Elliptic String Solutions in AdS_3 and Elliptic Minimal Surfaces in AdS_4

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Section 1

Motivation

- Classical string solutions have shed light to several aspects of the AdS/CFT correspondence
  - They are connected to gluon scattering amplitudes
  - Spiky strings can be related to single trace operators of the dual CFT
- As Pohlmeyer reduction is concerned, the relation between the degrees of freedom of the NLSM and those of the reduced theory is highly non-local.
  - There is great difficulty to invert the procedure.
  - In particular, it is not clear how exactly a solution of the Pohlmeyer reduced theory corresponds to one or more physically distinct solutions of the original NLSM.
- The area of minimal surfaces are connected to Entanglement Entropy through the Ryu-Takayanagi conjecture
  - Gravity as quantum entropic force
  - Black hole entropy as entanglement entropy
  - Entanglement entropy as an order parameter for confinement
- Minimal surfaces are interesting from a purely mathematical point of view

Section 2

Reduction of String Actions in AdS_3 / Minimal Surfaces in AdS_4

Pohlmeyer reduction: The critical element of this approach is a non-local coordinate transformation that manifestly satisfies the Virasoro constraints, thus leaving only the physical degrees of freedom.

Embedding of the two-dimensional world-sheet (or minimal surface) into the **symmetric** target space of the NLSM,

which is in turn embedded in a higher-dimensional **flat** space.

AdS_3 and dS_3 can be dealt similarly ($s = +1$ corresponds to dS and $s = -1$ corresponds to AdS).

The action is

$$S = \int d\xi_+ d\xi_- \left(\partial_+ Y \cdot \partial_- Y + \lambda \left(Y \cdot Y - s\Lambda^2 \right) \right).$$

Dot is the inner product in the enhanced space, performed with the metric $\eta = diag\{-1, s, +1, +1\}$
Solution is subject to

- The geometric constraint $Y \cdot Y = s\Lambda^2$
- The Virasoro constraints $\partial_\pm Y \cdot \partial_\pm Y = 0$.
- The equations of motion $\partial_+ \partial_- Y = -s \frac{1}{\Lambda^2} \left(\partial_+ Y \cdot \partial_- Y \right) Y$.

We introduce a vector basis v_i, $i = 1, 2, 3, 4$ in the enhanced space, combining the vectors Y, $\partial_+ Y$ and $\partial_- Y$ with one more vector v_4 as

$$v_i = \{Y, \, \partial_+ Y, \, \partial_- Y, \, v_4\} \, .$$

v_4 is space-like and defined to be orthogonal to v_1, v_2 and v_3 .
We define the Pohlmeyer field

$$e^a := \partial_+ Y \cdot \partial_- Y .$$

We decompose the derivatives of the vectors v_i in the basis v_i using the 4×4 matrices $A^{\pm}$,

$$\partial_+ v_i = A^+_{ij} v_j, \quad \partial_- v_i = A^-_{ij} v_j .$$

They obey the zero-curvature condition

$$\partial_- A^+ - \partial_+ A^- + \left[A^+, A^-\right] = 0.$$

The zero-curvature condition combined with constraints and the equations of motion imposes the following equations for the Pohlmeyer field variable a,

$$\partial_+ \partial_- a = -s \frac{1}{\Lambda^2} e^a + f^{(+)}\left(\xi_+\right) f^{(-)}\left(\xi_-\right) e^{-a}.$$

Last equation can be brought to the form of the sinh- or cosh-Gordon equation defining

$$\varphi := a - \frac{1}{2} \ln\left(\Lambda^2 \left| f^{(+)}\left(\xi_+\right) f^{(-)}\left(\xi_-\right)\right|\right)$$

and

$$\frac{d{\xi_\pm}'}{d\xi_\pm} = \sqrt{\Lambda \left| f^{(\pm)}\left(\xi_\pm\right)\right|}.$$

The form of the final equation depends on the sign of the product $-sf^{(+)}(\xi_+) f^{(-)}(\xi_-)$. For AdS_3, we have

- If $f^{(+)} f^{(-)} < 0$ then $\partial_+ \partial_- \varphi = 2 \sinh \varphi / \Lambda^2$
- If $f^{(+)} f^{(-)} > 0$ then $\partial_+ \partial_- \varphi = 2 \cosh \varphi / \Lambda^2$

In a completely similar fashion, the Pohlmeyer reduction of static minimal surfaces in AdS_4 (or else minimal surfaces in H_3) results always in the Euclidean cosh-Gordon equation

$$\partial \bar{\partial} \varphi = 2 \cosh \varphi / \Lambda^2$$

Notice that the change of coordinates does not alter the expression for the action

$$\int dz d\bar{z} e^{a} = \int dz' d\bar{z}' e^{\varphi} .$$

Section 3

Elliptic Solutions of the Sinh- and Cosh-Gordon Equations

The usual approach for finding solutions of the sinh-Gordon equation, such as the kinks, is to use the corresponding Bäcklund transformation starting from the vacuum as seed solution.
This method, however, cannot be applied to the cosh-Gordon equation; although it possesses Bäcklund transformations similar to those of the sinh-Gordon equation, it does not admit a vacuum solution.
Consequently, some string solutions can be studied using these methods. However, as static minimal surfaces correspond always to the Euclidean cosh-Gordon equation, they cannot be studied at all using these techniques.

In this work, we focus on solutions of the sinh-Gordon or cosh-Gordon equations that depend on only one of the two world-sheet coordinates ξ_0 and ξ_1.
The motivation is provided by the inverse Pohlmeyer reduction, which requires to solve equations

$$\frac{\partial^2 Y^\mu}{\partial \xi_1^2} - \frac{\partial^2 Y^\mu}{\partial \xi_0^2} = -s \frac{1}{\Lambda^2} e^{\varphi(\xi_0,\xi_1)} Y^\mu,$$

plus the geometric and Virasoro constraints. The latter will be significantly simplified via separation of variables if $\varphi(\xi_0, \xi_1)$ depends only on ξ_0 or ξ_1.

