Geometric formulation of scattering amplitudes in N=4 sYM

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in memoriam of Ioannis Bakas
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Outline

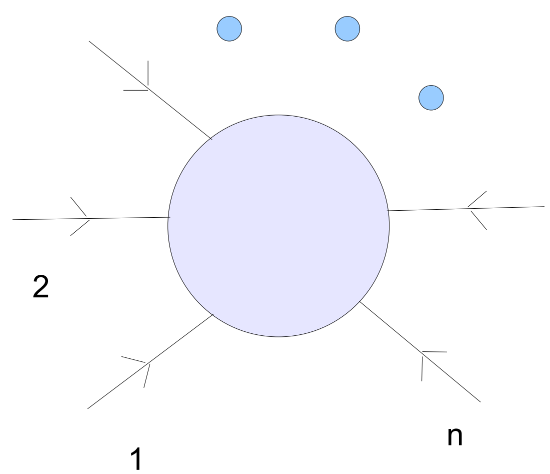
* Introduction
* Scattering amplitudes in N=4 sYM
* Symmetries of amplitudes in planar N=4
* Recent formulations for tree-level:
    * Grassmannian
    * Amplituhedron
* Symmetries of the amplituhedron:
    * New differential eqs
    * Volume at tree-level
* Open questions

Scattering amplitudes

Gauge theories are basis for every model of elementary particles

Scattering amplitudes: central objects in GTs describe interactions between particles

$\mathcal{A}_n =$



- huge number of diagrams
- high complexity already at two loops
- symmetries and good formalism can help

Scattering amplitudes

Scattering amplitudes: central objects in GTs
describe interactions between particles

QCD is the GT of strong interactions

Problems:
* at strong coupling
* at weak coupling

Can we use another theory?

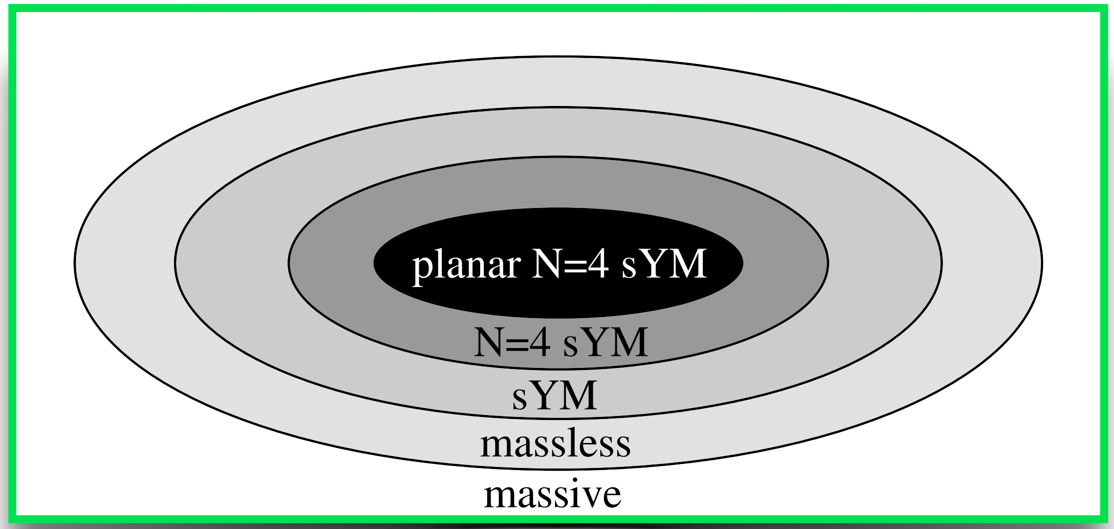
Scattering amplitudes

Scattering amplitudes: central objects in GTs

describe interactions between particles

QCD is the GT of strong interactions

Maximally supersymmetric Yang-Mills theory



(picture of L. Dixon)

Different Lagrangians but common properties

Scattering amplitudes in N=4 sYM

N=4 vs QCD

- at weak coupling it shares properties of QCD ampls but easier to compute

tree level: gluon ampls are the same

loop level: one-loop QCD=sum of susy

maximal transcendentality principle

- new computational methods can be transferred
- at high energies QCD -> a conformal limit
- at strong coupling AdS/CFT can be used

Scattering amplitudes in N=4 sYM

Scattering amplitudes: central objects in GTs
describe interactions between particles

N=4 sYM: interacting 4d QFT with
highest degree of symmetry

Features:

- ☑ scale invariant
- ☑ hidden symmetries in planar limit - integrable structure
- ☑ triality ampls/Wilson loops/correlation fcs
- ☑ AdS/CFT correspondence

Scattering amplitudes in N=4 sYM

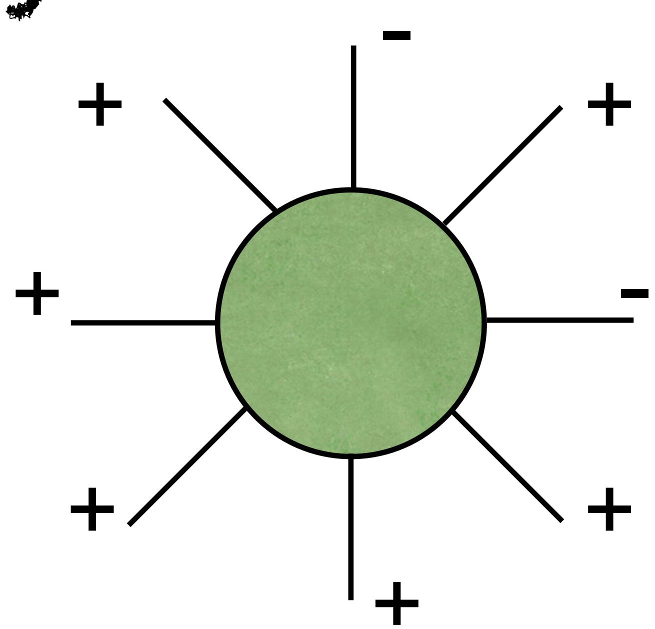
On-shell supermultiplet described by a superfield:

$$\Phi = G^+ + \eta^A \Gamma_A + \frac{1}{2!}\eta^A\eta^B S_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\overline{\Gamma}^D + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D \epsilon_{ABCD} G^-$$

