Calogero, spherically reduced and PT-deformed

Francisco Correa **Olaf Lechtenfeld**

Universidad Austral de Chile, Valdivia, Chile
Leibniz Universität Hannover, Germany

- Some history
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- Tetrahexahedric model: intertwiner & integrability
- Summary and outlook

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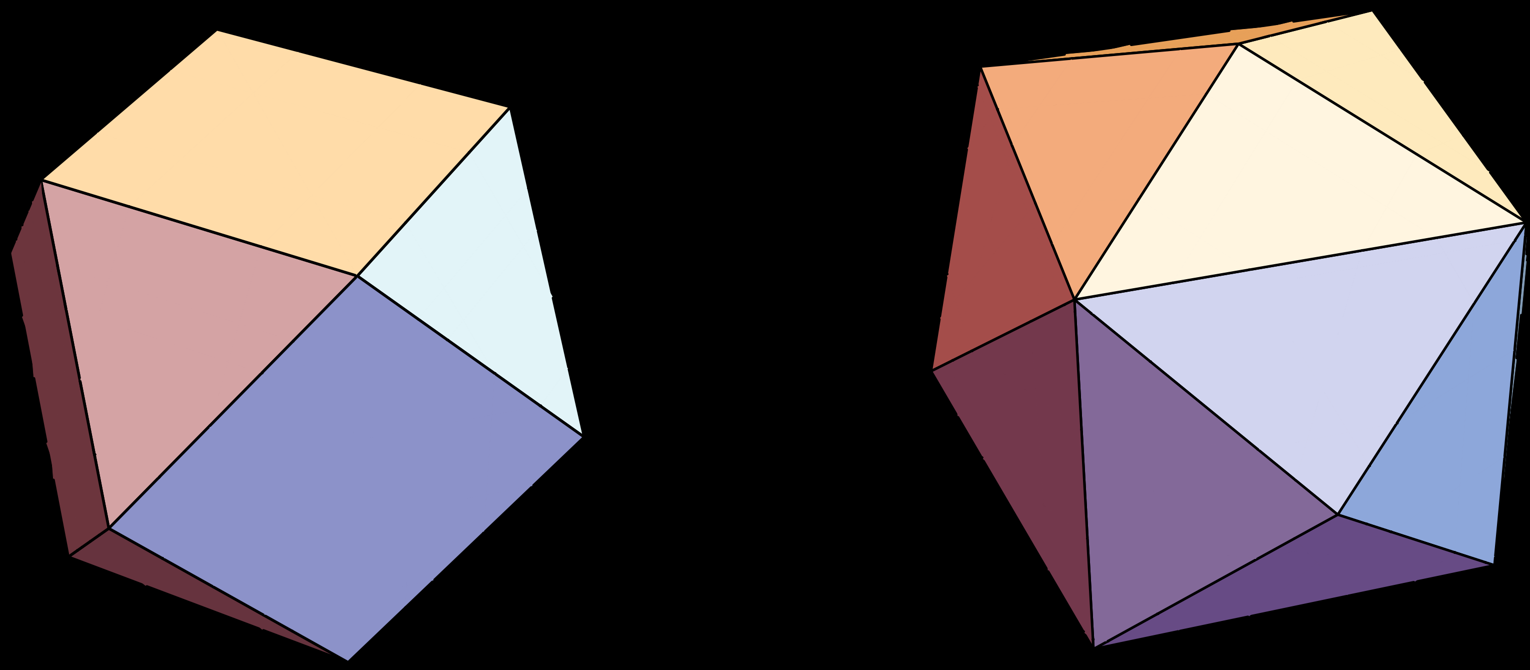
Some history

- 1971 Calogero:
  Solution of the one-dimensional N-body problem with . . . inversely quadratic pair potentials
- 1981 Olshanetsky & Perelomov:
  Classical integrable finite-dimensional systems related to Lie algebras (1983: quantum)
- 1983 Wojciechowski:
  Superintegrability of the Calogero–Moser system
- 1989 Dunkl:
  Differential-difference operators associated to reflection groups
- 1990 Veselov & Chalykh:
  Commutative rings of partial differential operators and Lie algebras, supercompleteness
- 1991 Heckman:
  Elementary construction for commuting charges and intertwiners (shift operators)
- 2003 M. Feigin:
  Intertwining relations for the spherical parts of generalized Calogero operators
- 2008 Fring, Znojil:
  $\mathcal{PT}$-symmetric deformations of Calogero models
- 2008 Hakobyan, Nersessian, Yeghikyan:
  The cuboctahedric Higgs oscillator from the rational Calogero model (classical)
- 2013 M. Feigin, Lechtenfeld, Polychronakos:
  The quantum angular Calogero–Moser model (spectra, eigenstates)
- 2014 M. Feigin, Hakobyan:

  On the algebra of Dunkl angular momentum operators



The angular (relative) Calogero model

A_{n-1} Calogero Hamiltonian: $$H = \sum_{\mu<\nu}^{n} \left\{ \frac{1}{2n}(p_\mu - p_\nu)^2 + \frac{g(g-1)}{(x^\mu - x^\nu)^2} \right\}$$

Quantization: $[\, x^\mu \,,\, p_\nu \,] = \mathrm{i}\, \delta^\mu_\nu$ with $\mu, \nu = 1, \ldots, n$

$$\frac{1}{n} \sum_{\mu<\nu} (x^\mu - x^\nu)^2 = r^2 \qquad \text{and} \qquad \frac{1}{n} \sum_{\mu<\nu} (p_\mu - p_\nu)^2 = p_r^2 + \frac{1}{r^2} L^2 + \frac{(n-2)(n-4)}{4\, r^2}$$

Introduce $n-1$ relative coordinates:

$$r^2 = \sum_{i=1}^{n-1} (y^i)^2 \quad , \quad p_i \equiv p_{y^i} \quad , \quad L_{ij} = -\mathrm{i}(y^i p_j - y^j p_i) \quad , \quad L^2 = -\sum_{i<j} L_{ij}^2$$

SL(2,$\mathbb{R}$) conformal algebra generated by

$$H = \frac{1}{2} p_r^2 + \frac{(n-2)(n-4)}{8\, r^2} + \frac{1}{r^2} H_\Omega \quad , \quad D = \frac{1}{2}(r\, p_r + p_r r) \quad , \quad K = \frac{1}{2} r^2$$

Angular Calogero Hamiltonian:

$$H_\Omega \;=\; \tfrac12 L^2 + U(\vec\theta) \;=\; C - \tfrac18(n{-}1)(n{-}5) \qquad \text{with} \qquad C \;=\; K\,H + H\,K - \tfrac12 D^2$$

$$U(\vec\theta) \;=\; r^2 \sum_{\mu<\nu} \frac{g(g{-}1)}{(x^\mu{-}x^\nu)^2} \;=\; r^2 \sum_{\alpha\in\mathcal{R}_+} \frac{g(g{-}1)}{(\alpha\cdot y)^2} \;=\; \frac{g(g{-}1)}{2} \sum_{\alpha\in\mathcal{R}_+} \cos^{-2}\theta_\alpha$$

Potential blows up on Weyl-chamber walls $\Rightarrow$ particle trapped in $(n{-}2)$-simplex

Position representation: $\quad p_i \;\mapsto\; -\mathrm{i}\partial_i \quad\Longrightarrow\quad p_r \;\mapsto\; -\mathrm{i}\big(\partial_r + \tfrac{n-2}{2\,r}\big)$

Hamiltonians:

$$H \;\mapsto\; -\tfrac12\Big(\partial_r^2 + \tfrac{n-2}{r}\partial_r\Big) \;+\; \tfrac{1}{r^2}H_\Omega \;=\; w^{-1}\Big[-\tfrac12\Big(\partial_r^2 - \frac{(n{-}2)(n{-}4)}{4\,r^2}\Big) \;+\; \tfrac{1}{r^2}H_\Omega\Big]\,w$$

$$H_\Omega \mapsto\; -\tfrac12 \sum_{i<j}\Big(y^i\partial_j - y^j\partial_i\Big)^2 \;+\; r^2 \sum_{\alpha\in\mathcal{R}_+} \frac{g(g{-}1)}{(\alpha\cdot y)^2} \qquad \text{with} \quad w = r^{\frac{n-2}{2}}$$

Radial-angular separation: $\varepsilon = \frac{1}{2}q\,(q+n-3)$

$$H\,\Psi = E\,\Psi \qquad \text{with} \qquad \Psi = R(r)\,v(\vec{\theta}) \qquad \text{and} \qquad H_\Omega\, v = \varepsilon\, v$$

$$w\,H\,w^{-1}\Big|_\varepsilon \mapsto -\tfrac{1}{2}\partial_r^2 + \tfrac{1}{2\,r^2}\Big[(\tfrac{n}{2}-1)(\tfrac{n}{2}-2) + q\,(q+n-3)\Big]$$
$$= -\tfrac{1}{2}\partial_r^2 + \tfrac{1}{2\,r^2}\Big(q+\tfrac{n}{2}-1\Big)\Big(q+\tfrac{n}{2}-2\Big)$$

Add harmonic confining potential $\frac{1}{2}\omega^2 r^2 \quad \Rightarrow \quad E\Big|_\varepsilon = \omega\,\Big(2\ell_2 + q + \frac{n-1}{2}\Big)\,,\ \ell_2 \in \mathbb{N}_0$

But harmonic Calogero spectrum is known:

$$E = \omega\,\Big(2\ell_2 + 3\ell_3 + \ldots + n\ell_n + \tfrac{1}{2}n(n-1)\,g + \tfrac{n-1}{2}\Big) \quad \text{with} \quad \ell_\mu \in \mathbb{N}_0$$

Comparison: ℓ_2 is missing!

$$q = \tfrac{1}{2}n(n-1)\,g + \ell \qquad \text{where} \qquad \ell = 3\ell_3 + 4\ell_4 + \ldots + n\ell_n \in \mathbb{N}_0$$

Angular spectrum: $\varepsilon_q = \frac{1}{2}q\,(q+n-3)$ with generalized angular momentum q

Degeneracies: $$\deg_n(\varepsilon_q) = p_n(\ell) - p_n(\ell-1) - p_n(\ell-2) + p_n(\ell-3)$$

Generating function: $$p_n(t) := \sum_{\ell=0}^{\infty} p_n(\ell)\, t^\ell = \prod_{m=1}^{n} \left(1 - t^m\right)^{-1}$$

Degeneracies for n=3, 4 and 5:

$$\deg_3(\ell) = \begin{cases} 0 & \text{for} \quad \ell = 1, 2 \bmod 3 \\ 1 & \text{for} \quad \ell = 0 \bmod 3 \end{cases}$$

$$\deg_4(\ell) = \left\lfloor \frac{\ell}{12} \right\rfloor + \begin{cases} 0 & \text{for} \quad \ell = 1, 2, 5 \bmod 12 \\ 1 & \text{for} \quad \ell = \text{else} \bmod 12 \end{cases}$$

$$\deg_5(\ell) = \left\lfloor \frac{6\ell^2 + 72\ell - 89}{720} \right\rfloor + \begin{cases} 0 & \text{for} \quad \ell = 2, 22, 26, 46 \bmod 60 \\ 2 & \text{for} \quad \ell = 0, 48 \bmod 60 \\ 1 & \text{for} \quad \ell = \text{else} \bmod 60 \end{cases}$$

Special cases g=0 and g=1 are free, but limits keep the infinite walls and deg's

Angular eigenfunctions: Weyl reflections $v_\ell{}^{(g)}(s_\alpha\vec\theta) = \mathrm{e}^{\mathrm{i}\pi g}\, v_\ell{}^{(g)}(\vec\theta)$

$$v_q(\vec\theta) \;\equiv\; v_\ell{}^{(g)}(\vec\theta) \;\sim\; r^{n-3+q}\Big(\prod_{\mu=3}^{n}\sigma_\mu\big(\{\mathcal{D}_i\}\big)^{\ell_\mu}\Big)\,\Delta^g\, r^{3-n-n(n-1)g}$$

Vandermonde and Dunkl operators:

$$\Delta \;=\; \prod_{\alpha\in\mathcal{R}_+}\alpha\cdot y \qquad \text{and} \qquad \mathcal{D}_i \;=\; \partial_i \;-\; g\sum_{\alpha\in\mathcal{R}_+}\frac{\alpha_i}{\alpha\cdot y}\,s_\alpha$$

Relation with Dunkl-deformed Weyl-symmetric harmonic polynomials:

$$v_\ell{}^{(g)}(\vec\theta) \;=\; r^{-q}\,\Delta^g\, h_\ell{}^{(g)} \qquad \text{with} \qquad H\left(\Delta^g\, h_\ell{}^{(g)}\right) \;=\; 0$$

‘Dunklize’ angular momenta:

$$L_{ij} \;\mapsto\; -(y^i\partial_j - y^j\partial_i) \qquad \Longrightarrow \qquad \mathcal{L}_{ij} \;=\; -(y^i\mathcal{D}_j - y^j\mathcal{D}_i)$$