All cases of equations for the reduced system can be rewritten in unified form

$$\partial_+ \partial_- \varphi = -s \frac{m^2}{2} \left(e^{\varphi} + t e^{-\varphi} \right) ,$$

where $t = -\mathrm{sgn} \left(s f^{(+)} f^{(-)} \right)$ and $m = \sqrt{2/\Lambda}$.
We start searching for static solutions, $\varphi \left(\xi_0, \xi_1 \right) = \varphi_1 \left(\xi_1 \right)$. The sinh- or cosh-Gordon equation reduces to the ODE

$$\frac{d^2 \varphi_1}{d {\xi_1}^2} = -s \frac{m^2}{2} \left(e^{\varphi_1} + t e^{-\varphi_1} \right) ,$$

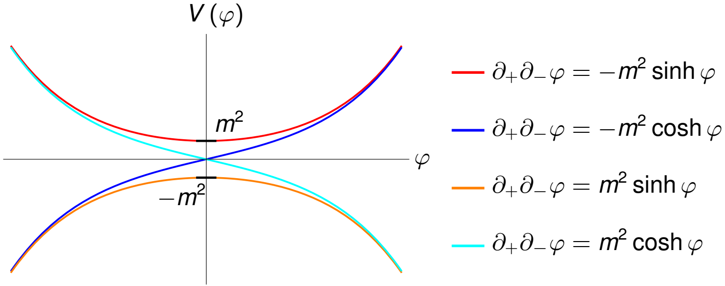
which can be integrated to

$$\frac{1}{2} \left(\frac{d \varphi_1}{d \xi_1} \right)^2 + s \frac{m^2}{2} \left(e^{\varphi_1} - t e^{-\varphi_1} \right) = E .$$

This can be viewed as the conservation of energy for an **effective one-dimensional mechanical problem** describing the motion of a particle with potential

$$U_1\left(\varphi_1\right) = s\frac{m^2}{2}\left(e^{\varphi_1} - te^{-\varphi_1}\right),$$

letting ξ_1 play the role of time and φ_1 the role of the particle coordinate.



The potential of the one-dimensional mechanical analogue

Considering this effective mechanical problem, we obtain a qualitative picture for the behaviour of the solutions.

- Sinh-Gordon equation with an overall minus sign:
  - oscillating solutions with energy $E > m^2$
  - no solutions for $E < m^2$
- Sinh-Gordon equation with an overall plus sign: two different classes of solutions
  - reflecting scattering solutions for $E < -m^2$
  - transmitting scattering solutions for $E > -m^2$
- Cosh-Gordon equation:
  - reflecting scattering solutions for all energies

Introducing the quantities

$$V_1 := -s\frac{m^2}{2}e^{\varphi}, \quad V_1 = 2y - \frac{E}{3},$$

we transform the 1-d problem of motion for a particle with energy E and a hyperbolic potential to yet another 1-d problem, describing the motion of a particle with zero energy and a cubic potential,

$$\left(\frac{dy}{d\xi_1}\right)^2 = 4y^3 - \left(\frac{1}{3}E^2 + t\frac{m^4}{4}\right)y + \frac{E}{3}\left(\frac{1}{9}E^2 + t\frac{m^4}{8}\right).$$

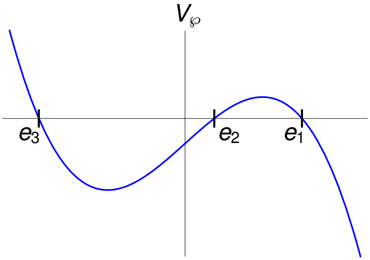
Energy conservation takes the **standard Weierstrass form**.

Translationally invariant solutions $\varphi(\xi_0, \xi_1) = \varphi_0(\xi_0)$ of the sinh- or cosh-Gordon equation are similar to the static ones. The reduced system equation is written as

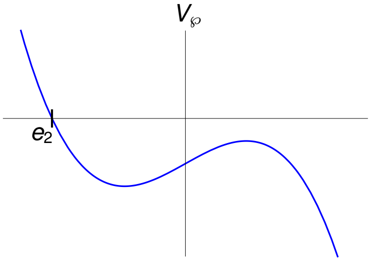
$$\frac{1}{2}\left(\frac{d\varphi_0}{d\xi_1}\right)^2 - s\frac{m^2}{2}\left(e^{\varphi_0} - te^{-\varphi_0}\right) = E.$$

As before, this equation can be viewed as energy conservation for a 1-d point particle problem with potential identical to the problem of static configurations, letting $s \to -s$. This implies that the static solutions of the Pohlmeyer reduced system for string propagation in AdS_3 are identical to the translationally invariant solutions of the Pohlmeyer reduced system for string propagation in dS_3 and vice versa.

We are looking for real solutions of Weierstrass equation defined in the real domain.
The latter can be visualized in terms of an 1-d mechanical problem, describing the motion of a point particle with $E = 0$ and a cubic potential $V_\wp(y) = -Q(y)$



$\Delta > 0$



$\Delta < 0$

When $Q(y)$ has three real roots, we expect to have two real solutions, one being unbounded with $y > e_1$ and a bounded one with $e_3 < y < e_2$.
When $Q(y)$ has one real root, we expect to have only one real solution which is unbounded with $y > e_2$.
Analytic properties of $\wp$ and the half period relations lead to

$$y_1(x) = \wp(x),$$
$$y_2(x) = \wp(x + \omega_2),$$

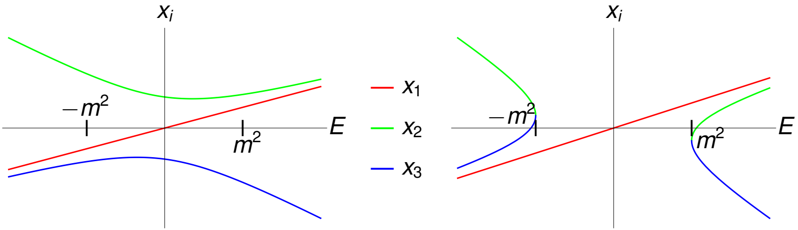
corresponding to the unbounded and bounded solutions respectively.

