Arithmetic Quantum Gravity

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Based on work with:

Michael Koehn and Hermann Nicolai (AEI)

[arXiv:0907.3048][arXiv:0912.0854]

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Context and Plan

Minisuperspace models for quantum gravity and quantum cosmology [DeWitt 1967; Misner 1969]

Hidden symmetries in supergravity [Cremmer, Julia 1978; Damour, Henneaux 2000; Damour, Henneaux, Nicolai 2002; West 2001]

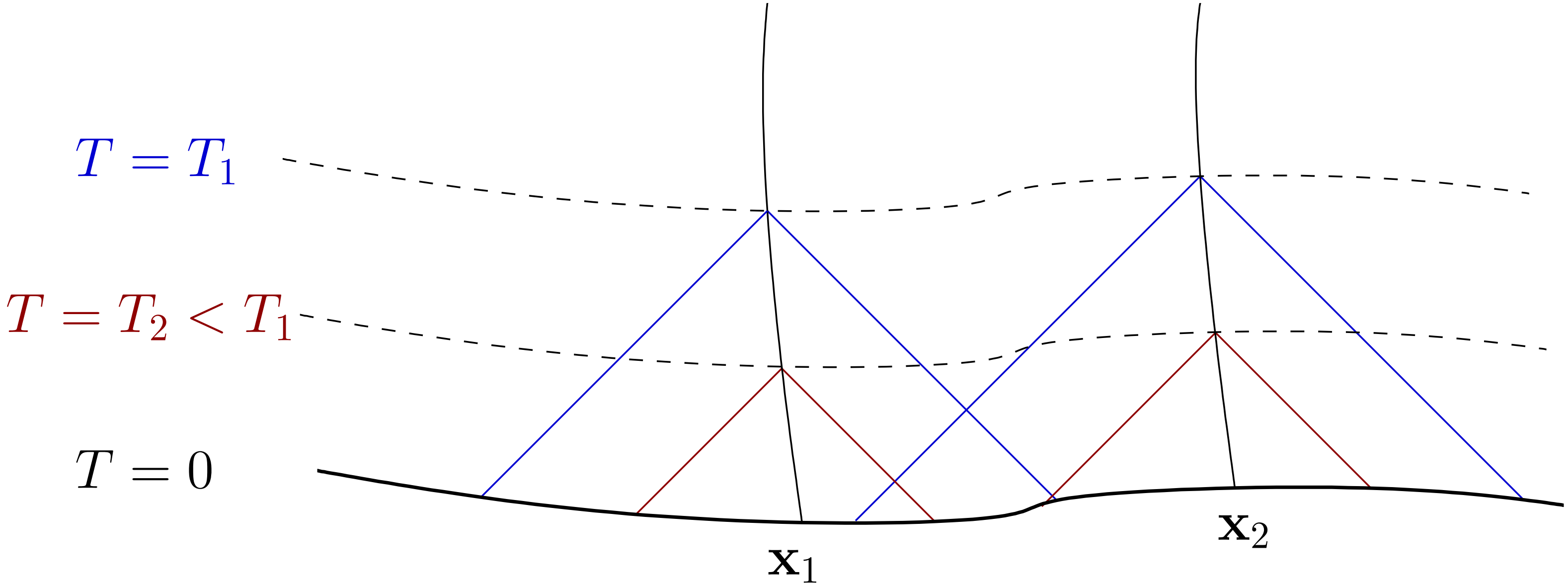
Plan

- Cosmological billiards
- Quantum cosmological billiards
- Arithmetic structure
- Interpretation
- Generalization and outlook

Cosmological billards: BKL

Supergravity dynamics near a space-like singularity simplify.

[Belinskii, Khalatnikov, Lifshitz 1970; Misner 1969; Chitre 1972]



Spatial points decouple $\Rightarrow$ dynamics becomes ultra-local.