* $p^2 = 0 \Longleftrightarrow p^{\alpha\dot\alpha} = \lambda^\alpha \tilde\lambda^{\dot\alpha}, \quad q^{\alpha A} = \lambda^\alpha \eta^A$

(Tree) Amplitudes labeled by two numbers:

* number of particles - n
* helicity - k



MHV tree level

[Parke-Taylor]

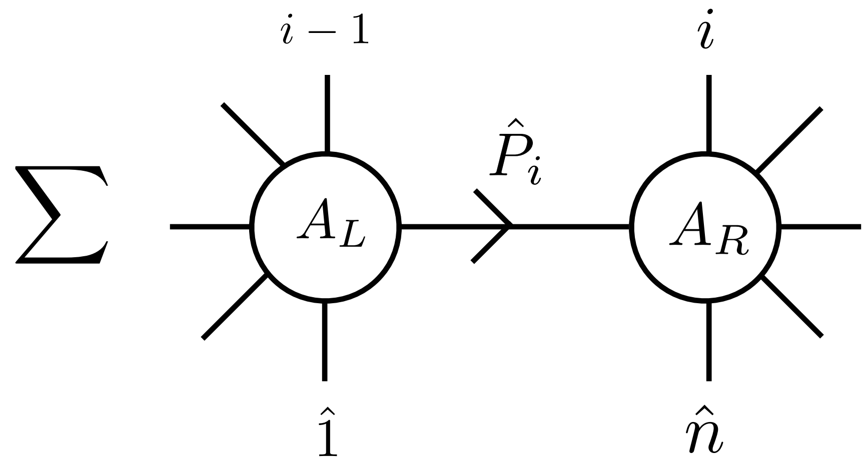
$$\mathcal{A}^{\text{tree}}_{n,2} = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle \dots \langle n1\rangle}, \qquad \langle \text{ij}\rangle = \lambda^\alpha_i \lambda_{j\alpha}$$

Scattering amplitudes in N=4 sYM

* On-shell methods

-> BCFW recursion relations:

$$\mathcal{A}_n = \sum_{i,h} A^{h}_{i-1} \frac{1}{P_i^2} A^{-h}_{n-i+1}$$



Example:

„R-invariants"

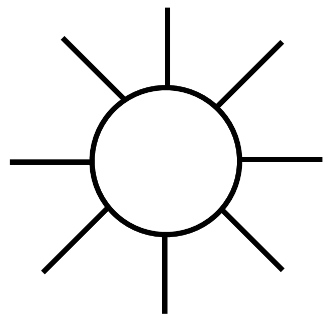
$$\mathcal{A}_n^{\mathrm{NMHV}} = \mathcal{A}_n^{\mathrm{MHV}} \sum_{j=2}^{n-3} \sum_{k=j+2}^{n-1} [n, j-1, j, k-1, k]$$

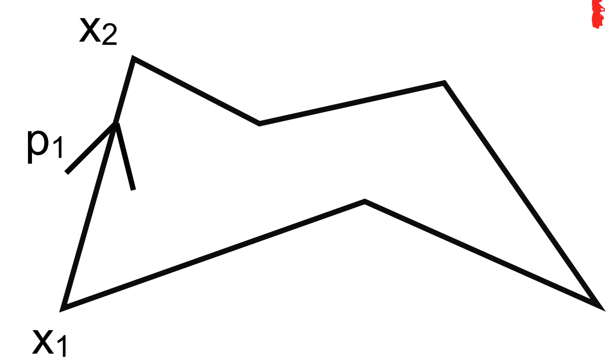
A_6^{NMHV}/A_6^{MHV} = [2, 3, 4, 6, 1] + [2, 3, 4, 5, 6] + [2, 4, 5, 6, 1]

= [3, 1, 6, 5, 4] + [3, 2, 1, 6, 5] + [3, 2, 1, 5, 4]

Scattering amplitudes in N=4 sYM

Amplitude duality $\longleftrightarrow$ **Wilson loop**





$$p_i^{\alpha\dot\alpha} = x_i^{\alpha\dot\alpha} - x_{i+1}^{\alpha\dot\alpha}$$
$$q_i^{\alpha A} = \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A}$$

on-shell superspace $\longleftrightarrow$ **dual superspace**

$(\lambda_i^\alpha, \tilde\lambda_i^{\dot\alpha}, \eta_i^A)$ $\qquad$ $(\lambda_i^\alpha, x_i^{\alpha\dot\alpha}, \theta_i^{\alpha A})$

Fourier transform on λ_i^α $\updownarrow$ $\qquad$ $\updownarrow$ Incidence relations

$$\tilde\mu_i^{\dot\alpha} := x_i^{\dot\alpha\alpha}\lambda_{i\alpha}$$
$$\chi_i^A := \theta_i^{\alpha A}\lambda_{i\alpha}$$

twistor superspace $\longleftrightarrow$ **momentum-twistor superspace**

$\mathcal{W}_i^{\mathcal{A}} = (\mu_i^\alpha, \tilde\lambda_i^{\dot\alpha}, \eta_i^A)$ $\qquad$ $\mathcal{Z}_i^{\mathcal{A}} = (\lambda_i^\alpha, \tilde\mu_i^{\dot\alpha}, \chi_i^A)$

Later on: bosonized

Symmetries and ampls in N=4 sYM

Important in discovering the characteristic of ampls

Superconformal symm. expected [Witten]

$$j_a \mathcal{A}_n^{\rm tree} = 0, \quad j_a \in \mathfrak{psu}(2,2|4)$$

Dual superconformal symm. (planar) hidden [DHKS]

$$J'_a \mathcal{A}_n^{\rm tree} = 0, \quad J'_a \in \mathfrak{psu}(2,2|4)^{\rm dual}$$

Yangian symmetry [Drummond,Henn Plefka]

infinite number of „levels" of generators

* level-zero generators: $[j_a, j_b] = f_{ab}{}^c j_c$ -> superconf. gens
* level-one generators: $[j_a, j_b^{(1)}] = f_{ab}{}^c j_c^{(1)}$ -> dual superconf. gens
* + Serre relations

