Generalised Geometry of Supergravity

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Based on work with André Coimbra and Daniel Waldram

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Introduction

- Reformulation of supergravity using:

  $O(d,d) \times \mathbb{R}^+$ generalised geometry $\leftrightarrow$ Type II
  $E_{d(d)} \times \mathbb{R}^+$ generalised geometry $\leftrightarrow$ 11D

- Bosonic sector purely geometrical (exactly like GR)
- SUSY and fermion equations naturally included
- Bosonic symmetries and degrees of freedom both unified
- Mathematically nice structure

Outline of talk

- Review: Features of ordinary geometry and gravity
- Internal sector of 11D supergravity and $E_{d(d)} \times \mathbb{R}^+$ group
- $E_{d(d)} \times \mathbb{R}^+$ Generalised Geometry
- Generalised metric and H_d structures
  - $\rightarrow$ Recovering supergravity equations
  - $\rightarrow$ Example: $d = 7$ with $SU(8)$ indices [de Wit & Nicolai '86]
- Concluding remarks
  - $\rightarrow$ Connections to other works
  - $\rightarrow$ Future directions

Ordinary Geometry and GR

- Manifold M of dimension d, with tangent bundle TM
- Frame bundle:

$$F = \{(x, \{\hat{e}_a\}) : \; x \in M \text{ and } \{\hat{e}_a\} \text{ is a basis for } T_x M\} .$$

is principal bundle with structure group $GL(d, \mathbb{R})$

Ordinary Geometry and GR

- Diffeomorphisms

  $\rightarrow$ generated by vector fields $v \in TM$

  $\rightarrow$ acts by Lie derivative $\delta_v = \mathcal{L}_v$

  $\rightarrow$ algebra $[\delta_{v_1}, \delta_{v_2}] = \delta_{[v_1,v_2]}$

- Lie derivative is derivative minus adjoint action of $GL(d,\mathbb{R})$

$$\mathcal{L}_v = \partial_v - (\partial \otimes v)\cdot$$

  thinking of $(\partial \otimes v)^\mu{}_\nu \sim \partial_\nu v^\mu$ as matrix in $\mathfrak{gl}(d,\mathbb{R})$

- E.g. $(\mathcal{L}_v w)^\mu = (v^\nu \partial_\nu) w^\mu - (\partial_\nu v^\mu) w^\nu$

Ordinary Geometry and GR

- Connection ∇ on TM; $\quad \nabla_\mu v^a = \partial_\mu v^a + \omega_\mu{}^a{}_b v^b$
- Torsion defined by

$$T(v) = \mathcal{L}_v^\nabla - \mathcal{L}_v, \qquad v \in TM$$

- Naively $T \in T^*M \otimes \mathrm{ad}(F)$ but in fact $T \in TM \otimes \Lambda^2 T^*M$
- All structure so far exists without introducing a metric!

Ordinary Geometry and GR

- Gravity field $\leftrightarrow$ metric $g_{\mu\nu}$ of signature $(d-1,1)$ on TM
- Metric equivalent to principal $O(d-1,1)$ sub-bundle

$$P = \{(x, \{\hat{e}_a\}) \in F : g(\hat{e}_a, \hat{e}_b) = \eta_{ab}\}$$

- $\exists\, \nabla$ torsion-free $O(d-1,1)$ compatible connection

  (Compatible $\Leftrightarrow \omega^a{}_b \in \mathrm{ad}(P) \Leftrightarrow \nabla g = 0$)
- Note: Levi-Civita connection exists uniquely
- Curvatures $[\nabla, \nabla]$ give action & eqns of motion
- All defined purely by the $O(d-1,1)$ structure

Generalised Geometry: The Plan

- Define generalised tangent bundle as generators of bosonic symmetries of SUGRA [Structure group: $O(d,d)$ or $E_{d(d)}$]
- Define analogues of Lie derivative, connections and torsion
- Define principal sub-bundle using bosonic fields of SUGRA [Structure group: $O(d) \times O(d)$ or H_d]
- Find analogue of Levi-Civita and resulting curvatures

11D Supergravity [Cremmer, Julia & Scherk '78]

- Field content $\{g_{\mu\nu}, \mathcal{A}_{\mu\nu\rho}, \psi_\mu\}$ with $\mathcal{F} = \mathrm{d}\mathcal{A}$
- Bosonic Action

$$S_{\mathsf{B}} \sim \int \left(\mathrm{vol}_g \, \mathcal{R} - \tfrac{1}{2}\mathcal{F} \wedge *\mathcal{F} - \tfrac{1}{6}\mathcal{A} \wedge \mathcal{F} \wedge \mathcal{F}\right)$$