Reduction of degress of freedom to spatial scale factors β^a

$$ds^2 = -N^2 dt^2 + \sum_{a=1}^{d} e^{-2\beta^a} dx_a^2 \qquad (t \sim -\log T)$$

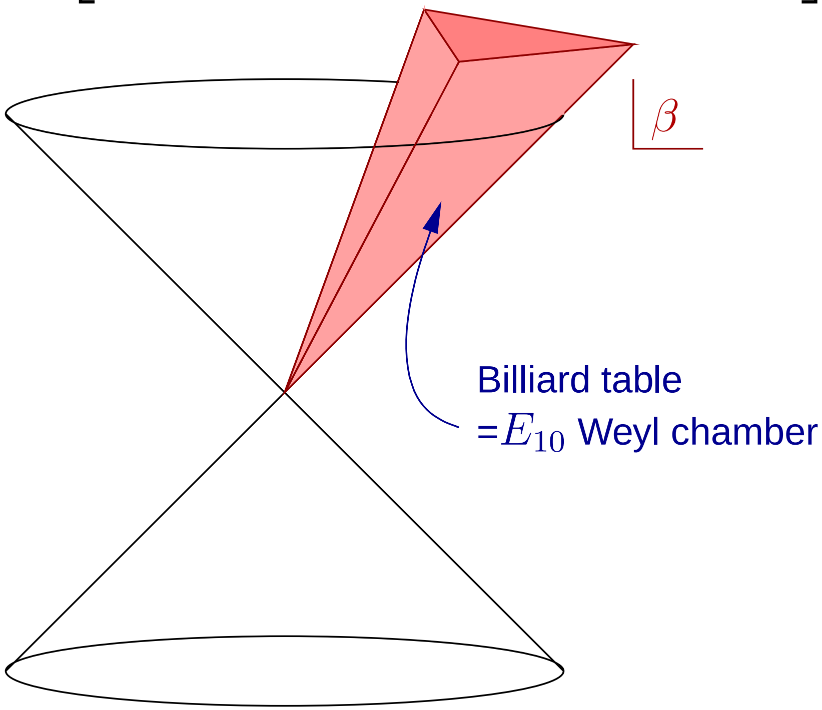
Cosmological billiards: Dynamics

Effective Lagrangian for $\beta^a(t)$ ($a = 1, \ldots, d$)

$$\mathcal{L} = \frac{1}{2} \sum_{a,b=1}^{d} n^{-1} G_{ab} \dot{\beta}^a \dot{\beta}^b + V_{\text{eff}}(\beta)$$

[G_{ab}: DeWitt metric (Lorentzian signature)]

Close to the singularity V_{eff} consists of infinite potentials walls, obstructing free null motion of β^a.



Cosmological billiards: Geometry

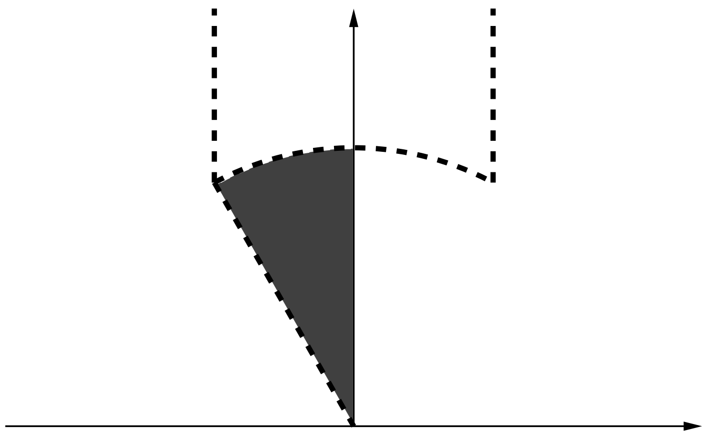
The E_{10} Weyl group $W(E_{10})$ is a discrete, arithmetic subgroup of $O(9,1;\mathbb{R})$: symmetries of the unique even self-dual lattice $\mathrm{II}_{9,1} = \Lambda_{E_8} \oplus \mathrm{II}_{1,1}$.

Norm-preserving $\Rightarrow$ restrict to hyperboloids. Tessellated by action of $W(E_{10})$.

Picture for ordinary gravity $W(E_{10}) \to W(AE_3) \cong PGL_2(\mathbb{Z})$.

Finite (hyperbolic) volume $\Rightarrow$ Chaos!