Returning to the problem of finding static and translationally invariant solutions of the sinh- and cosh-Gordon equations, we remind that the cubic polynomial is not arbitrary, but

$$Q\left(x\right) = 4x^3 - \left(\frac{1}{3}E^2 + t\frac{m^4}{4}\right)x + \frac{E}{3}\left(\frac{1}{9}E^2 + t\frac{m^4}{8}\right).$$

The roots of the cubic polynomial are

$$x_1 = \frac{E}{6}, \quad x_{2,3} = -\frac{E}{12} \pm \frac{1}{4}\sqrt{E^2 + tm^4}.$$



cosh-Gordon

sinh-Gordon

The roots of $Q(x)$ as function of the energy E

The Weierstrass function allows for a unifying description of the elliptic solutions of both sinh- and cosh-Gordon equations. Different classes of solutions simply correspond to different ordering of the roots x_i.

	reality of roots	ordering of roots
$t=+1$	3 real roots	$e_1=x_2,\ e_2=x_1,\ e_3=x_3$
$t=-1,\ E>m^2$	3 real roots	$e_1=x_1,\ e_2=x_2,\ e_3=x_3$
$t=-1,\ E<-m^2$	3 real roots	$e_1=x_2,\ e_2=x_3,\ e_3=x_1$
$t=-1,\ \lvert E\rvert<m^2$	1 real, 2 complex	$e_1=x_2,\ e_2=x_1,\ e_3=x_3$

All elliptic solutions of the sinh- and cosh-Gordon equations take the following form

$$V_1\left(\xi_1;E\right) = 2\wp\left(\xi_1 + \delta\xi_1; g_2\left(E\right), g_3\left(E\right)\right) - \frac{E}{3},$$

$$\varphi_1\left(\xi_1;E\right) = \ln\left[-s\frac{2}{m^2}\left(2\wp\left(\xi_1 + \delta\xi_1; g_2\left(E\right), g_3\left(E\right)\right) - \frac{E}{3}\right)\right],$$

for all choices of the overall sign s.

In particular, we choose

- $\delta\xi_1 = 0$ for the reflecting solutions of the cosh-Gordon equation as well as for the right incoming reflecting and transmitting solutions of the sinh-Gordon equation, having $s = -1$ in both cases
- $\delta\xi_1 = \omega_1$ for the left incoming transmitting solutions of the sinh-Gordon equation, having $s = -1$
- $\delta\xi_1 = \omega_2$ for the left incoming reflecting solutions of the sinh-Gordon equation with $s = -1$ as well as for the oscillating solutions of the sinh-Gordon equation with $s = +1$
- $\delta\xi_1 = \omega_1 + \omega_2$ for the reflecting solutions of the cosh-Gordon equation with $s = +1$.

Section 4

The Building Blocks of the String / Minimal Surface Solutions

Given a classical string configuration, it is straightforward to find the corresponding solution of the Pohlmeyer reduced system.
The inverse problem is highly non-trivial due to the non-local nature of the transformation relating the embedding functions Y^μ with the reduced field φ and because the Pohlmeyer reduction is a many-to-one mapping.

Such a construction requires the solution of the equations of motion for the embedding functions,

$$\frac{\partial^2 Y^\mu}{\partial \xi_1^2} - \frac{\partial^2 Y^\mu}{\partial \xi_0^2} = -s \frac{1}{\Lambda^2} e^\varphi Y^\mu ,$$

supplemented with the geometric constraint as well as the Virasoro constraints of the embedding problem,

$$Y \cdot Y = s\Lambda^2 ,$$
$$\partial_\pm Y \cdot \partial_\pm Y = 0.$$

Consider the case of a static solution of the reduced system $\varphi(\xi_0, \xi_1) = \varphi_1(\xi_1)$. We define

$$V_1(\xi_1) := -s \frac{1}{\Lambda^2} e^{\varphi_1}.$$

Then, the equations of motion can be rewritten as

$$\frac{d^2 Y^\mu}{d\xi_1^2} - \frac{d^2 Y^\mu}{d\xi_0^2} = V_1(\xi_1)\, Y^\mu.$$

Since V_1 depends solely on ξ_1, it is possible to separate the variables letting

$$Y^{\mu}\left(\xi_0,\xi_1\right) := \Sigma^{\mu}\left(\xi_1\right)\mathrm{T}^{\mu}\left(\xi_0\right).$$

We arrive at a pair of ODEs,

$$-\frac{d^2\mathrm{T}^{\mu}}{d\xi_0^2} = \kappa^{\mu}\mathrm{T}^{\mu},$$

$$-\frac{d^2\Sigma^{\mu}}{d\xi_1^2} + V_1\left(\xi_1\right)\Sigma^{\mu} = \kappa^{\mu}\Sigma^{\mu},$$

which can be viewed as two **effective Schrödinger problems** with **common** eigenvalues.

This pair of Schrödinger problems does not require any normalization condition for the effective wavefunction.

Translationally invariant solutions of the reduced system can be related to classical string configurations in a similar manner. The only difference is that $V_1(\xi_1)$ is replaced by

$$V_0(\xi_0) := s\frac{1}{\Lambda^2}e^{\varphi_0}.$$

We result in the same system of effective Schrödinger problems with $\Sigma \leftrightarrow \mathrm{T}$. Since the translationally invariant reduced solutions are identical to the static ones with the inversion $s \to -s$, it suffices to construct the string solutions for the static configurations. The translationally invariant solutions can then be obtained with the inversion $s \to -s$.

For all cases, the effective potential V_1 takes the special form

$$V_1 = 2\wp\left(\xi_1 + \delta\xi_1\right) - \frac{E}{3},$$

where $\delta\xi_1$ is either 0 or ω_2 depending on the use of the unbounded or the bounded real solution.

The special class of periodic potentials

$$V\left(x\right) = n\left(n+1\right)\wp\left(x\right)$$

are called Lamé potentials and they are analytically solvable.

The spectrum of the corresponding Schrödinger problem contains up to n finite allowed bands, plus one more continuous band extending to infinite energy.

Our case corresponds to the $n = 1$ Lamé problem,

$$-\frac{d^2 y}{dx^2} + 2\wp(x)\, y = \lambda y,$$

whose solutions are given in general by

$$y_\pm(x; a) = \frac{\sigma(x \pm a)}{\sigma(x)\,\sigma(\pm a)} e^{-\zeta(\pm\alpha)x}$$

with corresponding eigenvalues

$$\lambda = -\wp(a)\,.$$

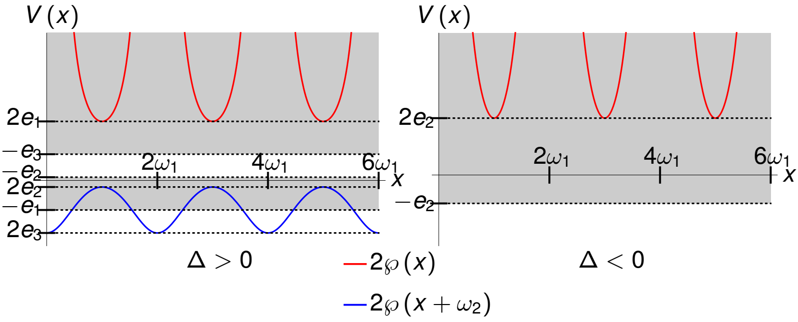
- If the cubic polynomial has three real roots and for $\lambda < -e_1$ or $-e_2 < \lambda < -e_3$ or if it has one real root and $\lambda < -e_2$, the eigenstates $y_\pm(x)$ are real and if they are shifted by a period $2\omega_1$ they will get multiplied by a real number. In those cases, the eigenfunctions diverge exponentially as $x \to \pm\infty$.
- If the cubic polynomial has three real roots and for $\lambda > -e_3$ or $-e_1 < \lambda < -e_2$ or if it has one real root and $\lambda > -e_2$, the eigenstates $y_\pm(x)$ are complex conjugate to each other and if they are shifted by a period $2\omega_1$ they will acquire a complex phase. These states are the familiar Bloch waves of periodic potentials.