From ‘pre-Hamiltonians’ to Hamiltonians:

$$\mathcal{H} \;=\; -\tfrac12\sum_i\mathcal{D}_i^2 \qquad \text{and} \qquad \mathcal{H}_\Omega \;=\; -\tfrac12\sum_{i<j}\mathcal{L}_{ij}^2 \;+\; \tfrac12\, g{\textstyle\sum_\alpha} s_\alpha\,\big(g{\textstyle\sum_\alpha} s_\alpha + n-3\big)$$

$$H \;=\; \mathrm{res}(\mathcal{H}) \qquad \text{and} \qquad H_\Omega \;=\; \mathrm{res}(\mathcal{H}_\Omega) \;=\; \tfrac12\mathrm{res}\big(\mathcal{L}^2\big) \;+\; \varepsilon_q(\ell{=}0)$$

Conserved charges of order t:

$$\mathcal{C}_t(\mathcal{L}_{ij}) \quad \text{Weyl-invariant} \qquad \Longrightarrow \qquad C_t = \text{res}(\mathcal{C}_t) \quad \text{commutes with } H_\Omega$$

Maximal superintegrability: $\exists$ $2n-5$ such charges ($C_2 = H_\Omega$), but not in involution

Angular intertwiners of order s:

$$\mathcal{M}_s(\mathcal{L}_{ij}) \quad \text{Weyl-}\underline{\text{anti}}\text{invariant} \qquad \Longrightarrow \qquad M_s \ = \ \text{res}(\mathcal{M}_s) \quad \text{intertwines with } H_\Omega$$

Since $[\mathcal{L}_{ij}, \mathcal{H}] = 0$ we have:

$$[\mathcal{M}_s, \mathcal{H}] = 0 \qquad \Longrightarrow \qquad M_s^{(g)} H^{(g)} \ = \ H^{(-g)} M_s^{(g)} \ = \ H^{(g+1)} M_s^{(g)}$$

$$\text{and} \qquad M_s^{(g)} \ : \ \left\{\Psi_{E,q}^{(g)}\right\} \ \rightarrow \ \left\{\Psi_{E,q}^{(g+1)}\right\}$$

$$[\mathcal{M}_s, \mathcal{H}_\Omega] = 0 \qquad \Longrightarrow \qquad M_s^{(g)} H_\Omega^{(g)} \ = \ H_\Omega^{(-g)} M_s^{(g)} \ = \ H_\Omega^{(g+1)} M_s^{(g)}$$

$$\text{and} \qquad M_s^{(g)} \ : \ \left\{v_\ell^{(g)}\right\} \ \rightarrow \ \left\{v_{\ell-n(n-1)/2}^{(g+1)}\right\}$$

Algebra generated by $\{\mathcal{D}_i, y^j\}$ and Weyl reflections is a rational Cherednik algebra

Algebra generated by $\{\mathcal{L}_{ij}\}$ and Weyl reflections is a subalgebra, containing H_Ω

$$[\mathcal{L}_{ij}, \mathcal{L}_{k\ell}] \;=\; \mathcal{L}_{i\ell}\mathcal{S}_{jk} - \mathcal{L}_{ik}\mathcal{S}_{j\ell} - \mathcal{L}_{j\ell}\mathcal{S}_{ik} + \mathcal{L}_{jk}\mathcal{S}_{i\ell}$$

with $$\mathcal{S}_{ij} \;=\; \begin{cases} -g\, s_{ij} & \text{for} \quad i \neq j \\ 1 + g\sum_{k(\neq i)} s_{ik} & \text{for} \quad i = j \end{cases}$$

$$[\mathcal{S}_{ij}, \mathcal{L}_{k\ell}] = 0 \quad , \qquad \{\mathcal{S}_{ij}, \mathcal{L}_{ij}\} = 0 \quad , \qquad \mathcal{S}_{ij}\mathcal{L}_{ik} \;=\; \mathcal{L}_{jk}\mathcal{S}_{ij}$$

It is a ‘Dunkl deformation’ of $so(n{-}1)$, with H_Ω being the Casimir invariant

$\boxed{\text{Complex } \mathcal{PT} \text{ deformation}}$

Quantum mechanics achieves $E \in \mathbb{R}$ by $H^\dagger = H$ but $H^\dagger = \rho\, H\, \rho^{-1}$ suffices

Such non-hermitian H is related to a hermitian H_0 by a similarity transformation

Real spectrum assured by (unbroken) invariance under a combined involution $\mathcal{PT}$, where $\mathcal{P}$ is linear and $\mathcal{T}$ is <u>anti</u>linear (usually $\mathcal{T} =$ complex conjugation $\mathrm{i} \mapsto -\mathrm{i}$)

$\mathcal{PT}$ deformation: a non-hermitian $\mathcal{PT}$-invariant family H_ϵ smoothly deforming $H_0 = H_0^\dagger$

Induce $H = H_0 \mapsto H_\epsilon$ from a complex coordinate deformation $\Gamma(\epsilon) : \mathbb{R}^{n-1} \to \mathbb{C}^{n-1}$

If $\Gamma(\epsilon)$ is compatible with Weyl invariance, then integrability will be preserved

Simple possibility here: $\mathcal{P}$ = order-2 element s from Weyl group (e.g. reflection s_α)

Conditions on a complex angular coordinate deformation $\Gamma(\epsilon)$ (debatable!):

- it should be linear $\qquad\qquad\qquad\qquad \searrow$
- it should not change the kinetic term L^2 $\quad\Rightarrow\quad \Gamma(\epsilon) \in \mathrm{SO}(n-1), \mathbb{C})$
- it should render $U_\epsilon(\vec{\theta}) := U\big(\Gamma(\epsilon)\vec{\theta}\big)$ $\mathcal{PT}$-invariant $\quad\Rightarrow\quad \mathcal{PT}\,\Gamma(\epsilon) = \Gamma(\epsilon)$

Consequences:

$$\Gamma(\epsilon) \;=\; \exp\Big\{\sum_{i<j} \epsilon_{ij} G_{ij}\Big\} \qquad \text{with} \qquad G_{ij} : y^k \;\mapsto\; \mathrm{i}\big(\delta^{kj} y^i - \delta^{ki} y^j\big)$$

$$\mathcal{P}\,\Gamma(\epsilon) \;=\; s\,\Gamma(\epsilon)\,s \;\stackrel{!}{=}\; \Gamma(-\epsilon) \;=\; \Gamma(\epsilon)^* \;=\; \mathcal{T}\,\Gamma(\epsilon)$$

$\Rightarrow\quad \epsilon{:}G \equiv \sum_{i<j} \epsilon_{ij} G_{ij}$ intertwines between the $+1$ and -1 eigenspaces of s

Simplest case: $\mathcal{P}$ = root reflection s_γ $\quad\Rightarrow\quad \epsilon{:}G \;\sim\; \epsilon\,\gamma\wedge G\,\gamma \;\in su(1,1)$

$$\Rightarrow \quad \Gamma(\epsilon) \;=\; \mathrm{e}^{\epsilon:G} \;=\; \begin{pmatrix} \cosh(\epsilon) & -\mathrm{i}\sinh(\epsilon) & 0\cdots 0 \\ \mathrm{i}\sinh(\epsilon) & \cosh(\epsilon) & 0\cdots 0 \\ 0 & 0 & \\ \vdots & \vdots & \mathbb{1}_{n-3} \\ 0 & 0 & \end{pmatrix} \quad \text{in suitable coordinates}$$

$\Leftrightarrow$ complexifies the angle $\phi \mapsto \phi + \epsilon$ in the 2-plane $\gamma\wedge G\,\gamma$

Much more general $\mathcal{PT}$ deformations are possible; their classification is open

Benefit: partial de-singularization of the potential; singular loci obey

$\alpha\cdot y = 0 \quad\mapsto\quad \alpha\cdot\Gamma(\epsilon)\,y = 0 \quad\Rightarrow$ <u>two</u> real conditions for each root α

Singularities (generically) reduce from codimension-one to codimension-two $\Rightarrow$
Particle is liberated from its Weyl-chamber trap and can move everywhere on S^{n-2}

Explicitly:

$$U_\epsilon(\vec\theta) \;=\; r^2 \sum_{\alpha\in\mathcal{R}_+} \frac{g(g-1)}{(\alpha\cdot\Gamma(\epsilon)\,y)^2} \;=\; \frac{g(g-1)}{2} \sum_{\alpha\in\mathcal{R}_+} \cos^{-2}\theta_\alpha(\epsilon)$$

is less singular due to $\theta_\alpha(\epsilon) = \theta_\alpha + \mathrm{i}\eta_\alpha(\vec\theta,\epsilon))$

$$\frac{1}{\cos^2(\theta_\alpha + \mathrm{i}\eta_\alpha)} = \frac{\cosh^2\eta_\alpha\cos^2\theta_\alpha - \sinh^2\eta_\alpha\sin^2\theta_\alpha \;+\; \frac{\mathrm{i}}{2}\sinh 2\eta_\alpha \sin 2\theta_\alpha}{(\cosh^2\eta_\alpha\cos^2\theta_\alpha + \sinh^2\eta_\alpha\sin^2\theta_\alpha)^2}$$

Wave functions carry a factor

$$\Delta^g \;=\; \prod_{\alpha\in\mathcal{R}_+} (\alpha\cdot y)^g \quad\mapsto\quad \Delta_\epsilon^{\,g} \;=\; \prod_{\alpha\in\mathcal{R}_+} \big(\alpha\cdot\Gamma(\epsilon)\,y\big)^g$$

and remain unphysical (non-normalizable) for $g<0$, except at $n{=}3$ (see below)

$\exists$ (nonlinear) $\mathcal{PT}$ deformations which totally de-singularize Δ and U at $n{>}3$ $\Rightarrow$
then state tower for $g>1$ must be joined with new state tower for $g' = 1{-}g < 0$

$g\in\mathbb{Z}$: $\mathcal{PT}$ deformations may roughly double the degeneracy of the energy levels
and introduce new ‘odd’ conserved charges Q with $Q\,H_\epsilon{}^{(1-g)} = H_\epsilon{}^{(g)}\,Q$

Warmup: the hexagonal or Pöschl-Teller model

Jacobi relative coordinates on $\mathbb{R}^2 \perp$ center-of-mass X:

$$x^1 = X + \tfrac{1}{\sqrt{2}}\,y^1 + \tfrac{1}{\sqrt{6}}\,y^2 \quad , \qquad \partial_{x^1} = \tfrac{1}{3}\partial_X + \tfrac{1}{\sqrt{2}}\,\partial_{y^1} + \tfrac{1}{\sqrt{6}}\,\partial_{y^2}$$

$$x^2 = X - \tfrac{1}{\sqrt{2}}\,y^1 + \tfrac{1}{\sqrt{6}}\,y^2 \quad , \qquad \partial_{x^2} = \tfrac{1}{3}\partial_X - \tfrac{1}{\sqrt{2}}\,\partial_{y^1} + \tfrac{1}{\sqrt{6}}\,\partial_{y^2}$$

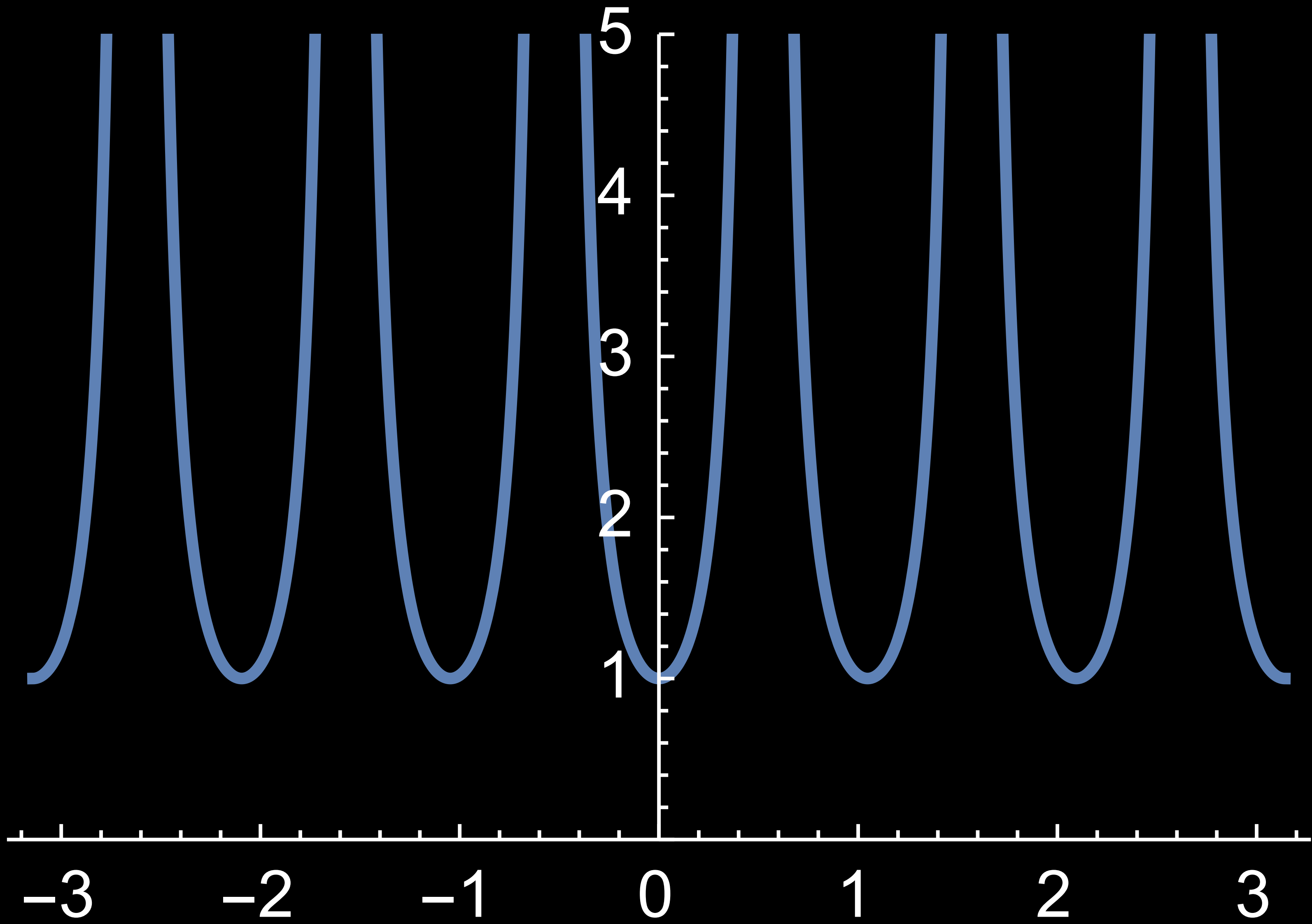
$$x^3 = X - \tfrac{2}{\sqrt{6}}\,y^2 \quad , \qquad \partial_{x^3} = \tfrac{1}{3}\partial_X - \tfrac{2}{\sqrt{6}}\,\partial_{y^2}$$

