Fermionic Entanglement Entropy

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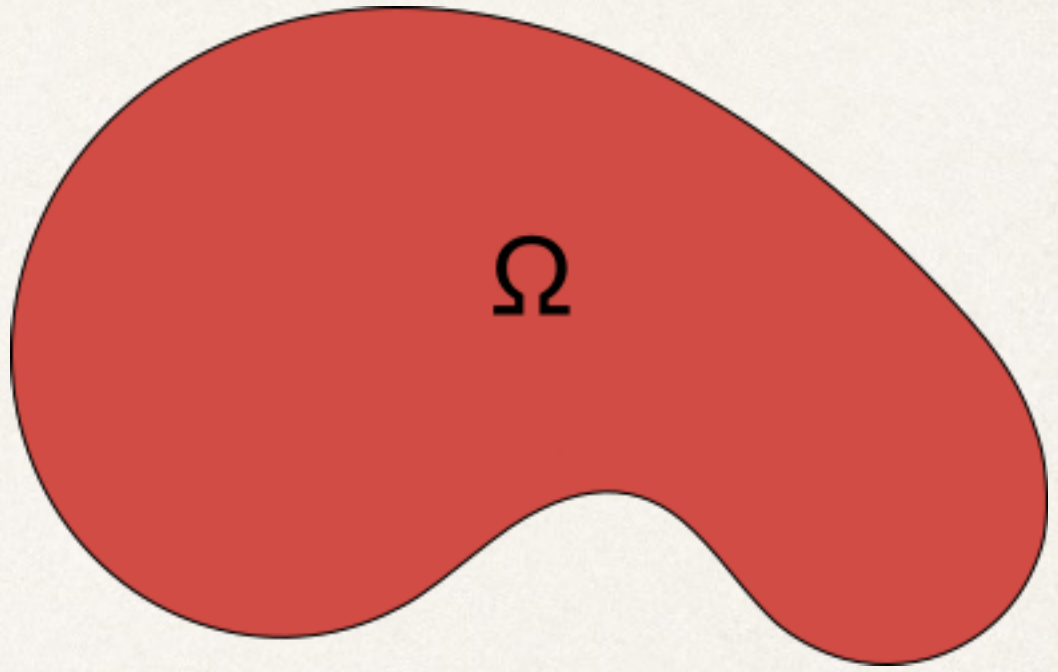
Entanglement Entropy

- Similarity to Black Hole Entropy: Area Law
- Quantum Information
- Quality of Numerics (Density Matrix Renormalization Group)
- Ryu-Takanayagi Holographic Computation
- Direct Computation possible

Definition

- Take a QFT with (quasi-local) operators $\mathcal{A}$
- Take a state $\omega\colon \mathcal{A} \to \mathbb{C}$ (think: ground state of local Hamiltonian $\omega(A) = \langle\Psi|A|\Psi\rangle$
- Restrict to operators localized in a spatial region $\mathcal{A}(\Omega)$
- This is also a state on $\mathcal{A}(\Omega)$ but in general it is mixed:

$$\omega|_{\mathcal{A}(\Omega)}(A) = \mathrm{tr}(\rho A) \quad \text{with} \quad \rho_\Omega = \mathrm{tr}_{\mathcal{F}(L^2(\mathbb{R}\setminus\Omega))}|\Psi\rangle\langle\Psi|$$



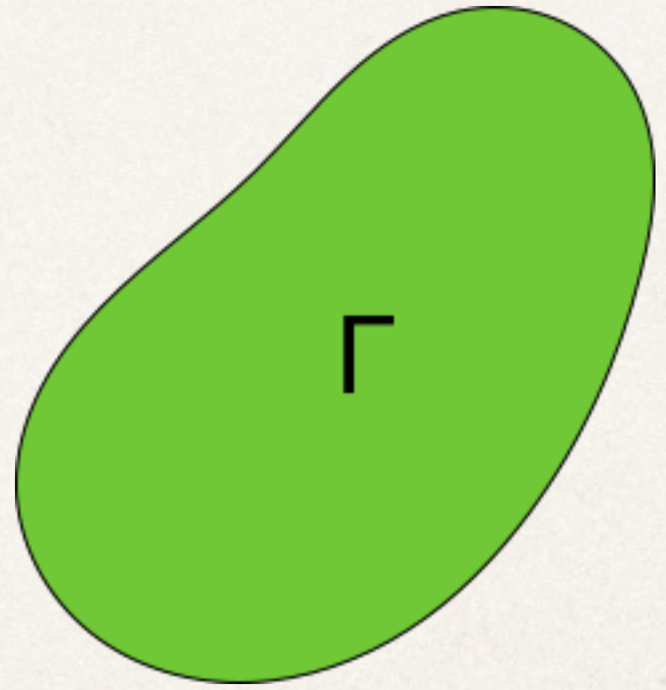
- This reduced state has entropy: $S_\Omega = -\mathrm{tr}(\rho_\Omega \log \rho_\Omega)$
- Scaling upon blowing up Ω by a factor R? Area law: $S_{R\Omega} = O(R^{n-1})$