Thus, the band structure of the $n = 1$ Lamé potential contains a finite "valence" band between the energies $-e_1$ and $-e_2$ an infinite "conduction" band above $-e_3$ or only one infinite "conduction" band at energies higher than $-e_2$.

The whole process of finding the eigenstates and the band structure can be repeated for the potential $V = 2\wp(x + \omega_2)$. The results are the same apart from making a shift by ω_2 in the definition of the eigenfunctions and an appropriate choice of the normalization constant in order to absorb the complex phases,

$$y_{\pm}(x; a) = \frac{\sigma(x + \omega_2 \pm a)\, \sigma(\omega_2)}{\sigma(x + \omega_2)\, \sigma(\omega_2 \pm a)} e^{-\zeta(\pm a)x}.$$

As a result, the potentials $V = 2\wp(x)$ and $V = 2\wp(x + \omega_2)$ have the same band structure. The two potentials are quite dissimilar functions, the first one having poles and the other being smooth and bounded function



The band structure of the Lamé potential $2\wp$

Section 5

Construction of Classical String Solutions

It turns out that if all four eigenvalues κ^{μ} are equal there will be no string solution that is compatible with the constraints.
The simplest solution to obtain is provided by two distinct eigenvalues.
The form of the target space metrics suggests that AdS_3 favours the selection of eigenvalues of the same sign, which can be either positive or negative, whereas dS_3 favours the selection of eigenvalues of opposite sign.

For positive eigenvalues $\kappa = \ell^2$, the solution of the flat Schrödinger problem is

$$\mathrm{T}(\xi_0) = c_1 \cos(\ell\xi_0) + c_2 \sin(\ell\xi_0),$$

while for negative eigenvalues $\kappa = -\ell^2$, the corresponding solution is

$$\mathrm{T}(\xi_0) = c_1 \cosh(\ell\xi_0) + c_2 \sinh(\ell\xi_0).$$

Any of these solutions should be combined with the eigenfunctions $\Sigma(\xi_1)$ of the Lamé effective Schrödinger problem.
The relation between the eigenvalues of the pair of effective Schrödinger problems implies that κ should be $\kappa = -\wp(a) - 2x_1$.

As an indicative example, let us consider string solutions associated with two distinct positive eigenvalues $\kappa = \ell_{1,2}^2$ given by the ansatz

$$Y = \begin{pmatrix} c_1^+ \Sigma_1^+ (\xi_1) \cos (\ell_1 \xi_0) + c_1^- \Sigma_1^- (\xi_1) \sin (\ell_1 \xi_0) \\ c_1^+ \Sigma_1^+ (\xi_1) \sin (\ell_1 \xi_0) - c_1^- \Sigma_1^- (\xi_1) \cos (\ell_1 \xi_0) \\ c_2^+ \Sigma_2^+ (\xi_1) \cos (\ell_2 \xi_0) + c_2^- \Sigma_2^- (\xi_1) \sin (\ell_2 \xi_0) \\ c_2^+ \Sigma_2^+ (\xi_1) \sin (\ell_2 \xi_0) - c_2^- \Sigma_2^- (\xi_1) \cos (\ell_2 \xi_0) \end{pmatrix}.$$

The functions $\Sigma_{1,2}^{\pm} (\xi_1)$ are in general linear combinations of the Lamé eigenfunctions $y_\pm (\xi_1)$ with moduli equal to $a_{1,2}$, where

$$\ell_{1,2}^2 = -\wp (a_{1,2}) - 2 x_1 .$$

The geometric constraint implies

$$\left(c_1^+\Sigma_1^+\right)^2+\left(c_1^-\Sigma_1^-\right)^2-\left(c_2^+\Sigma_2^+\right)^2-\left(c_2^-\Sigma_2^-\right)^2=\Lambda^2.$$

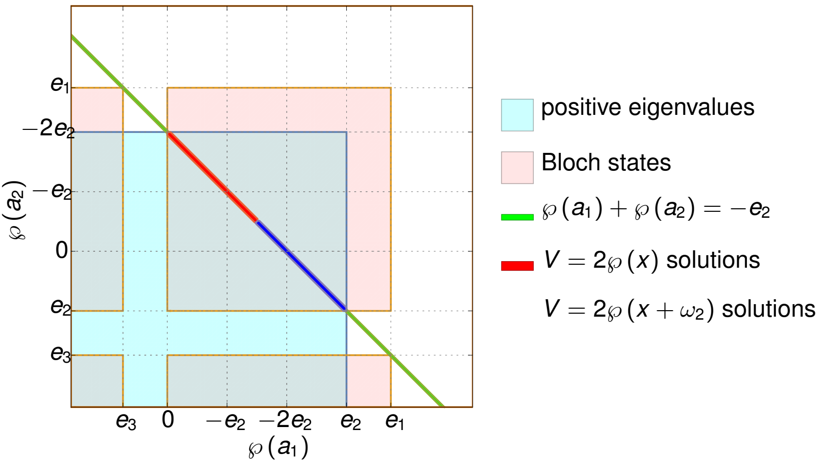
The Virasoro constraints imply

$$\ell_1 c_1^+ c_1^-\left({\Sigma_1^+}'\Sigma_1^- - {\Sigma_1^-}'\Sigma_1^+\right)=\ell_2 c_2^+ c_2^-\left({\Sigma_2^+}'\Sigma_2^- - {\Sigma_2^-}'\Sigma_2^+\right),$$