$$y^1 = r\,\cos\phi \quad \text{and} \quad y^2 = r\,\sin\phi \qquad \Longrightarrow \qquad w := y^1 + \mathrm{i}y^2 = r\,\mathrm{e}^{\mathrm{i}\phi}$$

Angular Hamiltonian:

$$H_\Omega = \tfrac{1}{2}\Big(w\partial_w - \bar{w}\partial_{\bar{w}}\Big)^2 + g(g{-}1)\frac{18\,(w\bar{w})^3}{(w^3+\bar{w}^3)^2}$$

$$U(\phi) = \frac{g(g{-}1)}{2}\sum_{k=0,1,2}\cos^{-2}(\phi{+}k\tfrac{2\pi}{3}) = \tfrac{9}{2}g(g{-}1)\,\cos^{-2}(3\phi)$$



5
4
3
2
1
−3
−2
−1
0
1
2
3

Spectrum and eigenfunctions:

$$\varepsilon_q = \tfrac{1}{2}q^2 \qquad \text{with} \qquad q = 3g + \ell = 3(g+\ell_3) \qquad \text{and} \qquad \deg(\varepsilon_q) = 1$$

$$\Psi_{E,q}(r,\phi) = J_q(\sqrt{2E}\,r)\,v_q(\phi) \qquad\qquad \ell = 3\ell_3$$

$$v_q(\phi) \equiv v_\ell{}^{(g)}(\phi) \sim r^q \left(\mathcal{D}_w^3 - \mathcal{D}_{\bar w}^3\right)^{\ell_3} \Delta^g\, r^{-6g} = r^{-q}\,\Delta^g\, h_\ell{}^{(g)}(w^3,\bar w^3)$$

$$\Delta \sim w^3 + \bar w^3 \sim r^3\,\cos(3\phi) \qquad \text{vanishes at } \phi = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5\pi}{6}$$

$$\mathcal{D}_w = \partial_w - g\left\{\frac{1}{w+\bar w}\,s_0 + \frac{\rho}{\rho w + \bar\rho\bar w}\,s_+ + \frac{\bar\rho}{\bar\rho w + \rho\bar w}\,s_-\right\} \quad \text{with} \quad \rho = \mathrm{e}^{2\pi\mathrm{i}/3}$$

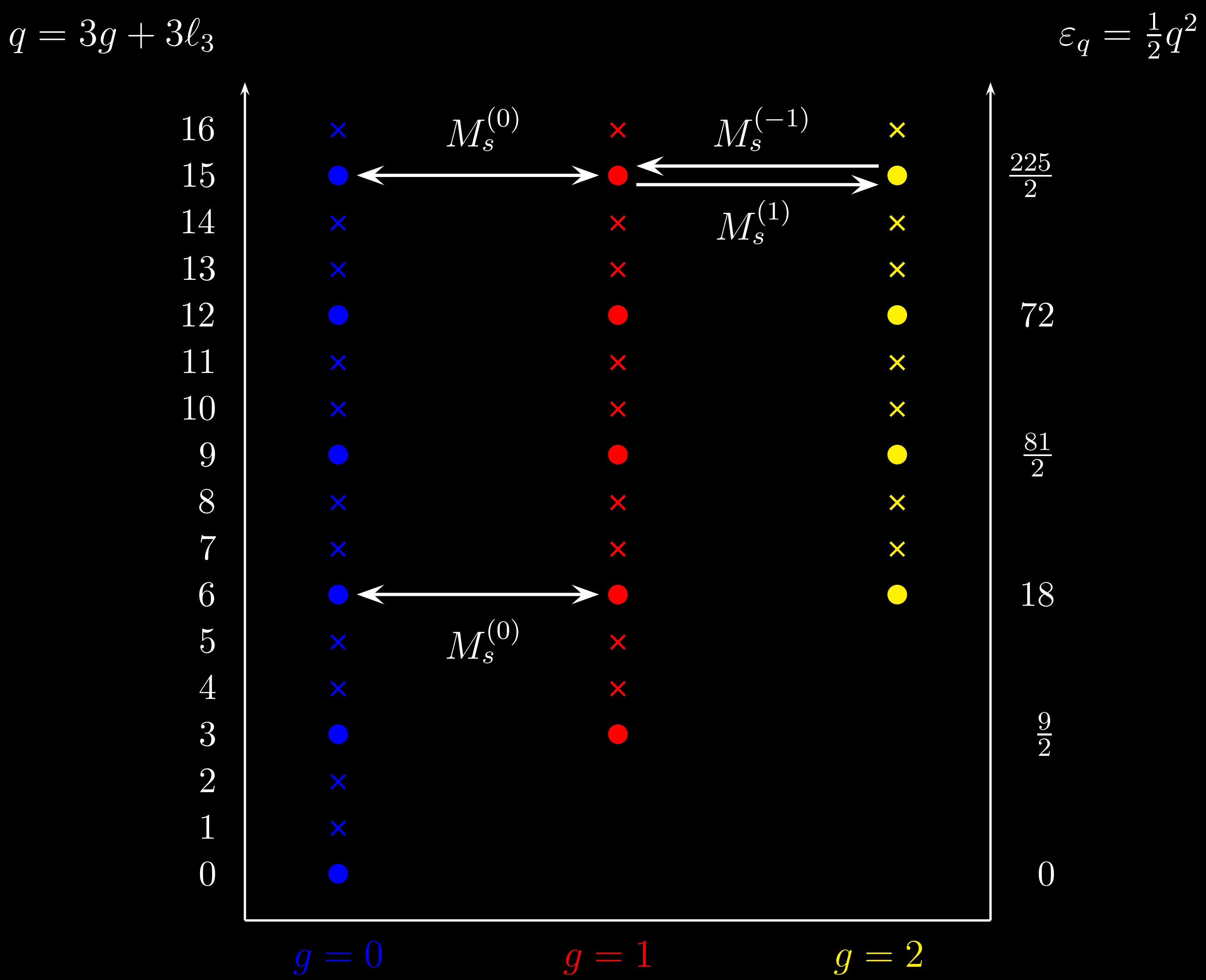
$$h_\ell{}^{(g)}(w^3,\bar w^3) = \sum_{k=0}^{\ell_3} (-1)^k \frac{\Gamma(1+\ell_3)\,\Gamma(g+k)\,\Gamma(g+\ell_3-k)}{\Gamma(2g+\ell_3)\,\Gamma(g)\,\Gamma(1+k)\,\Gamma(1+\ell_3-k)}\, w^{\ell-3k}\bar w^{3k}$$

Low-lying wave functions $v_\ell^{(g)} = r^{-\ell-3g}\,\Delta^g\, h_\ell^{(g)}$ of the Pöschl-Teller model

q	$h_\ell^{(0)}$	$h_\ell^{(1)}$	$h_\ell^{(2)}$	$\cdots$
0	(00)			
3	$(10)-(01)$	(00)		
6	$(20)+(02)$	$(10)-(01)$	(00)	$\ddots$
9	$(30)-(03)$	$(20)-(11)+(02)$	$(10)-(01)$	$\cdots$
12	$(40)+(04)$	$(30)-(21)+(12)-(03)$	$3(20)-4(11)+3(02)$	$\cdots$
15	$(50)-(05)$	$(40)-(31)+(22)-(13)+(04)$	$4(30)-6(21)+6(12)-4(03)$	$\cdots$
18	$(60)+(06)$	$(50)-(41)+(32)-(23)+(14)-(05)$	$5(40)-8(31)+9(22)-8(13)+5(04)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Notation: $(m\,\bar{m}) := w^{3m}\bar{w}^{3\bar{m}} = (y_1+\mathrm{i}y_2)^{3m}(y_1-\mathrm{i}y_2)^{3\bar{m}}$

Normalization is arbitrary $\quad \Delta = (10)+(01) \quad r^6 = (11)$


$q = 3g + 3\ell_3$
$\varepsilon_q = \frac{1}{2}q^2$
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
$M_s^{(0)}$
$M_s^{(-1)}$
$M_s^{(1)}$
$M_s^{(0)}$
$\frac{225}{2}$
72
$\frac{81}{2}$
18
$\frac{9}{2}$
0
$g = 0$
$g = 1$
$g = 2$

Angular intertwiner:

$$\mathcal{M}_1 \sim \mathrm{i}\Big(w\mathcal{D}_w - \bar{w}\mathcal{D}_{\bar{w}}\Big)$$

$$\sim \mathrm{i}\Big(w\partial_w - \bar{w}\partial_{\bar{w}}\Big) - \mathrm{i}\,g\left\{\frac{w-\bar{w}}{w+\bar{w}}\,s_0 + \frac{\rho w-\bar{\rho}\bar{w}}{\rho w+\bar{\rho}\bar{w}}\,s_+ + \frac{\bar{\rho} w-\rho\bar{w}}{\bar{\rho} w+\rho\bar{w}}\,s_-\right\}$$

$$M_1 \sim \mathrm{i}\Big(w\partial_w-\bar{w}\partial_{\bar{w}}\Big) - 3\mathrm{i}\,g\,\frac{w^3-\bar{w}^3}{w^3+\bar{w}^3} = \mathrm{i}\,\Delta^g\Big(w\partial_w-\bar{w}\partial_{\bar{w}}\Big)\Delta^{-g} = \partial_\phi + 3\,g\,\tan 3\phi$$

$$\Longrightarrow \qquad h_\ell^{(g+1)} \sim \mathrm{i}\,\Delta^{-1}\Big(w\partial_w - \bar{w}\partial_{\bar{w}}\Big)\,h_{\ell+3}^{(g)}$$

No further conserved charges:

$$\Big(M_1^\dagger M_1\Big)^{(g)} = -2\,H_\Omega^{(g)} + 9\,g^2 = -\mathrm{res}(\mathcal{L}^2) = -C_2^{(g)}$$

$\mathcal{PT}$ deformation:

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \longmapsto \frac{1}{3} \begin{pmatrix} 1+2\cosh\epsilon & 1-\cosh\epsilon-\mathrm{i}\sqrt{3}\sinh\epsilon & 1-\cosh\epsilon+\mathrm{i}\sqrt{3}\sinh\epsilon \\ 1-\cosh\epsilon+\mathrm{i}\sqrt{3}\sinh\epsilon & 1+2\cosh\epsilon & 1-\cosh\epsilon-\mathrm{i}\sqrt{3}\sinh\epsilon \\ 1-\cosh\epsilon-\mathrm{i}\sqrt{3}\sinh\epsilon & 1-\cosh\epsilon+\mathrm{i}\sqrt{3}\sinh\epsilon & 1+2\cosh\epsilon \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