Free Fermions (non-relativistic)

- Wick's theorem: Everything determined form 2-point function

$$\langle c_k^\dagger c_k \rangle = \chi_\Gamma(k) = \langle k | P_\Gamma | k \rangle$$

- Reduce to 1-particle space, projector onto Fermi sea



$$P_\Gamma(\mathbf{x}, \mathbf{x}') = \frac{1}{(2\pi)^n} \int_\Gamma e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{k}} \, d^n k$$

1 Particle Language

- Restrict to Ω by projection with $Q_\Omega = \chi_\Omega(\mathbf{x})$
- 1-particle effective density operator: $\varrho_{\Omega,\Gamma} = Q_\Omega P_\Gamma Q_\Omega$
- Entanglement entropy becomes

$$\begin{aligned} S_{\Omega,\Gamma} &= \operatorname{tr}(\varrho_{\Omega,\Gamma} \log \varrho_{\Omega,\Gamma} - (1-\varrho_{\Omega,\Gamma}) \log(1-\varrho_{\Omega,\Gamma})) \\ &\geq \operatorname{tr}(\varrho_{\Omega,\Gamma}(1-\varrho_{\Omega,\Gamma}) \end{aligned}$$

Violation of Area Law

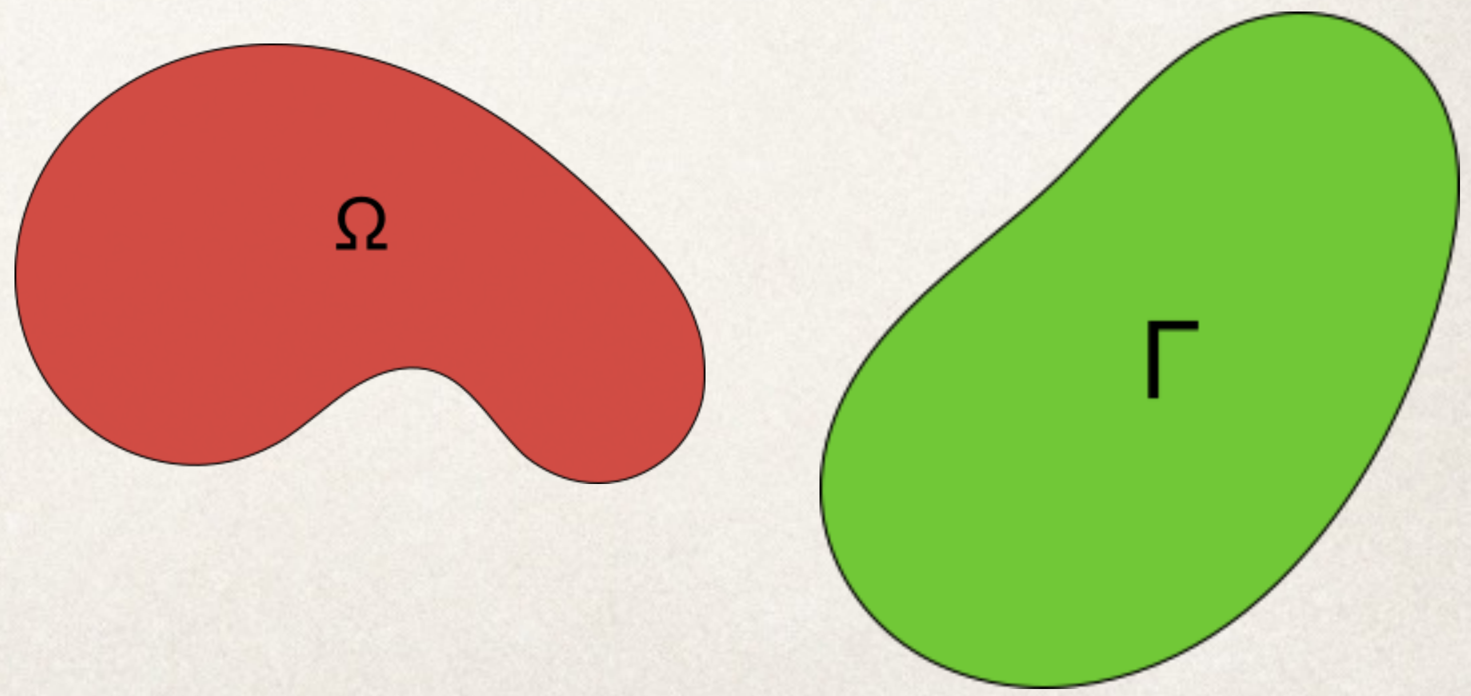
- I will show you how to compute

$$S_{R\Omega,\Gamma} \geq \frac{\ln 2}{\pi^2}\left(\frac{R}{2\pi}\right)^{n-1} \ln R \int_{\partial\Omega\times\partial\Gamma} d\sigma(\mathbf{x})d\sigma(\mathbf{p})\,|\mathbf{n_x}\cdot\mathbf{n_p}| + o(R^{n-1}\ln R)$$

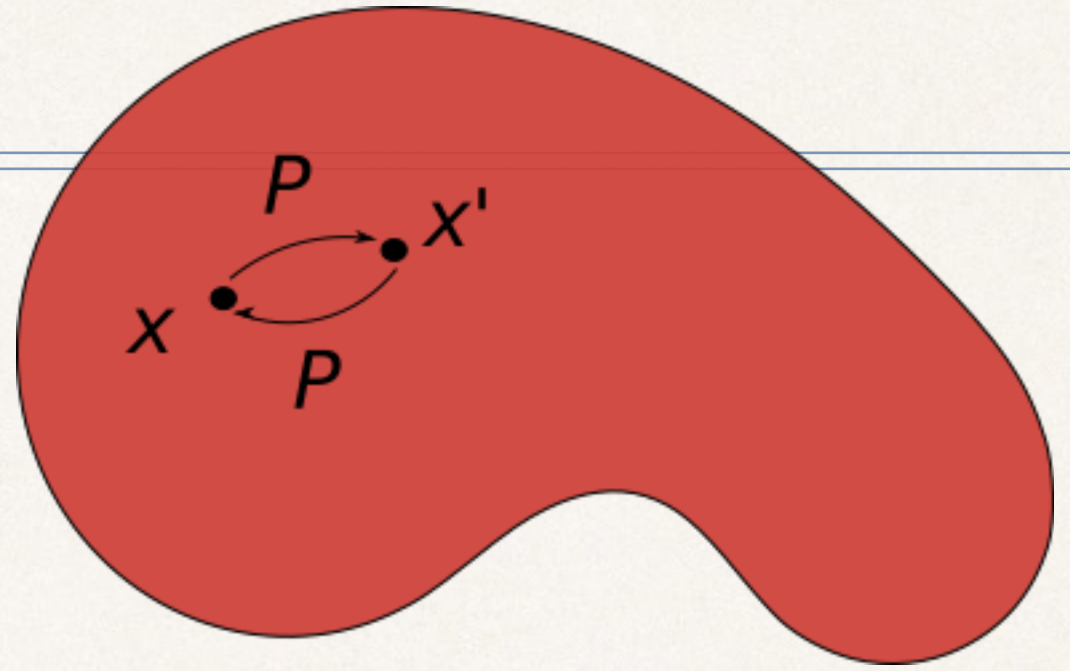
In fact, there is equality (for a slightly different coefficient).

