Classification and unitary realizations of higher spin (super)-algebras , AdS/CFT dualities and the unique supersymmetric higher spin theory in AdS_6

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In Memoriam Ioannis Bakas

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based mainly on MG, Skvortsov & Tran, 1608.07582; Fernando & MG , 0908.3624, 1005.3580, 1008.0702, 1409.2185 , 1015-09130 , and Govil & MG, 1312.2907, 1401.6930; MG 1603.02359.

- Covariant twistorial oscillator construction of positive energy unitary representations of space-time (super)-algebras and AdS/CFT dualities
- Geometric quasiconformal realizations of noncompact groups versus their conformal realizations
- Geometric quasiconformal realization of the group $SO(d,2)$ as the invariance group of a "quartic light-cone".
- Quantization of the geometric quasiconformal realizations and the minimal unitary representations
- Deformations of the minimal unitary representation of $SO(d,2)$ and massless conformal fields in d-dimensions.
- $AdS_{(d+1)}/CFT_d$ higher spin algebras as universal enveloping algebras of the minreps of $SO(d,2)$ obtained via the quasiconformal approach and their deformations and supersymmetric extensions.
- AdS/CFT dualities in higher spin theories at one loop.
- Unique exceptional higher spin theory in AdS_6 and Romans supergravity.
- Comments and open problems

Oscillator construction of the positive energy unitary representations of noncompact Lie groups

MG & Saclioglu (1982)

Simple noncompact Lie groups G that admit positive energy (lowest weight) unitary representations are in one to one correspondence with the irreducible noncompact Hermitian symmetric spaces $G/K \times U(1)$ with $K \times U(1)$ being the maximal compact subgroup. The Lie algebra g of G has a three graded decomposition with respect to the generator E of $U(1)$

$$g = g^{-1} \oplus g^{0} \oplus g^{+1}$$

$$[g^{(m)}, g^{(n)}] \subseteq g^{(m+n)} \qquad\qquad m, n = \mp 1, 0$$

Lie algebra of $H = K \times U(1) = \mathfrak{g}^0$. The generators of G are realized as bilinears of bosonic oscillators transforming in a certain representation of K. In the Fock space $\mathcal{F}$ one chooses a set of lowest energy states $|\Omega>$ which transforms irreducibly under H and which are annihilated by the generators in g^{-1} space

$$g^{-1}|\Omega >= 0 \ , \ g^{0}|\Omega >= |\Omega' > \leftrightarrow \mathfrak{irrep\ of\ g^o}$$

Then by acting on the lowest energy irrep $|\Omega>$ repeatedly with the generators in g^{+1} space one obtains an infinite set of states

$$|\Omega >, \quad g^{+1}|\Omega >, \quad g^{+1}g^{+1}|\Omega >, \ldots$$

which forms the basis of an irreducible unitary lowest weight representation of G. The irreducibility of the representation of G follows from the irreducibility of $|\Omega>$ under $K \times U(1)$.

Table: The complete list of simple non-compact groups G that admit positive energy unitary representations:

G	$K \times U(1)$
$SU(p,q)$	$S(U(p) \times U(q))$
$Sp(2n,R)$	$U(n)$
$SO^*(2n)$	$U(n)$
$SO(n,2)$	$SO(n) \times SO(2)$
$E_{6(-14)}$	$SO(10) \times U(1)$
$E_{7(-25)}$	$E_6 \times U(1)$

Oscillators form irreps under the maximal compact subgroup. If the minimal number of irreps needed is one or two the corresponding unitary representations of the noncompact group are called singletons or doubletons, respectively.
Special isomorphisms of conformal groups in 3,4 and 6 dimensions:

$$SO(3,2) \equiv Sp(4,\mathbb{R}) \quad SO(4,2) \equiv SU(2,2) \quad SO(6,2) \equiv SO^*(8)$$

Oscillator construction of the positive energy unitary representations of noncompact Lie superalgebras

Bars, MG (1983)

Consider a Lie superalgebra g with a three graded decomposition with respect to compact subsuperalgebra of maximal rank (determined by the generator E of $U(1) \in g^0$):

$$g = g^{-1} \oplus g^0 \oplus g^{+1}$$

$$[g^{(m)}, g^{(n)}] \subseteq g^{(m+n)} \qquad\qquad m, n = \mp 1, 0$$

The generators of g are realized as bilinears of superoscillators transforming in a certain representation of g^0. In the super Fock space $\mathcal{F}$ one chooses a set of lowest energy states $|\Omega >$ which transforms irreducibly under g^0 and which are annihilated by the generators in g^{-1} space

$$g^{-1}|\Omega >= 0 \ , \ g^0|\Omega >= |\Omega' >$$

Then the infinite set of states

$$|\Omega >, \quad g^{+1}|\Omega >, \quad g^{+1}g^{+1}|\Omega >, \ldots$$

form the basis of an irreducible unitary lowest weight representation (positive energy) of g.

Extension of the oscillator construction to Lie superalgebras that admit only a 5-graded decomposition with respect to a subalgebra of maximal rank. MG 1987

AdS/CFT in Kaluza-Klein supergravity Aspen 1984

- The Kaluza-Klein spectrum of IIB supergravity on $AdS_5 \times S^5$ was first obtained via the oscillator method by simple tensoring of the CPT self-conjugate doubleton supermultiplet (SCD) of $N = 8$ AdS_5 superalgebra $PSU(2,2\,|\,4)$ repeatedly and restricting to CPT self-conjugate representations.
- $(N = 4SYM) \times (N = 4SYM)|_{CPT} \Longrightarrow$ massless graviton supermultiplet in AdS_5
- $(N = 4SYM)^n|_{CPT} \Longrightarrow$ massive graviton supermultiplets in AdS_5 for $n > 2$
- The CPT self-conjugate doubleton supermultiplet of $PSU(2,2\,|\,4)$ symmetry of $AdS_5 \times S^5$ solution of IIB supergravity does not have a Poincaré limit in five dimensions and decouples from the Kaluza-Klein spectrum as gauge modes and the field theory of CPT self-conjugate doubleton supermultiplet of $PSU(2,2\,|\,4)$ lives on the boundary of AdS_5, which can be identified with $4D$ Minkowski space on which $SO(4,2)$ acts as a conformal group, and the unique candidate for this theory is the four dimensional $N = 4$ super Yang-Mills theory that was known to be conformally invariant. MG , Marcus (1984)
- The above results showed how to gauge the maximal supergravity in $d = 5$ which was subsequently done in MG, Romans and Warner and Pilch and van Nieuwenhuizen.
- Fields of maximal $N = 8$ supergravity in $d = 4$ fit into the CPT-self-conjugate doubleton supermultiplet of $SU(2,2|8)$ with even subgroup $SU(2,2) \times U(8)$. $N = 8$ conformal supergravity based on the fields of maximal supergravity in $d = 4$?! MG , Marcus (1984)