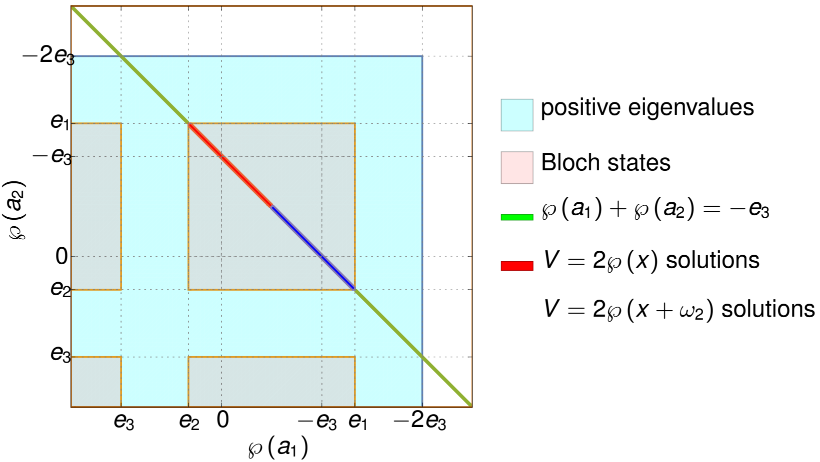
$$\left[\left(c_1^+\Sigma_1^+\right)^2+\left(c_1^-\Sigma_1^-\right)^2\right]\ell_1^2-\left[\left(c_2^+\Sigma_2^+\right)^2+\left(c_2^-\Sigma_2^-\right)^2\right]\ell_2^2=\Lambda^2\left(\wp\left(\xi_1\right)-x_1\right).$$

The solution of the constraints and the demand that the solution is real combined with the properties of the Lamé eigenfunctions result in the following

- Positive eigenvalues must correspond to Bloch wave solutions of the $n = 1$ Lamé problem, whereas negative eigenvalues must correspond to non-normalizable states in the gaps of the $n = 1$ Lamé potential.
- a_1 and a_2 must obey
$$\wp(a_1) + \wp(a_2) = -x_1 .$$
- $\wp(a_1) < \wp(a_2)$ for unbounded solutions, whereas $\wp(a_1) > \wp(a_2)$ for the bounded ones.



The allowed $\wp(a_{1,2})$ for classical string solutions when $x_1 = e_2$, $E < 0$



The allowed $\wp(a_{1,2})$ for classical string solutions when $x_1 = e_3$

To visualize the form of the solutions, we convert to global coordinates

$$Y = \Lambda \begin{pmatrix} \sqrt{1+r^2}\cos t \\ \sqrt{1+r^2}\sin t \\ r\cos\varphi \\ r\sin\varphi \end{pmatrix},$$

in which the AdS_3 metric takes the usual form

$$ds^2 = -\left(1+r^2\right)dt^2 + \frac{1}{1+r^2}dr^2 + r^2 d\varphi^2 .$$

The string solution associated to the unbounded configurations takes the form

$$\begin{aligned} r &= \sqrt{\frac{\wp\left(\xi_1\right) - \wp\left(a_2\right)}{\wp\left(a_2\right) - \wp\left(a_1\right)}}, \\ t &= \ell_1 \xi_0 - \arg \frac{\sigma\left(\xi_1 + a_1\right)}{\sigma\left(\xi_1\right)\sigma\left(a_1\right)} e^{-\zeta(a_1)\xi_1}, \\ \varphi &= \ell_2 \xi_0 - \arg \frac{\sigma\left(\xi_1 + a_2\right)}{\sigma\left(\xi_1\right)\sigma\left(a_2\right)} e^{-\zeta(a_2)\xi_1}. \end{aligned}$$

Likewise, for the bounded configurations, the corresponding string solution is

$$r = \sqrt{\frac{\wp\left(a_2\right) - \wp\left(\xi_1 + \omega_2\right)}{\wp\left(a_1\right) - \wp\left(a_2\right)}},$$

$$t = \ell_1 \xi_0 - \arg \frac{\sigma\left(\xi_1 + \omega_2 + a_1\right)\sigma\left(\omega_2\right)}{\sigma\left(\xi_1 + \omega_2\right)\sigma\left(a_1 + \omega_2\right)} e^{-\zeta(a_1)\xi_1},$$

$$\varphi = \ell_2 \xi_0 - \arg \frac{\sigma\left(\xi_1 + \omega_2 + a_2\right)\sigma\left(\omega_2\right)}{\sigma\left(\xi_1 + \omega_2\right)\sigma\left(a_2 + \omega_2\right)} e^{-\zeta(a_2)\xi_1}.$$

In both cases, the solution corresponds to a rigidly rotating spiky string with constant angular velocity $\omega = \ell_2/\ell_1$, which

$$\omega < 1, \quad \text{when} \quad \wp(a_1) < \wp(a_2),$$
$$\omega > 1, \quad \text{when} \quad \wp(a_1) > \wp(a_2).$$

ω is smaller than one for the unbounded solution and larger than one for the bounded one, since the radial coordinate r is also unbounded or bounded, respectively depending on the form of the solution.

The periodic sinh-Gordon configurations exhibit an interesting limit as

$$\wp\left(a_{1,2}\right) \to e_{1,2} \quad \text{or} \quad \wp\left(a_{1,2}\right) \to e_{2,1}.$$

In this limit, the functions $y_{\pm}\left(\xi_1; a_{1,2}\right)$ both tend to $\sqrt{\wp\left(\xi_1\right) - e_{1,2}}$. These eigenfunctions are real, and, thus, we have the relation $\phi - \omega t = 0$ and the solution degenerates to a straight string rotating like a rigid rod around its center. This limit gives rise to the Gubser-Klebanov-Polyakov solution.

If one considers the translationally invariant solutions of the cosh-Gordon equation, ξ_0 and ξ_1 will be interchanged and the solution will be written as

$$
\begin{aligned}
r &= \sqrt{\frac{\wp\left(a_2\right) - \wp\left(\xi_0 + \omega_2\right)}{\wp\left(a_1\right) - \wp\left(a_2\right)}}, \\
t &= \ell_1 \xi_1 - \arg \frac{\sigma\left(\xi_0 + \omega_2 + a_1\right) \sigma\left(\omega_2\right)}{\sigma\left(\xi_0 + \omega_2\right) \sigma\left(a_1 + \omega_2\right)} e^{-\zeta(a_1)\xi_0}, \\
\varphi &= \ell_2 \xi_1 - \arg \frac{\sigma\left(\xi_0 + \omega_2 + a_2\right) \sigma\left(\omega_2\right)}{\sigma\left(\xi_0 + \omega_2\right) \sigma\left(a_2 + \omega_2\right)} e^{-\zeta(a_2)\xi_0}.
\end{aligned}
$$