The Yangian Y(g) of the Lie algebra g is generated by j and $j^{(1)}$

* tree-level collinear singularities
* broken at loop level

Grassmannian

(Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Cheung, Goncharov, Hodges, Kaplan,Postnikov,Trnka)(Mason,Skinner)

In momentum twistor space: $\mathcal{Z}_i^A = (\lambda_i^\alpha, \tilde{\mu}_i^{\dot{\alpha}}, \chi_i^A)$

$$A^{\text{tree}}_{n,\tilde{k}}(\mathcal{Z}) = \oint \frac{d^{\tilde{k}\times n}c}{(12\ldots\tilde{k})(23\ldots\tilde{k}+1)\ldots(n1\ldots n+\tilde{k}-1)} \prod_{\alpha=1}^{\tilde{k}} \delta^{4|4}\left(\sum_{a=1}^{n} c_{\alpha a}\mathcal{Z}_a\right)$$

* c's: complex parameters forming a $\tilde{k}$xn matrix
* $(i\ i{+}1\ldots i{+}\tilde{k}{-}1)$: determinant of $\tilde{k}$x$\tilde{k}$ submatrix of c's
* GL($\tilde{k}$) invariance
* space of $\tilde{k}$-planes in n dimensions = Gr($\tilde{k}$,n)
* Yangian invariant

Later on...

Bosonized momentum twistors

(picture of A. Gilmore)

Amplituhedron

Conjecture

the volume of the amplituhedron

=

amplitude in planar N=4 sYM

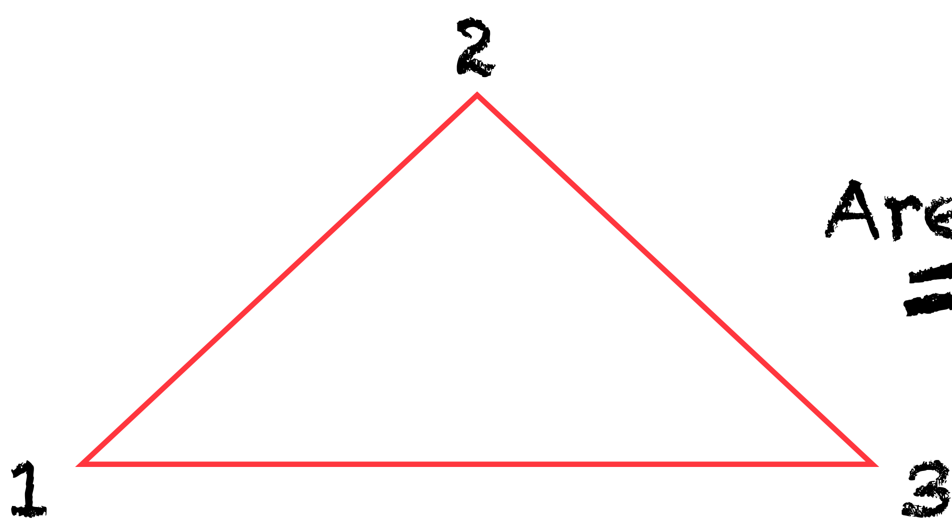
(N. Arkani-Hamed, J. Trnka)

The idea: NMHV tree-level amplitudes = volume of polytope in dual momentum twistor space (Hodges)

different triangulations = different recursion rel.s

Amplituhedron

The idea: area of triangle in 2d plane



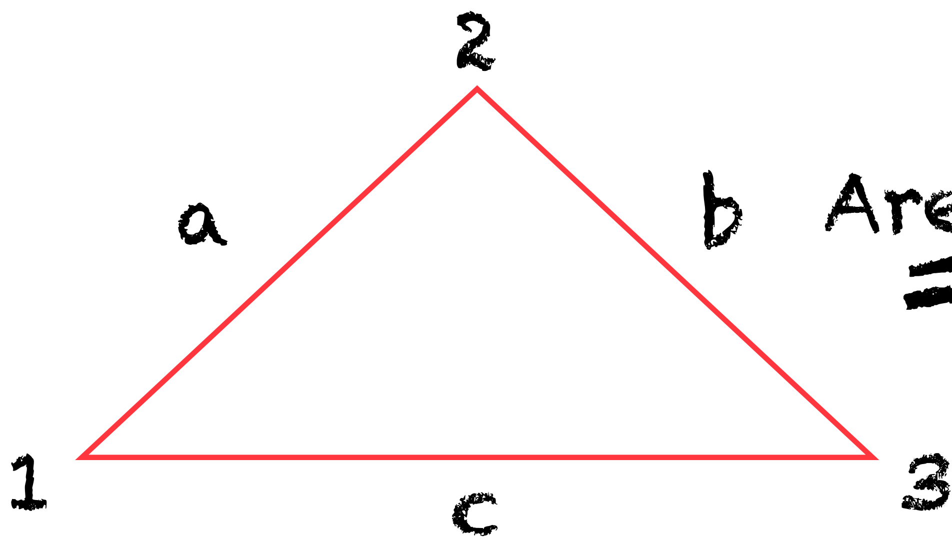
Area = $\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$ = $\frac{1}{2}\frac{\langle 123\rangle}{(Z_0\cdot W_1)(Z_0\cdot W_2)(Z_0\cdot W_3)}$

$$W_{iI} = \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} \quad Z_0^I = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \langle 123\rangle = \epsilon^{IJK} W_{1I} W_{2J} W_{3K}$$

$$I = 1, 2, 3$$

Amplituhedron

The idea: area of triangle in 2d plane



Area $= \frac{1}{2}\frac{\langle abc\rangle^2}{\langle 0bc\rangle\langle 0ab\rangle\langle 0ca\rangle} := [abc]$

„R-invariants"