- The spectrum of $11D$ supergravity over $AdS_7 \times S^4$ were fitted into supermultiplets of the symmetry superalgebra $OSp(8^*|4)$ with even subalgebra $SO(6,2) \oplus USp(4)$. The entire Kaluza-Klein spectrum was obtained by tensoring $(2,0)$ doubleton supermultiplet $OSp(8^*|4)$, and restricting to CPT self-conjugate supermultiplets: MG, van Nieuwenhuizen, Warner (1984)
- $[(2,0)CFT] \times [(2,0)CFT]|_{CPT} \Longrightarrow$ Massless graviton supermultiplet of maximal sugra in AdS_7
- $[(2,0)CFT]^n_{CPT} \Longrightarrow$ massive graviton supermultiplets for $n > 2$.
- The spectrum of $11D$ supergravity over over $AdS_4 \times S^7$ was fitted into supermultiplets of $OSp(8\,|\,4,\mathbb{R})$ with even subalgebra $SO(8) \oplus SO(3,2)$. The entire spectrum was obtained by tensoring the singleton supermultiplet of $OSp(8\,|\,4,\mathbb{R})$ and restricting to the CPT self-conjugate supermutiplets.

  MG, Warner (1984)
- The relevant singleton supermultiplet of $OSp(8\,|\,4,\mathbb{R})$ and doubleton supermultiplet of $OSp(8^*|4)$ do not have a Poincaré limit in four and seven dimensions, respectively, and decouple from the respective spectra as gauge modes. Again it was pointed out that field theories of the singleton and $(2,0)$ doubleton supermutiplets live on the boundaries of AdS_4 and AdS_7 as superconformally invariant theories.
- The above results on the oscillator construction of the spectra of 11-d supergravity and type IIB supergravity represent the earliest work on AdS/CFT dualities within the framework of K-K supergravity in a true Wignerian sense.
- Modern era of AdS/CFT dualities in M/Superstring theory was ushered in by the famous paper of Maldacena followed by the work of Witten and Gubser, Klebanov and Polyakov (1998).

Higher spin algebras and superalgebras and the oscillator method

Early work on the connection between high spin (super)algebras and the universal enveloping algebras of singleton representations of AdS Lie(super)algebras: Fradkin-Vasiliev higher spin algebra in AdS_4 as the enveloping algebra of the singletonic realization of $Sp(4,\mathbb{R})$ and proposal to extend it to AdS_5 and AdS_7 HS (super)-algebras using the doubletonic realizations of $SU(2,2)$ and $SO^*(8)$ and their supersymmetric extensions. MG (1989)

AdS_3 higher spin algebras and universal enveloping algebras. Konstein and Vasiliev (1989)

The quotient of the universal enveloping algebra (UEA) $\mathcal{U}(o(d,2))$ of $o(d,2)$ by its annihilator on the scalar "singleton module" is the $AdS_{(d+1)}/CFT_d$ higher-spin algebra. Vasiliev (2003)

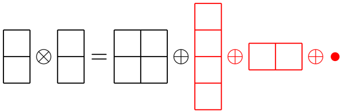
The ideal with which to quotient the UEA is the Joseph ideal that annihilates the minrep. Eastwood (2005)

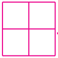
I will adopt this definition and define the $AdS_{(d+1)}/CFT_d$ higher spin algebras as the universal enveloping algebras $\mathcal{U}(SO(d,2))$ of $SO(d,2)$ quotiented by their Joseph ideals $\mathcal{J}(SO(d,2))$:

$$HS(SO(d,2)) \equiv \frac{\mathcal{U}(SO(d,2)}{\mathcal{J}(SO(d,2)}$$

The generators J_{ABCD} of Joseph ideal vanish identically as operators for the singletonic realization of the Lie algebra $SO(3,2) = Sp(4,\mathbb{R})$ as bilinears of covariant twistorial oscillators. Therefore the AdS_4/CFT_3 HS algebra is given simply by the enveloping algebra of the singletonic realization of $Sp(4,\mathbb{R})$.

Poincare-Birkhoff-Witt theorem: The enveloping algebra $\mathscr{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ decomposes into symmetric tensor products of the adjoint representation of $\mathfrak{g}$. For $\mathfrak{so}(d,2)$ the symmetric product of the adjoint representation decomposes as:



where the singlet $\bullet$ is the quadratic Casimir of $SO(d,2)$.
Vasiliev: the higher spin algebra $HS(\mathfrak{g})$ must be a quotient of $\mathscr{U}(\mathfrak{g})$ since the higher spin gauge fields are described by traceless two row Young tableaux. Hence the relevant ideal should quotient out all the diagrams except the first one in the above decomposition.
The Joseph ideal generators include all the diagrams indicated in red on the RHS and does not include the "window" diagram . Therefore by quotienting $\mathscr{U}(\mathfrak{g})$ by the Joseph ideal generated by J_{ABCD} we get rid of all the "unwanted" diagrams and obtain the standard higher spin algebra $HS(d,2)$ a la Eastwood & Vasiliev .

Minimal unitary representation and the Joseph ideal

Among all the unitary representations of a noncompact Lie group the minimal one is distinguished by the fact that it is annihilated by the Joseph ideal inside its universal enveloping algebra. Denoting the generators of the Lie algebra of $SO(n-2,2)$ as G_{ab} the Joseph ideal is generated by the following elements of the enveloping algebra:

Eastwood et.al.(2005)

$$J_{abcd} = G_{ab}G_{cd} - G_{ab} \circledcirc G_{cd} - \frac{1}{2}[G_{ab}, G_{cd}] + \frac{n-4}{4(n-1)(n-2)}.\langle G_{ab}, G_{cd}\rangle$$

where $\langle G_{ab}, G_{cd}\rangle$ is the Killing form, $G_{ab} \circledcirc G_{cd}$ is the Cartan product:

$$\begin{aligned} G_{ab} \circledcirc G_{cd} \quad \equiv \quad & \frac{1}{3}G_{ab}G_{cd} + \frac{1}{3}G_{dc}G_{ba} + \frac{1}{6}G_{ac}G_{bd} - \frac{1}{6}G_{ad}G_{bc} + \frac{1}{6}G_{db}G_{ca} - \frac{1}{6}G_{cb}G_{da} \\ & - \frac{1}{2(n-2)}\left(G_{ae}G_{c}^{e}\eta_{bd} - G_{be}G_{c}^{e}\eta_{ad} + G_{be}G_{d}^{e}\eta_{ac} - G_{ae}G_{d}^{e}\eta_{bc}\right) \\ & - \frac{1}{2(n-2)}\left(G_{ce}G_{a}^{e}\eta_{bd} - G_{ce}G_{b}^{e}\eta_{ad} + G_{de}G_{b}^{e}\eta_{ac} - G_{de}G_{a}^{e}\delta_{bc}\right) \\ & + \frac{1}{(n-1)(n-2)}G_{ef}G^{ef}\left(\eta_{ac}\eta_{bd} - \eta_{bc}\eta_{ad}\right) \end{aligned}$$

where η_{ab} is the $SO(n-2,2)$ invariant metric.
For the singletonic realization of $Sp(4,R) = SO(3,2)$, J_{abcd} vanish identically as operators. However for doubletonic realizations of $SO(4,2)$ and $SO(6,2)$ in terms of twistorial oscillators they do not vanish as operators.