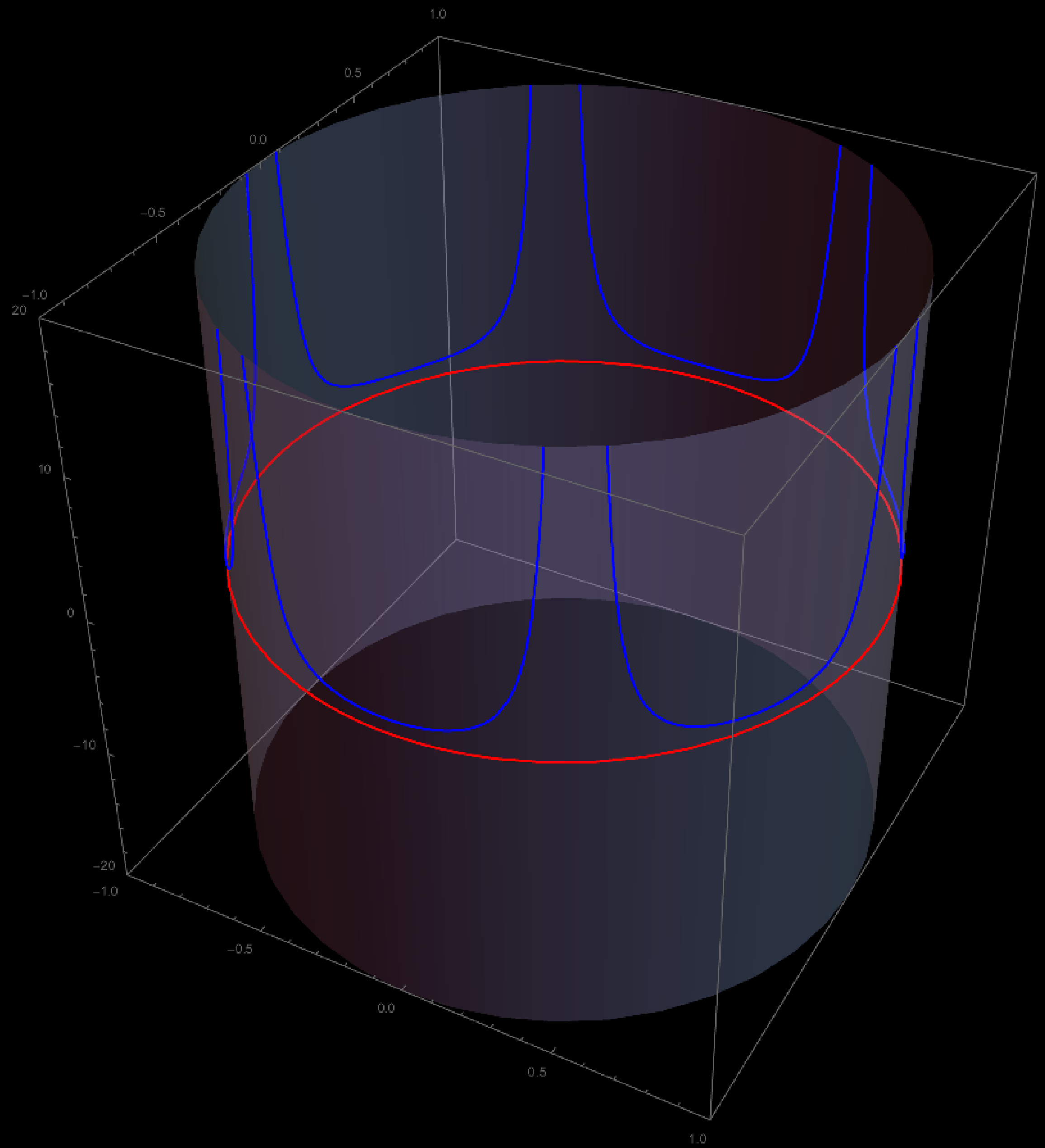
or

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \longmapsto \begin{pmatrix} \cosh\epsilon & -\mathrm{i}\sinh\epsilon \\ \mathrm{i}\sinh\epsilon & \cosh\epsilon \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = r \begin{pmatrix} \cos(\phi+\mathrm{i}\epsilon) \\ \sin(\phi+\mathrm{i}\epsilon) \end{pmatrix}$$

$\Longleftrightarrow$ $\quad \phi \mapsto \phi + \mathrm{i}\epsilon$ or $(w, \bar{w}) \mapsto (\mathrm{e}^{-\epsilon} w, \mathrm{e}^{\epsilon} \bar{w})$

Complex potential:

$$U_\epsilon(\phi) = 9\, g(g-1)\, \frac{(1+\cosh 6\epsilon \cos 6\phi) + 2\mathrm{i} \sinh 6\epsilon \sin 6\phi}{(\cosh 6\epsilon + \cos 6\phi)^2}$$


1.0
0.5
0.0
−0.5
−1.0
20
10
0
−10
−20
−1.0
−0.5
0.0
0.5
1.0

Spectrum: independent of ϵ but previously singular states for $g<0$ appear!

$$\varepsilon_q = \tfrac{1}{2}q^2 \qquad \text{with} \qquad q = 3g+\ell = 3(g+\ell_3) \qquad \text{and} \qquad \ell_3 \geq \min(-g,0)$$

$$\triangle_{\epsilon} \sim \mathrm{e}^{-3\epsilon}w^3 + \mathrm{e}^{3\epsilon}\bar{w}^3 \sim r^3\Big(\cosh(3\epsilon)\cos(3\phi) - \mathrm{i}\sinh(3\epsilon)\sin(3\phi)\Big) \neq 0$$

Eigenfunction formula extends to $g<0$ with proper $\frac{\infty}{\infty}$ regularization

$$h_{\ell}^{\epsilon(g)}(w^3,\bar{w}^3) = \sum_{k=0}^{\ell_3}(-1)^k \frac{\Gamma(1+\ell_3)\,\Gamma(g+k)\,\Gamma(g+\ell_3-k)}{\Gamma(2g+\ell_3)\,\Gamma(g)\,\Gamma(1+k)\,\Gamma(1+\ell_3-k)}(\mathrm{e}^{-\epsilon}w)^{\ell-3k}(\mathrm{e}^{\epsilon}\bar{w})^{3k}$$

Low-lying wave functions $v_\ell^{\epsilon(g)} = r^{-\ell-3g} \underset{\epsilon}{\triangle}^{g} h_\ell^{\epsilon(g)}$ of the $\mathcal{PT}$-Pöschl-Teller model

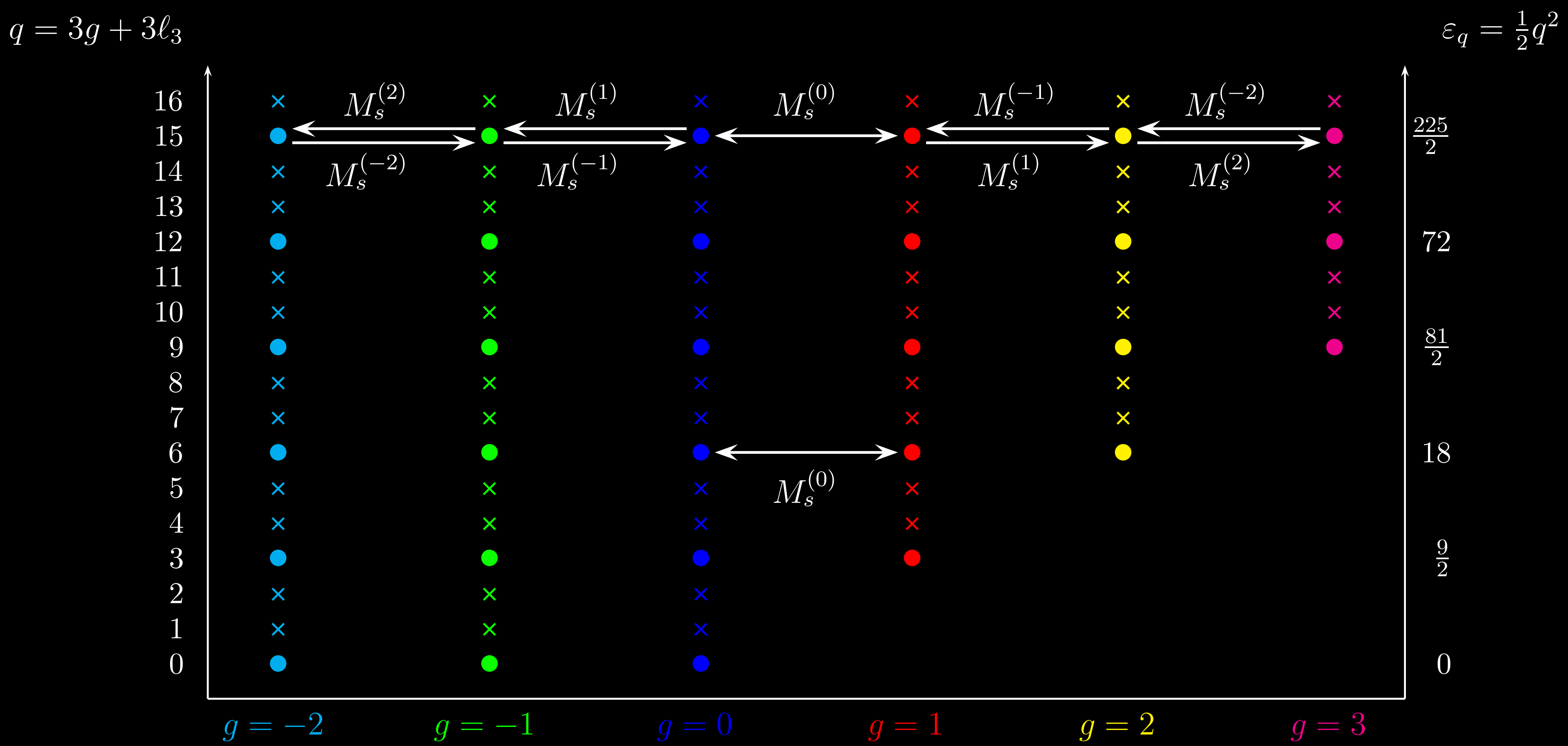
q	$h_\ell^{\epsilon(-1)}$	$h_\ell^{\epsilon(0)}$	$h_\ell^{\epsilon(1)}$	$h_\ell^{\epsilon(2)}$
0	$(10)-(01)$	(00)		
3	(11)	$(10)-(01)$	(00)	
6	$(30)+3(21)-3(12)-(03)$	$(20)+(02)$	$(10)-(01)$	(00)
9	$2(40)+4(31)+4(13)+2(04)$	$(30)-(03)$	$(20)-(11)+(02)$	$(10)-(01)$
12	$3(50)+5(41)-5(14)-3(05)$	$(40)+(04)$	$(30)-(21)+(12)-(03)$	$3(20)-4(11)+3(02)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

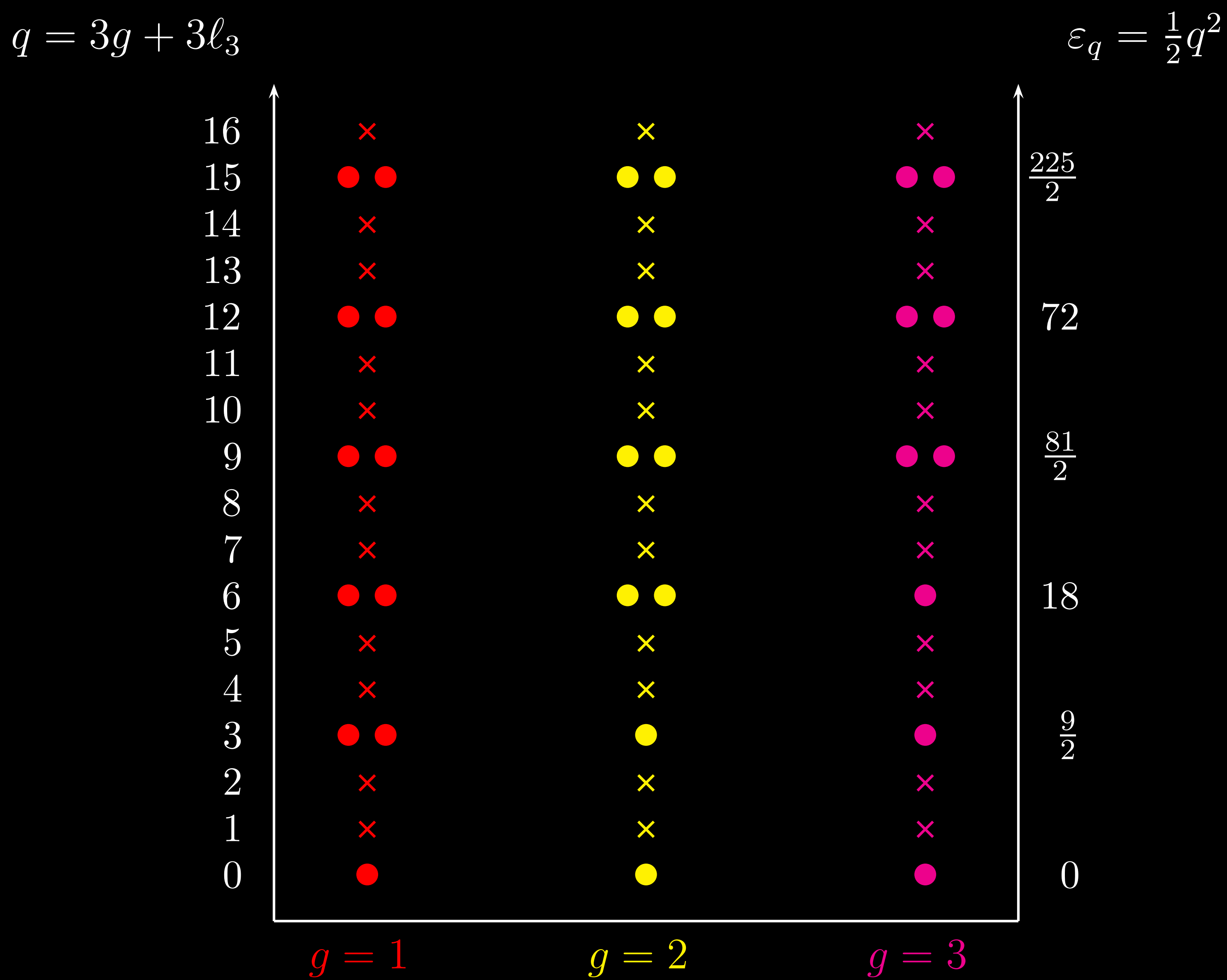
Notation: $(m\,\bar{m}) := \mathrm{e}^{-3(m-\bar{m})\epsilon}\, w^{3m}\bar{w}^{3\bar{m}}$

