New results about the Vainshtein mechanism in massive gravity

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1. Pauli-Fierz theory and the vDVZ discontinuity.

2. Non linear Pauli-Fierz theory, the Vainshtein Mechanism.

3. The problem we solved.

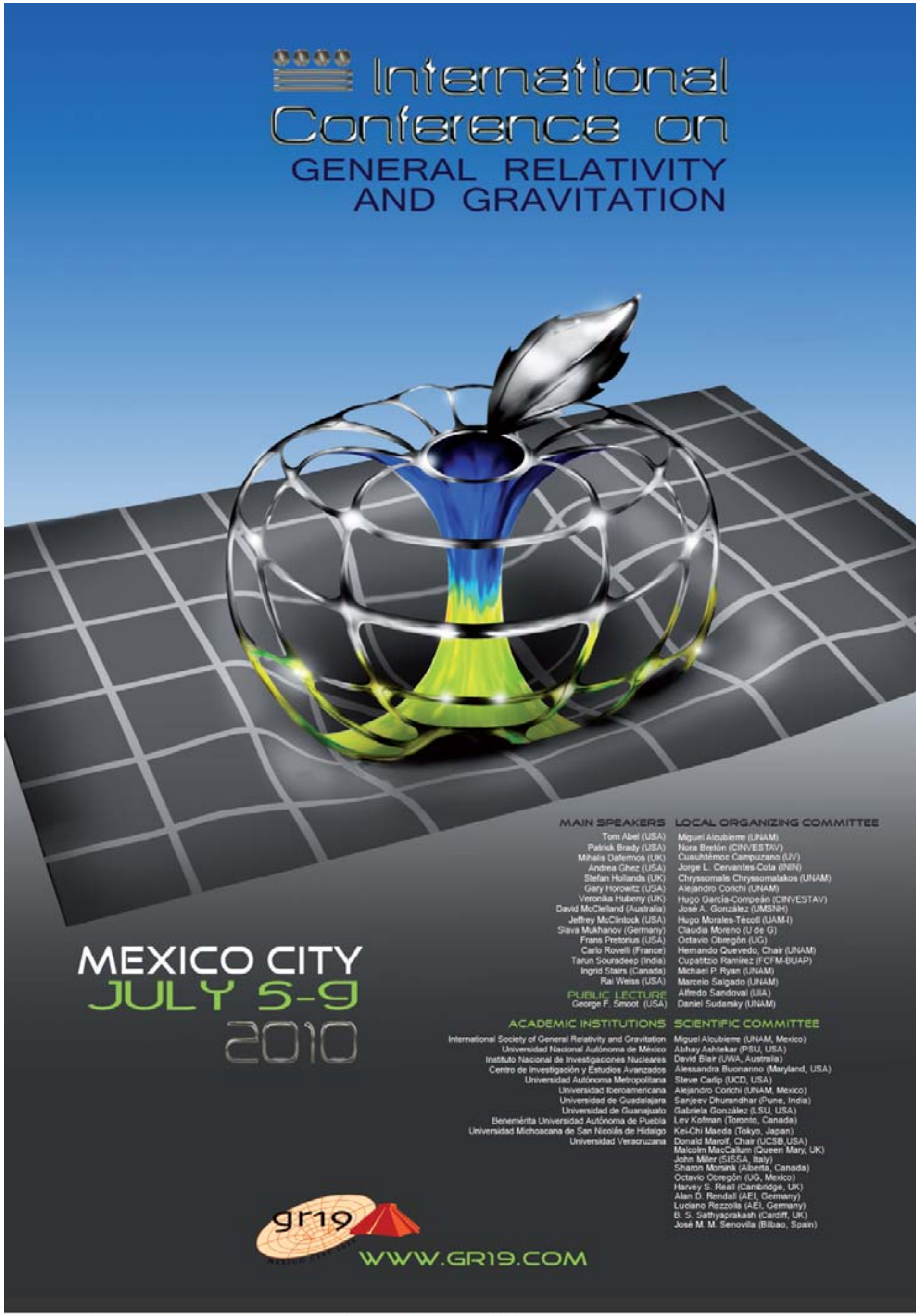
4. Use of the « decoupling limit » of massive gravity.

5. Numerical results.

6. k-Mouflage

In collaboration with E. Babichev and R. Ziour

- JHEP 0905:098,2009. (arXiv:0901.0393)
- PRL 103.201102(arXiv:0907.4103)
- To appear (arXiv:10soon.?)



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The **Vainshtein mechanism** is widely used in various attempts to modify gravity in the IR

- DG
- Ma
- De
- Ca
- Ga
- GI
- k-I

e.g. in DGP:

Various arguments in favour of a working Vainshtein mechanism,

Including
- some exact cosmological solutions
C.D., Dvali, Gabadadze, Vainshtein ‘02
- Sphericall symmetric solution on the brane
Gabadadze, Iglesias ‘04
- Approximate solutions
Gruzinov ‘01, Tanaka ‘04

Go

… However no definite proof (in the form of an exact solution) that this is indeed the case in particular for the phenomenlogically interesting case of static spherically symmetric solutions !

1. Quadratic massive gravity: the Pauli-Fierz theory and the vDVZ discontinuity

Pauli-Fierz action: second order action for a massive spin two $h_{\mu\nu}$

$$\int d^4x \underbrace{\sqrt{g}R_g} + m^2 \int d^4x\, h_{\mu\nu}h_{\alpha\beta}\left(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\nu}\eta^{\alpha\beta}\right)$$

second order in $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$

Only Ghost-free (quadratic) action for a Lorentz invariant massive spin two   Pauli, Fierz

(NB: breaks explicitly gauge invariance)

The propagators read

propagator for $m=0$ $\quad D_0^{\mu\nu\alpha\beta}(p) = \frac{\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\alpha}\eta^{\nu\alpha}}{2p^2} - \frac{\eta^{\mu\nu}\eta^{\alpha\beta}}{2p^2} + \mathcal{O}(p)$

propagator for $m\neq 0$ $\quad D_m^{\mu\nu\alpha\beta}(p) = \frac{\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\alpha}\eta^{\nu\alpha}}{2(p^2-m^2)} - \frac{\eta^{\mu\nu}\eta^{\alpha\beta}}{3(p^2-m^2)} + \mathcal{O}(p)$

Coupling the graviton with a conserved energy-momentum tensor

$$S_{int} = \int d^4x \sqrt{g}\, h_{\mu\nu} T^{\mu\nu}$$

$$h^{\mu\nu} = \int D^{\mu\nu\alpha\beta}(x - x') T_{\alpha\beta}(x') d^4x'$$

The amplitude between two conserved sources T and S is given by

$$\mathcal{A} = \int d^4x\, S^{\mu\nu}(x) h_{\mu\nu}(x)$$

For a massless graviton: $\mathcal{A}_0 = \left(\hat{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\hat{T}\right)\hat{S}^{\mu\nu}$

For a massive graviton: $\mathcal{A}_m = \left(\hat{T}_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}\hat{T}\right)\hat{S}^{\mu\nu}$

In Fourier space

e.g. amplitude between two non relativistic sources:

$$\left.\begin{aligned}\hat{T}^{\mu}_{\nu} &\propto \mathrm{diag}(\hat{m}_1, 0, 0, 0) \\ \hat{S}^{\mu}_{\nu} &\propto \mathrm{diag}(\hat{m}_2, 0, 0, 0)\end{aligned}\right\} \quad \mathcal{A} \sim \tfrac{2}{3}\hat{m}_1 \hat{m}_2$$

Instead of $\mathcal{A} \sim \frac{1}{2}\hat{m}_1 \hat{m}_2$

Rescaling of Newton constant $G_{\mathrm{Newton}} = \frac{4}{3} G_{(4)}$

defined from Cavendish experiment

appearing in the action

but amplitude between an electromagnetic probe and a non-relativistic source is the same as in the massless case (the only difference between massive and massless case is in the trace part) ⇨ wrong light bending! (factor ¾)

2. Non linear Pauli-Fierz theory and the Vainshtein Mechanism

Can be defined by an action of the form

$$S = \int d^4x \sqrt{-g} \left(\frac{M_P^2}{2} R_g + L_g \right) + S_{int}[f,g],$$

Einstein-Hilbert action for the *g* metric

Matter action (coupled to metric *g*)

Interaction term coupling the metric *g* and the non dynamical metric *f*

Leads to the e.o.m. $M_P^2 G_{\mu\nu} = \left(T_{\mu\nu} + T^g_{\mu\nu}(f,g) \right)$

Matter energy-momentum tensor

Effective energy-momentum tensor (*f,g* dependent)

The interaction term $S_{int}[f,g],$ is chosen such that

- It is invariant under diffeomorphisms
- It has flat space-time as a vacuum
- When expanded around a flat metric ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $f_{\mu\nu} = \eta_{\mu\nu}$)
  It gives the Pauli-Fierz mass term

Some working examples

$$S_{int}^{(2)} = -\frac{1}{8} m^2 M_P^2 \int d^4x \sqrt{-f}\, H_{\mu\nu} H_{\sigma\tau} \left(f^{\mu\sigma} f^{\nu\tau} - f^{\mu\nu} f^{\sigma\tau} \right)$$

(Boulware Deser)

$$S_{int}^{(3)} = -\frac{1}{8} m^2 M_P^2 \int d^4x \sqrt{-g}\, H_{\mu\nu} H_{\sigma\tau} \left(g^{\mu\sigma} g^{\nu\tau} - g^{\mu\nu} g^{\sigma\tau} \right)$$

(Arkani-Hamed, Georgi, Schwartz)

with $H_{\mu\nu} = g_{\mu\nu} - f_{\mu\nu}$

(infinite number of models with similar properties)

NB: similar theory were investigated in various contexts in particular also

- « Strong gravity » Salam et al. 71
- « bigravity » Damour, Kogan 03
- « Higgs for gravity » Chamseddine, Mukhanov 10

⇒ Look for static spherically symmetric solutions

With th

f_{AB}

Interest: in this form the g metric can easily be compared to standard Schwarzschild form

Gauge transformation

$$\begin{cases} g_{\mu\nu}dx^\mu dx^\nu &= -e^{\nu(R)}dt^2 + e^{\lambda(R)}dR^2 + R^2 d\Omega^2 \\ f_{\mu\nu}dx^\mu dx^\nu &= -dt^2 + \left(1 - \frac{R\mu'(R)}{2}\right)^2 e^{-\mu(R)}dR^2 + e^{-\mu(R)}R^2 d\Omega^2 \end{cases}$$

Then look for an expansion in G_N (or in $R_S \propto G_N M$) of the would be solution

(For $R \ll m^{-1}$)

$$\nu(R) = -\frac{R_S}{R}\left(1 + \frac{7}{32}\epsilon + \ldots\right)$$

$$\lambda(R) = +\frac{1}{2}\frac{R_S}{R}\left(1 - \frac{21}{8}\epsilon + \ldots\right)$$

with $\epsilon = \frac{R_S}{m^4 R^5}$

Vainshtein '72

In some kind of non linear PF

Wrong light bending!

This coefficient equals +1 in Schwarzschild solution

Introduces a new length scale R_v in the problem below which the perturbation theory diverges!

For the sun: bigger than solar system! ⇐ with $R_v = (R_S m^{-4})^{1/5}$

So, what is going on at smaller distances?