Quasiconformal Realizations of Non-compact Groups

- Not all groups have conformal realizations. The exceptional groups G_2, F_4 and E_8 do not admit a three-graded decomposition with respect to any subalgebra of maximal rank.
- However all simple Lie algebras $\mathfrak{g}$ admit a 5-graded decomposition with respect to a subalgebra $\mathfrak{g}^0$ of maximal rank of the form

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{1} \oplus \mathfrak{g}^{2}$$

  such that $\dim(\mathfrak{g}^{\pm 2}) = 1$.
- The Lie algebra $\mathfrak{g}$ admits a geometric quasiconformal realization on a $2n+1$ dimensional space, where $2n = dim(\mathfrak{g}^1)$, which leaves invariant light-like separations with respect to a quartic distance function ("quartic light-cone").
  MG, Koepsell, Nicolai, 2000
- The quasiconformal realization of a noncompact Lie algebra $\mathfrak{g}$ corresponds to an extension of conformal realizations of certain subgroups of $\mathfrak{g}$.
- Conformal $SO(d,2) \Longrightarrow$ quasiconformal $SO(d+2,4)$.
  Conformal $E_{7(7)} \Longrightarrow$ quasiconformal $E_{8(8)}$.

Geometric realization of $SO(d,2)$ as a quasiconformal group over a $(2d-3)$-dimensional space $\mathcal{T}$ with coordinates $\mathcal{X} = \left(X^{i,a}, x\right)$,

MG, Koepsell, Nicolai (GKN) (2000) & MG, Pavlyk (2005)

$$\mathfrak{so}(d,2) = K_- \oplus U_{i,a} \oplus \left[\Delta \oplus \mathcal{L}_{ij} \oplus \mathcal{M}_{ab}\right] \oplus \widetilde{U}_{i,a} \oplus K_+$$

where $\mathcal{L}_{ij}$ and $\mathcal{M}_{ab}$ are the generators of $SO(d-2)$ and $SU(1,1)$ subgroups.

$$K_+ = \frac{1}{2}\left(2x^2 - \mathcal{I}_4\right)\frac{\partial}{\partial x} - \frac{1}{4}\frac{\partial \mathcal{I}_4}{\partial X^{i,a}}\eta^{ij}\epsilon^{ab}\frac{\partial}{\partial X^{j,b}} + x\, X^{i,a}\frac{\partial}{\partial X^{i,a}}$$

$$U_{i,a} = \frac{\partial}{\partial X^{i,a}} - \eta_{ij}\epsilon_{ab}\, X^{j,b}\frac{\partial}{\partial x}$$

$$\mathcal{L}_{ij} = \eta_{ik}X^{k,a}\frac{\partial}{\partial X^{j,a}} - \eta_{jk}X^{k,a}\frac{\partial}{\partial X^{i,a}}$$

$$\mathcal{M}_{ab} = \epsilon_{ac}X^{i,c}\frac{\partial}{\partial X^{i,b}} + \epsilon_{bc}X^{i,c}\frac{\partial}{\partial X^{i,a}}$$

$$K_- = \frac{\partial}{\partial x} \quad , \qquad \Delta = 2\,x\frac{\partial}{\partial x} + X^{i,a}\frac{\partial}{\partial X^{i,a}} \quad , \qquad \widetilde{U}_{i,a} = \left[U_{i,a}\,,\, K_+\right]$$

$$\mathcal{I}_4(X) = \delta_{ij}\delta_{kl}\epsilon_{ac}\epsilon_{bd}X^{i,a}X^{j,b}X^{k,c}X^{l,d}$$

$$\epsilon_{ab} = -\epsilon_{ba} \qquad i,j,\cdots = 1,\ldots,d-2;\;\; a,b,\cdots = 1,2$$

The quartic norm (length) of a vector $\mathcal{X} = \left(X^{i,a}, x\right) \in \mathcal{T}$ is defined as

$$\mathcal{Q}\left(\mathcal{X}\right) = \mathcal{I}_4\left(X\right) + 2\,x^2 .$$

To see the geometric picture behind the above nonlinear realization, one defines a quartic distance function between any two points $\mathcal{X}$ and $\mathcal{Y}$ in the $(2d-3)$ dimensional space $\mathcal{T}$ as

$$d\left(\mathcal{X}, \mathcal{Y}\right) = \mathcal{Q}\left(\delta\left(\mathcal{X}, \mathcal{Y}\right)\right)$$

where the "symplectic" difference $\delta\left(\mathcal{X}, \mathcal{Y}\right)$ is defined as

$$\delta\left(\mathcal{X}, \mathcal{Y}\right) = \left(X^{i,a} - Y^{i,a}\,,\, x - y - \eta_{ij}\epsilon_{ab}\,X^{i,a}Y^{j,b}\right) = -\delta\left(\mathcal{Y}, \mathcal{X}\right) .$$

where $\eta_{ij}\epsilon_{ab}\,X^{i,a}Y^{j,b}$ a skew symmetric bilinear form.
The lightlike separations between any two points with respect to the quartic distance function are left invariant under the quasiconformal action of $SO(d,2)$. In other words, $SO(d,2)$ acts as the invariance group of a "light-cone" with respect to a quartic distance function in a $(2d-3)$-dimensional space.

$$\mathcal{I}_4(X) = \delta_{ij}\delta_{kl}\epsilon_{ac}\epsilon_{bd}X^{i,a}X^{j,b}X^{k,c}X^{l,d}$$

$$\epsilon_{ab} = -\epsilon_{ba} \qquad i,j,\cdots = 1,\ldots,d-2;\;\; a,b,\cdots = 1,2$$

Minimal Unitary Representations and Quasiconformal Realizations of Groups: GKN (2001) & MG, Pavlyk (2005)

- Quantization of the quasiconformal realization of a non-compact Lie group leads directly to its minimal unitary representation $\Rightarrow$ Unitary representation over an Hilbert space of square integrable functions of smallest number of variables possible.