Normalization is arbitrary

State spaces at g and $1-g$ to be joined $\Rightarrow$ $\deg(\varepsilon_q) \overset{g\geq 0}{=} \begin{cases} 1 & \text{for } q < 3g \\ 2 & \text{for } q \geq 3g \end{cases}$

q	$\triangle_{\epsilon}^{-1} h_{\ell}^{\epsilon(-1)}$	$h_{\ell}^{\epsilon(0)}$	$\triangle_{\epsilon} h_{\ell}^{\epsilon(1)}$	$\triangle_{\epsilon}^{2} h_{\ell}^{\epsilon(2)}$
0	$\dfrac{(10)-(01)}{(10)+(01)}$	(00)		
3	$\dfrac{(11)}{(10)+(01)}$	$(10)-(01)$	$(10)+(01)$	
6	$\dfrac{(30)+3(21)-3(12)-(03)}{(10)+(01)}$	$(20)+(02)$	$(20)-(02)$	$(20)+2(11)+(02)$
9	$\dfrac{2(40)+4(31)+4(13)+2(04)}{(10)+(01)}$	$(30)-(03)$	$(30)+(03)$	$(30)+(21)-(12)-(03)$
12	$\dfrac{3(50)+5(41)-5(14)-3(05)}{(10)+(01)}$	$(40)+(04)$	$(40)-(04)$	$3(40)+2(31)-2(22)+2(13)+3(04)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$


$q = 3g + 3\ell_3$
$\varepsilon_q = \frac{1}{2}q^2$
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
$M_s^{(2)}$
$M_s^{(-2)}$
$M_s^{(1)}$
$M_s^{(-1)}$
$M_s^{(0)}$
$M_s^{(-1)}$
$M_s^{(1)}$
$M_s^{(-2)}$
$M_s^{(2)}$
$M_s^{(0)}$
$\frac{225}{2}$
72
$\frac{81}{2}$
18
$\frac{9}{2}$
0
$g=-2$
$g=-1$
$g=0$
$g=1$
$g=2$
$g=3$



$q = 3g + 3\ell_3$
$\varepsilon_q = \frac{1}{2}q^2$
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
$\frac{225}{2}$
72
$\frac{81}{2}$
18
$\frac{9}{2}$
0
$g = 1$
$g = 2$
$g = 3$

Tetrahexahedric model: the spectrum

Walsh-Hadamard coordinates ($A_3 \simeq D_3$!):

$$
\begin{aligned}
x^1 &= X + \tfrac{1}{2}(+x + y + z) \quad , \qquad \partial_{x^1} = \tfrac{1}{4}\partial_X + \tfrac{1}{2}(+\partial_x + \partial_y + \partial_z)\\
x^2 &= X + \tfrac{1}{2}(+x - y - z) \quad , \qquad \partial_{x^2} = \tfrac{1}{4}\partial_X + \tfrac{1}{2}(+\partial_x - \partial_y - \partial_z)\\
x^3 &= X + \tfrac{1}{2}(-x + y - z) \quad , \qquad \partial_{x^3} = \tfrac{1}{4}\partial_X + \tfrac{1}{2}(-\partial_x + \partial_y - \partial_z)\\
x^4 &= X + \tfrac{1}{2}(-x - y + z) \quad , \qquad \partial_{x^4} = \tfrac{1}{4}\partial_X + \tfrac{1}{2}(-\partial_x - \partial_y + \partial_z)
\end{aligned}
$$

$$
x = r \sin\theta \cos\phi \quad , \qquad y = r \sin\theta \sin\phi \quad , \qquad z = r \cos\theta
$$

Angular momenta:

$$
L_x = -(y\partial_z - z\partial_y) \,, \qquad L_y = -(z\partial_x - x\partial_z) \,, \qquad L_z = -(x\partial_y - y\partial_x)
$$

S^2 Laplacian: $$L^2 = -(L_x^2 + L_y^2 + L_z^2) = -\frac{1}{\sin\theta}\partial_\theta \sin\theta\, \partial_\theta - \frac{1}{\sin^2\theta}\partial_\phi^2$$

Hamiltonian:

$$H = -\tfrac{1}{2}(\partial_x^2 + \partial_y^2 + \partial_z^2) + 2\,g(g-1)\Big(\frac{x^2+y^2}{(x^2-y^2)^2} + \frac{y^2+z^2}{(y^2-z^2)^2} + \frac{z^2+x^2}{(z^2-x^2)^2}\Big)$$

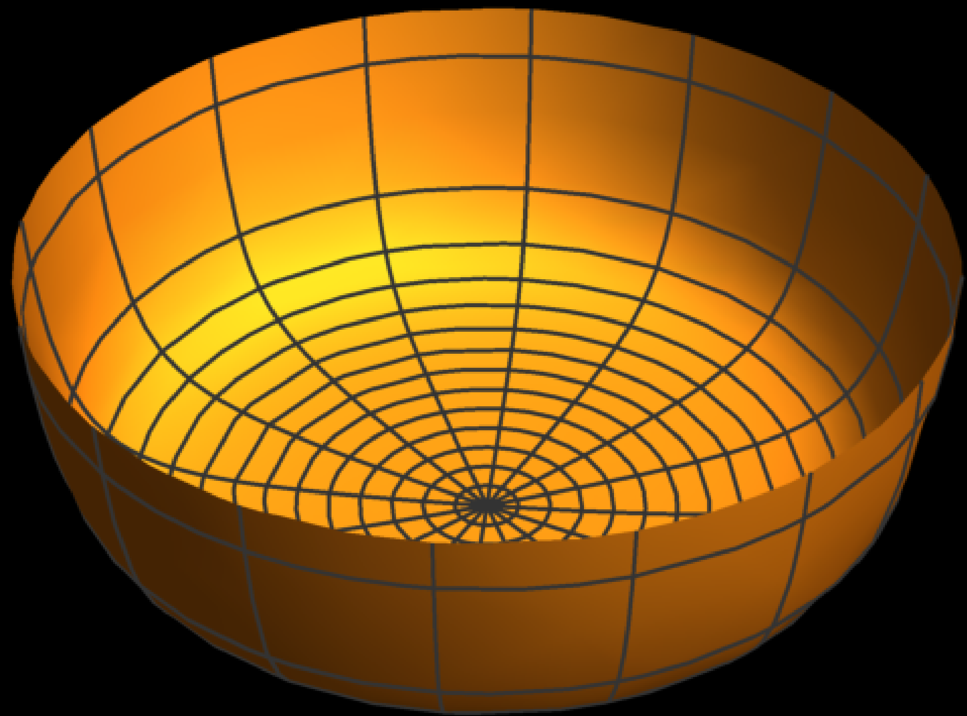
$$U(\theta,\phi) = 2g(g-1)\Big\{\frac{1}{\sin^2\theta\,\cos^2 2\phi} + \frac{\cos^2\theta + \sin^2\theta\cos^2\phi}{(\cos^2\theta - \sin^2\theta\cos^2\phi)^2} + \frac{\cos^2\theta + \sin^2\theta\sin^2\phi}{(\cos^2\theta - \sin^2\theta\sin^2\phi)^2}\Big\}$$

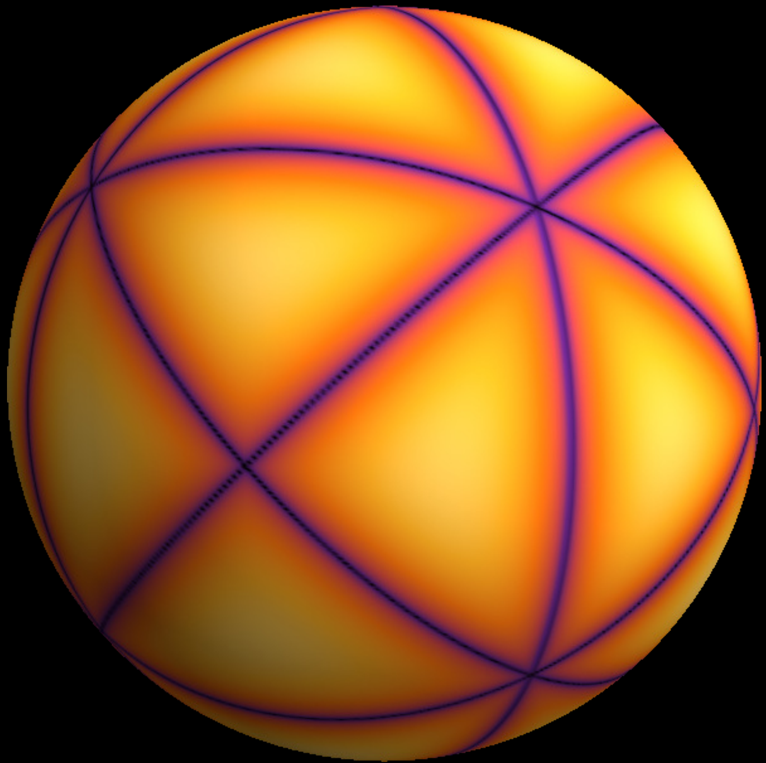
S_4 Weyl group action (elementary reflections):

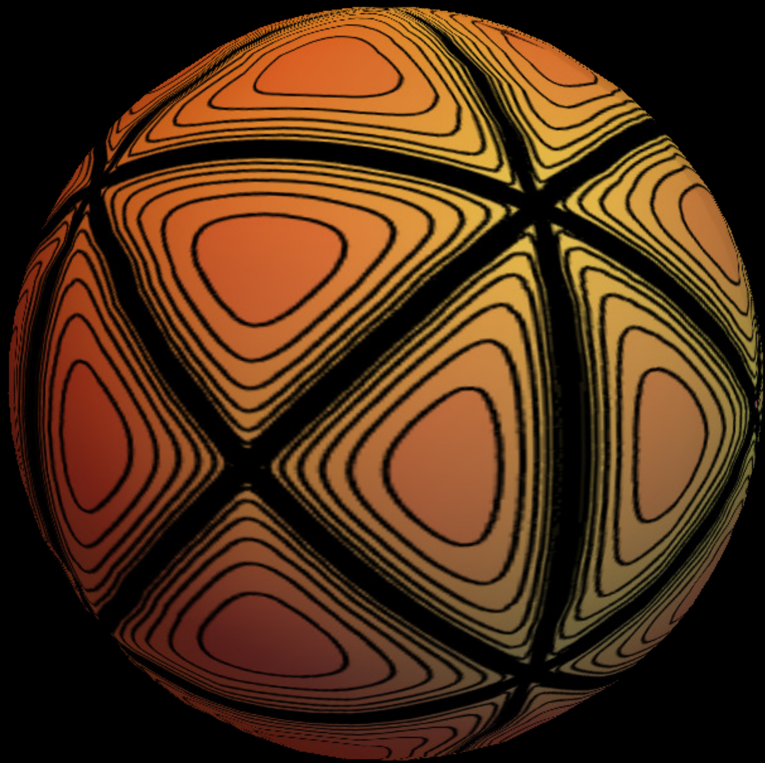
$$s_{x+y}: (x,y,z) \mapsto (-y,-x,+z) \quad , \quad s_{x-y}: (x,y,z) \mapsto (+y,+x,+z)$$

$$s_{y+z}: (x,y,z) \mapsto (+x,-z,-y) \quad , \quad s_{y-z}: (x,y,z) \mapsto (+x,+z,+y)$$

$$s_{z+x}: (x,y,z) \mapsto (-z,+y,-x) \quad , \quad s_{z-x}: (x,y,z) \mapsto (+z,+y,+x)$$







Spectrum: $\varepsilon_q = \frac{1}{2}q\,(q+1)$ with $q = 6g+\ell = 6g+3\ell_3+4\ell_4$

Wave functions:

$$\Psi_{E,q}(r,\theta,\phi) = j_q(\sqrt{2E}\,r)\,v_q(\theta,\phi)$$

$$v_\ell^{(g)}(\theta,\phi) \sim r^{q+1}\big(\mathcal{D}_x\mathcal{D}_y\mathcal{D}_z\big)^{\ell_3}\big(\mathcal{D}_x^4+\mathcal{D}_y^4+\mathcal{D}_z^4\big)^{\ell_4}\Delta^g\,r^{1-12g} = r^{-q}\Delta^g\,h_\ell^{(g)}(x,y,z)$$

$$\Delta \sim (x^2-y^2)(y^2-z^2)(x^2-z^2)$$

Linear Dunkl operators:

$$\mathcal{D}_x = \partial_x - \frac{g}{x+y}s_{x+y} - \frac{g}{x-y}s_{x-y} - \frac{g}{z+x}s_{x+z} - \frac{g}{x-z}s_{z-x}$$

$$\mathcal{D}_y = \partial_y - \frac{g}{y+x}s_{x+y} - \frac{g}{y-x}s_{x-y} - \frac{g}{y+z}s_{y+z} - \frac{g}{y-z}s_{y-z}$$