- Supersymmetry

$$\delta\psi_\mu = \nabla_\mu \epsilon + \tfrac{1}{288}\left(\Gamma_\mu{}^{\nu_1 \dots \nu_4} - 8\delta_\mu{}^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}\right)\mathcal{F}_{\nu_1 \dots \nu_4}\epsilon,$$

Restricting to d dimensions

- Warped metric ansatz $(m, n = 1, \ldots, d)$

$$\mathrm{d}s_{11}^2 = \mathrm{e}^{2\Delta(x)}\eta_{\mu\nu}\mathrm{d}y^\mu \mathrm{d}y^\nu + g_{mn}(x)\mathrm{d}x^m\mathrm{d}x^n$$

- Internal gauge field $A_{mnp} = \mathcal{A}_{mnp}$ and field strength $F = \mathrm{d}A$
- (If $d \geq 7$) Dual field strength $\tilde{F}_{m_1 \ldots m_7} \sim *_{(11)}\mathcal{F}_{m_1 \ldots m_7}$
- Introduce 6-form potential $\tilde{A}_{m_1 \ldots m_6}$ s.t. $\tilde{F} = \mathrm{d}\tilde{A} - \frac{1}{2}A \wedge F$
- Gauge transformation: $(\Lambda \in \Lambda^2 T^* M$, $\tilde{\Lambda} \in \Lambda^5 T^* M)$

$$A' = A + \mathrm{d}\Lambda \qquad \tilde{A}' = \tilde{A} + \mathrm{d}\tilde{\Lambda} - \tfrac{1}{2}\mathrm{d}\Lambda \wedge A$$

Restricting to d dimensions

- Fields are $\{g_{mn}, A_{mnp}, \tilde{A}_{m_1 \ldots m_6}, \Delta, \psi_m, \rho\}$
- Action $(c + d = 11)$

$$S_{\mathsf{B}} = \frac{1}{2\kappa^2} \int \sqrt{g}\, \mathrm{e}^{c\Delta} \left(\mathcal{R} + c(c-1)(\partial\Delta)^2 - \tfrac{1}{2}\tfrac{1}{4!} F^2 - \tfrac{1}{2}\tfrac{1}{7!} \tilde{F}^2 \right)$$

Action of $E_{d(d)} \times \mathbb{R}^+$ in $GL(d,\mathbb{R})$ representations

- Consider a vector space F of dimension $d \leq 7$, and set

$$W_1 = F \oplus \Lambda^2 F^* \oplus \Lambda^5 F^* \oplus (F^* \otimes \Lambda^7 F^*)$$

$$W_2 = \mathbb{R} \oplus (F \otimes F^*) \oplus \Lambda^3 F^* \oplus \Lambda^6 F^* \oplus \Lambda^3 F \oplus \Lambda^6 F$$

- These are $GL(d,\mathbb{R})$ decompositions

$$W_1 \sim \begin{cases} \mathbf{10}_{+1} & E_{4(4)} \times \mathbb{R}^+ \\ \mathbf{16}_{+1} & E_{5(5)} \times \mathbb{R}^+ \\ \mathbf{27}_{+1} & E_{6(6)} \times \mathbb{R}^+ \\ \mathbf{56}_{+1} & E_{7(7)} \times \mathbb{R}^+ \end{cases} \qquad W_2 \sim \mathrm{ad}(E_{d(d)} \times \mathbb{R}^+)$$

Action of $\Lambda^3 F^* \oplus \Lambda^6 F^*$

- Take $a \in \Lambda^3 F^*$ and $\tilde{a} \in \Lambda^6 F^*$ and

$$V = v + \omega + \sigma + \tau \in W_1$$

- Then action of $a + \tilde{a}$ on V given by

$$(a+\tilde{a}) \cdot V = (0) + (v \lrcorner a) + (v \lrcorner \tilde{a} + a \wedge \omega) + (ja \wedge \sigma - j\tilde{a} \wedge \omega)$$

- This exponentiates to

$$\begin{aligned} e^{a+\tilde{a}} \cdot V = v &+ (\omega + i_v a) \\ &+ \big(\sigma + a \wedge \omega + \tfrac{1}{2} a \wedge i_v a + i_v \tilde{a}\big) \\ &+ \big(\tau + ja \wedge \sigma - j\tilde{a} \wedge \omega + \tfrac{1}{2} ja \wedge a \wedge \omega \\ &\quad + \tfrac{1}{2} ja \wedge i_v \tilde{a} - \tfrac{1}{2} j\tilde{a} \wedge i_v a + \tfrac{1}{6} ja \wedge a \wedge i_v a\big), \end{aligned}$$