[Damour, Henneaux 2000; Damour et al. 2002]



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Quantum cosmological billiards

Setting $n = 1$ one has to quantize

$$\mathcal{L} = \frac{1}{2}\sum_{a,b=1}^{d} \dot\beta^a G_{ab} \dot\beta^b = \frac{1}{2}\left[\sum_{a=1}^{d} (\dot\beta^a)^2 - \left(\sum_{a=1}^{d} \dot\beta^a\right)^2\right]$$

with null constraint $\dot\beta^a G_{ab} \dot\beta^b = 0$ on billiard domain.

Canonical momenta: $\pi_a = G_{ab}\dot\beta^b \quad \Rightarrow \quad \mathcal{H} = \frac{1}{2}\pi_a G^{ab}\pi_b$.

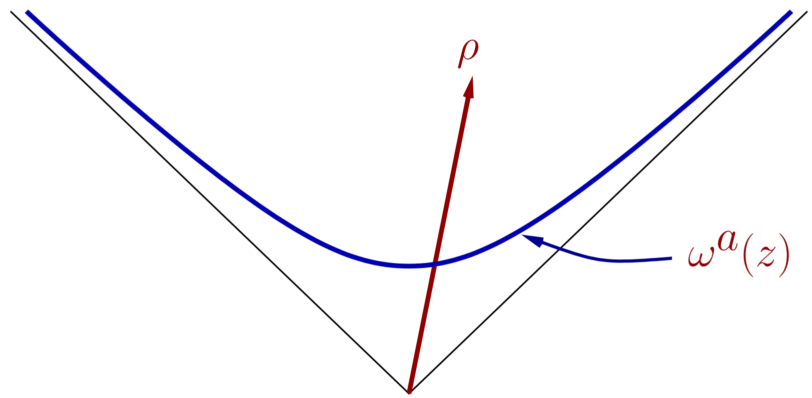
Wheeler–DeWitt (WDW) equation in canonical quantization

$$\mathcal{H}\Psi(\beta) = -\frac{1}{2}G^{ab}\partial_a\partial_b\Psi(\beta) = 0$$

Klein–Gordon ‘inner product’.

Quantum cosmological billiards (II)

Introduce new coordinates ρ and $\omega^a(z)$ from ‘radius’ and co-ordinates z on unit hyperboloid



$$\beta^a = \rho\omega^a \,, \qquad \omega^a G_{ab}\omega^b = -1$$
$$\rho^2 = -\beta^a G_{ab}\beta^b$$

Timeless WDW equation in these variables

$$\left[-\rho^{1-d}\frac{\partial}{\partial\rho}\left(\rho^{d-1}\frac{\partial}{\partial\rho}\right) + \rho^{-2}\Delta_{\text{LB}}\right]\Psi(\rho, z) = 0$$

Laplace–Beltrami operator on unit hyperboloid

Solving the WDW equation

$$\left[-\rho^{1-d}\frac{\partial}{\partial\rho}\left(\rho^{d-1}\frac{\partial}{\partial\rho}\right)+\rho^{-2}\Delta_{\mathsf{LB}}\right]\Psi(\rho,z)=0$$

Separation of variables: $\Psi(\rho,z)=R(\rho)F(z)$

For

$$-\Delta_{\mathsf{LB}}F(z)=EF(z)$$

get

$$R_{\pm}(\rho)=\rho^{-\frac{d-2}{2}\pm i\sqrt{E-\left(\frac{d-2}{2}\right)^2}}$$

[Positive frequency coming out of singularity is $R_-(\rho)$.]

Left with spectral problem on hyperbolic space.

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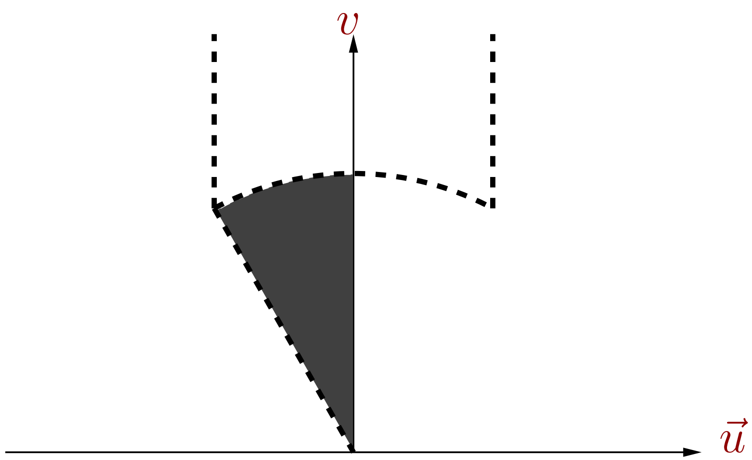
$\triangle_{\mathsf{LB}}$ and boundary conditions