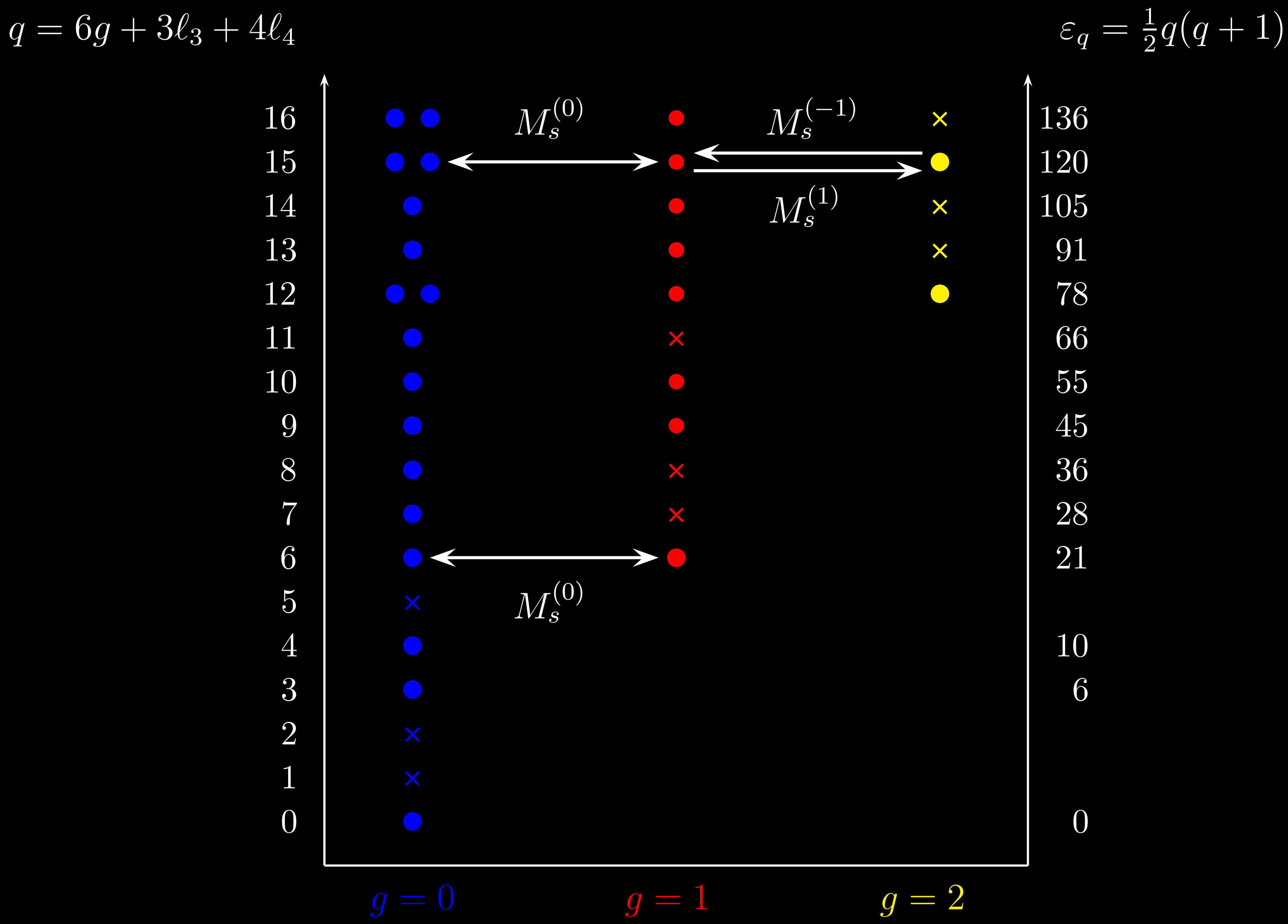
$$\mathcal{D}_z = \partial_z - \frac{g}{z+x}s_{z+x} - \frac{g}{z-x}s_{z-x} - \frac{g}{z+y}s_{y+z} - \frac{g}{z-y}s_{y-z}$$

Low-lying wave functions $v_\ell^{(g)} = r^{-\ell-6g}\triangle^g h_\ell^{(g)}$ of the tetrahexahedric model

ℓ	ℓ_3	ℓ_4	$h_\ell^{(0)}$
0	0	0	$\{000\}$
3	1	0	$\{111\}$
4	0	1	$\{400\} - 3\{220\}$
6	2	0	$\{600\} - 15\{420\} + 30\{222\}$
7	1	1	$3\{511\} - 5\{331\}$
8	0	2	$\{800\} - 28\{620\} + 35\{440\}$
9	3	0	$9\{711\} - 63\{531\} + 70\{333\}$
10	2	1	$\{1000\} - 45\{820\} + 42\{640\} + 504\{622\} - 630\{442\}$
11	1	2	$5\{911\} - 60\{731\} + 63\{551\}$
12	4	0	$36\{1200\} - 2376\{1020\} + 2445\{840\} + 46125\{822\} + 4893\{660\} - 215250\{642\} + 179375\{444\}$
12	0	3	$101\{1200\} - 6666\{1020\} + 47100\{840\} + 8685\{822\} - 42609\{660\} - 40530\{642\} + 33775\{444\}$

ℓ	ℓ_3	ℓ_4	$h_\ell^{(1)}$
0	0	0	$\{000\}$
3	1	0	$\{111\}$
4	0	1	$3\{400\} - 11\{220\}$
6	2	0	$3\{600\} - 39\{420\} + 196\{222\}$
7	1	1	$5\{511\} - 13\{331\}$
8	0	2	$\{800\} - 20\{620\} + 23\{440\} + 12\{422\}$
9	3	0	$3\{711\} - 27\{531\} + 56\{333\}$
10	2	1	$15\{1000\} - 425\{820\} + 576\{640\} + 7568\{622\} - 14454\{442\}$
11	1	2	$35\{911\} - 476\{731\} + 477\{551\} + 204\{533\}$
12	4	0	$12\{1200\} - 456\{1020\} + 657\{840\} + 13581\{822\} + 1137\{660\} - 88842\{642\} + 114007\{444\}$
12	0	3	$813\{1200\} - 30894\{1020\} + 165652\{840\} + 72131\{822\} - 147943\{660\} - 169702\{642\} + 57527\{444\}$

Notation: $\{rst\} := x^r y^s z^t + x^r y^t z^s + x^s y^t z^r + x^s y^r z^t + x^t y^r z^s + x^t y^s z^r$


$q = 6g + 3\ell_3 + 4\ell_4$
$\varepsilon_q = \frac{1}{2}q(q+1)$
$M_s^{(0)}$
$M_s^{(-1)}$
$M_s^{(1)}$
$M_s^{(0)}$
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
136
120
105
91
78
66
55
45
36
28
21
10
6
0
$g = 0$
$g = 1$
$g = 2$

Simplest linear $\mathcal{PT}$ deformation:

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \longmapsto \begin{pmatrix} 1+\cosh\epsilon & \mathrm{i}\sinh\epsilon & -\mathrm{i}\sinh\epsilon & 1-\cosh\epsilon \\ -\mathrm{i}\sinh\epsilon & 1+\cosh\epsilon & 1-\cosh\epsilon & \mathrm{i}\sinh\epsilon \\ \mathrm{i}\sinh\epsilon & 1-\cosh\epsilon & 1+\cosh\epsilon & -\mathrm{i}\sinh\epsilon \\ 1-\cosh\epsilon & -\mathrm{i}\sinh\epsilon & \mathrm{i}\sinh\epsilon & 1+\cosh\epsilon \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}$$

or

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \cosh\epsilon & -\mathrm{i}\sinh\epsilon & 0 \\ \mathrm{i}\sinh\epsilon & \cosh\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} \sin\theta\cos(\phi+\mathrm{i}\epsilon) \\ \sin\theta\sin(\phi+\mathrm{i}\epsilon) \\ \cos\theta \end{pmatrix}$$

$$\Longleftrightarrow \qquad \phi \mapsto \phi+\mathrm{i}\epsilon \quad \text{or} \quad \big(x\pm\mathrm{i}y, z\big) \mapsto \big(\mathrm{e}^{\mp\epsilon}(x\pm\mathrm{i}y), z\big)$$

Complex potential:

$$\frac{U_\epsilon(\theta,\phi)}{2g(g-1)} = \frac{1}{\sin^2\theta\,\cos^2 2(\phi+\mathrm{i}\epsilon)} + \frac{\cos^2\theta+\sin^2\theta\cos^2(\phi+\mathrm{i}\epsilon)}{(\cos^2\theta-\sin^2\theta\cos^2(\phi+\mathrm{i}\epsilon))^2} + \frac{\cos^2\theta+\sin^2\theta\sin^2(\phi+\mathrm{i}\epsilon)}{(\cos^2\theta-\sin^2\theta\sin^2(\phi+\mathrm{i}\epsilon))^2}$$

Still singular at five antipodal pairs of points $\Rightarrow$ $g<0$ states remain unphysical

Nonlinear $\mathcal{PT}$ deformation (for $\epsilon_1 \neq 0$):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto r \begin{pmatrix} \sin(\theta+\mathrm{i}\epsilon_1)\cos(\phi+\mathrm{i}\epsilon_2) \\ \sin(\theta+\mathrm{i}\epsilon_1)\sin(\phi+\mathrm{i}\epsilon_2) \\ \cos(\theta+\mathrm{i}\epsilon_1) \end{pmatrix} = r \begin{pmatrix} c_1 c_2\, x - \mathrm{i} c_1 s_2\, y + s_1 s_2 \frac{z\,y}{\rho} + \mathrm{i} s_1 c_2 \frac{z\,x}{\rho} \\ c_1 c_2\, y + \mathrm{i} c_1 s_2\, x - s_1 s_2 \frac{z\,x}{\rho} + \mathrm{i} s_1 c_2 \frac{z\,y}{\rho} \\ c_1\, z - \mathrm{i} s_1\, \rho \end{pmatrix}$$

with $\quad c_i = \cosh(\epsilon_i) \quad , \quad s_i = \sinh(\epsilon_i) \quad , \quad \rho = \sqrt{x^2 + y^2}$

deforms both L^2 and U, with U_ϵ completely nonsingular (if both ϵ_i nonzero)

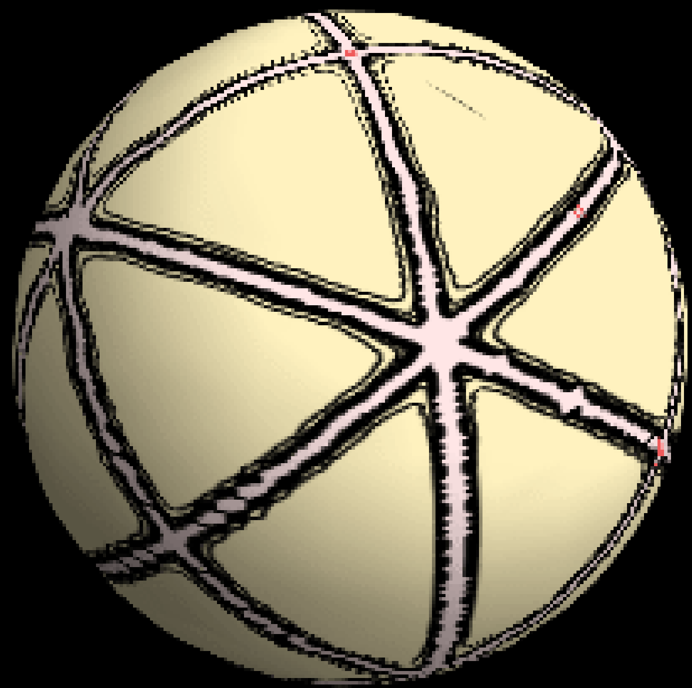
Vandermonde

$$\triangle_\epsilon \sim r^6 \sin^2(\theta+\mathrm{i}\epsilon_1)\, \cos^4(\theta+\mathrm{i}\epsilon_1)\, \cos^2(2\phi+2\mathrm{i}\epsilon_2)$$
$$\times \left(\tan^2(\theta+\mathrm{i}\epsilon_1)\cos^2(\phi+\mathrm{i}\epsilon_2) - 1\right)\left(\tan^2(\theta+\mathrm{i}\epsilon_1)\sin^2(\phi+\mathrm{i}\epsilon_2) - 1\right)$$

is nowhere vanishing $\quad \Longrightarrow \quad$ $g < 0$ wave functions now nonsingular

Linear involution: $\quad \mathcal{P}: \ (\theta, \phi) \mapsto (-\theta, -\phi) \quad \Leftrightarrow \quad (x, y, z) \mapsto (-x, y, z)$

together with complex conjugation $\mathcal{T}$ leaves the deformed Hamiltonian H_ϵ invariant