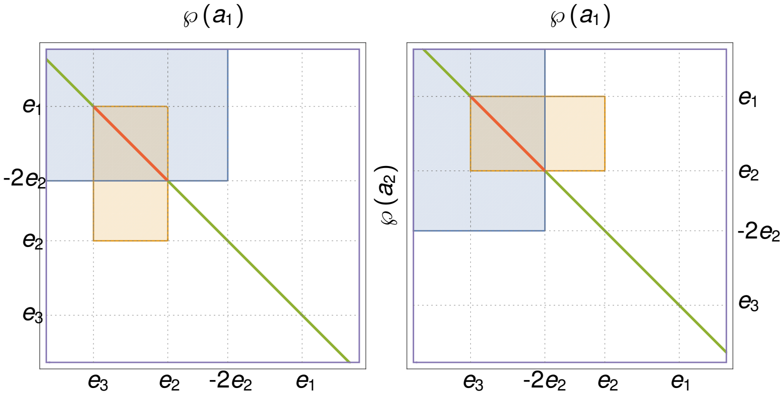
This describes the space-time "dual" picture of a finite spiky string. This solution is a circular string that rotates with angular velocity and radius that vary periodically in time. In this solutions, the radius of the string oscillates between two extremes. When it reaches the maximum value the string moves with the speed of light. Then, it is reflected towards smaller radii and starts shrinking until it reaches the minimum and it keeps oscillating. From the point of view of the enhanced space, the coordinates Y^{-1} and Y^0 have a periodic dependence on the global coordinate t with period equal to 2π. Thus, demanding that these solutions are single valued in the enhanced space enforces the oscillatory behaviour of the circular strings to have period equal to $2\pi/n$, where $n \in \mathbb{N}$.

Section 6

Static Minimal Surfaces in AdS_4

The construction of solutions of elliptic static minimal surfaces is identical to that of elliptic classical string solutions, with some trivial variations

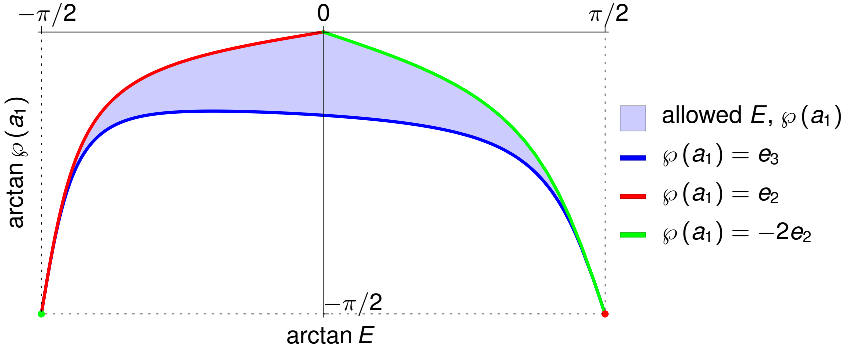
- There is no distinction between static and translationally invariant solutions, as both world-sheet coordinates are space-like.
- Solutions correspond always to the cosh-Gordon equation solutions
- Every pair of effective Schrödinger problems must have opposite eigenvalues instead of equal.
- Bounded solutions are excluded (they are not real and this is physically expected since surfaces not anchored at the boundary are shrinkable to a point)
- The form of the metric enforces the two distinct eigenvalues in the ansatz to be of opposite sign.
- The latter together with the correspondence between bands/gaps and signs of eigenvalues results in the positive eigenvalue corresponding to the finite band and the negative one to the finite gap.



$E < 0$ $\quad$ $E > 0$

- $\kappa_1 > 0$, $\kappa_2 < 0$
- κ_1 in gap, κ_2 in band
- $\wp(a_1) + \wp(a_2) = -e_2$
- minimal surface solutions

The family of elliptic minimal surfaces contains two free parameters. One of those is the constant of integration E, the other is the parameter $\wp(a_1)$, which takes values between e_3 and $\min(e_2, -2e_2)$.



The minimal surface reaches the boundary at the points the Weierstrass elliptic function diverges, namely $u = 2n\omega_1$. Thus, an appropriately anchored at the boundary minimal surface is spanned by

$$u \in (2n\omega_1, 2\,(n+1)\,\omega_1)\,, \quad v \in \mathbb{R},$$

where $n \in \mathbb{Z}$.
In Poincaré coordinates, denoting as $r_\pm$ and $\varphi_\pm$ the angular coordinates of the trace of the extremal surface at the boundary as $u \to 2n{\omega_1}^+$ and as $u \to 2\,(n+1)\,{\omega_1}^-$, respectively

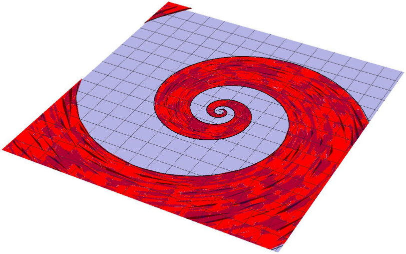
$$\begin{aligned} r_+ &= \Lambda e^{\frac{\ell_1}{\ell_2}\varphi_+ + 2n\left(\frac{\ell_1}{\ell_2}\mathrm{Im}\delta_2 + \mathrm{Re}\delta_1\right)} = \Lambda e^{\omega(\varphi_+ + \varphi_0)}, \\ r_- &= \Lambda e^{\frac{\ell_1}{\ell_2}(\varphi_- - \pi) + 2(n+1)\left(\frac{\ell_1}{\ell_2}\mathrm{Im}\delta_2 + \mathrm{Re}\delta_1\right)} = \Lambda e^{\omega(\varphi + \varphi_0 - \delta\varphi)}, \end{aligned}$$

where

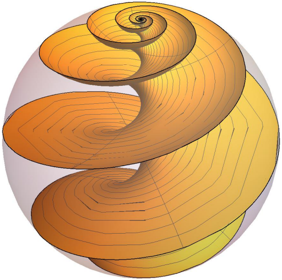
$$\omega = \frac{\ell_1}{\ell_2}, \quad \delta\varphi = \pi - 2\left(\mathrm{Im}\delta_2 + \frac{\ell_2}{\ell_1}\mathrm{Re}\delta_1\right),$$

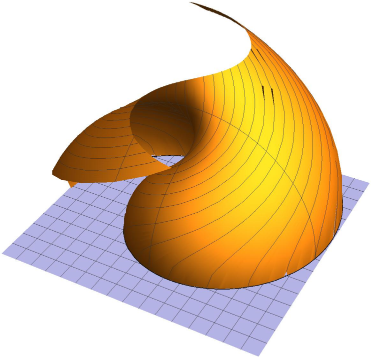
$$\delta_1 \equiv \zeta\,(\omega_1)\,a_1 - \zeta\,(a_1)\,\omega_1, \quad \delta_2 \equiv \zeta\,(\omega_1)\,a_2 - \zeta\,(a_2)\,\omega_1.$$

So in Poincaré coordinates, the trace of the minimal surface in the boundary is the union of two logarithmic spirals.