Vainshtein'72

There exists an other perturbative expansion at smaller distances, defined around (ordinary) Schwarzschild and reading:

$$\left.\begin{aligned} \nu(R) &= -\frac{R_S}{R}\left\{1+\mathcal{O}\left(R^{5/2}/R_v^{5/2}\right)\right\} \\ \lambda(R) &= +\frac{R_S}{R}\left\{1+\mathcal{O}\left(R^{5/2}/R_v^{5/2}\right)\right\} \end{aligned}\right\} \quad \text{with} \quad R_v^{-5/2} = m^2 R_S^{-1/2}$$

- This goes smoothly toward Schwarzschild as *m* goes to zero

- This leads to corrections to Schwarzschild which are non analytic in the Newton constant

To summarize: 2 regimes

$$\nu(R) = -\frac{R_S}{R}\left(1 + \frac{7}{32}\epsilon + \ldots\right) \quad \text{with} \quad \epsilon = \frac{R_S}{m^4 R^5}$$

Valid for $R \gg R_V$ with $R_V = \left(R_S m^{-4}\right)^{1/5}$

Standard perturbation theory around flat space

Crucial question: can one join the two regimes in a single existing non singular (asymptotically flat) solution? (Boulware Deser 72)

Expansion around Schwarzschild solution

$$\nu(R) = -\frac{R_S}{R}\left(1 + \mathcal{O}\left(R^{5/2}/R_V^{5/2}\right)\right)$$

Valid for $R \ll R_V$

This was investigated (by numerical integration) by Damour, Kogan and Papazoglou '03

No non-singular solution found matching the two behaviours (always singularities appearing at finite radius) and hence failure of the « Vainshtein mechanism »

(see also Jun, Kang '86)

In the rest of this talk:

A new look on this problem (using in particular the « Goldstone picture » of massive gravity in the « Decoupling limit. »)

(in collaboration with E. Babichev and R.Ziour)

3. The problem we solved !

Framework: non linear Pauli-Fierz theory

$$S = \int d^4x \sqrt{-g} \left(\frac{M_P^2}{2} R_g + L_g \right) + S_{int}[f,g]$$

Leads to the e.o.m. $$M_P^2 G_{\mu\nu} = \left(T_{\mu\nu} + T^g_{\mu\nu}(f,g) \right)$$

Matter energy-momentum tensor

Effective energy-momentum tensor (f,g) dependent

Bianchi indentity $\Rightarrow$ $\nabla^\mu T^g_{\mu\nu} = 0$

$$S^{(3)}_{int} = -\frac{1}{8} m^2 M_P^2 \int d^4x \sqrt{-g}\, H_{\mu\nu} H_{\sigma\tau} \left(g^{\mu\sigma} g^{\nu\tau} - g^{\mu\nu} g^{\sigma\tau} \right)$$

(Arkani-Hamed, Georgi, Schwartz)

Ansatz (« λ, μ, ν » gauge)

$$g_{\mu\nu}dx^\mu dx^\nu = -e^{\nu(R)}dt^2 + e^{\lambda(R)}dR^2 + R^2 d\Omega^2$$

$$f_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \left(1 - \frac{R\mu'(R)}{2}\right)^2 e^{-\mu(R)}dR^2 + e^{-\mu(R)}R^2 d\Omega^2$$

With this ansatz the e.o.m (+ Bianchi) read

$$\text{"}G_{tt}\text{"} \Longrightarrow e^{\nu-\lambda}\left(\frac{\lambda'}{R} + \frac{1}{R^2}(e^\lambda - 1)\right) = 8\pi G_N \left(T^g_{tt} + \rho e^\nu\right)$$

$$\text{"}G_{RR}\text{"} \Longrightarrow \frac{\nu'}{R} + \frac{1}{R^2}\left(1 - e^\lambda\right) = 8\pi G_N \left(T^g_{RR} + P e^\lambda\right)$$

$$\text{"}Bianchi\text{"} \Longrightarrow \nabla^\mu T^g_{\mu R} = 0$$

$$T^g_{tt} = m^2 M_P^2\, f_t, \quad T^g_{RR} = m^2 M_P^2\, f_R, \quad \nabla^\mu T^g_{\mu R} = -m^2 M_P^2 f_g,$$

$$f_t = \frac{e^{-\lambda-2\mu}}{4}$$

$$\times \left[\left(3e^{\mu+\nu} + e^{\mu} - 2e^{\nu}\right) \left(1 - \frac{R\mu'}{2}\right)^2 + e^{\lambda}\left(2e^{\mu} - e^{\nu}\right) - 3e^{\lambda+\mu}\left(2e^{\mu+\nu} + e^{\mu} - 2e^{\nu}\right) \right]$$

$$f_R = \frac{e^{-\nu-2\mu}}{4}$$

$$\times \left[\left(3e^{\mu+\nu} - e^{\mu} - 2e^{\nu}\right) \left(1 - \frac{R\mu'}{2}\right)^2 + e^{\lambda}\left(2e^{\mu} + e^{\nu}\right) - 3e^{\lambda+\mu}\left(-2e^{\mu+\nu} + e^{\mu} + 2e^{\nu}\right) \right]$$

$$f_g = -\left(1 - \frac{R\mu'}{2}\right) \frac{e^{-\lambda-2\mu-\nu}}{8R}$$

$$\times \Big[8\left(e^{\lambda} - 1\right)\left(3e^{\mu+\nu} - e^{\mu} - e^{\nu}\right) + 2R\left(\left(3e^{\mu+\nu} - 2e^{\nu}\right)\left(\lambda' + 4\mu' - \nu'\right) - e^{\mu}\left(\lambda' + 4\mu' + \nu'\right)\right)$$

$$-R^2\left(\left(3e^{\mu+\nu} - 2e^{\nu}\right)\left(\lambda'\mu' - 2\mu'' - \mu'\nu' + (\mu')^2\right) - e^{\mu}\left(\lambda'\mu' - 2\mu'' + \mu'\nu' + (\mu')^2\right) - 2e^{\nu}(\mu')^2\right)\Big]$$

To obtain our solutions, we used the Decoupling Limit, we first…

« shooted »

Then « relaxed »

We used a combination of shooting and relaxation methods
+ some analytic insight relying on (asymptotic) expansions,

with appropriate Boundary conditions
(asymptotic flatness, no singularity in R=0)

For setting boundary (or initial) conditions for the numerical integration, and better understand the result, we used crucially the Decoupling Limit.

4. The « Decoupling Limit »

4.1. How to get this Decoupling limit (DL) and why is it interesting ?

4.2. Solving the DL at large distance and lessons for the full non linear case

4.3. The DL at smaller distances

4.1. How to get the Decoupling limit (DL) ?

Originally proposed in the analysis of Arkani-Hamed, Georgi and Schwartz using « Stückelberg » fields …

and leads to the cubic action in the scalar sector (helicity 0) of the model

$$\frac{1}{2}(\nabla\tilde{\phi})^2 - \frac{1}{M_P}\tilde{\phi}T + \frac{1}{\Lambda^5}\left\{(\nabla^2\tilde{\phi})^3 + \underbrace{\ldots}\right\}$$

Other cubic terms omitted

With $\Lambda = (m^4 M_P)^{1/5}$

« Strong coupling scale »
(hidden cutoff of the model ?)

The Goldstone picture and Stückelberg trick

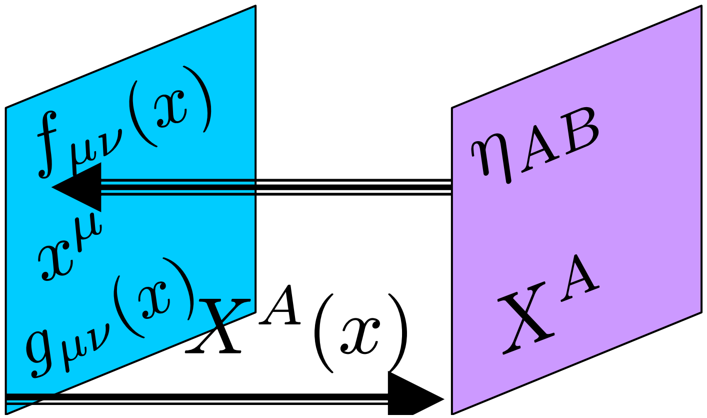
The theory considered has the usual diffeo invariance

$$\left\{ \begin{array}{lcl} g_{\mu\nu}(x) & = & \partial_\mu x'^\sigma(x) \partial_\nu x'^\tau(x) g'_{\sigma\tau}\left(x'(x)\right) \\ f_{\mu\nu}(x) & = & \partial_\mu x'^\sigma(x) \partial_\nu x'^\tau(x) f'_{\sigma\tau}\left(x'(x)\right) \end{array} \right.$$

This can be used to go from a « unitary gauge » where

$$f_{AB} = \eta_{AB}$$

To a « non unitary gauge » where some of the d.o.f. of the *g* metric are put into *f* thanks to a gauge transformation of the form



$$\begin{array}{lcl} f_{\mu\nu}(x) & = & \partial_\mu X^A(x) \partial_\nu X^B(x) \eta_{AB}\left(X(x)\right) \\ g_{\mu\nu}(x) & = & \partial_\mu X^A(x) \partial_\nu X^B(x) g_{AB}\left(X(x)\right) \end{array}$$

One (trivial) example: our spherically symmetric ansatz

$$
\left\{
\begin{array}{rcl}
g_{AB}dx^A dx^B & = & -J(r)dt^2 + K(r)dr^2 + L(r)r^2 d\Omega^2 \\
f_{AB}dx^A dx^B & = & -dt^2 + dr^2 + r^2 d\Omega^2
\end{array}
\right.
$$

Gauge transformation

$$
\left\{
\begin{array}{rcl}
g_{\mu\nu}dx^\mu dx^\nu & = & -e^{\nu(R)}dt^2 + e^{\lambda(R)}dR^2 + R^2 d\Omega^2 \\
f_{\mu\nu}dx^\mu dx^\nu & = & -dt^2 + \left(1 - \frac{R\mu'(R)}{2}\right)^2 e^{-\mu(R)}dR^2 + e^{-\mu(R)}R^2 d\Omega^2
\end{array}
\right.
$$

Expand the theory around the unitary gauge as

$$\begin{cases} X^A(x) = \delta^A_\mu x^\mu + \pi^A(x) \\ \pi^A(x) = \delta^A_\mu \left(A^\mu(x) + \eta^{\mu\nu}\partial_\nu \phi \right). \end{cases}$$

Unitary gauge coordinates ($\delta^A_\mu x^\mu$)

« pion » fields ($\pi^A(x)$)

The interaction term $S_{int}[f,g]$ expanded at quadratic order in the new fields A^μ and ϕ reads

$$\frac{M_P^2 m^2}{8}\int d^4x \qquad \left[h^2 - h_{\mu\nu}h^{\mu\nu} - F_{\mu\nu}F^{\mu\nu} \right.$$
$$\left. -4(h\partial A - h_{\mu\nu}\partial^\mu A^\nu) - 4(h\partial^\mu\partial_\mu\phi - h_{\mu\nu}\partial^\mu\partial^\nu\phi)\right]$$

A^μ gets a kinetic term via the mass term

ϕ only gets one via a mixing term

One can demix ϕ from *h* by defining

$$h_{\mu\nu} = \hat{h}_{\mu\nu} - m^2 \eta_{\mu\nu} \phi$$

And the interaction term reads then at quadratic order

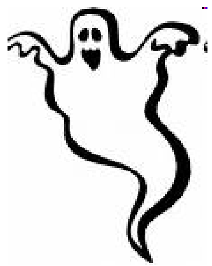
$$S = \frac{M_P^2 m^2}{8} \int d^4x \quad \Big\{ \hat{h}^2 - \hat{h}_{\mu\nu}\hat{h}^{\mu\nu} - F_{\mu\nu}F^{\mu\nu} - 4(\hat{h}\partial A - \hat{h}_{\mu\nu}\partial^\mu A^\nu)$$
$$+6m^2 \Big[\phi(\partial_\mu \partial^\mu + 2m^2)\phi - \hat{h}\phi + 2\phi\partial A \Big] \Big\}$$