Amplitude= area of the triangle in dual space

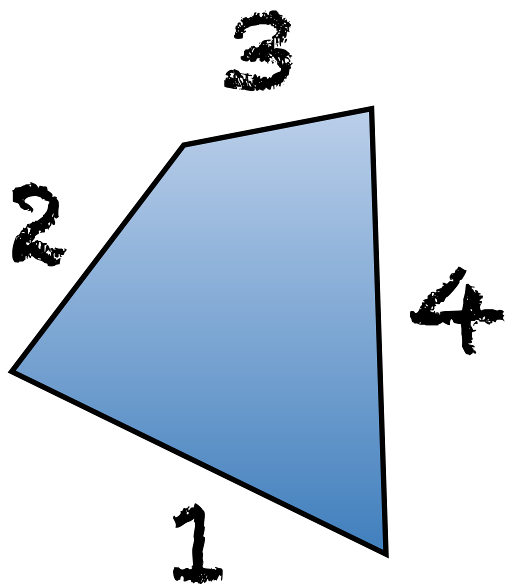
Lines in W-space → Points in Z-space

$$Z_a^I W_{1I} = Z_c^I W_{1I} = 0 \qquad W_1 = \langle *\; Z_c Z_a\rangle$$
$$Z_a^I W_{2I} = Z_b^I W_{2I} = 0 \quad \rightarrow \quad W_2 = \langle *\; Z_a Z_b\rangle$$
$$Z_b^I W_{3I} = Z_c^I W_{3I} = 0 \qquad W_3 = \langle *\; Z_b Z_c\rangle$$

$$\langle abc\rangle = \epsilon_{IJK} Z_a^I Z_b^J Z_c^K$$

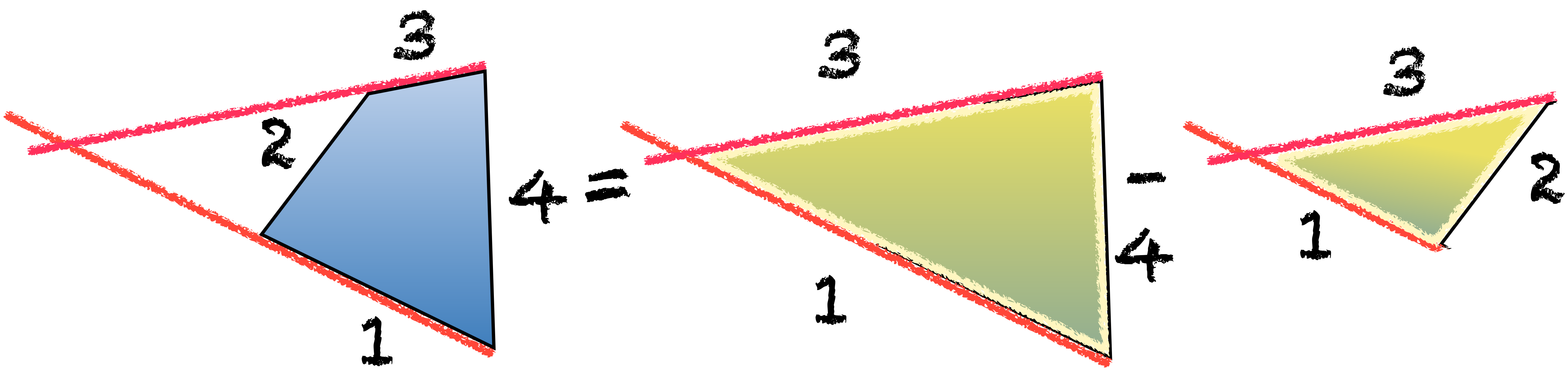
Amplituhedron

The idea: area of polytope in 2d plane



Amplituhedron

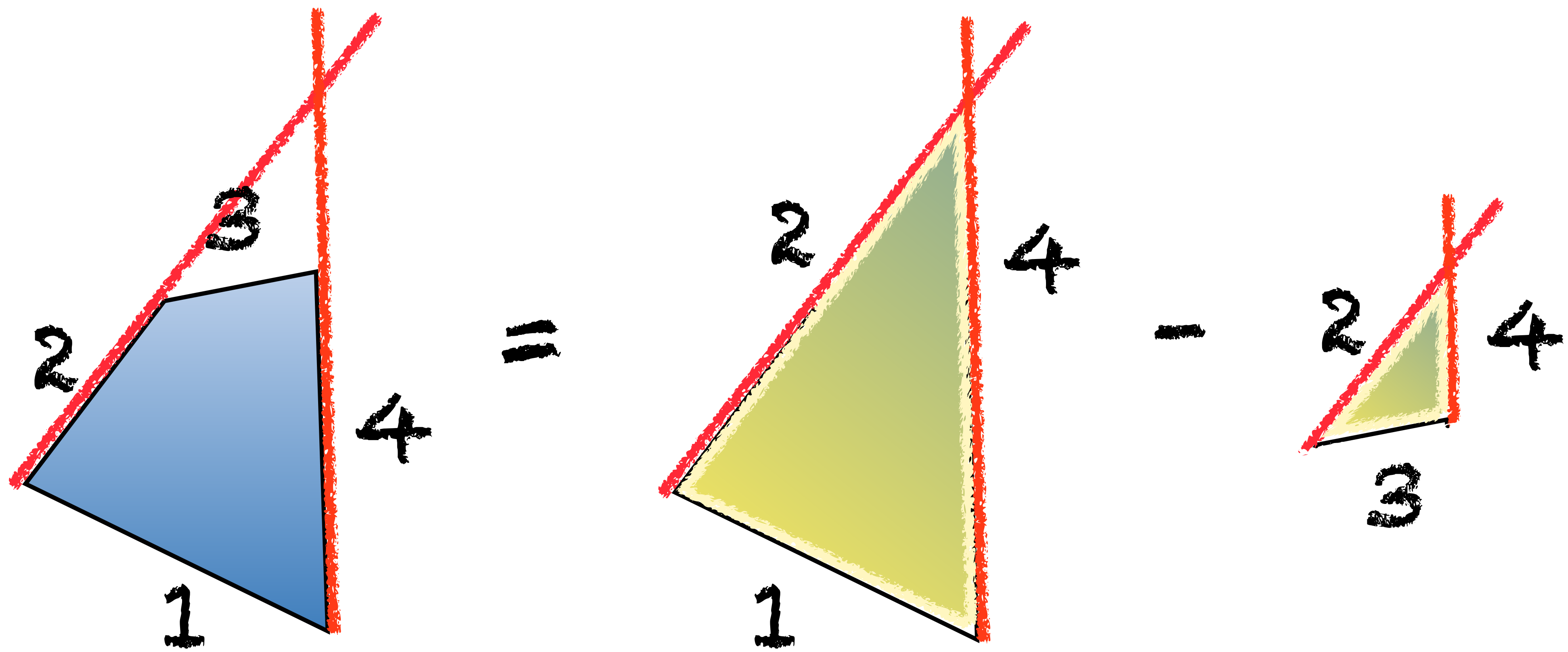
The idea: area of polytope in 2d plane



= [413] − [213]