Generators of Supergravity Symmetries

- Have the following actions of symmetries

Symmetry	Generator	Action
Diffeo	$v \in TM$	$\delta_v = \mathcal{L}_v$
Gauge	$\omega \in \Lambda^2 T^* M$	$\begin{cases} \delta A = \mathrm{d}\omega \\ \delta \tilde{A} = -\frac{1}{2}\mathrm{d}\omega \wedge A \end{cases}$
Gauge	$\sigma \in \Lambda^5 T^* M$	$\delta \tilde{A} = \mathrm{d}\sigma$

- But A, $\tilde{A}$ only locally defined on $U_{(i)} \subset M$

$$\begin{aligned} A_{(i)} &= A_{(j)} + \mathrm{d}\Lambda_{(ij)}, \\ \tilde{A}_{(i)} &= \tilde{A}_{(j)} + \mathrm{d}\tilde{\Lambda}_{(ij)} - \tfrac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge A_{(j)}. \end{aligned}$$

- This $\Rightarrow$ patching rules for v, ω, σ

Generalised Tangent Bundle [Hull '07; Pacheco & Waldram '08]

- Consider a bundle

$$E \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M)$$

- Define s.t. on patches $U_{(i)} \subset M$ represent section as

$$V_{(i)} \in \Gamma(TU_i \oplus \Lambda^2 T^*U_i \oplus \Lambda^5 T^*U_i \oplus (T^*U_i \otimes \Lambda^7 T^*U_i))$$

- Twisted by gauge transformations between patches

$$V_{(i)} = \mathrm{e}^{\mathrm{d}\Lambda_{(ij)} + \mathrm{d}\tilde{\Lambda}_{(ij)}} \cdot V_{(j)}$$

- This ensures sections of E $\leftrightarrow$ generators of symmetries

Generalised Tangent Bundle [Hull '07; Pacheco & Waldram '08]

- Crucially $\mathrm{e}^{\mathrm{d}\Lambda_{(ij)}+\mathrm{d}\tilde{\Lambda}_{(ij)}} \in E_{d(d)}\times\mathbb{R}^+$
- Parabolic ("geometric") subgroup

$$GL(d,\mathbb{R}) \ltimes \text{"Gauge"} \subset E_{d(d)}\times\mathbb{R}^+$$

- E contains topological data of TM and gauge fields

Generalised Frame Bundle

- Can choose coordinates x^m on patch $U_{(i)}$
- Construct coordinate basis for E as

$$\{\hat{E}_M\} = \left\{\frac{\partial}{\partial x^m}\right\} \oplus \{\tfrac{1}{2}\mathrm{d}x^m \wedge \mathrm{d}x^n\} \oplus \dots$$

- Coordinate index $M = 1, \dots, \dim(E)$ for $V \in \Gamma(E)$

$$V_{(i)} = V^M \hat{E}_M = v^m_{(i)} \frac{\partial}{\partial x^m} + \tfrac{1}{2}\omega_{(i)mn}\mathrm{d}x^m \wedge \mathrm{d}x^n + \dots$$

- Generalised frame bundle

$$F = \left\{(x, \{\hat{E}_A\}) : \; x \in M \;\&\; \{\hat{E}_A\} \text{ related to } \{\hat{E}_M\} \text{ by } E_{d(d)} \times \mathbb{R}^+\right\}$$

Partial Derivative

- Have embedding

$$T^*M \to E^* \simeq T^*M \oplus \Lambda^2 TM \oplus \dots$$

- So can write

$$\partial_M = \begin{cases} \partial_m & \text{M} = \text{m} \\ 0 & \text{otherwise} \end{cases}$$

Dorfman Derivative

- For $V \in \Gamma(E)$ define a derivative

$$L_V = V \cdot \partial - \big(\partial \times_{\mathrm{ad}(F)} V\big)$$

- Diffeo and gauge invariant $\Rightarrow$ well-defined on E
- Leibnitz property

$$[L_U, L_V] = L_{L_U V}$$

- $\delta_V = L_V$ generates the bosonic symmetries of SUGRA

$$L_V \sim \Big[\mathcal{L}_v - (\mathrm{d}\omega + \mathrm{d}\sigma)\,\cdot\,\Big]$$