The classical billiard ball is constrained to Weyl chamber with infinite potentials $\Rightarrow$ Dirichlet boundary conditions

Use upper half plane model

$$z = (\vec{u}, v)\,, \quad \vec{u} \in \mathbb{R}^{d-2}, v \in \mathbb{R}_{>0}$$

$$\Rightarrow \quad \Delta_{\mathsf{LB}} = v^{d-1}\partial_v(v^{3-d}\partial_v) + v^2\partial_{\vec{u}}^2$$



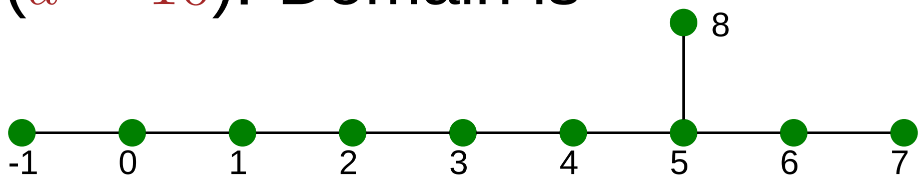
With *Dirichlet boundary conditions* ($d = 3$ in [Iwaniec])

$$-\Delta_{\mathsf{LB}}F(z) = EF(z) \quad \Rightarrow \quad E \geq \left(\frac{d-2}{2}\right)^2$$

Arithmetic structure (I)

Beyond general inequality details of spectrum depend on shape of domain. (‘Shape of the drum’ problem)

Focus on maximal supergravity ($d = 10$). Domain is determined by E_{10} Weyl group.



9-dimensional upper half plane with octonions: $u \equiv \vec{u} \in \mathbb{O}$

On $z = u + iv$ the ten fundamental Weyl reflections act by

$$w_{-1}(z) = \frac{1}{\bar{z}}\,, \;\; w_0(z) = -\theta\bar{z}\theta + \theta\,, \;\; w_j(z) = -\varepsilon_j\bar{z}\varepsilon_j$$

θ highest E_8 root; ε_j simple E_8 rts. [Feingold, AK, Nicolai 2008]

Arithmetic structure (II)

Iterated action of

$$w_{-1}(z) = \frac{1}{\bar{z}}\,,\;\; w_0(z) = -\theta\bar{z}\theta + \theta\,,\;\; w_j(z) = -\varepsilon_j\bar{z}\varepsilon_j$$

generates whole Weyl group $W(E_{10})$. No (very) simple octonionic representation of an arbitrary element known.

Restricting to the even Weyl group $W^+(E_{10})$ gives 'holomorphic' transformations and one obtains

$$W^+(E_{10}) = PSL_2(\mathtt{O})$$

that should be interpreted as a modular group over the integer 'octavians' $\mathtt{O}$. [Feingold, AK, Nicolai 2008]

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Modular wavefunctions (I)

Weyl reflections on wavefunction $\Psi(\rho, z)$

$$\Psi(\rho, w_I \cdot z) = \begin{cases} +\Psi(\rho, z) & \text{Neumann b.c.} \\ -\Psi(\rho, z) & \text{Dirichlet b.c.} \end{cases}$$

Use Weyl symmetry to *define* $\Psi(\rho, z)$ on the whole upper half plane, with Dirichlet boundary conditions $\Rightarrow$ $\Psi(\rho, z)$ is

- Sum of eigenfunctions of Δ_{LB} on UHP
- Invariant under action of $W^+(E_{10}) = PSL_2(\mathsf{O})$. Anti-invariant under extension to $W(E_{10})$.

$\Rightarrow$ Wavefunction is an odd Maass wave form of $PSL_2(\mathsf{O})$

[cf. [Forte 2008] for related ideas for Neumann conditions]

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Modular wavefunctions (II)

The spectrum of odd Maass wave forms is discrete but not known. For $PSL_2(\mathtt{O})$ the theory is not even developed (but see [Krieg]).