Amplituhedron

The idea: area of polytope in 2d plane



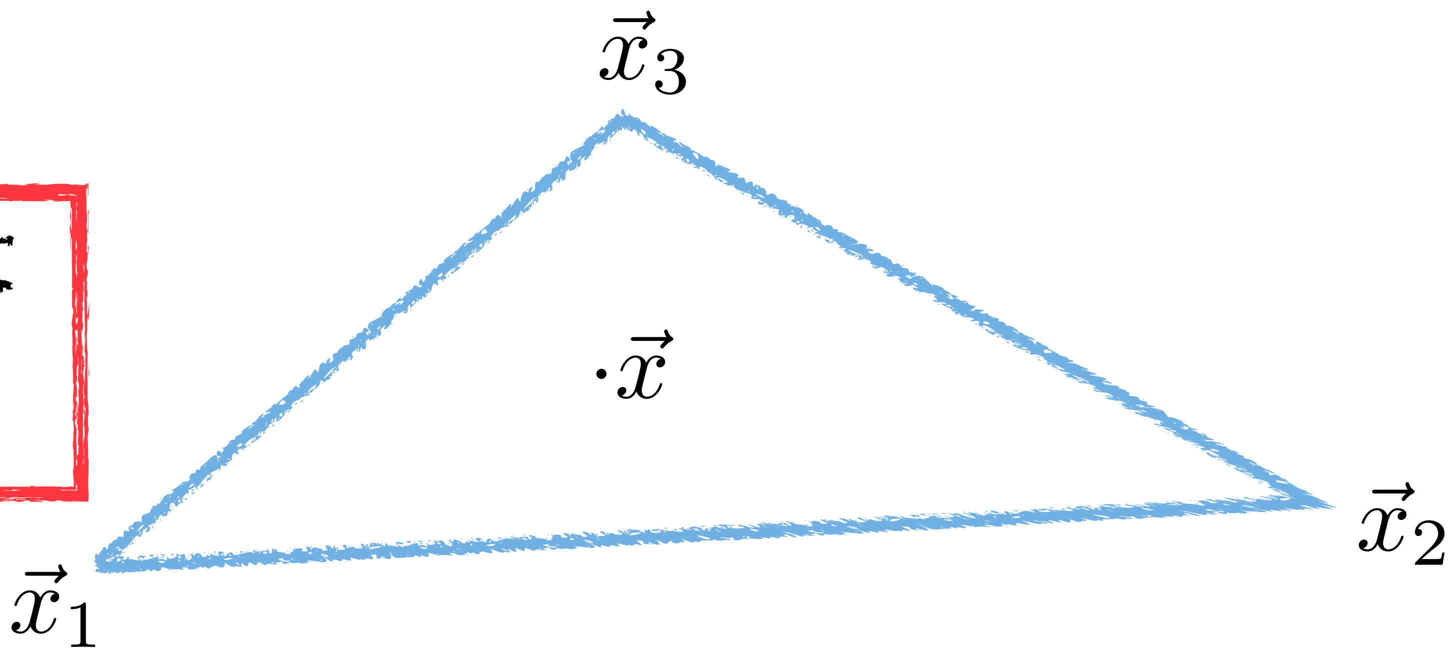
$$= [412] - [243]$$

$$= [413] - [213]$$

Amplituhedron

In a nutshell

center of mass of
3 points in 2d



$$\vec{x} = \frac{c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3}{c_1 + c_2 + c_3}$$

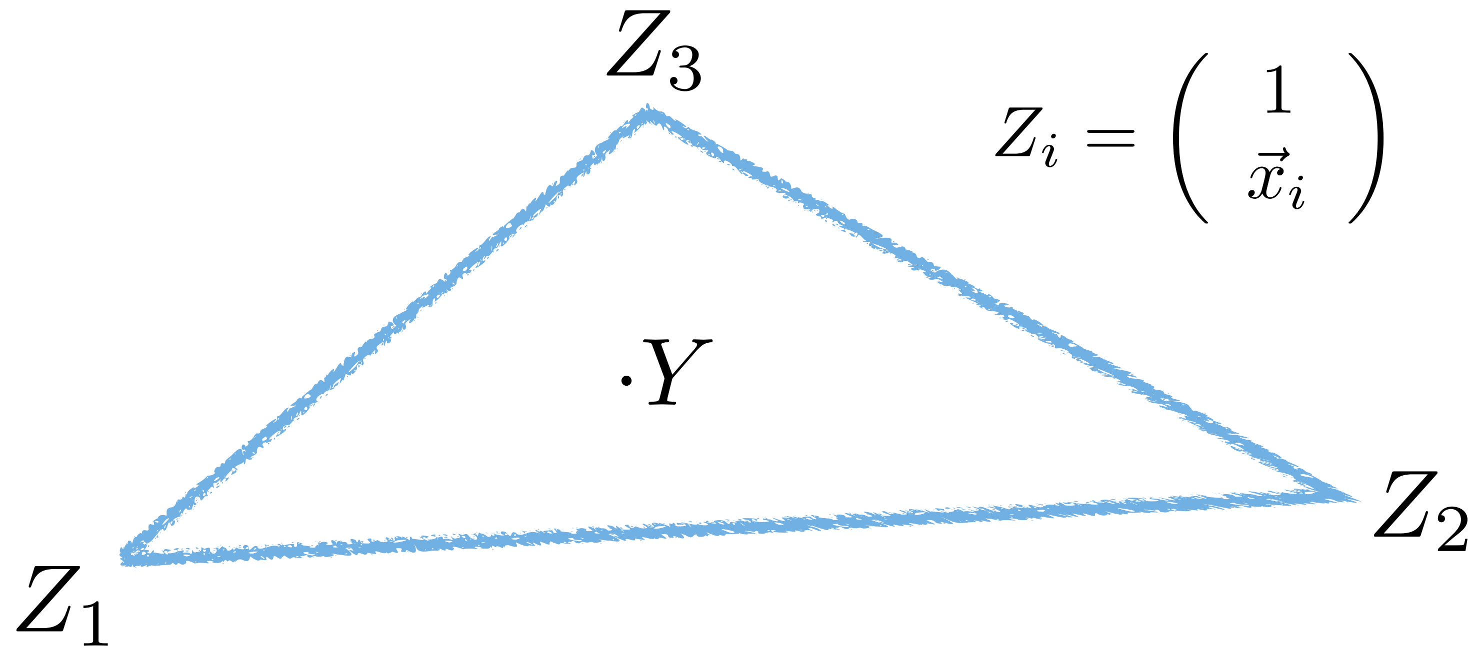
Interior: ranging over all positive c_1, c_2, c_3