The canonically normalized ϕ is given by $\tilde{\phi} = M_P m^2 \phi$

Taking then the
« **Decoupling Limit** »

$$\left\{ \begin{array}{rcl} M_P & \rightarrow & \infty \\ m & \rightarrow & 0 \\ \Lambda = (m^4 M_P)^{1/5} & \sim & const \\ T_{\mu\nu}/M_P & \sim & const, \end{array} \right.$$

One is left with …

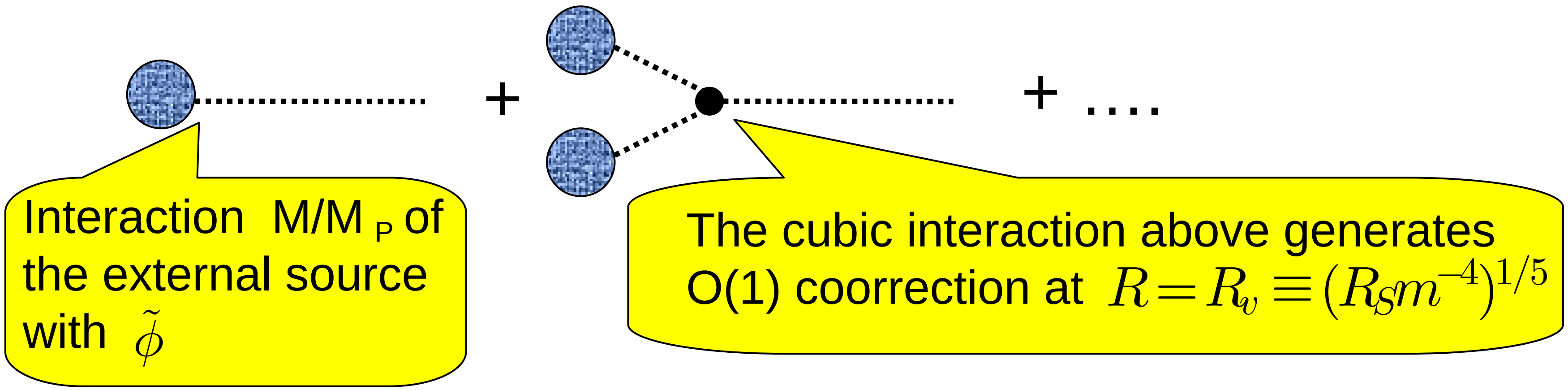



$$\frac{1}{2}\tilde{\phi}^{,\mu}\tilde{\phi}_{,\mu} - \frac{1}{M_P}\tilde{\phi}T - \frac{1}{\Lambda^5}\left\{\alpha\left(\Box\tilde{\phi}\right)^3 + \beta\left(\Box\tilde{\phi}\;\tilde{\phi}_{,\mu\nu}\;\tilde{\phi}^{,\mu\nu}\right)\right\}$$

With $\Lambda = (m^4 M_P)^{1/5}$ and α and β model dependent coefficients

In the decoupling limit, the Vainshtein radius is kept fixed, and one can understand the Vainshtein mechanism as

E.g. around a heavy source: of mass M



The cubic interaction is the strongest among all the others

$$\left\{ \begin{array}{l} \Lambda_{k_1,k_2,k_3,k_4}^{4-k_1-2k_2-3k_3-k_4} \tilde{h}^{k_1} \left(\partial \tilde{A}\right)^{k_2} \left(\partial\partial\tilde{\phi}\right)^{k_3} \tilde{\phi}^{k_4} \\ \Lambda_{k_1,k_2,k_3,k_4} = \Lambda \left(\frac{M_P}{m}\right)^{\frac{4k_1+3k_2+2k_3+4k_4-6}{5(k_1+2k_2+3k_3+k_4-4)}} \end{array} \right.$$

NB:

- Those interactions will all each have their own « Vainshtein Radius », which is much smaller than **THE** Vainshtein radius

- Can be seen to be negligible all the way to the Schwarzschild radius R_S

Here we take a different route, doing first the rescaling

$$\left\{ \begin{array}{rcl} \tilde{\nu} & \equiv & M_P \nu \\ \tilde{\lambda} & \equiv & M_P \lambda \\ \tilde{\mu} & \equiv & m^2 M_P \mu \end{array} \right.$$

And taking the « decoupling » limit

$$\left\{ \begin{array}{rcl} M_P & \rightarrow & \infty \\ m & \rightarrow & 0 \\ \Lambda = (m^4 M_P)^{1/5} & \sim & const \\ T_{\mu\nu}/M_P & \sim & const, \end{array} \right.$$

The full (non linear) system of e.o.m collapses to

$$\begin{array}{rcl} \frac{\tilde{\lambda}'}{R} + \frac{\tilde{\lambda}}{R^2} & = & -\frac{1}{2}(3\tilde{\mu} + R\tilde{\mu}') + \tilde{\rho} \\ \frac{\tilde{\nu}'}{R} - \frac{\tilde{\lambda}}{R^2} & = & \tilde{\mu} \\ \frac{\tilde{\lambda}}{R^2} & = & \frac{\tilde{\nu}'}{2R} + \frac{Q(\tilde{\mu})}{\Lambda^5} \end{array}$$

System of equations to be solved in the DL

$$\begin{aligned} \frac{\tilde{\lambda}'}{R} + \frac{\tilde{\lambda}}{R^2} &= -\frac{1}{2}(3\tilde{\mu} + R\tilde{\mu}') + \tilde{\rho} \\ \frac{\tilde{\nu}'}{R} - \frac{\tilde{\lambda}}{R^2} &= \tilde{\mu} \\ \frac{\tilde{\lambda}}{R^2} &= \frac{\tilde{\nu}'}{2R} + \frac{Q(\tilde{\mu})}{\Lambda^5} \end{aligned}$$

System of equations to be solved in the DL

$$\Downarrow$$

$$\frac{1}{\Lambda^5}\left[6Q(\tilde{\mu}) + 2RQ(\tilde{\mu})'\right] + \frac{9}{2}\tilde{\mu} + \frac{3}{2}R\,\tilde{\mu}' = \tilde{\rho}$$

Which can be integrated once to yield the first integral

$$\frac{2}{\Lambda^5}Q(\tilde{\mu}) + \frac{3}{2}\,\tilde{\mu} = -\frac{K}{R^3}$$

This first integral $-\frac{3}{2}\,\tilde{\mu} - \frac{2}{\Lambda^5} Q(\tilde{\mu}) = \frac{K}{R^3}$

upon the substitution

$$\tilde{\mu} = -\frac{2}{R}\tilde{\phi}'$$

$$f_{AB}dx^A dx^B \quad = \quad -dt^2 + dr^2 + r^2 d\Omega^2$$

Recall that μ is encoding the gauge transformation

$$f_{\mu\nu}dx^\mu dx^\nu \quad = \quad -dt^2 + \left(1 - \frac{R\mu'(R)}{2}\right)^2 e^{-\mu(R)} dR^2 + e^{-\mu(R)} R^2 d\Omega^2$$

Yields exactly one which is obtained using the Stückelberg field in the scalar sector $\tilde{\phi}$

$$3\,\frac{\tilde{\phi}'}{R} \quad + \quad \frac{2}{\Lambda^5}\left\{3\alpha\,\left(-4\frac{\tilde{\phi}'}{R^4} + 2\frac{\tilde{\phi}'\tilde{\phi}''}{R^3} + 2\frac{\tilde{\phi}''^2}{R^2} + 2\frac{\tilde{\phi}'\tilde{\phi}^{(3)}}{R^2} + \frac{\tilde{\phi}''\tilde{\phi}^{(3)}}{R}\right) + \right.$$

$$\left. +\beta\left(-6\frac{\tilde{\phi}'^2}{R^4} + 2\frac{\tilde{\phi}'\tilde{\phi}''}{R^3} + 4\frac{\tilde{\phi}''^2}{R^2} + 4\frac{\tilde{\phi}'\tilde{\phi}^{(3)}}{R^2} + 3\frac{\tilde{\phi}''\tilde{\phi}^{(3)}}{R}\right)\right\} = \frac{K}{R^3}$$

To summarize, in the decoupling limit the full non linear system reduces to

$$\frac{\tilde{\lambda}'}{R} + \frac{\tilde{\lambda}}{R^2} = -\frac{1}{2}(3\tilde{\mu} + R\tilde{\mu}') + \tilde{\rho}$$

$$\frac{\tilde{\nu}'}{R} - \frac{\tilde{\lambda}}{R^2} = \tilde{\mu}$$

$$\tfrac{2}{\Lambda^5} Q(\tilde{\mu}) + \tfrac{3}{2}\ \tilde{\mu} = -\tfrac{K}{R^3}$$

Which can be shown to give the leading behaviour of the solution in the range $R_S \ll R \ll m^{-1}$

The Vainshtein radius is in this range

4.2 Solving the DL (one only needs to solve the non linear ODE)

$$\frac{3}{2}\ \tilde{\mu} + \underbrace{\frac{2}{\Lambda^5} Q(\tilde{\mu})} = -\frac{K}{R^3}$$

Depends on the interaction term $S_{int}[f,g]$

E.g. in the Case of the two interaction terms (α+β=0)

$$S^{(2)}_{int} \quad = \quad -\frac{1}{8} m^2 M_P^2 \int d^4x \sqrt{-f}\ H_{\mu\nu} H_{\sigma\tau} \left(f^{\mu\sigma} f^{\nu\tau} - f^{\mu\nu} f^{\sigma\tau} \right)$$

(Boulware Deser)

$$S^{(3)}_{int} \quad = \quad -\frac{1}{8} m^2 M_P^2 \int d^4x \sqrt{-g}\ H_{\mu\nu} H_{\sigma\tau} \left(g^{\mu\sigma} g^{\nu\tau} - g^{\mu\nu} g^{\sigma\tau} \right)$$

(Arkani-Hamed, Georgi, Schwarz)

This equation boils down to the simple form

$$3w - s\left(\dot{w}^2 + 2w\ddot{w} + 8\frac{w\dot{w}}{\xi}\right) = \frac{2c_0}{\xi^3}$$

With s = $\pm$ 1 and the dimensionless quantities

$$\left\{ \begin{array}{rcl} w & = & (R_v m)^{-2} \mu \\ \xi & = & R/R_V \\ c_0 & = & \dfrac{K}{R_V^2 \Lambda^5} \end{array} \right.$$

$$3w - s\left(\dot{w}^2 + 2w\ddot{w} + 8\frac{w\dot{w}}{\xi}\right) = \frac{2c_0}{\xi^3}$$

With s = $\pm$ 1 and the dimensionless quantities

$$\begin{cases} w &= (R_v m)^{-2}\mu \\ \xi &= R/R_V \\ c_0 &= \dfrac{K}{R_V^2\Lambda^5} \end{cases}$$

How to read the Vainshtein mechanism and scalings ?