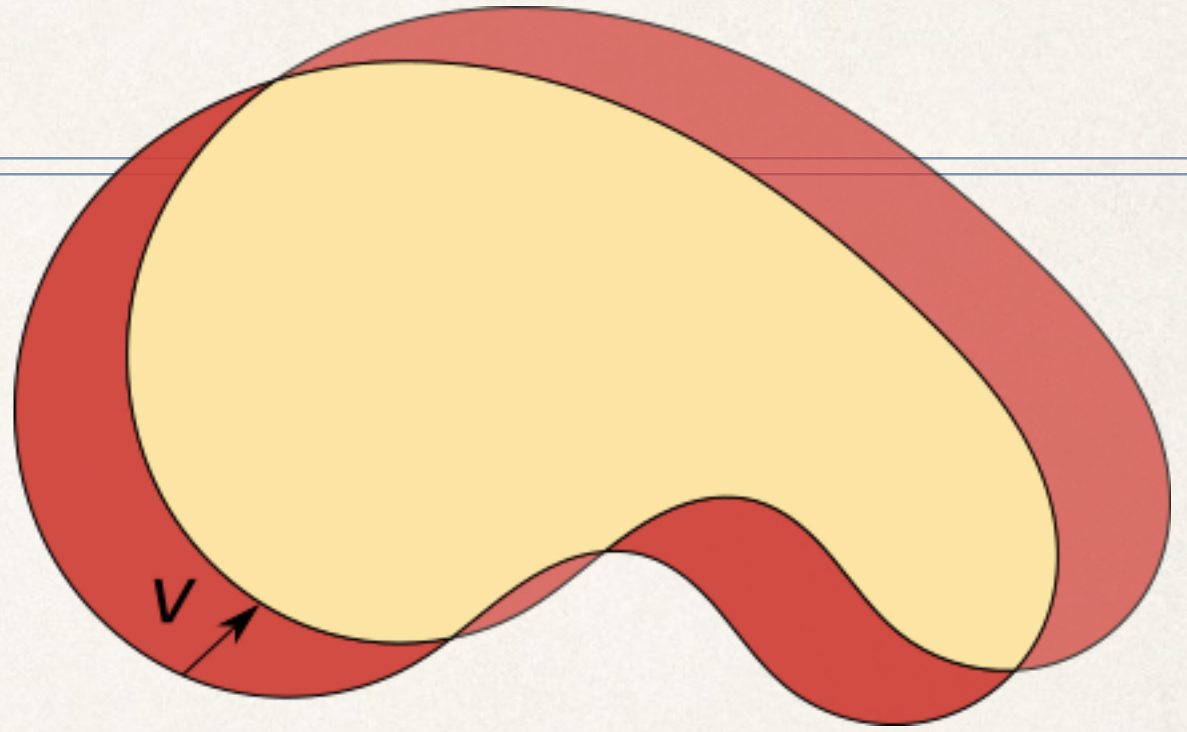
- We need Reyni-entropies $\mathrm{tr}(\varrho^k_{\Omega,\Gamma})$ for k=1 and k=2.

- k=1 is simple: $\mathrm{tr}(\varrho^1_{R\Omega,\Gamma}) = \left(\frac{R}{2\pi}\right)^n \int_\Omega d\mathbf{x} \int_\Gamma d\mathbf{p} 1 = \left(\frac{R}{2\pi}\right)^n |\Omega||\Gamma|$



k=2



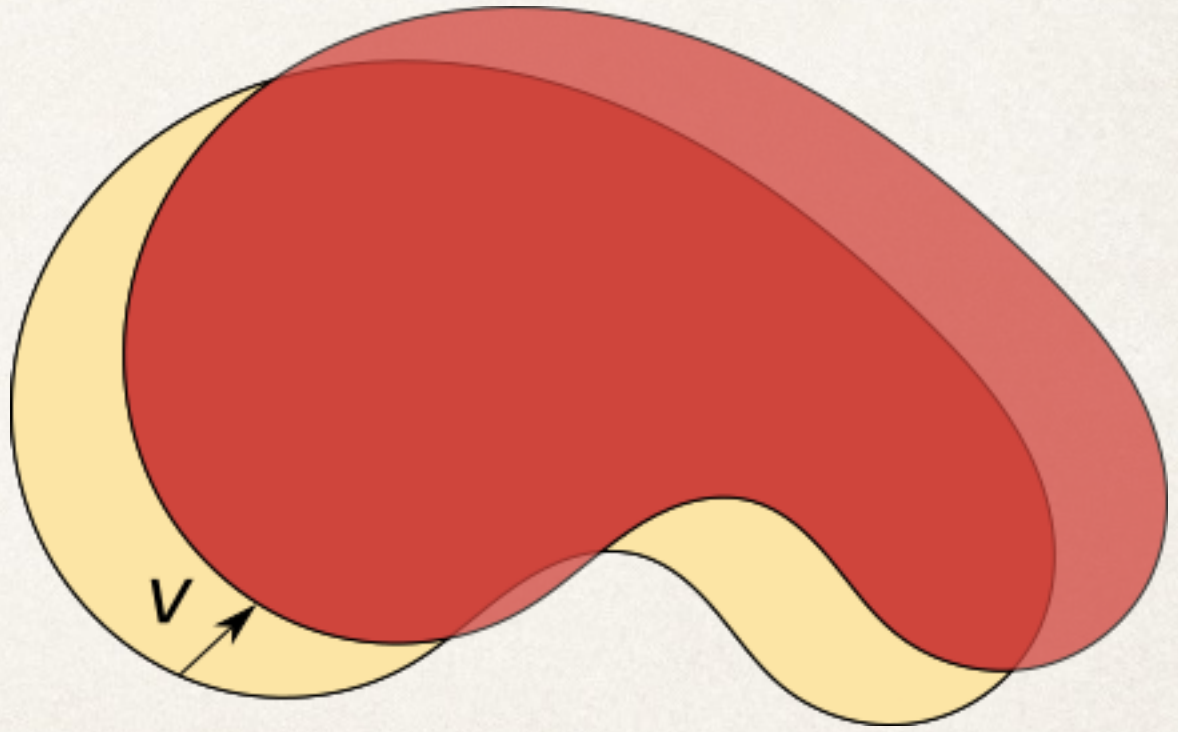


- This is more work:

- $$\mathrm{tr}(Q_{R\Omega}P_{\Gamma}Q_{R\Omega}P_{\Gamma}) = \int_{R\Omega} d\mathbf{x} \int_{R\Omega} d\mathbf{x}' \, |P_{\Gamma}(\mathbf{x}-\mathbf{x}')|^2 = \int_{R(\Omega-\Omega)} d\mathbf{v} \, |P_G(\mathbf{v})|^2 |R\Omega \cap (R\Omega - \mathbf{v})|$$

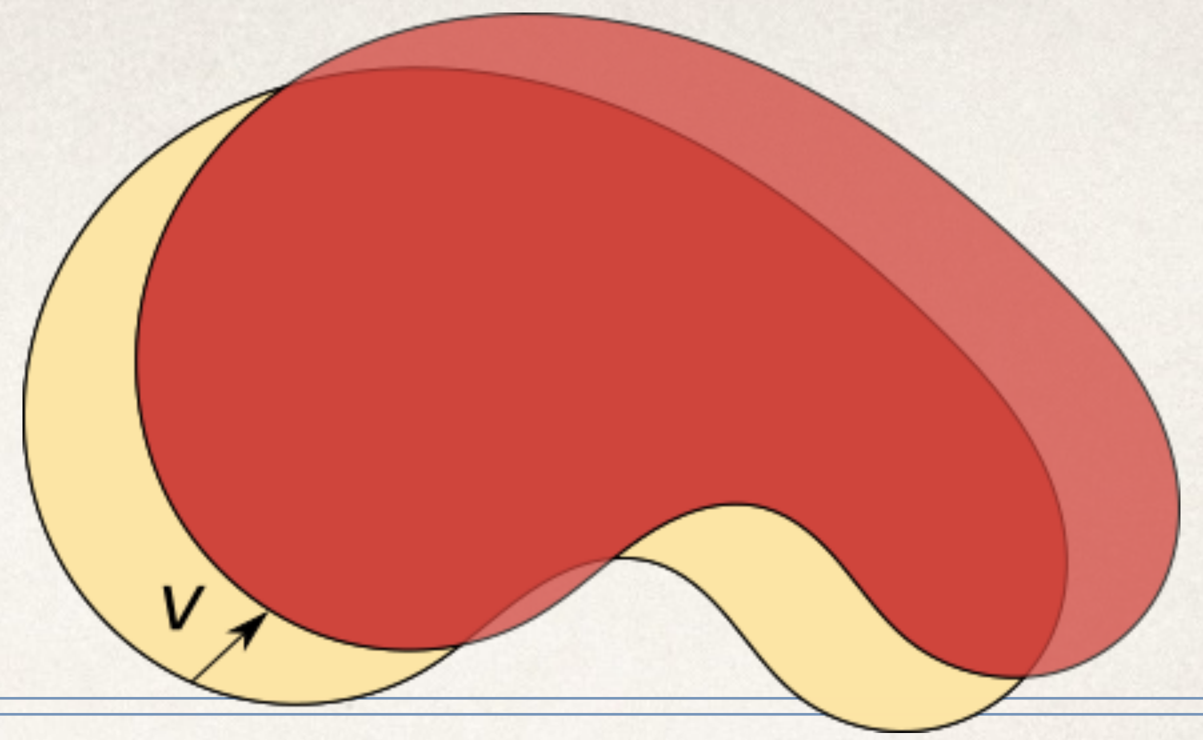
- Since $P_{\Gamma}(v) \sim \frac{1}{v^{(n-1)/2}}$

$$|R\Omega \cap (R\Omega - v)| = R^n|\Omega| + R^{n-1}\int_{\partial\Omega} d\sigma(\mathbf{x}) \max(0, \mathbf{v}\cdot\mathbf{n}_{\mathbf{x}}) + R^{n-2}O(|v|^2)$$



- First term yields $\left(\frac{R}{2\pi}\right)^n |\Omega||\Gamma|$ which cancels k=1 term.

k=2 (cont.)