- Lie algebra $\mathfrak{g}$ of a quantized quasiconformal realization of a group G :

$$\mathfrak{g} = E \oplus E^{\alpha} \oplus (J^{a} + \Delta) \oplus F^{\alpha} \oplus F$$

$\Delta = -\frac{i}{2}(yp + py)$ $([y, p] = i)$ determines the 5-grading and $\Omega^{\alpha\beta}$ is the symplectic invariant tensor of $\mathfrak{h}$ generated by J^{a} and $\left[\xi^{\alpha}, \xi^{\beta}\right] = \Omega^{\alpha\beta}$ $(\alpha, \beta, .. = 1, 2, ..., 2n)$

$$E = \frac{1}{2}y^{2} \qquad E^{\alpha} = y\,\xi^{\alpha}, \qquad J^{a} = -\frac{1}{2}\lambda^{a}{}_{\alpha\beta}\xi^{\alpha}\xi^{\beta}$$

$$F = \frac{1}{2}p^{2} + \frac{\kappa I_{4}(\xi^{\alpha})}{y^{2}}, \qquad F^{\alpha} = [E^{\alpha}, F]$$

$I_{4}(\xi^{\alpha}) = S_{\alpha\beta\gamma\delta}\xi^{\alpha}\xi^{\beta}\xi^{\gamma}\xi^{\delta} \Leftrightarrow$ *quartic invariant of* $\mathfrak{h}$
Choosing a polarization $\xi^{\alpha} = (x^{i}, p_{j})$ one has $[x^{i}, p_{j}] = i\delta^{i}_{j}$ $(i, j = 1, 2, .., n)$

- $(E, F, \Delta) \Longrightarrow SL(2, \mathbb{R})$ of Calogero model or conformal quantum mechanics with the quartic invariant I_{4} playing the role of coupling constant.
$E + F = \frac{1}{2}(y^{2} + p^{2}) + \frac{\kappa I_{4}(\xi^{\alpha})}{y^{2}} \leftrightarrow$ Calogero Hamiltonian.

The minimal unitary realization of $\mathfrak{so}(d,2)$ from its quasiconformal realization:

$$\begin{aligned}\mathfrak{so}(d,2) &= \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus [\,\Delta \oplus \mathfrak{so}(d-2) \oplus \mathfrak{su}(1,1)\,] \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{+2)} \\ & \quad K_- \oplus \begin{pmatrix} U_i \\ U_i^\dagger \end{pmatrix} \oplus (\Delta + L_{ij} + M_{ab}) \oplus \begin{pmatrix} W_i \\ W_i^\dagger \end{pmatrix} \oplus K_+\end{aligned}$$

$K_- = x^2/2 \quad , \quad \Delta = \frac{1}{2}(xp+px) \quad , \quad K_+ = \frac{1}{2}p^2 + \frac{1}{x^2}\mathcal{G} \Longrightarrow$ Calogero $SL(2,\mathbb{R})$

$$\begin{pmatrix} U_i \\ U_i^\dagger \end{pmatrix} = \begin{pmatrix} x\, a_i \\ x\, a_i^\dagger \end{pmatrix} \quad , \quad \begin{pmatrix} W_i \\ W_i^\dagger \end{pmatrix} = -i \begin{pmatrix} [U_i, K_+] \\ [U_i^\dagger, K_+] \end{pmatrix}$$

$$L_{ij} = i\left(a_i^\dagger a_j - a_j^\dagger a_i\right) \quad , \qquad M_+ = \frac{1}{2} a_i^\dagger a_i^\dagger \quad M_- = \frac{1}{2} a_i a_i \quad M_0 = \frac{1}{4}\left(a_i^\dagger a_i + a_i a_i^\dagger\right)$$

$$SO(d-2) \qquad , \qquad SU(1,1)$$

$$\mathcal{L}^2 = L_{ij} L_{ij} \qquad \mathcal{M}^2 = {M_0}^2 - \frac{1}{2}\left(M_+ M_- + M_- M_+\right)$$

$$\mathcal{L}^2 = 8\,\mathcal{M}^2 - \frac{1}{2}(d-2)(d-6)$$

$$\mathcal{G} = \frac{1}{4}\mathcal{L}^2 + \frac{1}{8}(d-3)(d-5) = 2\mathcal{M}^2 + \frac{3}{8}$$

$\mathcal{G}$ plays the role analogous coupling constant in conformal quantum mechanics or Calogero model.

$i, j, \dots = 1, 2, \dots, (d-2)$

$$
\begin{array}{ccccc}
 & & K_- & & \\
 & T_i^P & | & T_i^X & \\
L_- & --- & (D^X \oplus L_{ij} \oplus D^P) & --- & L_+ \\
 & K_i^X & | & K_i^P & \\
 & & K_+ & &
\end{array}
$$

Table: The 5×5 grading of the Lie algebra of $\mathfrak{so}(d,2)$ in an Hermitian basis. The vertical 5-grading is determined by $\Delta = 1/2(xp + px) = 1/2(D^X + D^P)$ and the horizontal 5-grading is determined by $L_0 = (X_i P_i + P_i X_i) = 1/2(D^X - D^P)$.

$$
\begin{aligned}
[T_i^X, K_j^X] &= -2i\delta_{ij} D^X + 2iL_{ij} \\
[T_i^P, K_j^P] &= 2i\delta_{ij} D^P - 2iL_{ij} \\
[T_i^X, K_j^P] &= iL_+ \\
[T_i^P, K_j^X] &= iL_-
\end{aligned}
$$

where $L_+ = P_i P_i$ and $L_- = X_i X_i$.
Therefore the quasiconformal realization of $SO(d,2)$ can be interpreted as the minimal Lie algebra containing the Euclidean conformal Lie algebra acting on transverse coordinates and the dual Euclidean conformal Lie algebra acting on the corresponding transverse momenta. The common subgroup of these two Euclidean conformal groups is $SO(d-2)$.

Considered as a conformal group $SO(d,2)$ has a three-graded decomposition determined by the dilatation generator $\mathcal{D}$:

$$\mathfrak{so}(d,2) = K_\mu \oplus (M_{\mu\nu} + \mathcal{D}) \oplus P_\mu$$

For the minrep the Poincare mass operator vanishes identically : $P_\mu P_\nu \eta^{\mu\nu} = 0$
By going to the compact three graded decomposition determined by the conformal Hamiltonian $H = \frac{1}{2}(K_+ + K_-) + M_0$:

$$\mathfrak{so}(d,2) = \mathfrak{C}^- \oplus (\mathfrak{so}(d) + H) \oplus \mathfrak{C}^+ \,.$$

one finds that the minrep is a unitary lowest weight (positive energy) representation with the lowest weight vector

$$H\, \psi_0^{\alpha_g}(x)\, |0\rangle = \frac{1}{4}(d-2)\, \psi_0^{\alpha_g}(x)\, |0\rangle$$

$$\psi_0^{(\alpha_g)}(x) = C_0\, x^{\alpha_g} e^{-x^2/2} \quad , \quad \alpha_g = \frac{(d-3)}{2} \quad , \quad a_i|0\rangle = 0$$

The Hilbert space of the minrep is spanned by states that are in the tensor product of Fock space of $(d-2)$ ordinary bosonic oscillators and the states of the singular oscillator that form irrep of $SU(1,1)_K$ subgroup with the lowest weight vector $\psi_0^{(\alpha_g)}(x)$.
The minrep describes a massless conformal scalar field in d dimensional Minkowski space.