For $\xi \gg 1$ Keep the linear part $3w = \frac{2c_0}{\ldots}$

For ξ ...

$$\ldots \frac{\ldots}{\xi}\Big) = \frac{2c_0}{\xi^3}$$

...ssume a power law scaling $\Rightarrow w \propto \xi^{-1/2}$

However, numerical integration (and mathematical properties of the non linear ODE) shows that the situation is much more complicated !

Indeed …

$$3w - s\left(\dot{w}^2 + 2w\ddot{w} + 8\,\frac{w\dot{w}}{\xi}\right) = \frac{2c_0}{\xi^3}$$

At large ξ (expect $w \propto 1/\xi^3$)

A power law expansion of the would-be solution to this problem can be found (here with $c_0 = 1$ and $s = +1$)

$$w(\xi) = \frac{2}{3\xi^3} + s\frac{4}{3\xi^8} + \frac{1024}{27\xi^{13}} + s\frac{712960}{243\,\xi^{18}} + \frac{104910848}{243\,\xi^{23}} + s\frac{225030664192}{2187\,\xi^{28}} + \ldots$$

Unique « solution » of perturbation theory

However… this series is divergent….

… but seems to give a good asymptotic expansion of the numerical solution at large ξ

- This can easily been checked numerically for
s= -*1* (Boulware Deser)
(where the Vainshtein solution does not exist at small ξ, becoming complex !)

- For *s*=+*1* (Arkani-Hamed et al.) solution is numerically highly unstable, singularities are seemingly arising at finite ξ…

However by using a combination of relaxation method / Runge-Kutta/ Asymptotic expansion ,

one can see that solutions (infinitely many !) with Vainshtein asymptotics at large ξ do exist.

In our case, using the « resurgence theory »
(J. Ecalle) extending Borel resummation

$\sum_k \tilde{w}_k(\tilde{\xi})$

Formal
(divergent) serie

$\sum_k a_k \xi^{-k}$ — Borel transform → $\sum_k \frac{a_k}{(k-1)!} \tilde{\xi}^{k-1}$

Laplace transform or rather
« convolution average»
extension

Solution of the ODE

(proof provided to
us by J. Ecalle)

s = -1

Unique solution
with *w* $\propto$ 1/ ξ^3
decay at infinity

The difference
between any two
solutions is given
(asymptotically) by
$\xi^{3/2} \exp\left(-k\ 3/5\ \xi^{5/2}\right)$
(with integer *k* !)

s = 1

Infinitely many
solutions with
w $\propto$ 1/ ξ^3 decay
at infinity

So, in the s=+1, the perturbation theory does not uniquely fix the solution of the DL at infinity !

A toy example with similar properties

Consider the two (linear) ODE

$$y''(x) + y(x) = \frac{1}{x} \qquad (1)$$

$$-y''(x) + y(x) = \frac{1}{x} \qquad (2)$$

And the Cauchy problem $y_{1,2}(x) \to \frac{1}{x}, \text{ when } x \to \infty$

This problem can be solved explicitely

$$y_1(x) = \frac{\pi}{2}\cos(x) + \mathrm{Ci}(x)\sin(x) - \mathrm{Si}(x)\cos(x)$$

$$y_2(x) = \bar{C}_2 e^{-x} - \frac{1}{2}\left(e^x \mathrm{Ei}(-x) - e^{-x}\mathrm{Ei}(x)\right)$$

In the second case, one can add freely an homogeous solution

Both solutions have the following (divergent) power serie expansion

$$y_1(x) = \frac{1}{x} - \frac{2}{x^3} + \frac{24}{x^5} - \frac{720}{x^7} + O\left(\frac{1}{x^9}\right)$$

$$y_2(x) \quad = \quad \frac{1}{x} + \frac{2}{x^3} + \frac{24}{x^5} + \frac{720}{x^7} + O\left(\frac{1}{x^9}\right)$$

Where the homogeous mode is not seen !

Typical from asymptotic expansions

Back to the full non linear case

Flat space perturbation theory,
Starting with
($z = R\, m^{-1}$ and $\epsilon \propto G_N$)

$$\begin{cases} \nu_0 &= -\frac{4\epsilon}{3z} e^{-z} \\ \lambda_0 &= \frac{2\epsilon}{3}\left(1+\frac{1}{z}\right) e^{-z} \\ \mu_0 &= \frac{2\epsilon}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}\right) e^{-z} \end{cases}$$

$$\begin{cases} \lambda &= \lambda_0 + \lambda_1 + \dots \\ \nu &= \nu_0 + \nu_1 + \dots \\ \mu &= \mu_0 + \mu_1 + \dots \end{cases}$$

where λ_i, ν_i, μ_i are assumed to be proportional to ϵ^{i+1}

One finds the
unique expansion
At large z (large R)

$$\begin{cases} \mu_n &= \epsilon^{n+1} e^{-(n+1)z} \sum_{i=-\infty}^{i=0} \mu_{n,i} z^i \\ \lambda_n &= \epsilon^{n+1} e^{-(n+1)z} \sum_{i=-\infty}^{i=0} \lambda_{n,i} z^i \\ \nu_n &= \epsilon^{n+1} e^{-(n+1)z} \sum_{i=-\infty}^{i=0} \nu_{n,i} z^i \end{cases}$$

However, this misses a subdominant (non perturbative) correction of the form

$$
\left\{
\begin{aligned}
\delta\mu &= F_\infty(z)\exp\left(-\frac{3}{\sqrt{\epsilon}}\,z\,e^{z/2}\right)\\
\delta\lambda &= -F_\infty(z)\frac{z^2}{2}\exp\left(-\frac{3}{\sqrt{\epsilon}}\,z\,e^{z/2}\right)\\
\delta\nu &= -F_\infty(z)\frac{\sqrt{\epsilon}}{3}\exp\left(-\frac{z}{2}-\frac{3}{\sqrt{\epsilon}}\,z\,e^{z/2}\right)
\end{aligned}
\right.
$$

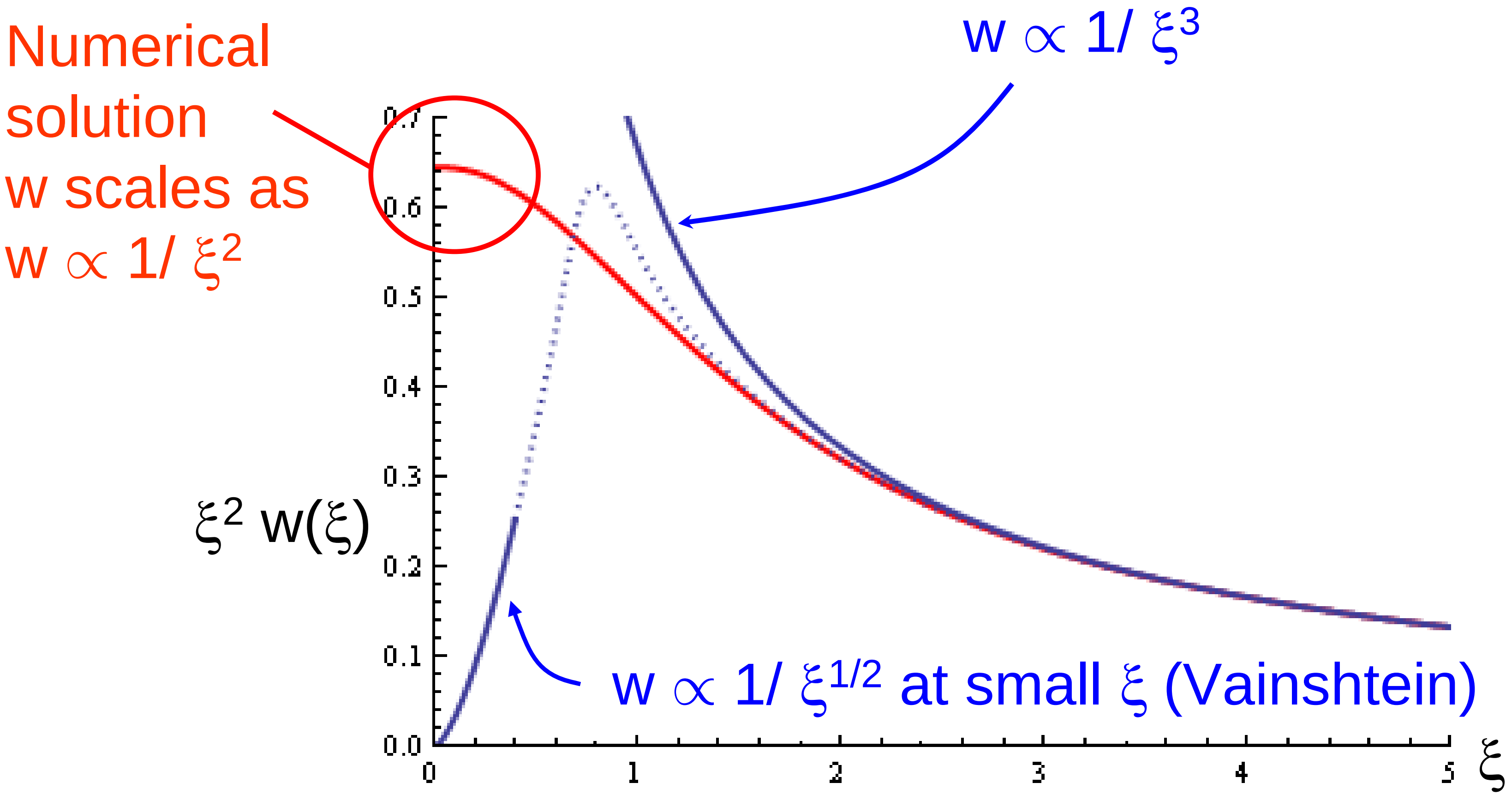
With $F_\infty(z) \sim \mathcal{O}\left(e^{z/4} z^{-3/2}\right)$

Hence, the solution at large z is not unique !