- Write $\max(0, \mathbf{v}\cdot\mathbf{n_x}) = \theta(\mathbf{v}\cdot\mathbf{n}_x)\mathbf{v}\cdot\mathbf{n}_x$ and use Gauß′ theorem

$$(2\pi)^n \mathbf{v} P_{\Gamma}(\mathbf{v}) = \mathbf{v}\int_{\Gamma} d\mathbf{p}\, e^{i\mathbf{v}\cdot\mathbf{p}} = -i\int_{\partial\Gamma} d\sigma(\mathbf{p})\,\mathbf{n_p} e^{i\mathbf{v}\cdot\mathbf{p}}$$

- We still need to compute

$$\int_{R(\Omega-\Omega)} d\mathbf{v}\,\theta(\mathbf{v}\cdot\mathbf{n_x})P_{\Gamma}(-\mathbf{v})e^{i\mathbf{v}\cdot\mathbf{p}}$$

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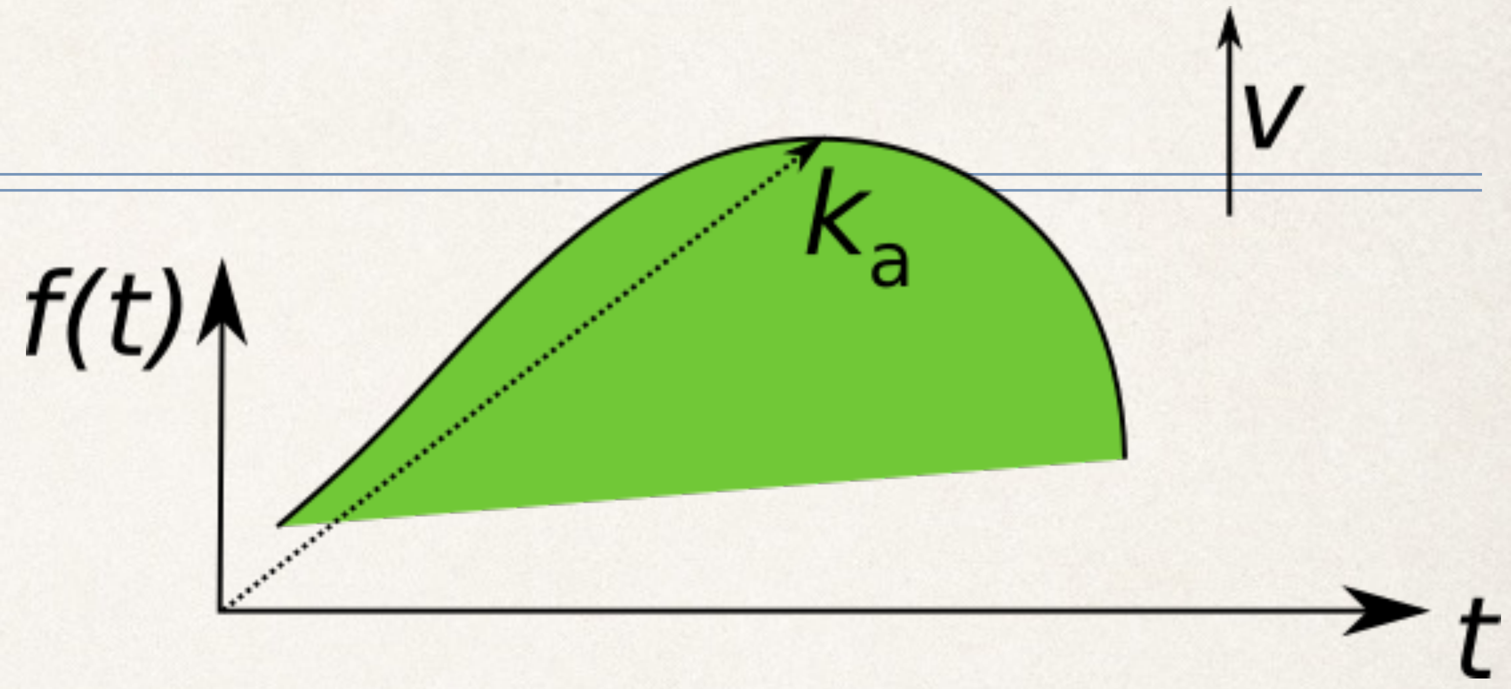
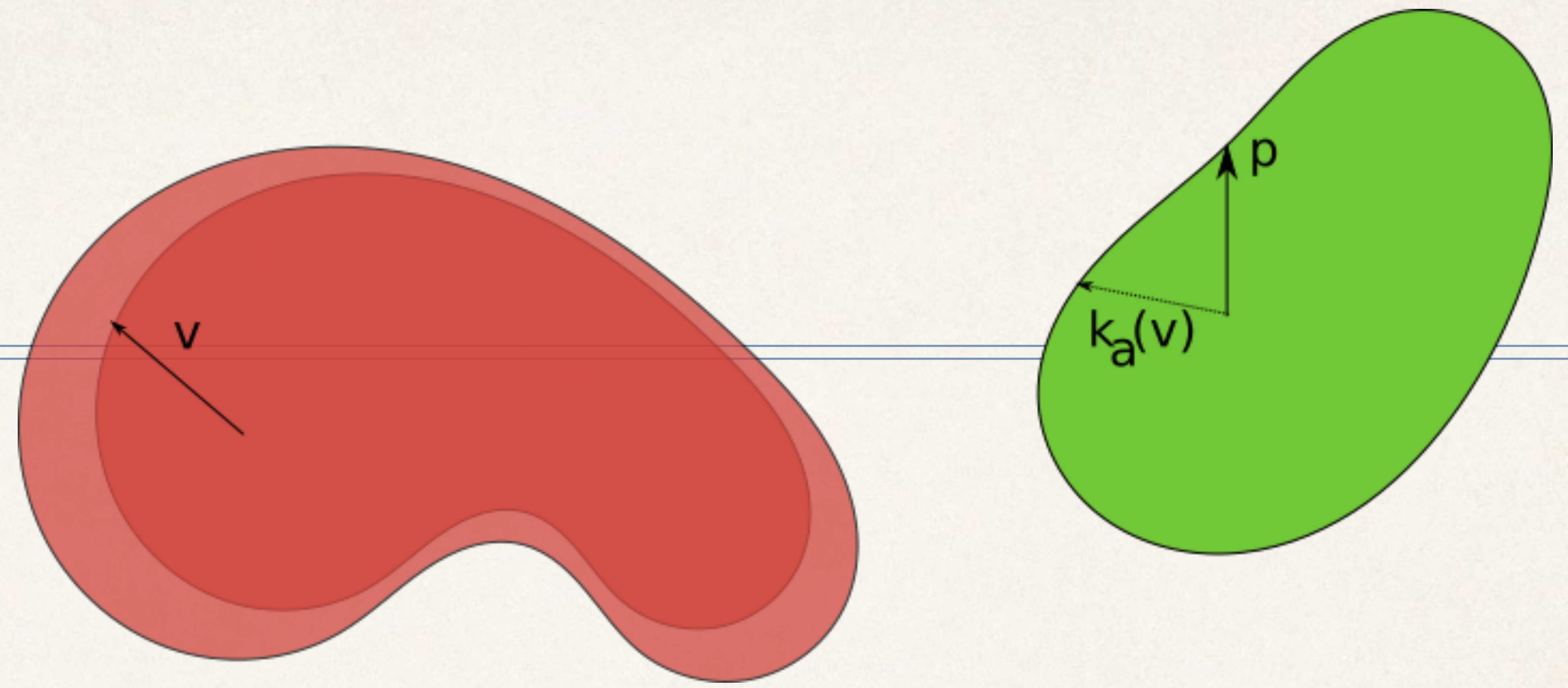