Deformations of the minimal unitary realization of $\mathfrak{so}(d,2)$:

MG, Fernando 2015

$$\mathfrak{so}(d,2) = \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus [\Delta \oplus \mathfrak{so}(d-2) \oplus \mathfrak{su}(1,1)] \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{+2)}$$

$$K_- \oplus \begin{pmatrix} U_i \\ U_i^\dagger \end{pmatrix} \oplus (\Delta + L_{ij} + M_{ab}) \oplus \begin{pmatrix} W_i \\ W_i^\dagger \end{pmatrix} \oplus K_+$$

$$K_- = x^2/2 \quad , \quad \Delta = \tfrac{1}{2}(xp+px) \quad , \quad K_+ = \tfrac{1}{2}p^2 + \tfrac{1}{x^2}\mathcal{G} \Longrightarrow \text{Calogero } SL(2,\mathbb{R})$$

$$\begin{pmatrix} U_i \\ U_i^\dagger \end{pmatrix} = \begin{pmatrix} x\, a_i \\ x\, a_i^\dagger \end{pmatrix} \quad , \quad \begin{pmatrix} W_i \\ W_i^\dagger \end{pmatrix} = -i \begin{pmatrix} [U_i, K_+] \\ [U_i^\dagger, K_+] \end{pmatrix}$$

$$J_{ij} = L_{ij} + S_{ij} \quad , \qquad M_+ = \frac{1}{2} a_i^\dagger a_i^\dagger \quad M_- = \frac{1}{2} a_i a_i \quad M_0 = \frac{1}{4}\left(a_i^\dagger a_i + a_i a_i^\dagger\right)$$

$S_{ij} =$ " Spin Generators of little group" , $\quad \mathcal{S}^2 = S_{ij}S_{ij}, \quad \mathcal{J}^2 = J_{ij}J_{ij} \quad , \mathcal{L}^2 = L_{ij}L_{ij}$

Coupling "constant" $\quad \mathcal{G} = \left(\frac{1}{2}\mathcal{J}^2 - \frac{1}{4}\mathcal{L}^2 - \frac{(d-6)}{2(d-2)}\mathcal{S}^2 + \frac{1}{8}(d-3)(d-5) \right)$

Jacobi identities require

$$\Delta_{ij} = S_{ik}S_{jk} + S_{jk}S_{ik} - \frac{2}{(d-2)}\mathcal{S}^2\,\delta_{ij} = 0$$

Remarkably these are precisely the identities satisfied by the massless representations of Poincaré group in d dimensions that extend to the unitary representations of conformal group! (Angelopoulos & Laoues 1997).

Therefore there exists a one-to-one correspondence between the minrep of $SO(d,2)$ and its deformations and massless conformal fields in d dimensions.

Deformations of the minrep of $SO(d, 2)$ for odd d:

There is a unique deformation of the minrep given by realizing the spin generators S_{ij} of $SO(d-2)$ as:

$$S_{ij} = \frac{1}{4}[\gamma_i, \gamma_j]$$

where γ_i are Euclidean gamma matrices in $(d-2)$ dimensions.

$$\mathcal{G} = \frac{1}{4}\mathcal{L}^2 + \epsilon\,\mathcal{L}\cdot\mathcal{S} + \epsilon\,\frac{1}{2}(d-3) + \frac{1}{8}(d-3)(d-5)$$

$\epsilon = 0$ for the minrep $\Longrightarrow$ Scalar singleton (massless conformal field)
Lowest energy irrep (K-type) is a singlet of $SO(d)$.
$\epsilon = 1$ deformed minrep $\Longrightarrow$ spinor singleton (massless spinor field).
Lowest energy irrep (K-type) is a spinor of $SO(d)$.
They are the analogs of the remarkable representations of $SO(3, 2)$ discovered by Dirac.

Deformations of the minrep of $SO(d, 2)$ for even d:

There exist infinitely many deformations of the minrep of $SO(d, 2)$ for even d. One can realize the generators S_{ij} of the little group $SO(d-2)$ in the Fock space of fermionic oscillators transforming irreducibly under the subgroup $U((d-2)/2)$.

Construction of the representations of the little group $SO(d-2)_S$ of massless particles, generated by S_{ij} over the Fock space of Fermionic oscillators using its 3-grading w.r.t $\mathfrak{u}((d-2)/2)$:

$$\mathfrak{so}(d-2)_S = \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)} = Z_{rs} \oplus T_{rs} \oplus Z^{\dagger}_{rs}$$

$$Z_{rs} = \vec{\alpha}_r \cdot \vec{\beta}_s - \vec{\alpha}_s \cdot \vec{\beta}_r + \varepsilon\, \xi_r \xi_s$$
$$T_{rs} = \vec{\alpha}^{\dagger}_r \cdot \vec{\alpha}_s - \vec{\beta}_s \cdot \vec{\beta}^{\dagger}_r + \frac{\varepsilon}{2}\left(\xi^{\dagger}_r \xi_s - \xi_s \xi^{\dagger}_r\right)$$
$$Z^{\dagger}_{rs} = -\vec{\alpha}^{\dagger}_r \cdot \vec{\beta}^{\dagger}_s + \vec{\alpha}^{\dagger}_s \cdot \vec{\beta}^{\dagger}_r - \varepsilon\, \xi^{\dagger}_r \xi^{\dagger}_s$$

where $r, s, \dots = 1, 2, \dots, (d-2)/2$; $\epsilon = 0, 1$ and $\vec{\alpha}_r \cdot \vec{\beta}_s = \sum_{K=1}^{P} \alpha_r(K)\beta_s(K)$.
The representations of even orthogonal groups $SO(d-2)$ that satisfy the constraint $\Delta_{ij} = 0$ are

$$(0, \dots, 0, 0, f)_D = \left(\frac{f}{2}, \dots, \frac{f}{2}, \frac{f}{2}\right)_{GZ} \qquad (0, \dots, 0, f, 0)_D = \left(\frac{f}{2}, \dots, \frac{f}{2}, -\frac{f}{2}\right)_{GZ}$$

where $f = 2P + \epsilon$ is the number of colors of fermionic oscillators.
They have the following lowest weight vectors of $SO(d-2)$ in the fermionic Fock space:

$$(0, \dots, 0, 0, f)_D \iff |0\rangle$$
$$(0, \dots, 0, f, 0)_D \iff \alpha^{\dagger}_{(r_1}(1)\beta^{\dagger}_{s_1}(1)\alpha^{\dagger}_{r_2}(2)\beta^{\dagger}_{s_2}(2)\dots\alpha^{\dagger}_{r_P}(P)\beta^{\dagger}_{s_P}(P)\xi^{\dagger}_{t)}\,|0\rangle \equiv |Symvac\rangle$$

and $(r_1, s_1, \dots, t)$ denotes complete symmetrization of indices.

