}; z\right] $$

$$ \varphi_2(y) := y\frac{d}{dy}\,\varphi_1 $$

$$ y = \frac{\sqrt{z-1}-\sqrt{z}}{\sqrt{z-1}+\sqrt{z}} $$

fullfils Fuchs system:

$$ \frac{d\Phi}{dy} = \left(\sum_{i=0}^{2} \frac{A_i}{y-y_i}\right)\,\Phi $$

with the four singularities $y_0 = 0,\ y_1 = 1,\ y_2 = -1$ and ∞

$$ A_0 = \begin{pmatrix} 0 & 1 \\ -ab & a+b \end{pmatrix} \quad , \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & -2c \end{pmatrix} \quad , \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2c-2a-2b \end{pmatrix} $$

$$
\begin{aligned}
\Phi_1(x) = 1 &+ A_0 \;\ln x + A_1 \;\ln(x-1) + A_2 \;[\,-\ln 2 + \ln(x+1)\,] \\
&+ A_0 A_1 \;[\;\zeta_2 - \mathcal{L}i_2(x) + i\pi\delta_x \;\ln x\;] + A_1 A_0 \;[\;-\zeta_2 + \mathcal{L}i_2(x) + \ln x \ln(1-x)\;] \\
&+ A_0 A_2 \;[\;-\frac{1}{2}\zeta_2 - \mathcal{L}i_2(-x) - \ln 2 \ln x\;] + A_2 A_0 \;[\;\frac{1}{2}\zeta_2 + \mathcal{L}i_2(-x) + \ln x \;\ln(1+x)\;] \\
&+ A_1 A_2 \;[\;-\frac{1}{2}\zeta_2 + \frac{1}{2}\ln^2 2 - \ln 2 \ln(1-x) + \mathcal{L}i_{1,1}(x,-1)\;] \\
&+ A_2 A_1 \;[\;\frac{1}{2}\zeta_2 - \frac{1}{2}\ln^2 2 + \mathcal{L}i_{1,1}(-x,-1) - i\pi\delta_x \;\{\ln 2 - \ln(1+x)\}\;] \\
&+ \frac{1}{2}\;\ln^2 x \; A_0^2 + \frac{1}{2}\;\ln^2(x-1)\; A_1^2 + \frac{1}{2}\;[\ln 2 - \ln(1+x)]^2 \; A_2^2 + \ldots
\end{aligned}
$$

$$
Z^{(+1)}(A_0, A_1, A_2) = \Phi_1(x)^{-1}\;\Phi_0(x)
$$

$$
\begin{aligned}
Z^{(+1)}(A_0, A_1, A_2) = 1 &- i\pi\; A_1 + \ln 2\; A_2 - \frac{\pi^2}{2}\; A_1^2 + \frac{1}{2}\;\ln^2 2\; A_2^2 \\
&- \zeta_2\;[A_0, A_1] + \frac{1}{2}\zeta_2\;[A_0, A_2] - \frac{1}{2}(\ln^2 2 - \zeta_2)\;[A_1, A_2] \\
&- i\pi \ln 2\; A_1 A_2 + \ldots
\end{aligned}
$$

$$Z^{(-1)}(A_0, A_1, A_2) = \Phi_2(x)^{-1}\ \Phi_0(x)$$

$$Z^{(-1)}(A_0, A_1, A_2) = 1 + i\pi\ A_0 + \ln 2\ A_1 - \frac{\pi^2}{2}\ A_0^2 + \frac{1}{2}\ \ln^2 2\ A_1^2$$
$$- \zeta_2\ [A_0, A_2] + \frac{1}{2}\zeta_2\ [A_0, A_1] + \frac{1}{2}(\ln^2 2 - \zeta_2)\ [A_1, A_2]$$
$$+ i\pi \ln 2\ A_1 A_0 + \ldots$$

$$\Phi_1(-1)_{reg.} = Z^{(-1)}\ \ (Z^{(+1)})^{-1}$$

Result:

$$\boxed{\begin{aligned} {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2} + c \end{matrix}; z\right] &= \Phi_1[A_0, A_1, A_2](y)\ |_{1,1} \\ {}_2F_1\left[\begin{matrix} a, b \\ c + \frac{1}{2} \end{matrix}; 1\right] &= Z^{(-1)}\ \ (Z^{(+1)})^{-1}\ \Big|_{1,1} \end{aligned}}$$

$$
{}_2F_1\left[\begin{matrix} a, b \\ c+\frac{1}{2} \end{matrix}; z\right] = 1 + \frac{1}{2}\ \ln^2 y\ A_0^2 + \frac{1}{6}\ \ln^3 y\ A_0^3
$$
$$
+ [\ 2\mathcal{L}i_3(y) - \ln y \mathcal{L}i_2(y) - \zeta_2 \ln z - 2\zeta_3\ ]\ \ A_0 A_1 A_0
$$
$$
+ \left[\ 2\mathcal{L}i_3(-y) - \ln y \mathcal{L}i_2(-y) + \frac{1}{2}\zeta_2\ \ \ln y + \frac{3}{2}\zeta_3\ \right]\ A_0 A_2 A_0 + \ldots
$$

- universal result: ϵ - expansion is identified with normalized solution of the underlying Fuchs system

- compact and simple result

- just matrix multiplication of k matrices at degree k

- each order in ϵ given by some **combinations of MPLs** and group-like **matrix products** carrying the information on parameter

generalized hypergeometric functions:
generically: singularity structure described by three regular points:
KZ equation

monodromies are explicitly furnished
by the Drinfeld associator

specific parameters: after suitable coordinate transformation,
properly assigning Lie-algebra and monodromy representations:
Schlesinger system with q+2 regular singular points

q+1 associators (regularized zeta-series)

generalized hypergeometric functions:
singularity structure described by three regular points:
KZ equation

multiple Gaussian hypergeometric functions:

singularity structure is generically given by divisors
whose monodromy is described by **generalized KZ equations**
of more variables