- Once more Gauß: $\frac{1}{(2\pi)^n}P_\Gamma(-\mathbf{v}) = \frac{i\mathbf{v}}{|\mathbf{v}|^2}\cdot\int_{\partial\Gamma} d\sigma(\mathbf{p}')\,\mathbf{n}_{\mathbf{p}'}e^{-i\mathbf{v}\cdot\mathbf{p}'}$
- Use coordinates with $v=(0,0,\ldots,0,V)$ and the boundary $\partial\Gamma\ni\mathbf{p}'=(\mathbf{t},f(\mathbf{t}))$
- Then $d\sigma(\mathbf{p}')=\sqrt{1+|\nabla f|^2}d\mathbf{t}$ and $n_{\mathbf{p}'}=\mathrm{sgn}(\mathbf{v}\cdot\mathbf{p}')(-\nabla f,1)/\sqrt{1+|\nabla f|^2}$
- Using stationary phase we find

Gauß curvature

$$P_\Gamma(-\mathbf{v}) = -(2\pi)^n\frac{1}{v}\int d\mathbf{t}\,\mathrm{sgn}(f(\mathbf{t}))e^{-ivf(\mathbf{t})}$$
$$= -i(2\pi v)^{-(n+1)/2}\sum_{\mathbf{k}_a}\frac{\mathrm{sgn}(\mathbf{v}\cdot\mathbf{k}_a)}{\sqrt{|\det f_{ij}(\mathbf{t}_a)|}}e^{-i\mathbf{v}\cdot\mathbf{k}_a-i\frac{\pi}{4}\mathrm{sgn}(f_{ij}(\mathbf{t}_a))}+o(v^{-(n+1)/2})$$

$\int_{R(\Omega-\Omega)} d\mathbf{v}\, \theta(\mathbf{v}\cdot\mathbf{n_x}) P_\Gamma(-\mathbf{v}) e^{i\mathbf{v}\cdot\mathbf{p}}$ (cont.)