Amplituhedron

In a nutshell

triangle in projective space

$$Z_i = \begin{pmatrix} 1 \\ \vec{x}_i \end{pmatrix}$$



$$Y^I = c_1 Z_1^I + c_2 Z_2^I + c_3 Z_3^I$$

$$I = 1, 2, 3$$

Interior: ranging over all positive c_1, c_2, c_3 (GL(1))

Amplituhedron

In a nutshell

Generalizations:

- triangle -> m-dim.l simplex $Y^I = \sum_{a=1}^{m+1} c_a Z_a^I$ $I = 1, 2, ..., m+1$

↓

space of k-planes in (k+m) dimensions $Y_\alpha^I = \sum_{a=1}^{k+m} c_{\alpha a} Z_a^I$ $I = 1, 2, ..., k+m$

↓

more vertices than (dims+1)

„Internal" positivity = interior

Amplituhedron

In a nutshell

Generalizations:

- most general:

$$Y_\alpha^I = \sum_{a=1}^{n} c_{\alpha a} Z_a^I \qquad I = 1, 2, ..., k+m$$

Geometric requirements {
„Internal" positivity (c) = interior
„External" positivity (Z) = convexity

↖ ordered minors > 0

Amplituhedron

In a nutshell

Generalizations:

Positive grassmannian -> tree amplituhedron $\Omega_{n,k}$

$$Y_\alpha^I = \sum_{a=1}^{n} c_{\alpha a} Z_a^I$$

$G(k, k+m)$ → Y_α^I

$G^+(k, n)$ → $c_{\alpha a}$

$G^+(k+m, n)$ → Z_a^I

Amplituhedron coords : Bosonized momentum twistors

$$Z = \begin{pmatrix} z_a \\ \varphi_1^A \chi_{aA} \\ \vdots \\ \varphi_k^A \chi_{aA} \end{pmatrix}$$

Amplituhedron

In a nutshell

Generalizations:

Positive grassmannian -> tree amplituhedron $\Omega_{n,k}$

$$Y_\alpha^I = \sum_{a=1}^{n} c_{\alpha a} Z_a^I$$

$G(k, k+m)$ → Y_α^I; $G^+(k, n)$ → $c_{\alpha a}$; $G^+(k+m, n)$ → Z_a^I

- physics: m=4
- tree: k=1 polytope, k>1 more complicated object
- loops: similar, more complicated formulae

Amplituhedron

(LF, T. Lukowski, A. Orta, M. Parisi)

$$A^{\text{tree}}_{n,k}(\mathcal{Z}) = \int d^4\varphi_1 \ldots d^4\varphi_k \int \delta^{4k}(Y, Y_0) \underbrace{\int \frac{d^{k\times n}c}{(12\ldots k)(23\ldots k+1)\ldots(n1\ldots n+k-1)} \prod_{\alpha=1}^{k} \delta^{k+4}\Big(Y_\alpha - \sum_a c_{\alpha a} Z_a\Big)}_{\Omega^{\text{tree}}_{n,k}}$$

$$Y_0 = \begin{pmatrix} \mathbb{O}_{4\times k} \\ - - - \\ \mathbb{I}_{k\times k} \end{pmatrix}$$

Open questions we want to address

* how to compute volume
* how to perform the integral
* symmetries - Yangian?

Amplituhedron

$$A_{n,k}^{\text{tree}}(\mathcal{Z}) = \int d^{m}\varphi_1 \ldots d^{m}\varphi_k \int \delta^{mk}(Y, Y_0) \int \frac{d^{k\times n}c}{(12\ldots k)(23\ldots k+1)\ldots(n1\ldots n+k-1)} \prod_{\alpha=1}^{k} \delta^{k+m}(Y_\alpha - \sum_a c_{\alpha a} Z_a)$$

(the last factor, from $\int \frac{d^{k\times n}c}{\ldots}$ onwards, is marked by a brace as $\Omega_{n,k}^{(m)}$)

Symmetries of Ω

(LF, T. Lukowski, A. Orta, M. Parisi)

- ☆GL(m+k) covariance
- ☆GL(1) invariance for Z's and GL(k) covariance for Y's
- ☆Capelli differential equations

$$\det\left(\frac{\partial}{\partial W_{a_\mu}^{A_\nu}}\right)_{\substack{1\le\nu\le k+1\\ 1\le\mu\le k+1}} \Omega_{n,k}^{(m)}(Y,Z) = 0$$

Amplituhedron

$$A^{\text{tree}}_{n,k}(\mathcal{Z}) = \int d^{m}\varphi_1 \ldots d^{m}\varphi_k \int \delta^{mk}(Y, Y_0) \underbrace{\int \frac{d^{k\times n}c}{(12\ldots k)(23\ldots k+1)\ldots(n1\ldots n+k-1)} \prod_{\alpha=1}^{k} \delta^{k+m}\Big(Y_\alpha - \sum_a c_{\alpha a} Z_a\Big)}_{\Omega^{(m)}_{n,k}}$$

Symmetries of Ω

(LF, T. Lukowski, A. Orta, M. Parisi)

- ⭐GL(m+k) covariance
- ⭐GL(1) invariance for Z's and GL(k) covariance for Y's
- ⭐Capelli differential equations