For lower dimensional cases like pure $(3+1)$-dimensional Einstein gravity with $PSL_2(\mathbb{Z})$ there are many numerical investigations. [Graham, Szépfalusy 1990; Steil 1994; Then 2003]

The result relevant here later is the inequality $E \geq \left(\frac{d-2}{2}\right)^2$.

Summary of analysis so far:

Quantum billiard wavefunction $\Psi(\rho, z)$ is an odd Maass wave form (Dirichlet b.c.) for $PSL_2(\mathtt{O})$.

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Interpretation (I)

'Wavefunction of the universe' in this set-up formally

$$|\Psi_{\mathsf{full}}\rangle = \prod_{\mathbf{x}} |\Psi_{\mathbf{x}}\rangle$$

Product of quantum cosmological billiard wavefunctions, one for each spatial point (ultra-locality). [Also [Kirillov 1995]]

Each factor contains a Maass wave form of the type $\Psi_{\mathbf{x}}(\rho, z) = \sum R_{\pm}(\rho) F(z)$ with

$$-\Delta_{\mathsf{LB}} F(z) = E F(z)\,, \quad R_{\pm}(\rho) = \rho^{-\frac{d-2}{2} \pm i\sqrt{E - \left(\frac{d-2}{2}\right)^2}}$$

Since $E \geq \left(\frac{d-2}{2}\right)^2$: $\Psi_{\mathbf{x}}(\rho, z) \to 0$ for $\rho \to \infty$

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Interpretation (II)

- Absence of potential: $\exists$ a well-defined Hilbert space with positive definite metric.
- Complexity and notion of positive frequency $\Rightarrow$ Arrow of time? [Isham 1991; Barbour 1993]
- The wavefunction vanishes at the singularity!
- But it remains oscillating and complex. It cannot be continued analytically past the singularity.
- Vanishing wavefunctions on singular geometries are one possible boundary condition. [DeWitt 1967]
- No way of going through the singularity. No bounce.
- ‘Semi-classical’ states are expected to spread (quantum ergodicity). [Non-relativistic intuition]

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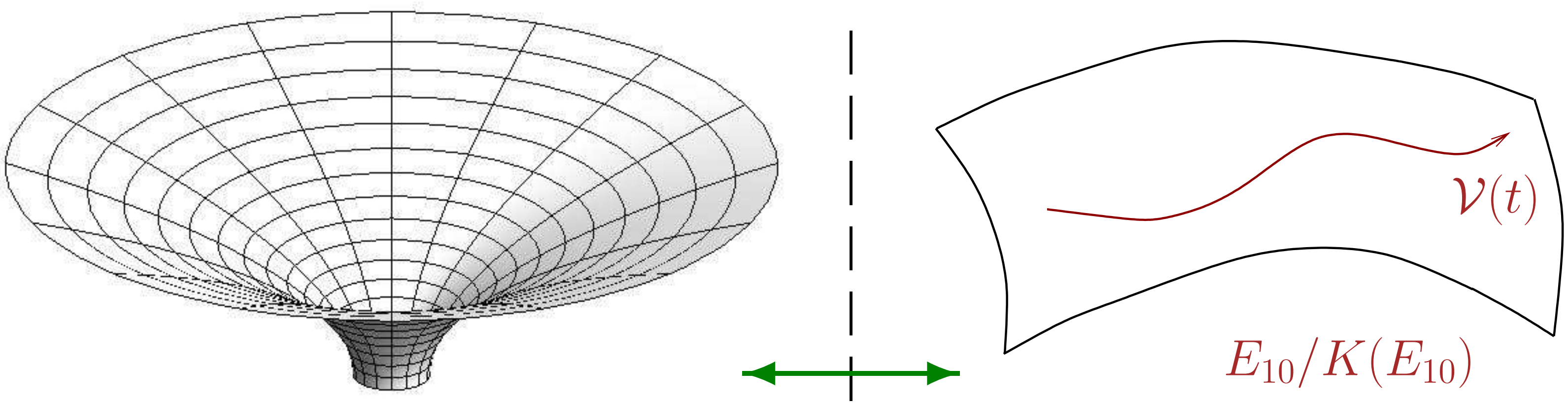
Generalization (I)

Classical cosmological billiards led to the E_{10} conjecture.