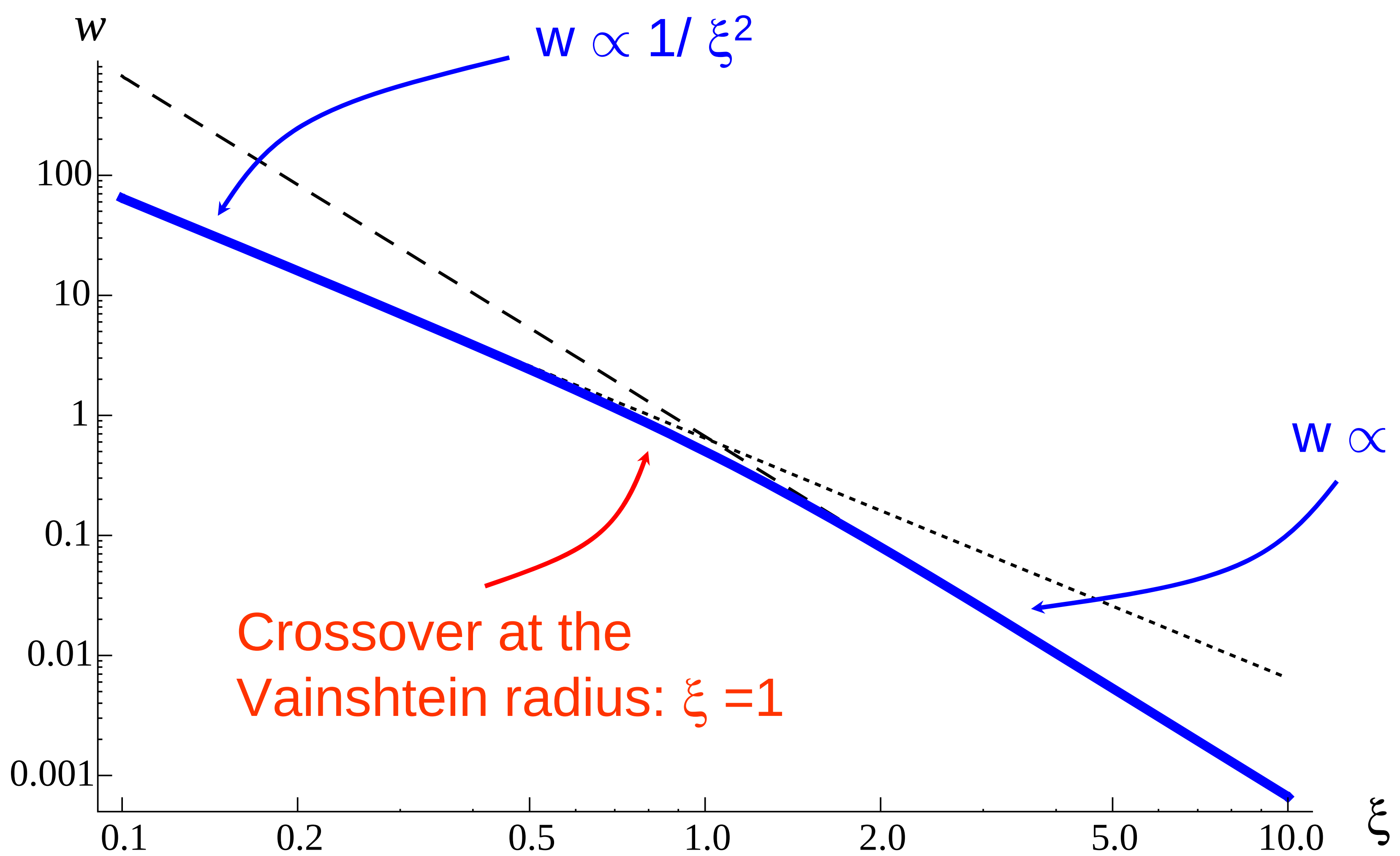
At small ξ (expect $w \propto 1/\xi^{1/2}$, when the solution is real)

Let us first discuss the *s*= *-1* case (Boulware Deser)

In this case: no real Vainshtein solution with $w \propto 1/\xi^{1/2}$



Another way to see the same



How to obtain such a scaling from

$$3w - s\left(\dot{w}^2 + 2w\ddot{w} + 8\,\frac{w\dot{w}}{\xi}\right) = \frac{2c_0}{\xi^3}$$

Which reduces at small distances to

$$-s\left(\dot{w}^2 + 2w\ddot{w} + 8\,\frac{w\dot{w}}{\xi}\right) = \frac{2c_0}{\xi^3}$$

Plug $w = A\,\xi^{-p}$ into this equation and get

$$\underbrace{-3sA^2\underbrace{(p-2)p}}\;\underbrace{\xi^{-2(1+p)}} = 2c_0\underbrace{\xi^{-3}}$$

Such a solution exists only if this factor is positive (requires s=+1)

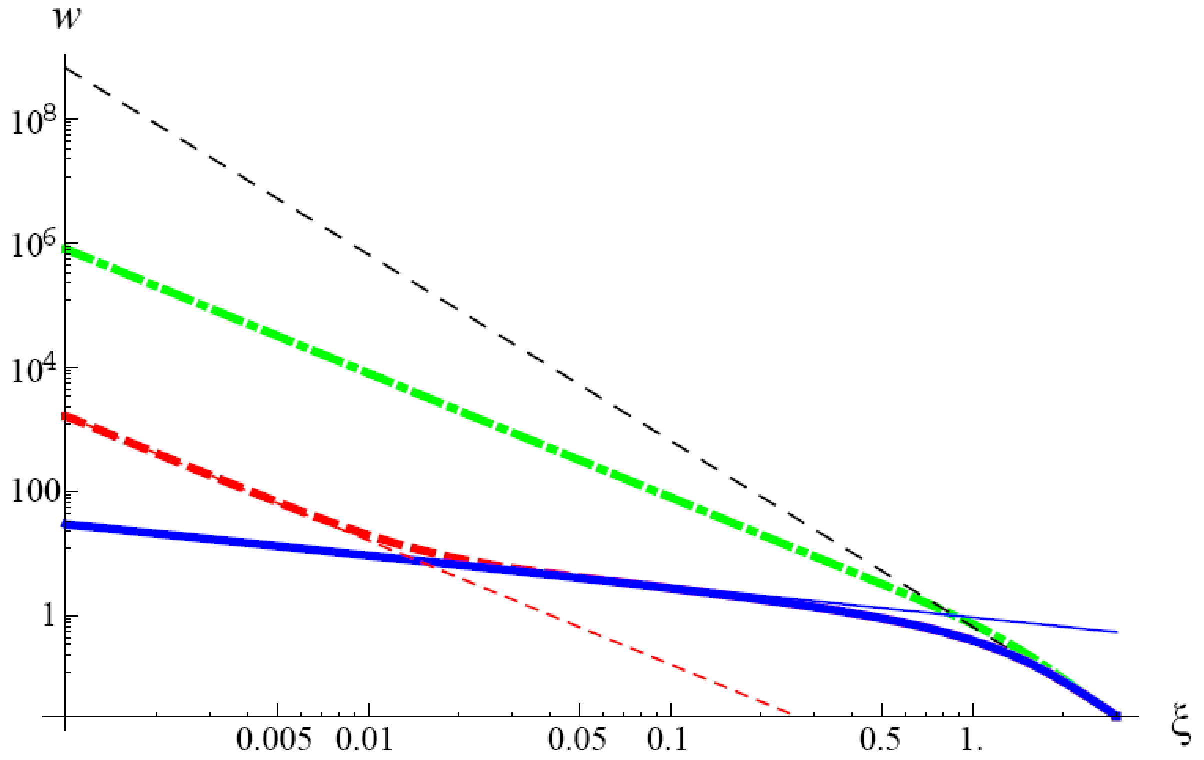
Equating those terms leads to p = ½ (Vainshtein)

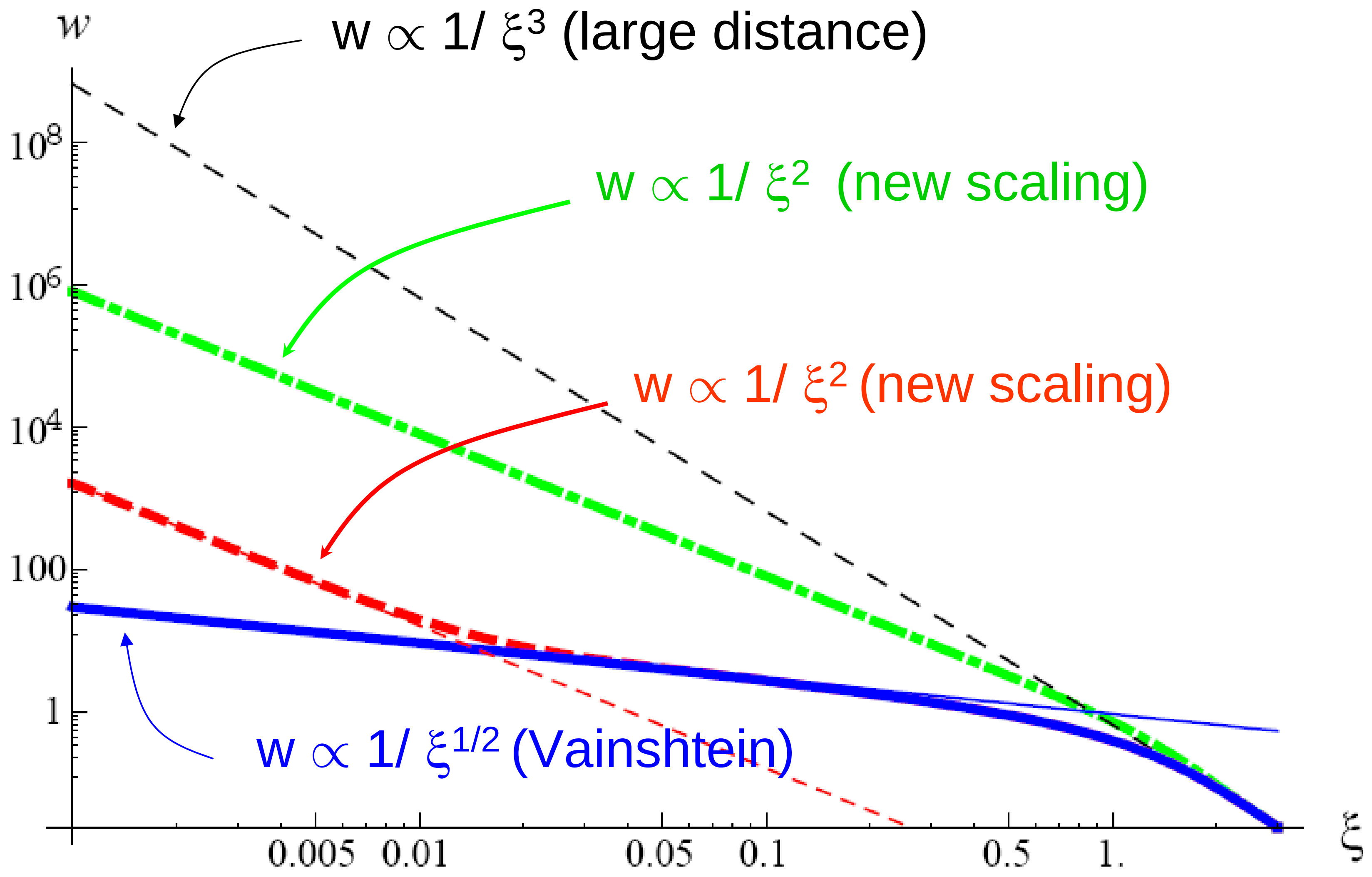
However *p* = 2 is a zero mode of the l.h.s

Let us now discuss the *s=+1* case (Arkani-Hamed et al.)