Minimal unitary realization of $SO(5,2)$ **Fernando, MG 2014**

$$\mathfrak{so}(5,2) = \bar{1} \oplus \overline{(3,2)} \oplus [\,\mathfrak{so}(1,1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(1,1)\,] \oplus (3,2) \oplus 1$$

$$= K_- \oplus (U_i,\ U_i^\dagger) \oplus (L_i\,, \Delta\,, (M_+, M_-\,, M_0)) \oplus (W_i + W_i^\dagger) \oplus K_+$$

$$K_- = \frac{1}{2}x^2 \qquad U_i = x\, a_i \qquad U_i^\dagger = x\, a_i^\dagger\,. \quad i = 1,2,3$$

$$\mathfrak{su}(1,1) \Rightarrow \qquad M_+ = \frac{1}{2} a_i^\dagger a_i^\dagger \qquad M_- = \frac{1}{2} a_i a_i \qquad M_0 = \frac{1}{4}\left(a_i^\dagger a_i + a_i a_i^\dagger\right)$$

$$\mathfrak{su}(2) \Rightarrow L_i = \epsilon_{ijk} a_j^\dagger a_k \qquad \mathcal{L}^2 = L_i L_i$$

$$K_+ = \frac{1}{2}p^2 + \frac{1}{4\,x^2}\left(8\,\mathcal{M}^2 + \frac{3}{2}\right) = \frac{1}{2}p^2 + \frac{1}{2\,x^2}\mathcal{L}^2$$

The minrep of $SO(5,2)$ describes a massless conformal scalar field in $d = 5$.
Deformation of the minimal unitary representation of $SO(5,2)$:

$$L_i \Longrightarrow J_i = L_i + S_i \qquad S_i = \frac{1}{2}\zeta^\dagger \sigma_i \zeta \qquad , \zeta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$K_+ = \frac{1}{2}p^2 + \frac{1}{2\,x^2}\left(2\,\mathcal{J}^2 - \mathcal{L}^2 + \frac{2}{3}\,\mathcal{S}^2\right)$$

Deformed minrep describes a massless conformal spinor field in $d = 5$. No other deformations!. Scalar and spinor singletons in $d = 5$ similar to the situation in $d = 3$.

Minimal unitary representation of the $5d$ superconformal algebra $F(4)$ with the even subalgebra $SO(5,2) \oplus SU(2)$

Fernando, MG 2014

$$\begin{aligned}\mathfrak{f}(4) &= \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{(+2)} \\ &= 1_B \oplus (6_B \oplus 4_F) \oplus [d(2,1;2) \oplus \Delta] \oplus (6_B \oplus 4_F) \oplus 1_B\end{aligned}$$

The $d(2,1;2)$ has the even subalgebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(1,1)$ and admits a 10-dimensional linear representation.

The conformal superalgebra $\mathfrak{f}(4)$ has a noncompact 5-grading in a manifestly $SO(4,1) = USp(2,2)$ covariant form:

$$\mathfrak{f}(4) = \mathcal{K}_\mu \ \oplus\ \mathfrak{S}_{\alpha r} \ \oplus\ [\,\mathcal{D}\,,\, \mathcal{M}_{\mu\nu}\,,\, T_{\pm,0}\,] \ \oplus\ \mathfrak{Q}_{\alpha r} \ \oplus\ \mathcal{P}_\mu$$

where $\mu, \nu = 0,1,2,3,4$; $\alpha = 1,2,3,4$; and $r = 1,2$.

$\mathcal{M}_{\mu\nu}$ are the generators of $Spin(4,1) \approx USp(2,2)$. $\mathcal{K}_\mu$ are the special conformal generators and $\mathcal{P}_\mu$ are the translations. The $\mathfrak{S}_{\alpha r}$ and $\mathfrak{Q}_{\alpha r}$ are the special conformal and Poincare supersymmetry generators.

Supersymmetry generators Ξ^r_α in the $(8,2)$ representation of $SO(5,2) \times SU(2)_T$ satisfy:

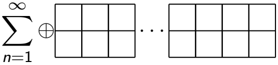
$$\left\{ \Xi^r_\alpha \,,\, \Xi^s_\beta \right\} = i\epsilon^{rs}\, M_{AB} \left(\Sigma^{AB} \mathcal{C}_7 \right)_{\alpha\beta} + 3i\, (\mathcal{C}_7)_{\alpha\beta}\, (i\sigma_2\sigma^i)^{rs}\, T_i$$

where $r,s = 1,2$ are the $SU(2)_T$ spinor indices, M_{AB} are the $SO(5,2)$ generators

The minimal unitary supermultiplet of $F(4)$ consists of two scalar singletons in a doublet of R-symmetry group $SU(2)_T$ and a spinor singleton which is a singlet of $SU(2)_T$.

Quasiconformal construction of the minrep and its deformations and higher spin algebras:

- For the minrep obtained by quantization of the quasiconformal realization of $SO(d,2)$ the generators of Joseph ideal vanish identically as operators ($J_{ABCD} \equiv 0$)
- The universal enveloping algebras of the minreps of $SO(d,2)$ obtained by quasiconformal methods yield directly the higher spin algebras in the respective dimensions: The resulting enveloping algebra decomposes into operators whose $SO(d,2)$ Young tableaux have only two rows corresponding to higher spin gauge fields described by traceless two row Y-Ts. The operators in the symmetric product of the generators with four rows and one column and one row and two columns vanish identically. K. Govil & MG for $d = 3,4,6$ and Fernando & MG for $d = 5$ and $d > 6$.



- **Deformations of the minrep as obtained via the quasiconformal approach are in one-to-one correspondence with the massless conformal fields. The enveloping algebras of the deformed minreps yield all the higher spin algebras in $d > 2$. They correspond to quotients**

$$HS(SO(d,2)) \equiv \frac{\mathcal{U}(SO(d,2))}{\mathcal{J}_{def}(SO(d,2))}$$

- **The enveloping algebras of minimal unitary realizations of AdS superalgebras and their deformations yield all the supersymmetric extensions of the higher spin algebras in $d \leq 6$.**

AdS_6/CFT_5 higher spin algebras and superalgebra

Fernando & MG (2014)

The Joseph ideal generators vanish identically as operators for the minrep of $SO(5,2)$ and a certain deformation of the Joseph ideal vanishes for the deformed minrep.

The enveloping algebra of the minrep (scalar singleton) of $SO(5,2)$ defines the AdS_6/CFT_5 bosonic higher spin algebra

$HS(5,2;t=0) = \frac{\mathcal{U}(SO(5,2))}{\mathcal{J}(SO(5,2))}$

and admits a single deformation

$HS(5,2;t=1/2) = \frac{\mathcal{U}(SO(5,2))}{\mathcal{J}_{(t=1/2)}(SO(5,2))}$

The enveloping algebra of the minimal unitary realization of $F(4)$ defines the unique AdS_6/CFT_5 higher spin superalgebra.