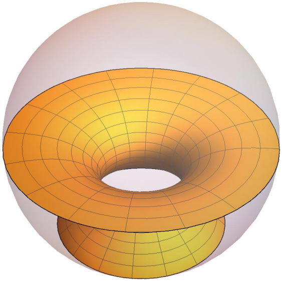


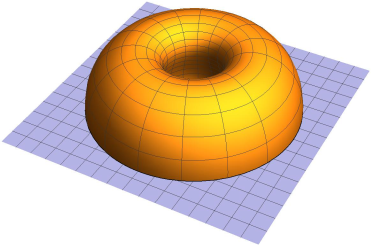
At the limit $\wp(a_1) = e_3$ and $\wp(a_2) = e_1$, we get the ruled surface limit (helicoids) for which $\delta\varphi_{\mathrm{helicoid}} = \pi$.



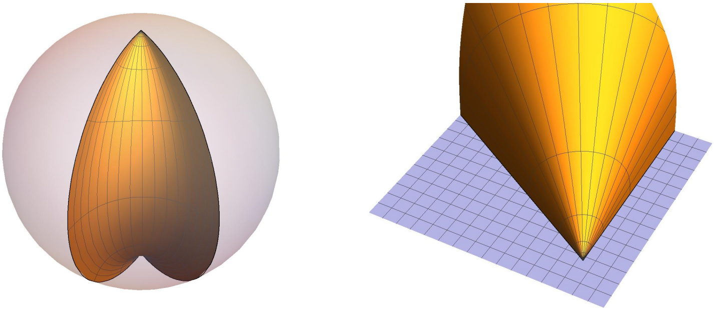


When $E > 0$, at the limit $\wp(a_1) = -2e_2$ and $\wp(a_2) = e_2$, we get the rotational surface limit (catenoids) for which $\omega_{\mathrm{catenoid}} = 0$, $\delta\varphi_{\mathrm{catenoid}} = +\infty$.





When $E < 0$, at the limit $\wp(a_1) = e_2$ and $\wp(a_2) = -2e_2$, we get the conical surface limit for which $\omega_{\mathrm{conical}} = \infty$.



The area of the minimal surface can be directly calculated with the use of formula

$$A = \Lambda^2 \int_{-\infty}^{+\infty} dv \int_{2n\omega_1}^{2(n+1)\omega_1} du \left(\wp\left(u\right) - e_2 \right).$$

The length of the entangling curve, due to the conformal parametrization can be expressed as

$$L = \lim_{u \to 2n\omega_1{}^+} \Lambda \int_{-\infty}^{+\infty} dv \sqrt{\wp\left(u\right) - e_2} + \lim_{u \to 2(n+1)\omega_1{}^-} \Lambda \int_{-\infty}^{+\infty} dv \sqrt{\wp\left(u\right) - e_2}.$$

It is straightforward to show that we recover the usual "area law"

$$A = \Lambda L - 2\Lambda^2 e_2 \omega_1 \int_{-\infty}^{+\infty} dv.$$

The universal constant term here diverges. This is due to the geometry of the entangling curve being infinite.

This divergence introduces a subtlety in the comparison of the areas of two distinct surfaces corresponding to the same entangling curve, as one may rescale v for each of those at will.

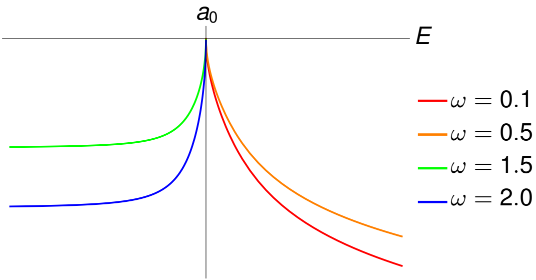
An appropriate regularization of the universal constant term must enforce that v is connected to the physical position of a given point on the entangling curve. The azimuthal angle φ specifies a unique point on the spiral entangling curve.

$$A = \Lambda L - \frac{\sqrt{2}}{3}\Lambda^2 \sqrt{E\left(1-\omega^2\right)}\omega_1 \int_{-\infty}^{+\infty} d\varphi .$$

We define

$$a_0\left(E,\omega\right) := -\frac{\sqrt{2}}{3}\Lambda^2 \sqrt{E\left(1-\omega^2\right)}\omega_1\left(E\right),$$

which can be used as a measure of comparison for the areas corresponding to the same entangling curve.


a_0
E
$\omega = 0.1$
$\omega = 0.5$
$\omega = 1.5$
$\omega = 2.0$

Unlike the general surface, where the parameter v has to take values in the whole real axis, in the catenoid limit the range of the coordinate v becomes finite. It is a direct consequence that the universal constant term in the area formula becomes finite and specifically,

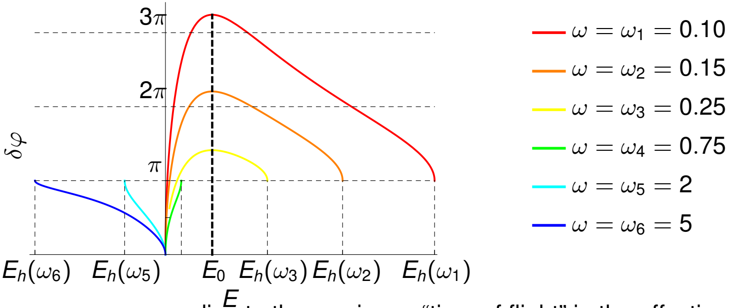
$$A_{\mathrm{catenoid}} = \Lambda L - 4\pi\Lambda^2 \sqrt{\frac{e_2}{3}}\omega_1 .$$