Spectrum: real and ϵ-independent but previously singular states for $g < 0$ appear!

$$\varepsilon_q \;=\; \tfrac{1}{2}q\,(q+1) \qquad \text{with} \qquad q \;=\; 6g + \ell \;=\; 6g + 3\ell_3 + 4\ell_4$$

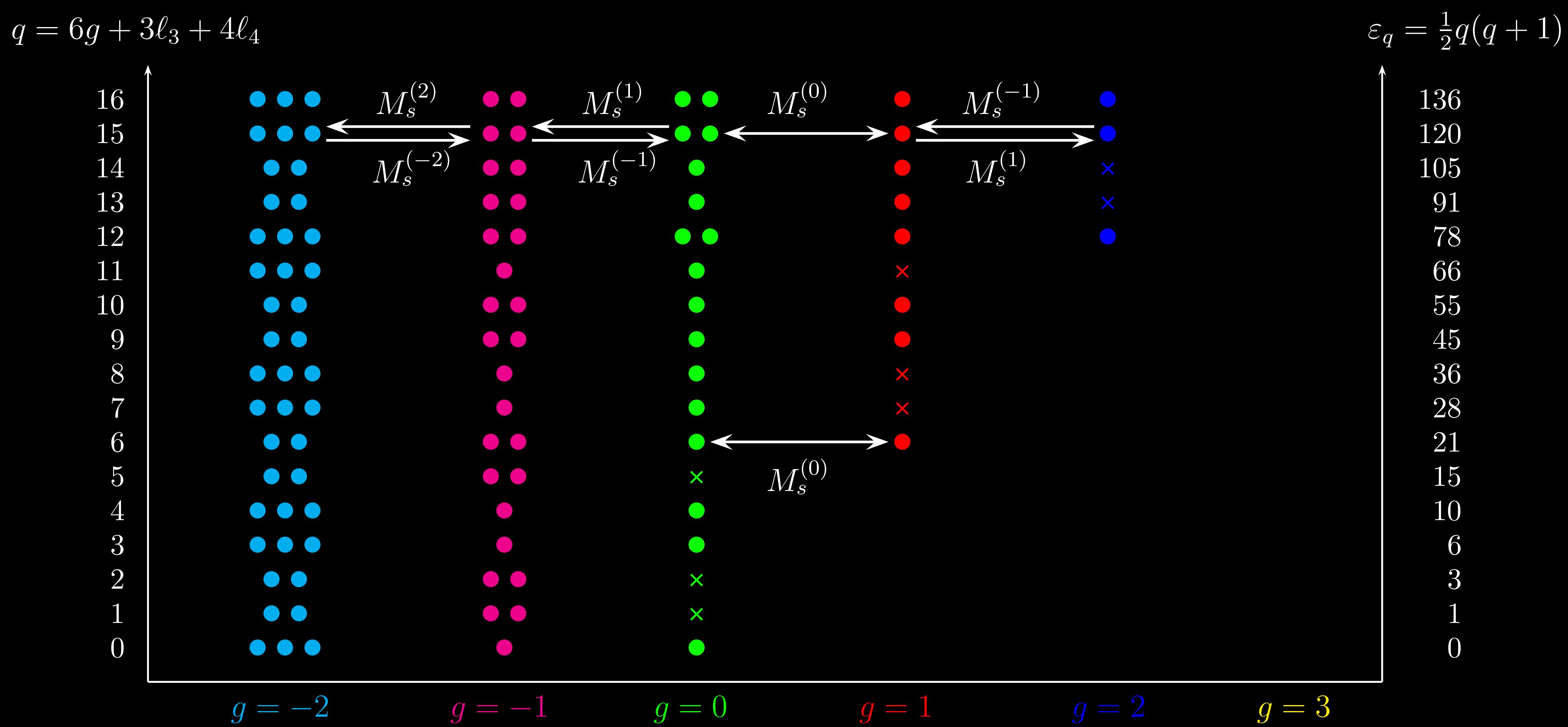
Eigenfunction construction extends to $g < 0$ with naive coordinate deformation

2nd branch for $g < 0$ $\quad\Longrightarrow\quad \deg(\varepsilon_q) \;=\; \deg_4(q-6g) + \deg_4(-q-6g-1)$

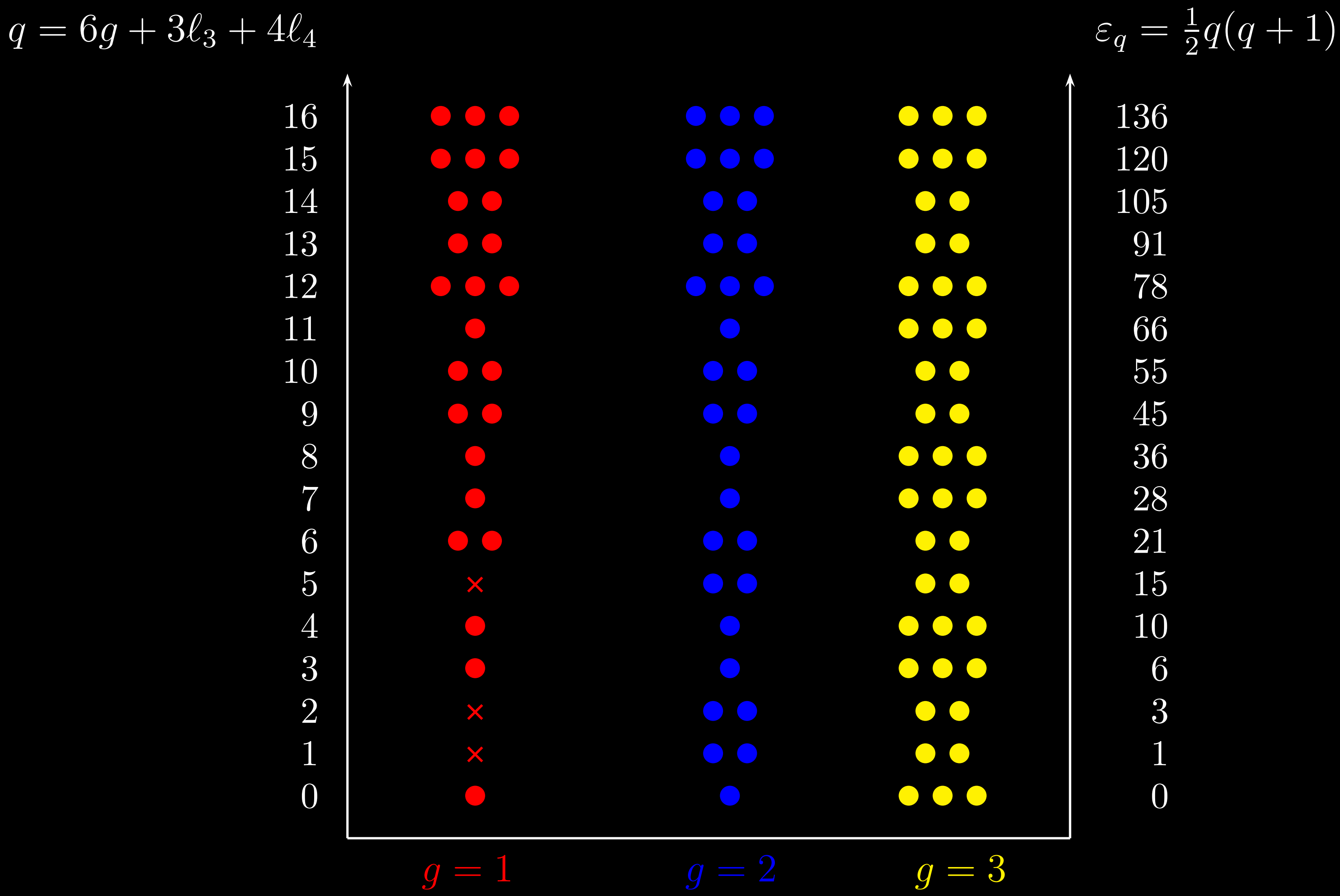
Join state spaces at g and $1-g$ $\quad\Longrightarrow\quad$ high-energy growth is g-independent:

$$\deg(\varepsilon_q) \;=\; \deg_4(q-6g) + \deg_4(q+6g-6) + \deg_4(-q+6g-7) \qquad g > 0$$

$$= \begin{cases} g-1 \;+\; \begin{Bmatrix} 0 & \text{for} & q+6g = 0,3,4,7,8,11 \bmod 12 \\ 1 & \text{for} & q+6g = 1,2,5,6,9,10 \bmod 12 \end{Bmatrix} & \text{if} \quad q < 6g-6 \\ \left\lfloor \frac{q}{6} \right\rfloor \;+\; \begin{Bmatrix} 0 & \text{for} & q = 1,2,5 \bmod 6 \\ 1 & \text{for} & q = 0,3,4 \bmod 6 \end{Bmatrix} & \text{if} \quad q \geq 6g-6 \end{cases}$$



$q = 6g + 3\ell_3 + 4\ell_4$
$\varepsilon_q = \frac{1}{2}q(q+1)$
16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
136 120 105 91 78 66 55 45 36 28 21 15 10 6 3 1 0
$M_s^{(2)}$
$M_s^{(-2)}$
$M_s^{(1)}$
$M_s^{(-1)}$
$M_s^{(0)}$
$M_s^{(-1)}$
$M_s^{(1)}$
$M_s^{(0)}$
$g=-2$
$g=-1$
$g=0$
$g=1$
$g=2$
$g=3$


$q = 6g + 3\ell_3 + 4\ell_4$
$\varepsilon_q = \frac{1}{2}q(q+1)$
16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
136 120 105 91 78 66 55 45 36 28 21 15 10 6 3 1 0
$g = 1$
$g = 2$
$g = 3$

Tetrahexahedric model: intertwiner & integrability

Angular Dunkl operators:

$$\mathcal{L}_x \;=\; L_x \;+\; g\Big\{\tfrac{z}{x-y}s_{x-y} - \tfrac{z}{x+y}s_{x+y} - \tfrac{y}{x-z}s_{z-x} + \tfrac{y}{z+x}s_{z+x} - \tfrac{y+z}{y-z}s_{y-z} + \tfrac{y-z}{y+z}s_{y+z}\Big\}$$

$$\mathcal{L}_y \;=\; L_y \;+\; g\Big\{\tfrac{x}{y-z}s_{y-z} - \tfrac{x}{y+z}s_{y+z} - \tfrac{z}{y-x}s_{x-y} + \tfrac{z}{y+x}s_{x+y} - \tfrac{z+x}{z-x}s_{z-x} + \tfrac{z-x}{z+x}s_{z+x}\Big\}$$

$$\mathcal{L}_z \;=\; L_z \;+\; g\Big\{\tfrac{y}{z-x}s_{z-x} - \tfrac{y}{z+x}s_{z+x} - \tfrac{x}{z-y}s_{y-z} + \tfrac{x}{z+y}s_{y+z} - \tfrac{x+y}{x-y}s_{x-y} + \tfrac{x-y}{x+y}s_{x+y}\Big\}$$

Conserved charges:

$$J_k \;:=\; \mathrm{res}\Big(\mathcal{L}_x^k + \mathcal{L}_y^k + \mathcal{L}_z^k\Big) \qquad \text{for} \quad k = (0,)2,4,6$$

$$J_0 \;=\; C_0 \;=\; 1 \qquad \text{and} \qquad J_2 \;=\; -C_2 \;=\; -2\,H_\Omega \;+\; 6g(6g{+}1)$$

Any word in $\{J_2, J_4, J_6\}$ is conserved

$$\text{Center} \;=\; \langle\langle J_0, J_2\rangle\rangle \qquad \Longrightarrow \qquad [J_2, J_k] \;=\; 0 \qquad \text{but} \qquad [J_4, J_6] \;\neq\; 0$$

$[J_4, J_6]$ and $\{J_4, J_6\}$ are new words, not linear combinations of others

Higher conserved charges are algebraically dependent:

$$\begin{aligned} 6J_8 \;=\;& 8J_6J_2 + 3J_4J_4 - 6J_4J_2J_2 + J_2J_2J_2J_2 \\ & - 12(8{+}5g{+}12g^2)J_6 + 4(34{+}23g{+}30g^2)J_4J_2 - 8(5{+}3g{+}3g^2)J_2J_2J_2 \\ & + 24(13{+}15g{-}102g^2{-}72g^3)J_4 - 4(43{+}70g{-}252g^2{-}144g^3)J_2J_2 \\ & - 48(1{+}3g)(1{+}4g)(1{-}12g)J_2 \end{aligned}$$