Existence of a scalar and a spinor singleton for $SO(5,2)$ is similar to $SO(3,2) = Sp(4,\mathbb{R})$. However only a unique R-symmetry group exists for supersymmetric extension, i.e $F(4)$ with even subalgebra $SO(5,2) \oplus SU(2)$.

$F(4)$ is the superconformal algebra of the unique exceptional superspace coordinatized by the exceptional Jordan superalgebra with 6 bosonic and 4 fermionic coordinates. It has no realization in terms of associative super matrices.

Minkowski spacetime is coordinatized by the Jordan algebra $J_2^{\mathbb{C}}$ of 2×2 Hermitian matrices.

$Rot(J_2^{\mathbb{C}} = SU(2) \qquad Lor(J_2^{\mathbb{C}}) = SL(2,C) \qquad Con(J_2^{\mathbb{C}}) = SU(2,2)$

THE EXCEPTIONAL SUPERSPACE:

- Rotation Lie superalgebra the exceptional superspace coordinatized by $JF(6/4)$ is
  $OSp(1/2) \times OSp(1/2) \supset SO(4) = SU(2) \times SU(2)$.
- Lorentz Lie superalgebra of $JF(6/4)$ is
  $OSp(2/4) \supset SO(2) \times Sp(4)$
- Superconformal Lie algebra of $JF(6/4)$ is
  $F(4) \supset SO(5,2) \times SU(2)$
  Non-linear action of $F(4)$ on the exceptional superspace can be obtained using the quadratic Jordan formulation. MG (1990).
- The exceptional $N = 2$ superconformal algebra $F(4)$ in five dimensions can not be embedded in any six dimensional super conformal algebra
  $OSp(8^*|2N) \supset SO(6,2) \times USp(2N)$ as expected from the exceptionality of the superspace defined by $JF(6/4)$.
- According to Nahm's classification $d = 6$ is the maximal dimension for the existence of superconformal field theories based on simple superconformal algebras!

Higher Spin Theories and $AdS_{(d+1)}/CFT_d$ Correspondence :

Free CFT of a massless scalar field satisfying $\Box\phi = 0$ has an infinite set of conserved currents :

$$J_{a_1\ldots a_s} = \phi\partial_{a_1}\cdots\partial_{a_s}\phi \Longrightarrow \partial^a J_{aa_2\ldots a_s} = 0$$

The corresponding charges Q_s form an infinite dimensional algebra which is the higher spin algebra and has $SO(d,2)$ as a subalgebra. On the AdS side these conserved charges correspond to gauge symmetries described by massless HS fields:

$$\partial^m J_{mabc\ldots} = 0 \qquad \Longleftrightarrow \qquad \delta\Phi_{\underline{mabc}\ldots} = \nabla_{\underline{m}}\xi_{\underline{abc}\ldots} + \ldots$$

- TYPE A HST: Minimal unitary representation of $SO(d,2)$ describes a massless scalar field and is sometimes called Rac. With one complex scalar one can construct conserved higher-spin currents, which are totally-symmetric tensors:

$$J_s = \bar{\phi}\partial^s\phi + \ldots\,, \qquad \Delta = d+s-2\,,$$
$$J_0 = \bar{\phi}\phi\,, \qquad \Delta = d-2\,,$$

  Operator product expansion (OPE) in a free CFT $\leftrightarrow$ the tensor product decomposition of the conformal algebra representations, which in the case of $\bar{\phi}\phi$ OPE leads to :

$$Rac \otimes Rac = \sum_s J_s\,.$$

The spectrum of the Type-A theory consists of bosonic totally-symmetric HS fields (Fronsdal fields) that are duals of J_s and an additional scalar field Φ_0 that is dual to ϕ^2. At the free level Fronsdal fields $s = 0, 1, 2, 3, ...$ obey

$$\left(-\nabla^2 + M_s^2\right)\left(\Phi^{\underline{\alpha}(s)} + \nabla^{\underline{\alpha}}\xi^{\underline{\alpha}(s-1)}\right) = 0\,, \qquad M_s^2 = (d+s-2)(s-2) - s\,,$$

where $\xi^{\underline{\alpha}(s-1)}$ represent the gauge parameters. The "mass-like term" is determined by the conformal weight of the conserved HS current it is dual to.

- TYPE B HST: Spinor singleton(doubleton) describes a massless fermion $\partial\!\!\!/\psi = 0 \leftrightarrow Di$. In a free theory of Di's the spectrum of single-trace operators have the symmetry of all hook Young diagrams $\mathbb{Y}(s, 1^p)$ with a single column and a single row:

  $$J_{s,p} = J_{a_1 \ldots a_s, m_1 \ldots m_p} = \bar{\psi}\gamma_{a_s m_1 \ldots m_p}\partial_{a_1 \ldots a_{s-1}}\psi + \ldots$$

  The spectrum of single-trace operators is given by the tensor product $Di \otimes Di$:

  $$Di \otimes Di = \sum_{s,p} J_{s,p}\,.$$

  Hence the spectrum of the Type-B theory consist of mixed-symmetry gauge fields with spin $\mathbb{Y}(s, 1^p)$, $s > 1, \forall p$ or $s = 1, p = 0$:

  $$\left(-\nabla^2 + M_{s,1^p}^2\right)\left(\Phi^{\underline{a}(s),\underline{m}[p]} + \nabla^{\underline{a}}\xi^{\underline{a}(s-1),\underline{m}[p]} + \ldots\right) = 0\,,$$
  $$M_{s,1^p}^2 = (d+s-2)(s-2) - s - p\,.$$

SUSY HS Theories: The simplest super-symmetric HS theories are dual to CFT's constructed out of a number of free scalars (Rac) and fermions (Di). The single-trace operators include those of Type-A and Type-B as well as the super-currents:

$$J_{s=m+\frac{1}{2}} = \phi\partial^m\psi + \dots \qquad \Longleftrightarrow \qquad J_{a(m);\alpha} = \phi\partial_{a_1}\dots\partial_{a_m}\psi_\alpha + \dots .$$

The super-currents correspond to representations in $Di \otimes Rac$:

$$Di \otimes Rac = \sum_{m=0} J_{s=m+\frac{1}{2}} .$$

The super-currents are dual to totally-symmetric spinorial HS fields (Fang-Fronsdal fields):

$$(\nabla\!\!\!/ + m)\left(\Phi^{\underline{a}(s);\underline{\alpha}} + \nabla^{\underline{a}}\xi^{\underline{a}(s-1);\underline{\alpha}}\right) = 0 , \qquad m^2 = -\left(s + \tfrac{d-4}{2}\right)^2 .$$

AdS/*CFT* implies a mapping between the correlation function of the conserved currents $\langle J\,J\,J \dots J\rangle$ and the interaction vertices in the dual HST theory in $AdS_{(d+1)}$ $\Longrightarrow$ constructive approach to HST.