Start with toy model: m=2

A toy model

→ m=2 instead of m=4

* k=1

$$\Omega(W) = \int \frac{d^n c}{c_1 c_2 \ldots c_n} \delta^3(Y - c \cdot Z)$$

$$W_i = Z_i, W_0 = Y$$

it satisfies:

$$\begin{cases} \frac{\partial^2 \Omega}{\partial W_i^A \partial W_j^B} = \frac{\partial^2 \Omega}{\partial W_i^B \partial W_j^A} & \text{~Yangian invariance} \\ \sum_{A=1}^3 W_j^A \frac{\partial \Omega}{\partial W_j^A} = \alpha_j \Omega & \text{homogeneity} \\ \sum_{j=0}^n W_j^A \frac{\partial \Omega}{\partial W_j^B} = -\delta_B^A \Omega & \text{invariance} \end{cases}$$

$$A, B = 1, 2, 3$$

$$i, j = 0, 1, ..., n$$

$$\alpha_0 = -3, \quad \alpha_1, ..., \alpha_n = 0$$

→ What if we start from the differential equations?

A toy model

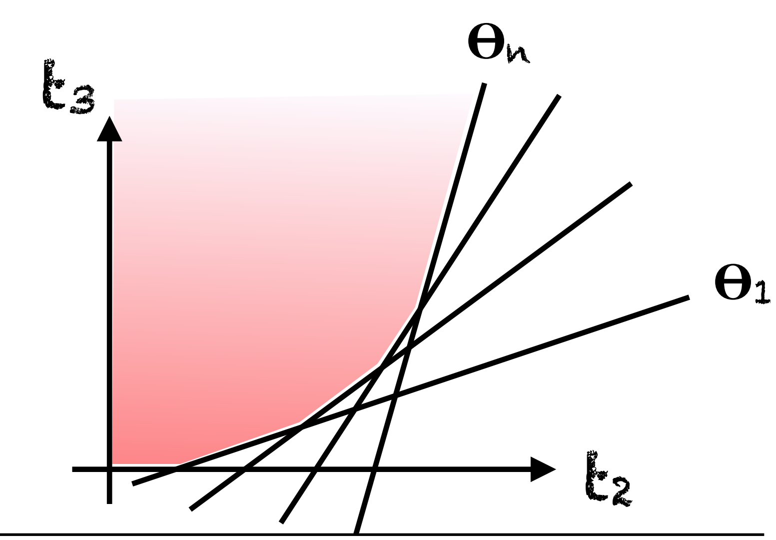
Solution for (m=2, k=1):
(following Gelfand, Graev, Retakh)

$$\Omega = \int_0^{+\infty} dt_2 dt_3 \frac{1}{(Y^1 + t_2 Y^2 + t_3 Y^3)^3} \prod_{i=4}^{n} \theta(Z_i^1 + t_2 Z_i^2 + t_3 Z_i^3)$$

t_2 and t_3: dual space

shape domain of integration with no singularities
$t \cdot Y = t \cdot (c_i Z_i) = c_i \ (t \cdot Z_i) > 0$

number of particles just in Θ-fc



-> Generalizes to generic m

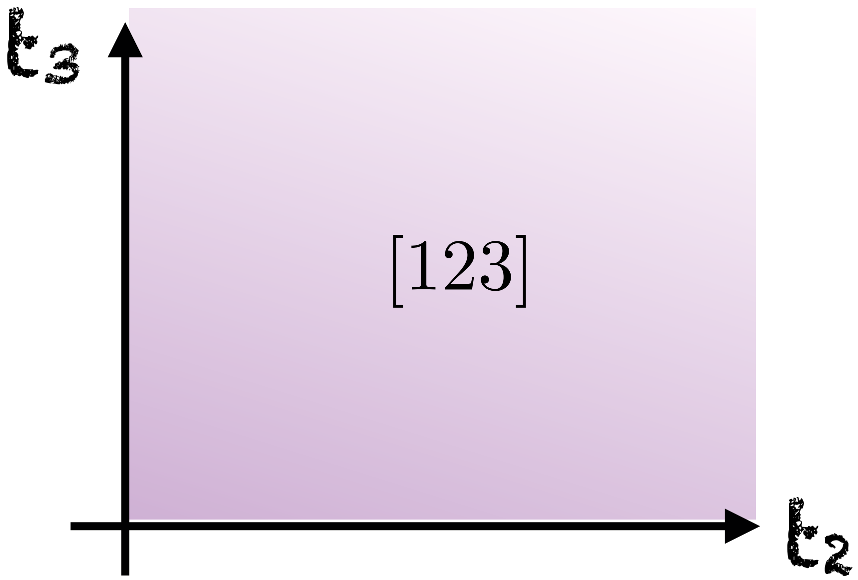
A toy model

Solution for (m=2, k=1):

$$\Omega = \int_0^{+\infty} dt_2 dt_3 \frac{1}{(Y^1 + t_2 Y^2 + t_3 Y^3)^3} \prod_{i=4}^{n} \theta(Z_i^1 + t_2 Z_i^2 + t_3 Z_i^3)$$

n=3

$$\Omega_3 = \int_0^{+\infty} dt_2 dt_3 \frac{1}{(Y^1 + t_2 Y^2 + t_3 Y^3)^3} = \frac{1}{Y^1 Y^2 Y^3} \equiv_{\text{gf}} \frac{\langle 123 \rangle^2}{\langle Y12 \rangle \langle Y23 \rangle \langle Y31 \rangle} = [123]$$