$D = 11$ supergravity can be mapped to a constrained null geodesic motion on infinite-dimensional $E_{10}/K(E_{10})$ coset space. [Damour, Henneaux, Nicolai 2002]



Correspondence

Symmetric space $E_{10}/K(E_{10})$ has $10 + \infty$ many directions.

Cartan subalgebra   pos. step operators

Generalization (II)

Features of the conjectured E_{10} correspondence

- Billiard corresponds to 10 Cartan subalgebra generators
- ∞ many step operators to remaining fields and spatial dependence. [Verified only at low 'levels' but for many different models]
- Space dependence introduced via *dual fields* (cf. Geroch group) — everything in terms of kinetic terms
- Space (de-)emergent via an algebraic mechanism
- Extension to E_{10} overcomes ultra-locality

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Generalization (III)

$$\mathcal{H}_{\text{Bill}} \to \mathcal{H} \equiv \mathcal{H}_{\text{Bill}} + \sum_{\alpha\in\Delta_+(E_{10})} e^{-2\alpha(\beta)} \sum_{s=1}^{\text{mult}(\alpha)} \Pi^2_{\alpha,s}$$

is the unique quadratic E_{10} Casimir. Formally like free Klein–Gordon; positive norm could remain consistent?

For the full theory there are more constraints than the Hamiltonian constraint $\mathcal{H}\Psi = 0$: diff, Gauss, etc.

- Global E_{10} symmetry provides ∞ conserved charges $\mathcal{J}$
- Evidence that constraints can be written as bilinears $\mathfrak{L} \sim \mathcal{J}\mathcal{J}$. [Damour, AK, Nicolai 2007; 2009]
- Analogy with affine Sugawara construction. Particularly useful for implementation as quantum constraints?

Aim: Quantize geodesic model. $E_{10}(\mathbb{Z})$ [Ganor 1999]?

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Supersymmetric extension (I)

$D = 11$ supergravity gravitino ψ_μ can be added to billiard analysis via $K(E_{10})$ representation. Work in supersymmetry gauge [Damour, AK, Nicolai 2005; de Buyl, Henneaux, Paulot 2005]

$$\psi_t = \Gamma_t \sum_{a=1}^{10} \Gamma^a \psi_a$$

Classically, separate billiard motion [Damour, Hillmann 2009]. Best in variable ($\Gamma_* = \Gamma^1 \cdots \Gamma^{10}$)

$$\varphi^a = g^{1/4} \Gamma_* \Gamma^a \psi^a \qquad \text{(no sum on } a\text{)}$$

Canonical Dirac bracket: $\left\{ \varphi^a_\alpha, \varphi^b_\beta \right\} = -iG^{ab} \delta_{\alpha\beta}$

Supersymmetric extension (II)

Quantize Clifford algebra using canonical anticommutators over a 2^{160}-dimensional Fock space vacuum $|\Omega\rangle$.

Have to implement supersymmetry constraint in quantum theory

$$\mathcal{S}_\alpha = i \sum_{a=1}^{10} \pi_a \varphi_\alpha^a \qquad (\alpha = 1, \dots, 32)$$

It obeys: $\{\mathcal{S}_\alpha, \mathcal{S}_\beta\} = \delta_{\alpha\beta}\mathcal{H}$ [Teitelboim 1977]

For quantum constraint choose 16 annihilation operators $\mathcal{S}_A$.

The state $|\Psi\rangle = \prod_{A=1}^{16} \mathcal{S}_A^\dagger \left(\Phi(\rho, z)|\Omega\rangle\right)$

solves the constraint iff $\Phi(\rho, z)$ solves the WDW equation.

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Summary and outlook

Done:

- Quantum cosmological billiards wavefunctions involve automorphic forms of $PSL_2(\mathtt{O})$
- Extendable to supersymmetric case
- Wavefunctions vanish at singularity (irrespective of susy) $\Rightarrow$ Singularity resolution?
- Non-computabitility (Penrose)?

To do:

- Construct wavefunctions? Behaviour of wavepackets?
- Include more variables $\Rightarrow$ E_{10} coset model? Constraints? Observables?

Thank you for your attention!

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