In this case the large distance behaviour $w(\xi) \sim \frac{2}{3\xi^3}$

Does not lead to a unique small distance ($\xi \ll 1$) behaviour (and solution)



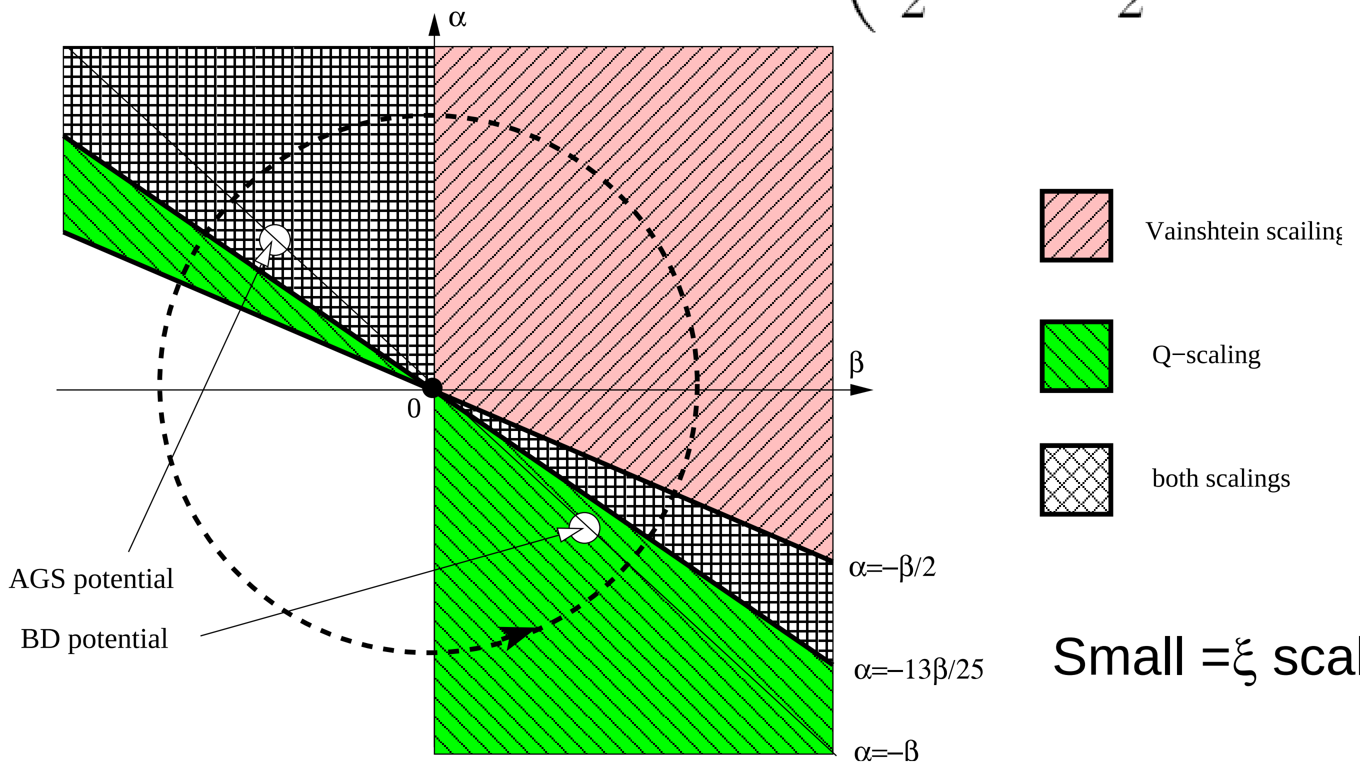


w
w ∝ 1/ ξ³ (large distance)
w ∝ 1/ ξ² (new scaling)
w ∝ 1/ ξ² (new scaling)
w ∝ 1/ ξ^1/2 (Vainshtein)
10⁸
10⁶
10⁴
100
1
0.005
0.01
0.05
0.1
0.5
1.
ξ

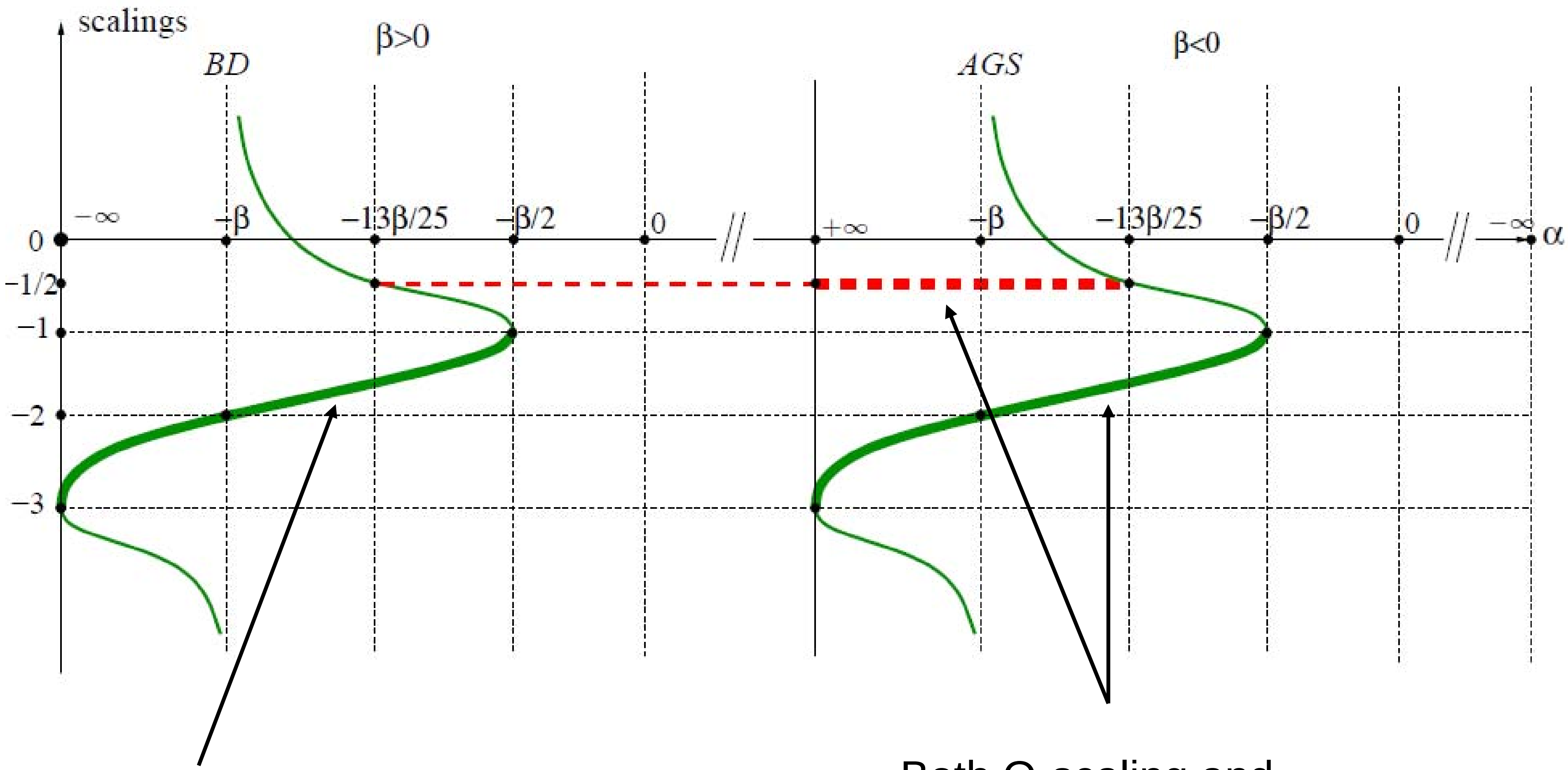
Most general case (general α, β)

We have to solve

$$\begin{cases} 2\,Q(w) + \dfrac{3}{2}w = \dfrac{1}{\xi^3} \\ \text{with} \\ Q(w) = -\dfrac{1}{2}\left\{ 3\alpha\left(\dfrac{\xi}{2}\dot{w}\ddot{w} + \dfrac{3}{2}w\ddot{w} + 2\dot{w}^2 + \dfrac{6w\dot{w}}{\xi}\right) + \beta\left(\dfrac{3\xi}{2}\dot{w}\ddot{w} + \dfrac{5}{2}w\ddot{w} + 5\dot{w}^2 + \dfrac{10w\dot{w}}{\xi}\right)\right\} \end{cases}$$



Small =ξ scalings



Only Q-scaling has the correct large R behaviour

Both Q-scaling and Vainshtein scaling have the correct large R behaviour

To summarize our DL findings

- One can find non singular solutions in the DL (but this can be hard because of numerical instabilities).

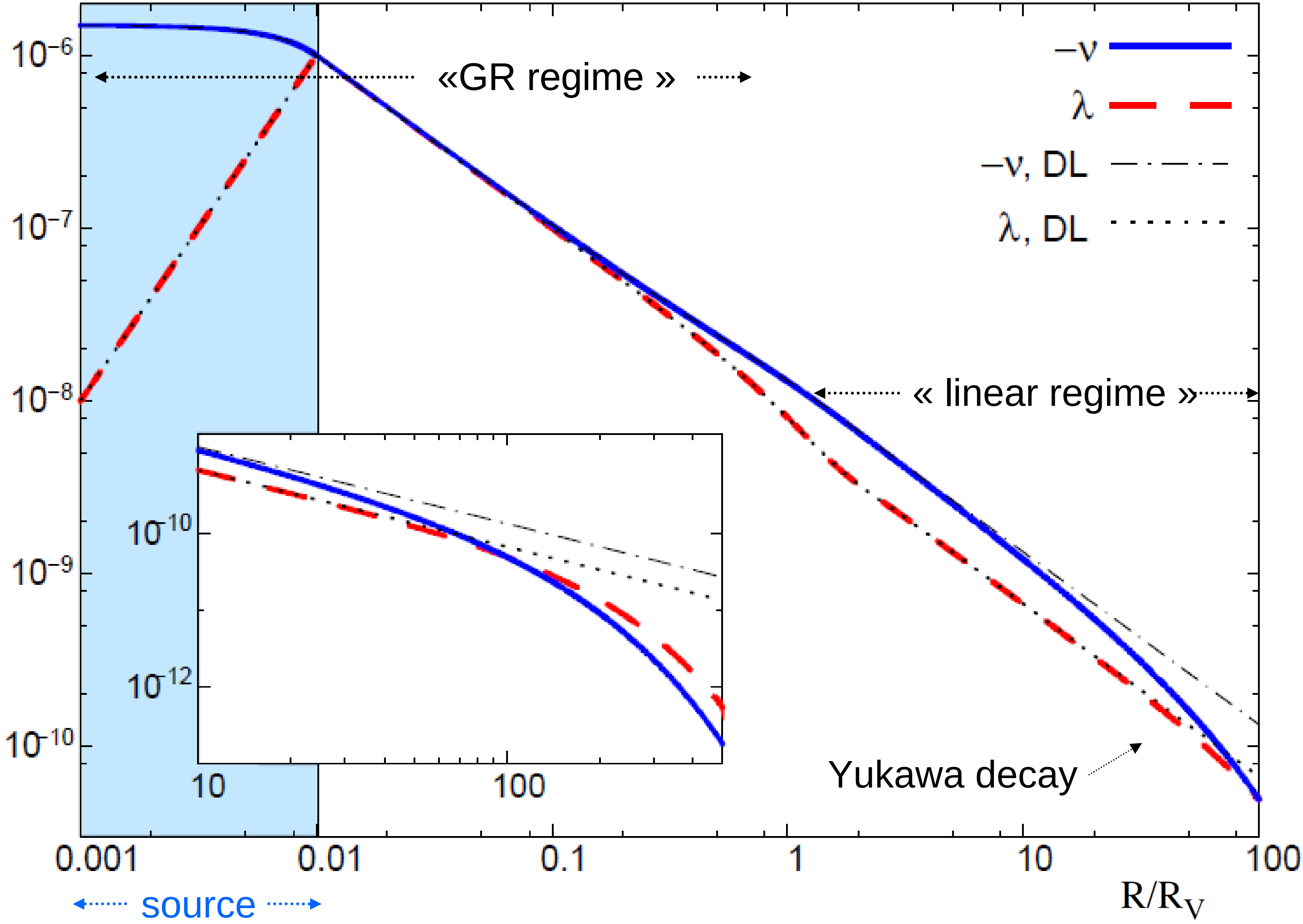
- The ghost does not prevent the existence of those solutions.

- The perturbative expansion (at large R) can be (depending on the potential) not enough to fix uniquely the solution.