In the case of catenoids it is convenient to define the quantity

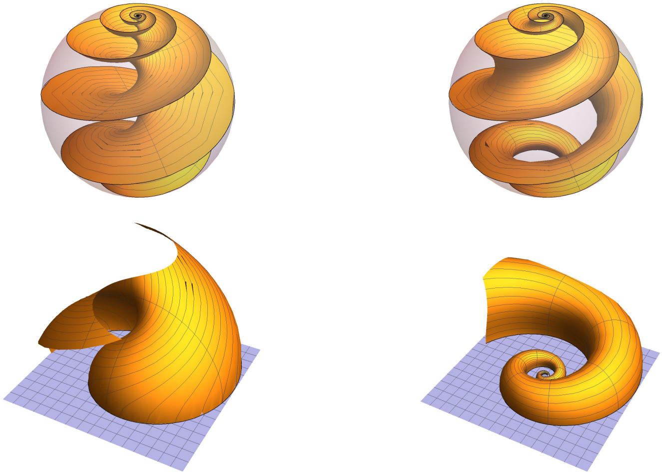
$$a_0^{\mathrm{cat}} := -4\pi\Lambda^2 \sqrt{\frac{e_2}{3}}\omega_1 ,$$

which can be used to compare the area of catenoids possessing the same entangling curve.

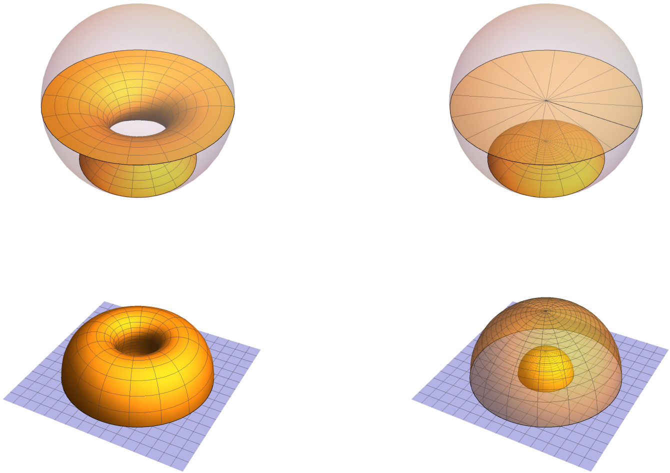
Possible geometric phase transitions may occur between minimal surfaces corresponding to the same boundary region. These are minimal surfaces with the same ω and $\delta\varphi$ equal or summing to 2π.
We plot $\delta\varphi$ along a constant ω curve in the moduli space.
These curves all start at $E = 0$, $\wp(a_1) = 0$ and end at a helicoid with $E = 1/\omega - \omega$.

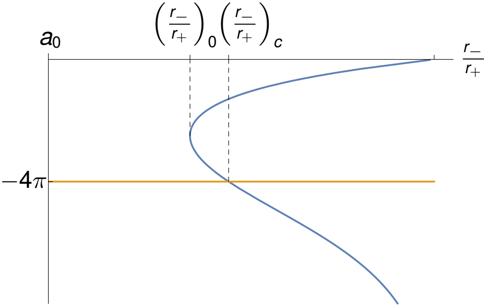


E_0 is the energy corresponding to the maximum “time of flight” in the effective one-dimensional mechanical problem.



In the case of catenoids, there is also the choice of a Goldschmidt solution. When the ratio of the radii of the boundary circles is smaller than a critical value $\left(\frac{r_-}{r_+}\right)_c \simeq 0.467209$ the disjoint surfaces are the preferred choice, whereas when the ratio of the radii is larger that this critical value the catenoid is preferred and specifically the catenoid corresponding to the larger value of E for the given ratio. The catenoid corresponding to the smaller value of E for a given ratio is never preferred in comparison to any of the two options




a_0
$\left(\frac{r_-}{r_+}\right)_0$
$\left(\frac{r_-}{r_+}\right)_c$
$\frac{r_-}{r_+}$
-4π

Section 7

Discussion

We developed a method to construct classical string solutions in AdS_3 and dS_3 from a specific family of solutions of the Pohlmeyer reduced theory that depend solely on one of the two world-sheet coordinates.

- These solutions admit a uniform description in terms of Weierstrass functions.
- They are characterized by an interesting interplay between static and translationally invariant solutions and string propagation in AdS_3 and dS_3 spaces.

Our construction is based on separation of variables leading to four pairs of effective Schrödinger problems

- The components of each pair of effective Schrödinger problems have the same (opposite for the Euclidean problem) eigenvalue.
- Each pair consists of a flat potential and a periodic $n = 1$ Lamé potential
- Consistent solutions fall within an ansatz that requires not one but two distinct eigenvalues.
- Relative size of the two eigenvalues corresponds to the selection between bounded and unbounded solutions (bounded are excluded in the Euclidean problem).
- The constraints select Bloch waves with positive eigenvalues and non-normalizable states in the gaps of the Lamé potential with negative eigenvalues.

- The class of elliptic string solutions that emerge in our study includes the spiky strings as well as several new solutions.
- They include rotating circular strings with periodically varying radius and angular velocity.
- Solutions corresponding to negative eigenvalues of the effective Schrödinger problems look like a periodic spiky structure translating with constant velocity in hyperbolic slicing of AdS_3

The inverse problem of Pohlmeyer reduction:

- For a given solution of the Pohlmeyer reduced equations, there is a continuously infinite set of distinct classical string solutions.
- Bounded solutions: A discrete but still infinite subset is single valued.

It would be interesting to study the extension to other target space geometries, such as the sphere. Spiky string solutions are known to exist on the sphere thus it is very probable that there is an analogous treatment for them.

In higher dimensional symmetric spaces, Pohlmeyer reduction results in multi-component generalizations of the sinh- or cosh-Gordon equations. An interesting question is whether there is an non-trivial extension of our techniques to those more general cases.

All these will be useful for applications to strings propagating in $AdS_5 \times S^5$ in the framework of holography.

The minimal surfaces constructed can lead to applications related to entanglement entropy in theories with holographic duals. Not much beyond sphere and strip were known so far.

- If Gravity is a quantum entropic force associated with quantum entanglement statistics (Raamsdonk), then equivalence between first law of entanglement thermodynamics and Einstein equations should hold for any entangling surface.
- The entangling surfaces have not trivial curvature (ulike the usual cases), so they are a good toy model to study dependence of entanglement entropy on the geometric characteristics of the entangling surface.
- From a purely mathematical point of view, we explicitly constructed a family of minimal surfaces interpolating between the catenoids, helicoids and conicals and reproduced the stability regions for the latter known only numerically so far.
- The geometric phase transitions discovered can provide further information about the role of entanglement entropy as an order parameter for the confinement/deconfinement phase transition.

Thank you!