- Exceptional Courant bracket: $[\![V, V']\!] = \frac{1}{2}(L_V V' - L_{V'} V)$

Generalised Connections and Torsion [Alekseev & Xu '01; Gualtieri '07]

- Take $\Omega_M \in \mathrm{ad}(E_{d(d)} \times \mathbb{R}^+)$ and set

$$D_M W^A = \partial_M W^A + \Omega_M{}^A{}_B W^B$$

- For $E_{d(d)} \times \mathbb{R}^+$ tensor α, define $T(V) \in \mathrm{ad}(E_{d(d)} \times \mathbb{R}^+)$

$$T(V) \cdot \alpha = L_V^{(\partial \to D)} \alpha - L_V \alpha$$

- Find that

$$T_C{}^A{}_B \in K \oplus E^* \subset E^* \otimes \mathrm{ad}(E_{d(d)} \times \mathbb{R}^+)$$

- K matches the embedding tensor, e.g. $\mathbf{912}_{-1}$ for $d = 7$

The bundle N

- Another $E_{d(d)} \times \mathbb{R}^+$ bundle is given by $N \subset S^2 E$

$$N \simeq T^*M \oplus \Lambda^4 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M) \oplus (\Lambda^3 T^*M \otimes \Lambda^7 T^*M) \oplus (\Lambda^6 T^*M \otimes \Lambda^7 T^*M)$$

$$N \sim \begin{cases} \mathbf{133}_{+2} & d = 7 \\ \mathbf{27'}_{+2} & d = 6 \\ \mathbf{10}_{+2} & d = 5 \\ \mathbf{5'}_{+2} & d = 4 \end{cases}$$

- Find that $\partial f \times_{N^*} \partial g = 0$
- Section condition of approaches with extra coordinates

[Berman, H.& M. Godazger & Perry '11; Berman, Cederwall, Kleinschmidt & Thompson '12]

N, Jacobi and Curvature

- Another feature of N is that (for $d \leq 6$)

$$L_V V' + L_{V'} V = \partial \times_E \left(V \times_N V'\right)$$

- Jacobiator of Courant bracket

$$\mathrm{Jac}(V, V', V'') \sim \partial \times_E \left([V, V'] \times_N V''\right) + \text{“cyclic”}$$

- If $V \otimes_N V' = 0$ then we have linear curvature operator

$$[D_V, D_{V'}] - D_{[\![V,V']\!]}$$

- Projections to N measure the failure of all of these things

H_d Group

- Maximal compact subgroup of $E_{d(d)} \times \mathbb{R}^+$

$E_{d(d)}$	$\tilde{H}_d$	$h^\perp = \mathrm{ad}(E_{d(d)} \times \mathbb{R}^+)/\,\mathrm{ad}(H_d)$
$E_{7(7)}$	$SU(8)$	$\mathbf{35} + \bar{\mathbf{35}} + \mathbf{1}$
$E_{6(6)}$	$USp(8)$	$\mathbf{42} + \mathbf{1}$
$Spin(5,5)$	$Spin(5) \times Spin(5)$	$(\mathbf{5},\mathbf{5}) + (\mathbf{1},\mathbf{1})$
$SL(5,\mathbb{R})$	$Spin(5)$	$\mathbf{14} + \mathbf{1}$

Supergravity Fields and the Generalised Metric

- Well known that

$$\{g_{mn}, A_{mnp}, \tilde{A}_{m_1\dots m_6}, \Delta\} \in \frac{E_{d(d)}\times\mathbb{R}^+}{H_d}$$

- On patch $U_i \subset M$ can build generalised metric G from fields.
- Patching of gauge fields

$$\begin{aligned} A_{(i)} &= A_{(j)} + \mathrm{d}\Lambda_{(ij)}, \\ \tilde{A}_{(i)} &= \tilde{A}_{(j)} + \mathrm{d}\tilde{\Lambda}_{(ij)} - \tfrac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge A_{(j)}. \end{aligned}$$

ensures that $G(V, V')$ is well-defined scalar

Conformal Split Frame

- Special class of frames $\sim$ "vielbeins" of G
- Take $\{\hat{e}_a\}$ vielbein for g_{mn} and dual basis $\{e^a\}$ for T^*M
- Build "conformal split frame"