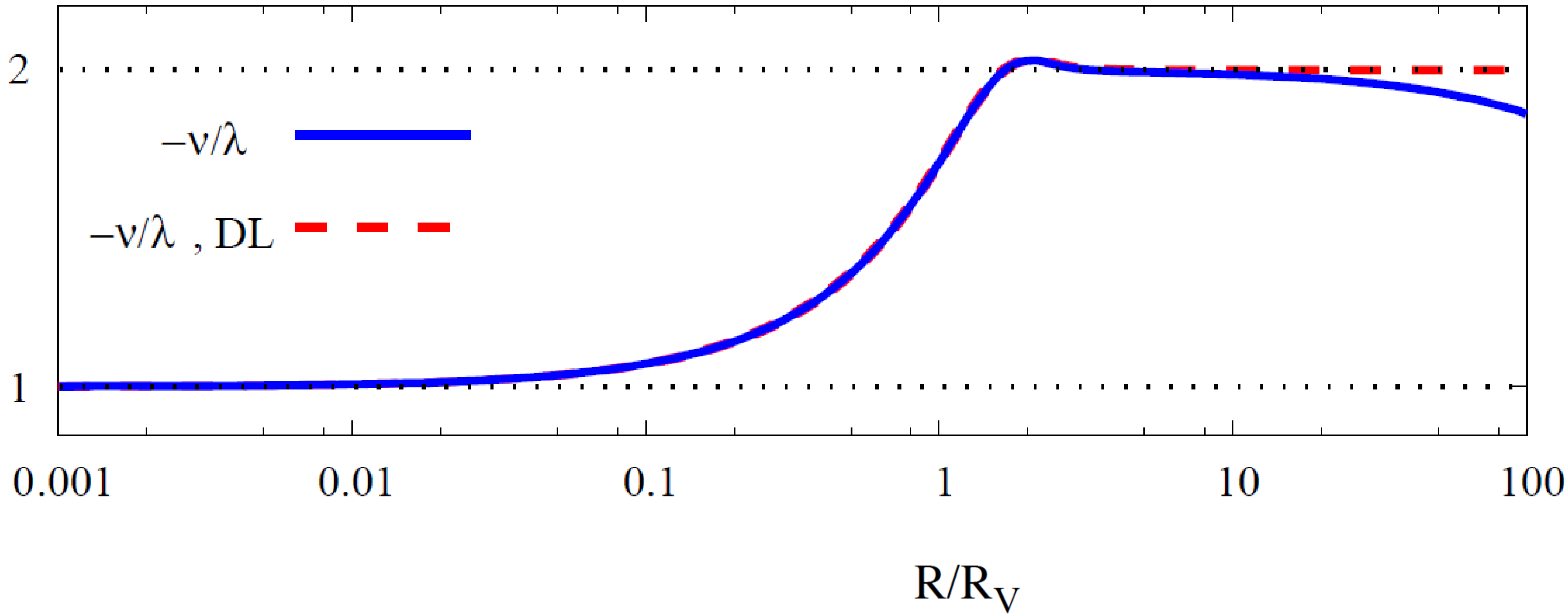
- There is a new possible scaling at small R

- Solution with the correct large R asymptotics cannot always be extended all the way to small R (depending on parameters α and β).

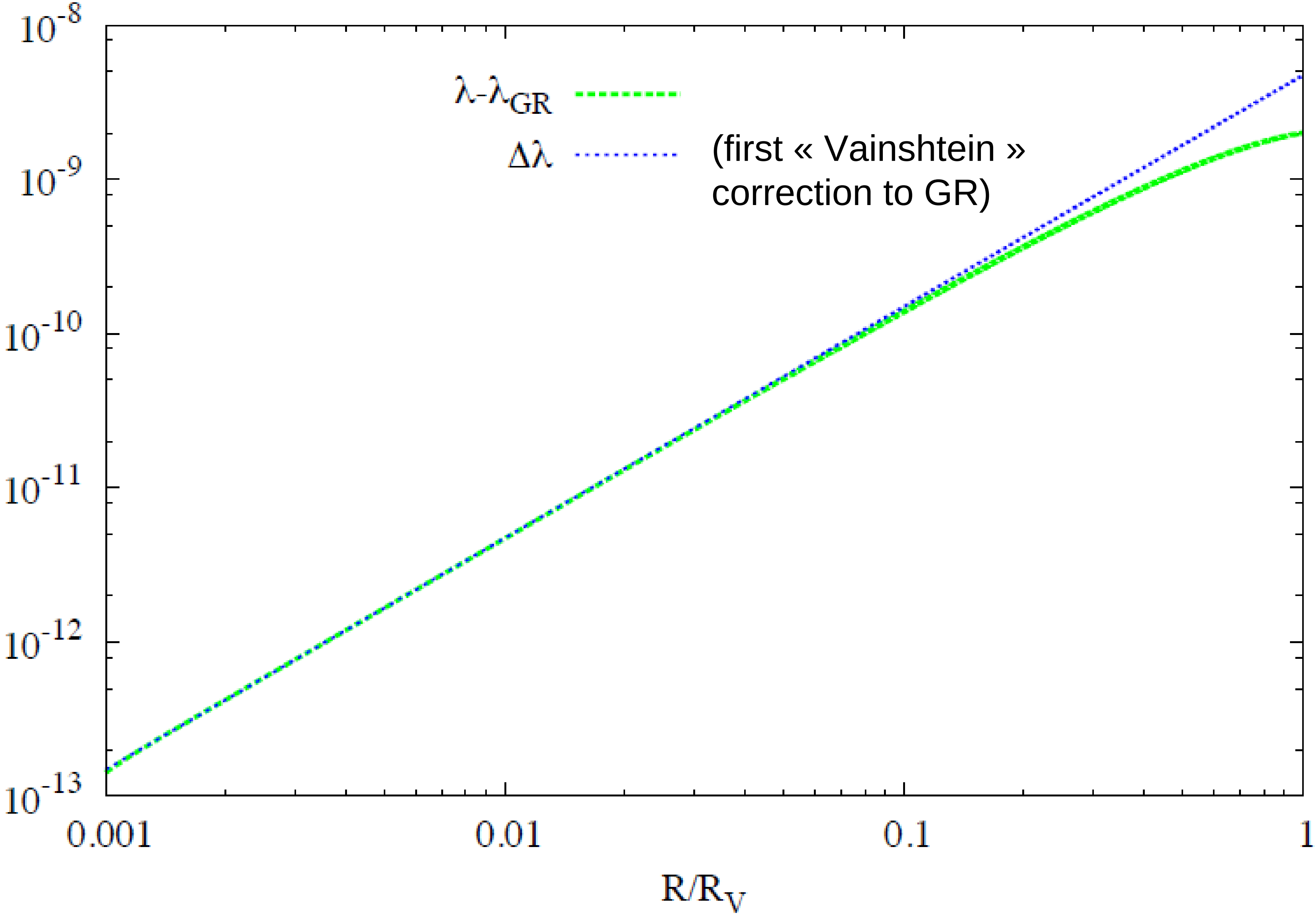
5. Numerical solutions of the full non linear system



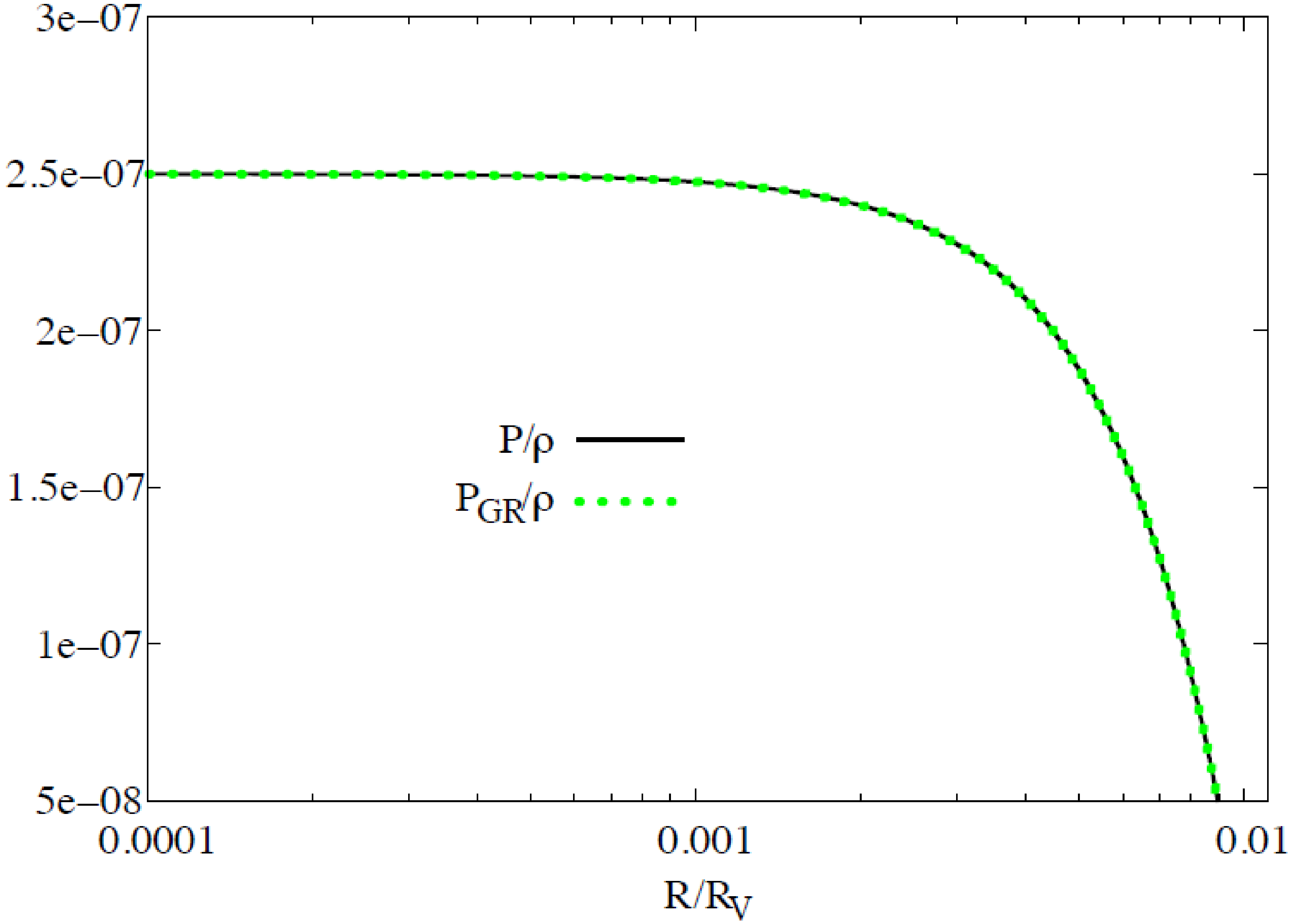
The vDVZ discontinuity gets erased for distances smaller than R_V as expected