First angular intertwiner:

$$\mathcal{M}_3 \;\sim\; \tfrac{1}{6}\Big(\mathcal{L}_x\mathcal{L}_y\mathcal{L}_z + \mathcal{L}_x\mathcal{L}_z\mathcal{L}_y + \mathcal{L}_y\mathcal{L}_z\mathcal{L}_x + \mathcal{L}_y\mathcal{L}_x\mathcal{L}_z + \mathcal{L}_z\mathcal{L}_x\mathcal{L}_y + \mathcal{L}_z\mathcal{L}_y\mathcal{L}_x\Big)$$

$$\begin{aligned}
M_3 \;\sim\;& y^2z\partial_{zxx} - yz^2\partial_{xxy} + \tfrac{1}{2}(y^2-z^2)\,\partial_{xx} + \;4g\,\tfrac{yz}{y^2-z^2}\big(yz\partial_{xx} + x^2\partial_{yz} - zx\partial_{xy}\big) \\
&+ g\,\Big[2g\,y^2z^2\big(\tfrac{8g}{(x^2-y^2)(z^2-x^2)} + \tfrac{16g}{(z^2-x^2)(y^2-z^2)} - \tfrac{2g-1}{(x^2-y^2)^2} + \tfrac{2g-1}{(z^2-x^2)^2}\big) \\
&\quad -\tfrac{2x^2y^2}{(z^2-x^2)^2} + \tfrac{2x^2z^2}{(x^2-y^2)^2} - \tfrac{2y^2}{x^2-y^2} - \tfrac{2z^2}{z^2-x^2} - 2\tfrac{y^2+z^2}{y^2-z^2}\Big]\,x\partial_x \\
&+ 2g(g-1)(g+2)\,x^2\,\Big[\tfrac{y^2+z^2}{(y^2-z^2)^2} + z\big(\tfrac{1}{(y-z)^3} - \tfrac{1}{(y+z)^3}\big)\Big] + g\,\big(2g^2+8g-1\big)\tfrac{y^2+z^2}{y^2-z^2} \\
&+ 2g^2(8+9g)\tfrac{x^2y^2z^2}{(x^2-y^2)(x^2-z^2)(y^2-z^2)} - \tfrac{2}{3}g^3\tfrac{x^6+y^6+z^6}{(x^2-y^2)(x^2-z^2)(y^2-z^2)} + \text{ cyclic permutations}
\end{aligned}$$

$$\begin{aligned}
\Delta^{-g}M_3\,\Delta^{g} \;\sim\;& y^2z\partial_{zxx} - yz^2\partial_{xxy} + \tfrac{1}{2}(y^2-z^2)\,\partial_{xx} + \;2g\,\frac{y^2z^2(y^2-z^2)}{(x^2-y^2)(x^2-z^2)}\,\partial_{xx} \\
&+ 4g\,\frac{xy^2z}{x^2-z^2}\,\partial_{xz} + 2g\,x\left[\frac{y^2(x^2+3z^2)}{(x^2-z^2)^2} - \frac{z^2(x^2+3y^2)}{(x^2-y^2)^2}\right]\partial_x \;+\; \text{cyclic permutations}
\end{aligned}$$

Second angular intertwiner:

$$\mathcal{M}_6 \sim \{\mathcal{L}_x^4, \mathcal{L}_y^2\} - \{\mathcal{L}_y^4, \mathcal{L}_x^2\} + \{\mathcal{L}_y^4, \mathcal{L}_z^2\} - \{\mathcal{L}_z^4, \mathcal{L}_y^2\} + \{\mathcal{L}_z^4, \mathcal{L}_x^2\} - \{\mathcal{L}_x^4, \mathcal{L}_z^2\}$$

$$M_6 = \text{rather lengthy expression}$$

$$\Delta^{-g} M_6 \, \Delta^{g} = \text{hopefully a bit shorter}$$

Higher angular intertwiners are reduced to M_3 and M_6

Basic intertwining relations:

$$
\begin{aligned}
M_3^{(g)} J_2^{(g)} &= \left(J_2^{(g+1)} - 6(7+12g)\right) M_3^{(g)} \\
M_3^{(g)} J_4^{(g)} &= \left(J_4^{(g+1)} - 4(11+12g) J_2^{(g+1)} + 48(26+73g+48g^2)\right) M_3^{(g)} \\
&+ 2\, M_6^{(g)} \\
M_3^{(g)} J_6^{(g)} &= \left(J_6^{(g+1)} - (35+36g) J_4^{(g+1)} - 3(7+4g) J_2^{(g+1)} J_2^{(g+1)}\right. \\
&+ 2(1111+2668g+1392g^2) J_2^{(g+1)} \\
&+ \left. 96(457+1933g+2717g^2+1368g^3+144g^4)\right) M_3^{(g)} \\
&+ \left(3 J_2^{(g+1)} - (115+200g+48g^2)\right) M_6^{(g)}
\end{aligned}
$$

For the nonlinear $\mathcal{PT}$ deformation:

The ladder $1{-}g \;\to\; 2{-}g \;\to\; \ldots \;\to\; g{-}2 \;\to\; g{-}1 \;\to\; g$

closes to a loop due to the identification of state spaces at $1{-}g$ and g

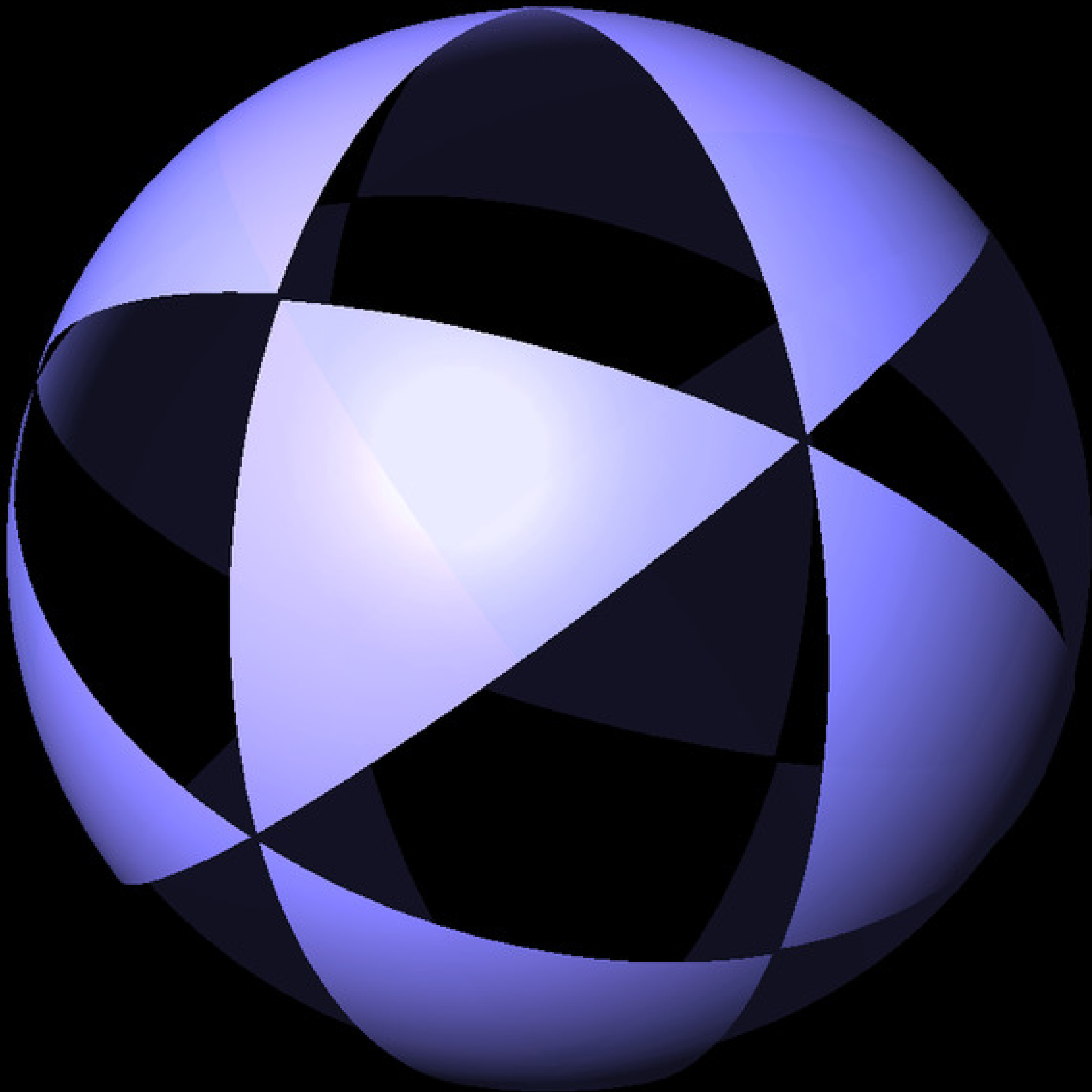
Additional 'odd' conserved charges ($* = 3$ or 6):

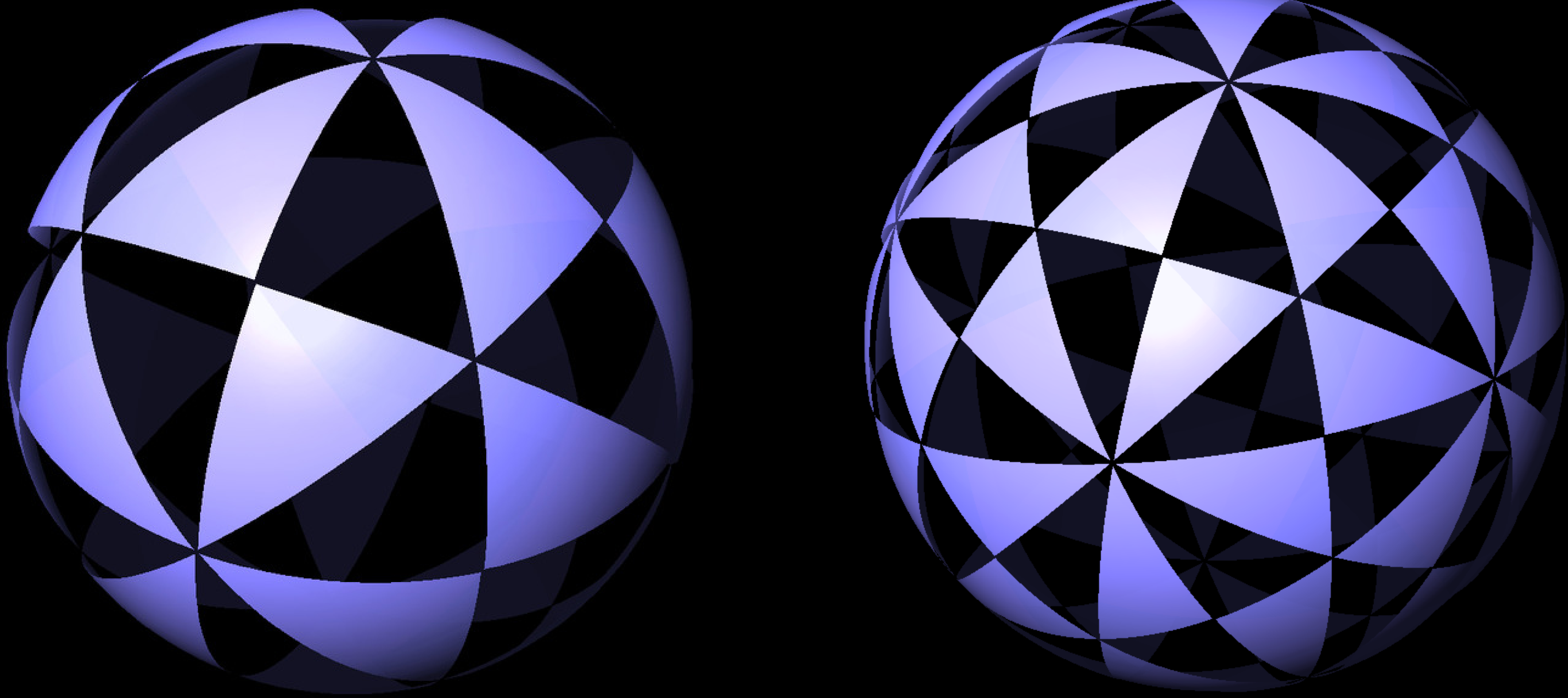
$$Q^{(g)} \;=\; M_*^{(g-1)} M_*^{(g-2)} \ldots M_*^{(1-g)}$$

$$\Longrightarrow \qquad Q^{(g)} H_\Omega^{(g)} \;=\; Q^{(g)} H_\Omega^{(1-g)} \;=\; H_\Omega^{(g)} Q^{(g)}$$

Independence: $Q^{(g)}$ is of odd order but $\left(Q^{(g)}\right)^2$ is a polynomial in the J_i

$\mathbb{Z}_2$ graded nonlinear algebra generated by $\{Q, J_2, J_4, J_6\}$





Summary and outlook

- Geometric picture of potential on S^{n-2}, superintegrable but not separable
- Characterization of the full set of conserved charges: Weyl invariants from $\mathcal{L}_{ij}$
- Characterization of the algebra generated by the conserved charges
- Are there more than two charges in involution? (need $n > 4$ to test)
- Characterization of the independent intertwiners: Weyl antiinvariants from $\mathcal{L}_{ij}$
- Intertwining relations of the conserved charges
- $\mathcal{PT}$ deformation: regularized potential, $g < 0$ states, degeneracy doubling
- Additional 'odd' conserved charges for integer coupling
- Generalization to trigonometric, hyperbolic, elliptic Calogero systems

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THANK YOU !

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