- Use coordinates in which p is vertical, $\partial(\Omega-\Omega) \ni (\mathbf{u}, h(\mathbf{u}))$ and write $\mathbf{v} = \lambda(\mathbf{u}, h(\mathbf{u}))$ and $\mathbf{k}_a((\mathbf{0}, h(\mathbf{0})) = \mathbf{p}$.
- Phase in dv-integration is $\mathbf{v}\cdot(\mathbf{p}-\mathbf{k}_a(\mathbf{v})) = \lambda h(\mathbf{0})(\mathbf{p}-\mathbf{k}_a(\mathbf{0}))_n + \lambda \frac{f_{ij}^{-1}(\mathbf{k}_a(0))}{2h(\mathbf{0})} u_i u_j$
- u-integration by stationary phase cancels Gauß curvature and leaves

$$\int d\lambda\, \frac{e^{i\lambda h(\mathbf{0})(\mathbf{p}-\mathbf{k}_a(\mathbf{0}))_n}}{\lambda}$$

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- This integral is over $\lambda \in [0, R]$ but up to an O(1) error, we can change it to $\lambda \in [1, R]$

$$\int_1^R d\lambda \, \frac{e^{i\lambda h(\mathbf{0})(\mathbf{p}-\mathbf{k}_a(\mathbf{0}))_n}}{\lambda} = \begin{cases} \ln R + O(1) & \text{for } \mathbf{p} - \mathbf{k}_a(\mathbf{0})_n = 0 \\ O(1) & \text{else} \end{cases}$$

- Collecting everything:

$$\mathrm{tr}(\varrho_{R\Omega,\Gamma}(1-\varrho_{R\Omega,\Gamma})) = \frac{\ln 2}{\pi^2}\left(\frac{R}{2\pi}\right)^{n-1} \ln R \int_{\partial\Omega\times\partial\Gamma} d\sigma(\mathbf{x}) d\sigma(\mathbf{p}) \, |\mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{p}}| + o(R^{n-1}\ln R)$$

As Quantization

$$\int_{\partial\Omega\times\partial\Gamma} d\sigma(\mathbf{x})d\sigma(\mathbf{p})\,|\mathbf{n_x}\cdot\mathbf{n_p}| = \int_{\partial\Omega\times\partial\Gamma} \omega^{\otimes(n-1)}$$

- Instead of scaling $R\Omega$, we can also place R more democratically in the exponent $e^{iR\mathbf{x}\cdot\mathbf{p}}$ (up to an overall factor).
- This shows that R actually plays the role of $1/\hbar$.
- In an informal, semi-classical expansion

$$\begin{aligned}\mathrm{tr}(Q_\Omega P_\Gamma Q_\Omega P_\Gamma) &= \mathrm{tr}(Q_\Omega Q_\Omega P_\Gamma P_\Gamma) + \mathrm{tr}(Q_\Omega [P_\Gamma, Q_\Omega] P_\Gamma) \\ &= \mathrm{tr}(Q_\Omega P_\Gamma) + \mathrm{tr}(Q_\Omega \hbar\{P_\Gamma, Q_\Omega\} P_\Gamma)\end{aligned}$$

- $\{P_\Gamma, Q_\Omega\} = \nabla\chi_\Gamma\cdot\nabla\chi_\Omega \sim \delta(\mathbf{p}\in\partial\Gamma)\delta(\mathbf{x}\in\partial\Omega)$ so for the discontinuous symbols we find a semi-classical term at $O(\log\hbar)$

log(R) term

- Double discontinuity in phase space is essential for area law violation.
- There is a simple extension when $\chi_\Omega(x)$ or $\chi_\Gamma(p)$ are multiplied by smooth functions.
- At finite temperature, the entropy has a bulk term (as we no longer start from a pure state) plus a strict surface term that goes as $\eta(T,\partial\Omega) = (1/12)J(\partial\Gamma_\mu,\partial\Omega)\ln(T_0/T) + \ldots$ and becomes the area law violating term at zero temperature
- The explicit form suggests there should be a more direct derivation (as an anomaly?).
- Holographic derivation from Fermi surface?