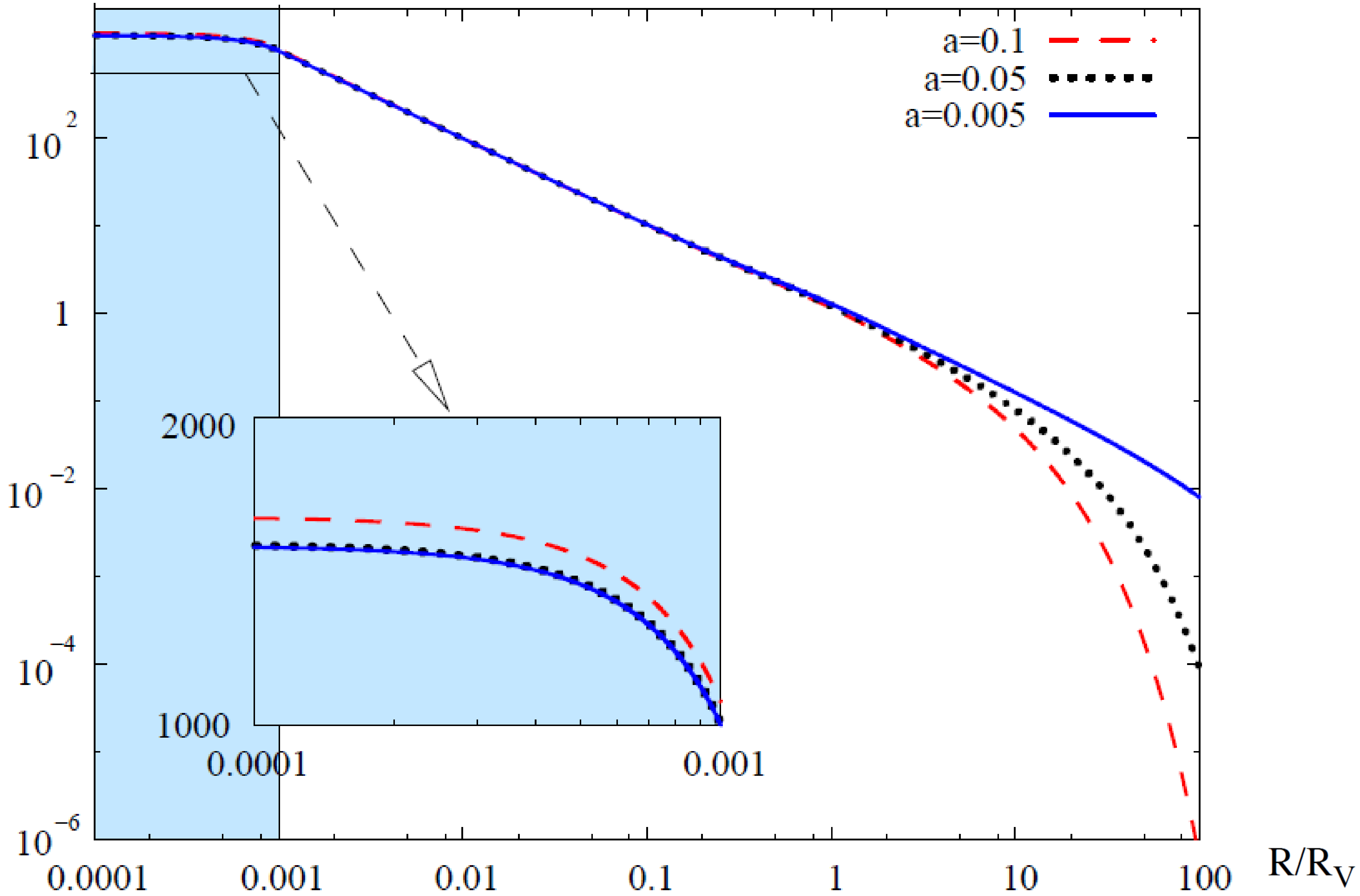
Corrections to GR in the $R \ll R_V$ regime



Pressure inside the source, and a comparison with GR



Capturing GR non linearities and Comparing with the Decoupling Limit



Solutions were obtained for very low density objects. We do not know what is happening for dense objects (and BHs).

The « Q-scaling » does not lead to a physical solution (singularities in R=0)

Conclusion (Vainshtein mechanism in massive gravity)

- It works !
- What is going on for dense object ?
- Black Holes ? (C.D. T. Jacobson to appear)
- In other models ?
- Gravitational collapse ?

5. k-Mouflage (Babichev, C.D., Ziour)

Idea: keep the qualitative structure of DL e.o.m.

$$
\begin{aligned}
\frac{\lambda'}{R} + \frac{\lambda}{R^2} &= -\frac{1}{2} m^2 (3\mu + R\mu') + \frac{\rho}{M_P^2}, \\
\frac{\nu'}{R} - \frac{\lambda}{R^2} &= m^2 \mu, \\
\frac{\lambda}{R^2} - \frac{\nu'}{2R} &= Q(\mu), \\
&\equiv -\frac{1}{2R} \left\{ 3\alpha \left(6\mu\mu' + 2R\mu'^2 + \frac{3}{2} R\mu\mu'' + \frac{1}{2} R^2 \mu' \mu'' \right) \right. \\
&\quad \left. + \beta \left(10\mu\mu' + 5R\mu'^2 + \frac{5}{2} R\mu\mu'' + \frac{3}{2} R^2 \mu' \mu'' \right) \right\},
\end{aligned}
$$

Obtained from the (DL) action

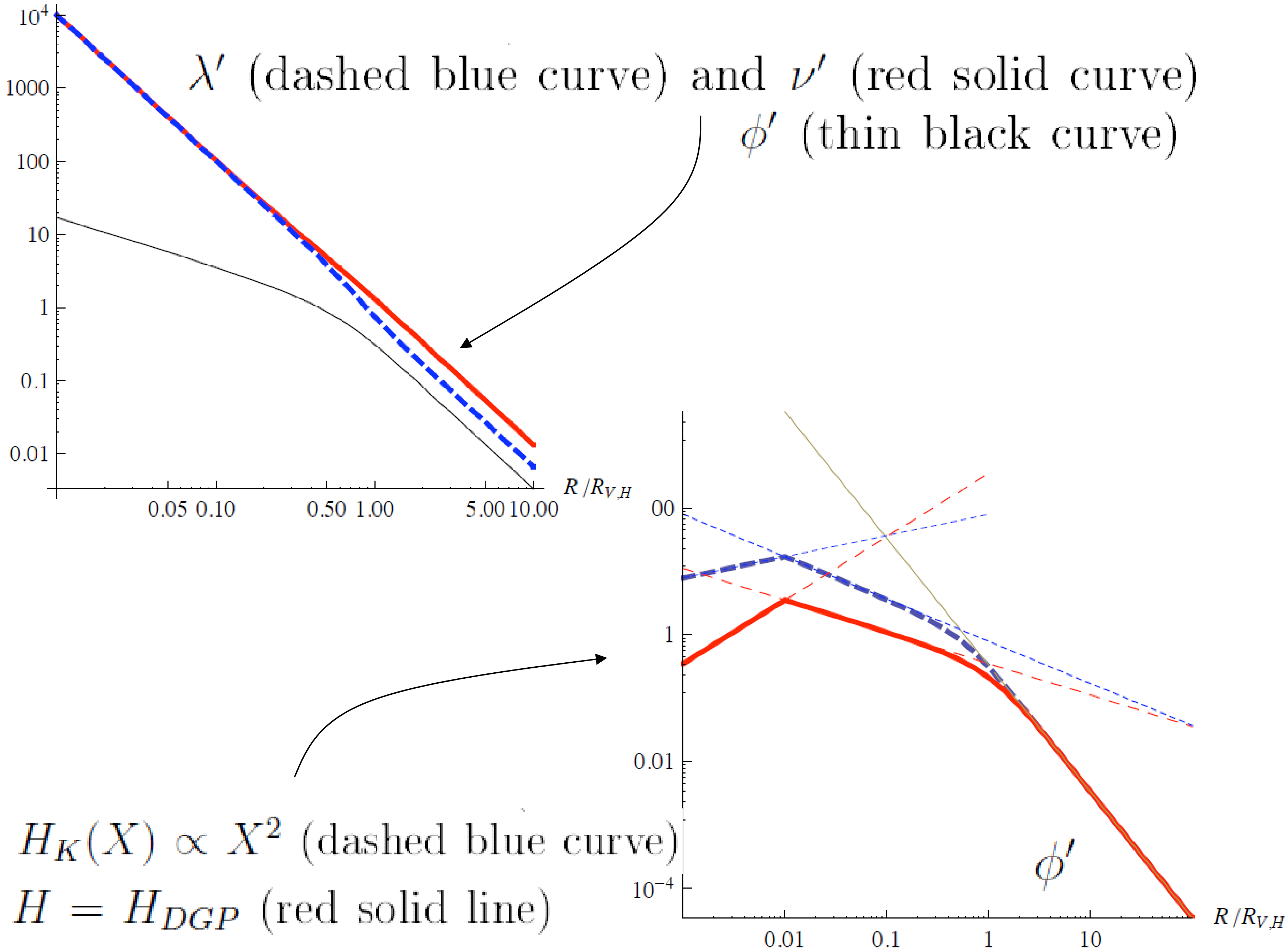
$$\left(\begin{array}{rcl} h_{\mu\nu} & \equiv & \{\lambda, \nu\}, \\ \mu & = & -2\phi'/R, \end{array} \right)$$

$$S = \frac{M_P^2}{8} \int d^4x \Big\{ 2h^{\mu\nu}\partial_\mu\partial_\nu h - 2h^{\mu\nu}\partial_\nu\partial_\sigma h^\sigma_\mu + h^{\mu\nu}\Box h_{\mu\nu} - h\Box h$$
$$+ m^2 \big[4(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\Box\phi) + 4\alpha\,(\Box\phi)^3 + 4\beta\,(\Box\phi\;\phi_{,\mu\nu}\;\phi^{,\mu\nu}) \big] \Big\}$$
$$+ \frac{1}{2}\int d^4x\; T_{\mu\nu}h^{\mu\nu}$$

N.L. completion (and extension)

$$S \;=\; M_P^2 \int d^4x \sqrt{-g} \left(\frac{R}{2} + \frac{\gamma}{2} m^2 \phi R + m^2 H(\phi) \right) + S_m,$$

$$\begin{array}{rcl} H(\phi)_{MG} & = & \frac{\alpha}{2}\left(\Box\phi^3\right) + \frac{\beta}{2}\left(\Box\phi\,\phi_{;\mu\nu}\phi^{;\mu\nu}\right), \\ H(\phi)_{DGP} & = & m^2\Box\phi\,\phi_{;\mu}\phi^{;\mu}, \\ H(\phi)_K & = & K(X), \quad \text{with} \quad X = m^2\phi_{;\mu}\phi^{;\mu}\;, \\ H(\phi)_{Gal} & = & m^2\left(\phi_{;\lambda}\,\phi^{;\lambda}\right)\left[2\left(\Box\phi\right)^2 - 2\left(\phi_{;\mu\nu}\,\phi^{;\mu\nu}\right)\right], \\ H(\phi)_{CovGal} & = & m^2\left(\phi_{;\lambda}\,\phi^{;\lambda}\right)\left[2\left(\Box\phi\right)^2 - 2\left(\phi_{;\mu\nu}\,\phi^{;\mu\nu}\right) - \frac{1}{2}\left(\phi_{;\mu}\,\phi^{;\mu}\right)R\right] \end{array}$$



λ' (dashed blue curve) and ν' (red solid curve)
ϕ' (thin black curve)
10^4
1000
100
10
1
0.1
0.01
0.05 0.10
0.50 1.00
5.00 10.00
$R/R_{V,H}$
00
1
0.01
10^{-4}
0.01
0.1
1
10
$R/R_{V,H}$
ϕ'
$H_K(X) \propto X^2$ (dashed blue curve)
$H = H_{DGP}$ (red solid line)

k-Mouflage

Nice (toy model) arena to explore to modify gravity in the IR

(Nicolis, Rattazzi and Trincherini; Chow, Khoury; Silva, Koyama… for Galileon)