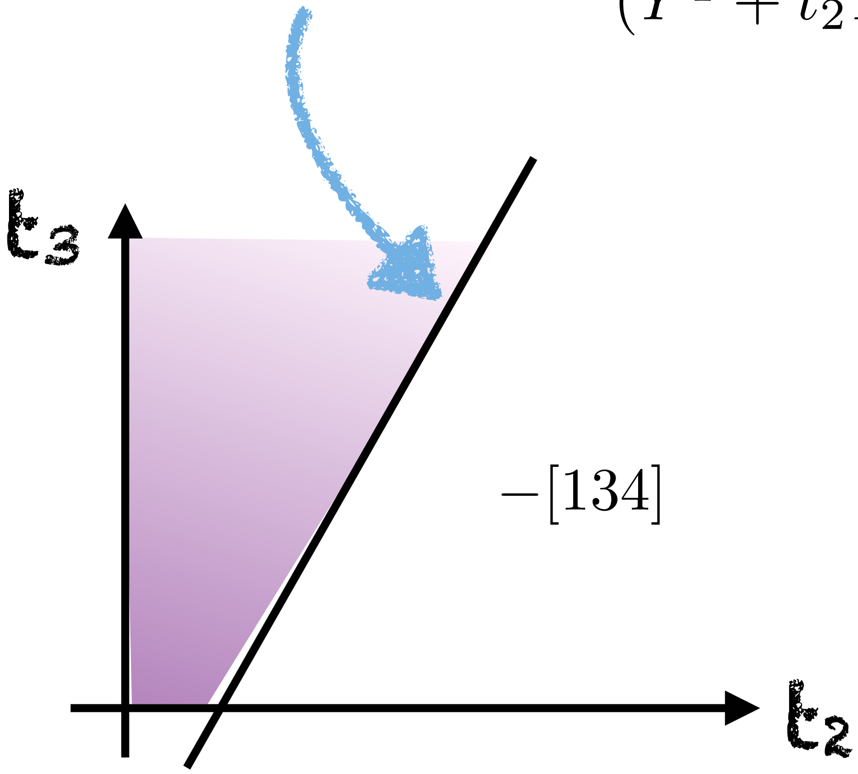
A toy model

Solution for (m=2, k=1):

$$\Omega = \int_0^{+\infty} dt_2 dt_3 \frac{1}{(Y^1 + t_2 Y^2 + t_3 Y^3)^3} \prod_{i=4}^{n} \theta(Z_i^1 + t_2 Z_i^2 + t_3 Z_i^3)$$

n=4

$$\Omega_4 = \int_0^{+\infty} dt_2 dt_3 \theta(Z_4^1 + t_2 Z_4^2 + t_3 Z_4^3) \frac{1}{(Y^1 + t_2 Y^2 + t_3 Y^3)^3}$$



$-[134]$

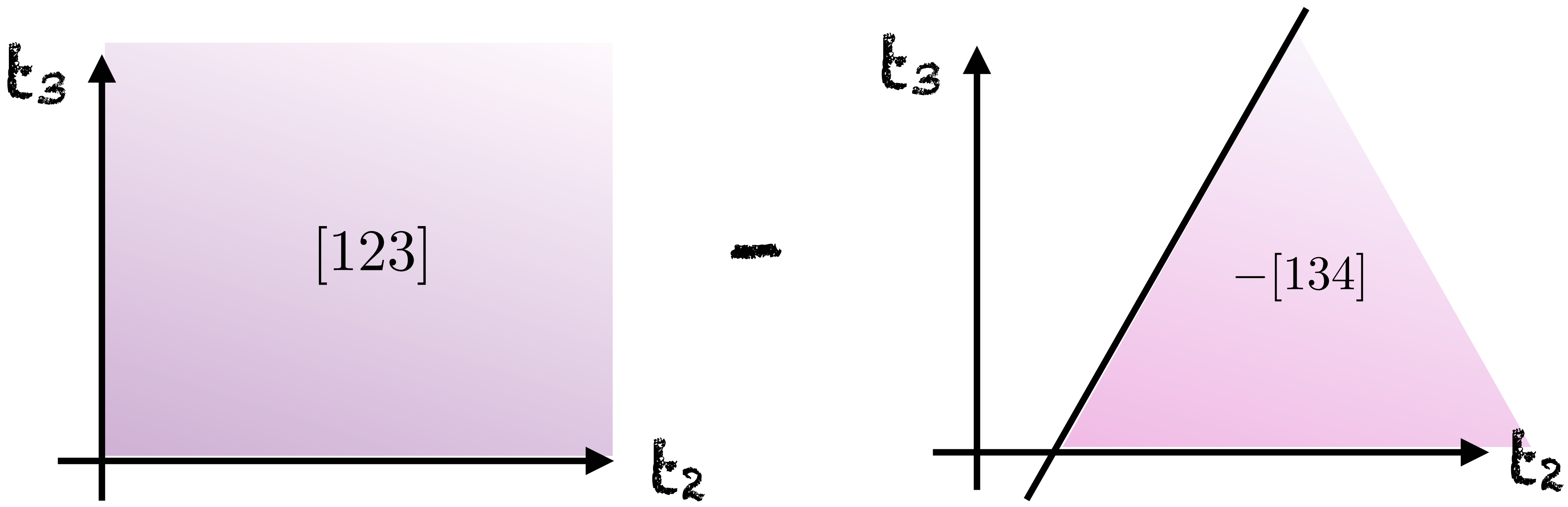
A toy model

Solution for (m=2, k=1):

$$\Omega = \int_0^{+\infty} dt_2 dt_3 \frac{1}{(Y^1 + t_2 Y^2 + t_3 Y^3)^3} \prod_{i=4}^{n} \theta(Z_i^1 + t_2 Z_i^2 + t_3 Z_i^3)$$

n=4

$$\Omega_4 = [123] + [134]$$



Extensions

* k=1, generic m

$$\Omega_{n,1}^{(m)} = \int_0^{+\infty} \left(\prod_{A=2}^{m+1} dt_A \right) \frac{m!}{(t \cdot Y)^{m+1}} \prod_{i=m+2}^{n} \theta\left(t \cdot Z_i\right)$$

- [x] m=4 <-> physics
- [x] no need to think about triangulation
- [x] directly in dual space

* Higher helicity

- [ ] integrand not fully fixed
- [ ] can Yangian symmetry help us?

General idea

In general:

Capelli differential equations:

$$\det\left(\frac{\partial}{\partial W_i^A}\right)\Omega_{n,k} = 0$$

+ invariance and scaling

Question: find a function satisfying the requirements

* Grassmannian integral satisfies the eqs for any k
* solve the diff. eqs directly in dual space
* no need to think about triangulation

Conclusions

Amplituhedron is conjectured to give a geometric interpretation of the amplitudes for planar N=4 sYM

- how to evaluate volume for k>1 and for loops?
- is Yangian symmetry preserved?
- do Capelli diff. eqs correspond to a symmetry?
- new recursion relations?

A lot of work still has to be done!