- For Type A theories nonlinear Vasiliev's equations are known for any d.
- For Type B Vasiliev's equations are not known for $d > 3$.
- Note in $d = 3$ higher spin gauge fields are pure symmetric tensors in both Type A and Type B theories
- Dimension $d = 5$ i.e AdS_6 is very special since the only massless $5d$ conformal fields are the scalar and spinor singletons and Type B theory has mixed symmetry gauge fields and there exists a unique conformal superalgebra $F(4)$ in AdS_6 .
- For even d there exit infinitely many massless conformal fields (doubletons) and supersymmetric extensions exist for any number of susy generators in $d = 4, 6$.
- Tensor product of singletons ($d = 2n + 1$) and doubletons ($d = 2n$) decompose into massless representations in $AdS_{(d+1)}$

One loop tests of AdS/CFT duality in HSTs:

The AdS partition function

$$Z_{AdS} = \int \prod_k D\Phi_k \, e^{S[\Phi_s]} \,,$$

G is the bulk coupling constant leads to the following expansion of the free energy F_{AdS}:

$$-\ln Z_{AdS} = F_{AdS} = \frac{1}{G} F^0_{AdS} + F^1_{AdS} + G F^2_{AdS} + \ldots \,,$$

where the first term is the classical action evaluated at an extremum. F^1 stands for one-loop corrections, etc. The large-N counting implies that $G^{-1} \sim N$. On the dual CFT side we have

$$-\ln Z_{CFT} = F_{CFT} = N F^0_{CFT} + F^1_{CFT} + \frac{1}{N} F^2_{CFT} + \ldots \,.$$

For a free CFT only the first term is non-zero, which should match F^0_{AdS}. Since the full classical action is not known, we do not know how to compute F^0_{AdS} and compare it to F^0_{CFT}. AdS/CFT requires that the second term, F^1_{AdS} either vanishes identically or produces a contribution proportional to F^0_{CFT}, that modifies the relation $G^{-1} = N$ to be of the form $G^{-1} = a(N + \text{integer})$

Giombi, Klebanov, Safdi, Tseytlin, Beccaria, Joung, Lal, Bekaert, Boulanger,....
MG, Skvortsov, Tran (2016) ; Giombi, Klebanov, Tan (2016).

HST in AdS_6 based on the Superalgebra $F(4) \supset SO(5,2) \oplus SU(2)$

MG, Skvortsov, Tran (2016)

F(4) supersingleton consist of two scalar singletons ($Rac = D(3/2;(0,0)_D)$ in doublet of $SU(2)_R$ and one spinor singleton ($Di = D(2;(0,1)_D)$

The tensor product of two $F(4)$ super-singletons contains an infinitely many AdS_6 massless $F(4)$ supermultiplets that have higher-spin fields which extend the Romans graviton supermultiplet in AdS_6 (Roman's Tower):

Scalar tower:	$D(3+s;(s,0)_D)$	$\boxed{\quad s \quad}$
Spinor tower:	$D^r(7/2+s;(s,1)_D)$	$\boxed{\quad s \quad}_{\frac{1}{2}}$
Tensor field tower:	$D(4+s;(s,2)_D)$	$\begin{array}{l}\boxed{\quad s+1 \quad}\\ \boxed{}\end{array}$
Vector field tower:	$D^a(4+s;(s+1,0)_D)$	$\boxed{\quad s+1 \quad}$
Gravitino tower:	$D^r(9/2+s;(s+1,1)_D)$	$\boxed{\quad s+1 \quad}_{\frac{1}{2}}$
Graviton tower:	$D(5+s;(s+2,0)_D)$	$\boxed{\quad s+2 \quad}$

where $r = 1,2$ and $a = 1,2,3$ are the spinor and adjoint indices of $SU(2)_R$ symmetry and $s = 0,1,2,....$ For each s they describe an irreducible unitary supermultiplet of $F(4)$.

$$[F(4) \text{ Super Singleton}]^2 = \text{Romans Tower} \oplus L(8|8)\,,$$

$$L(8|8) = D(4;(1,0)_D) \oplus D^r(7/2;(0,1)_D) \oplus D^a(3;(0,0)) \oplus D(4;(0,0))\,,$$

$$L(8|8) = A_\mu \oplus \chi^r \oplus \phi^a \oplus \phi$$

- The full spectrum of the $F(4)$ HST obtained by squaring the $F(4)$ super-singleton $(2Rac \oplus Di)$ passes the one-loop tests (modulo a known puzzle in type B theories of all even dimensional AdS spaces) . The contribution from the short supermultiplet $L(8|8)$ is critical to pass the one loop tests. Hence AdS/CFT requires that in a consistent formulation of the full nonlinear $F(4)$ HS theory the Romans tower must be coupled to the fields of the $L(8|8)$ supermultiplet. The supermultiplet $L(8|8)$ is the linear supermultiplet that plays an important role in conformal supergravity in $5d$.
- The graviton supermultiplet as obtained by the quasiconformal approach involves a massive anti-symmetric tensor which sits at the bottom of an infinite tower of mixed symmetry fields $\begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{\qquad s+1 \qquad} \\ \hline \ \ & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$.

  The fields of Romans gauged supergravity are e_μ^m, ψ_μ^I, A_μ^a, a_μ, $B_{\mu\nu}$, ξ_i, ϕ The field $B_{\mu\nu}$ becomes massive by "eating" a_μ in a Higgslike mechanism and in the supersymmetric ground state $g = 3m$ where g is the $SU(2)_R$ gauge coupling constant.

Further comments and open problems

- The above results show that there is a one-to-one correspondence between massless conformal fields and massless conformal supermultiplets and higher spin algebras and their deformations and supersymmetric extensions in all dimensions ($d > 2$). As such they extend the results of Maldacena and Zhiboedov about the duality between free conformal field theories in $d = 3$ and higher spin theories in AdS_4 with unbroken higher spin symmetry to all the deformations of the $AdS_{(d+1)}/CFT_d$ higher spin theories and their susy extensions.
- The quasiconformal construction of the minrep and its deformations are non-linear , except for $d = 3$. This raises the question whether there exist interacting ,but integrable conformal field theories that are dual to unbroken higher spin gauge theories in higher dimensions.
- The quasiconformal construction of the minrep of $D(2, 1 : \alpha)$ and its deformations describe the spectra of $N = 8$ supersymmetric interacting quantum mechanical models obtained using harmonic superspace techniques by Fedoruk, Ivanov and Lechtenfeld,... Govil & MG
- Question: could these interacting yet integrable theories correspond to some dimensionally reduced CFTs ?
- Reformulation of nonlinear quasiconformal realizations of higher spin algebras and their supersymmetric extensions in terms of covariant fields.
- Reformulation of Vasiliev's equations for Type A theories within the quasiconformal approach. MG, E. Skvortsov

THANK YOU