$$\begin{aligned}
\hat{E}_a &= \mathrm{e}^{\Delta}\Big(\hat{e}_a + i_{\hat{e}_a}A + i_{\hat{e}_a}\tilde{A} + \tfrac{1}{2}A\wedge i_{\hat{e}_a}A \\
&\qquad\qquad + jA\wedge i_{\hat{e}_a}\tilde{A} + \tfrac{1}{6}jA\wedge A\wedge i_{\hat{e}_a}A\Big), \\
\hat{E}^{ab} &= \mathrm{e}^{\Delta}\left(e^{ab} + A\wedge e^{ab} - j\tilde{A}\wedge e^{ab} + \tfrac{1}{2}jA\wedge A\wedge e^{ab}\right), \\
\hat{E}^{a_1\dots a_5} &= \mathrm{e}^{\Delta}\left(e^{a_1\dots a_5} + jA\wedge e^{a_1\dots a_5}\right), \\
\hat{E}^{a,a_1\dots a_7} &= \mathrm{e}^{\Delta}e^{a,a_1\dots a_7},
\end{aligned}$$

Conformal Split Frame

- In this frame write

$$V = v^a \hat{E}_a + \tfrac{1}{2}\omega_{ab}\hat{E}^{ab} + \tfrac{1}{5!}\sigma_{a_1\ldots a_5}\hat{E}^{a_1\ldots a_5} + \tfrac{1}{7!}\tau_{a,a_1\ldots a_7}\hat{E}^{a,a_1\ldots a_7}$$

- Set $v = v^a \hat{e}_a \in TM$, $\omega = \frac{1}{2}\omega_{ab}e^{ab} \in \Lambda^2 T^* M$, etc...
- Remark: These are patch independent!
- In this frame

$$G(V,V) = |v|^2 + |\omega|^2 + |\sigma|^2 + |\tau|^2$$

- General H_d frame defined by this
- H_d frames define principal sub-bundle $P \subset F$ with fibre H_d

Volume form

- H_d structure provides volume form related to $\det(G)$
- In coordinate frame this evaluates as

$$\mathrm{vol}_G = \sqrt{g}\mathrm{e}^{(c-2)\Delta}$$

Compatible Connections

- H_d compatible connection defined by

$$DG = 0$$

- Can build from Levi-Civita for g_{mn}

$$D_M V^A = D_M^{(\nabla)} V^A + \Sigma_M{}^A{}_B V^B$$

- $\exists$ family of Σ s.t. D torsion-free compatible (Not unique!!!)

- $T = 0 \Rightarrow$ Some cpts of Σ fixed to be $F, \tilde{F}, \mathrm{d}\Delta$

Torsion-free compatible connection [Hilmann '09; Grana & Orsi '11]

- H_d algebra $\sim \Lambda^2 T^* M \oplus \Lambda^3 T^* M \oplus \Lambda^6 T^* M$ under $SO(d)$
- $\Sigma(V) = V^M \Sigma_M \in \mathrm{ad}(H_d)$ then acts as

$$\Sigma(V)_{ab} = \mathrm{e}^{\Delta}\Big(2\left(\tfrac{7-d}{d-1}\right) v_{[a}\partial_{b]}\Delta + \tfrac{1}{4!}\omega_{cd}F^{cd}{}_{ab} + \tfrac{1}{7!}\sigma_{c_1\dots c_5}\tilde{F}^{c_1\dots c_5}{}_{ab} + C(V)_{ab}\Big)$$

$$\Sigma(V)_{abc} = \mathrm{e}^{\Delta}\left(\tfrac{6}{(d-1)(d-2)}(\mathrm{d}\Delta\wedge\omega)_{abc} + \tfrac{1}{4}v^d F_{dabc} + C(V)_{abc}\right)$$

$$\Sigma(V)_{a_1\dots a_6} = \mathrm{e}^{\Delta}\left(\tfrac{1}{7}v^b\tilde{F}_{ba_1\dots a_6} + C(V)_{a_1\dots a_6}\right)$$

- $C(V)$ the undetermined parts

Spinors and $\tilde{H}_d$ representations

- Actually take double cover $\tilde{H}_d$
- Can embed $\tilde{H}_d$ algebra in $\mathrm{Cliff}(d)$ as $\{\gamma^{(2)}, \pm\gamma^{(3)}, \gamma^{(6)}\}$
- Spinor of $Spin(d)$ becomes $S^\pm$, fundamentals of $\tilde{H}_d$
- Gravitino ψ_m becomes representations $J^\pm$ of $\tilde{H}_d$

Even d	Odd d
$S \simeq S^+ \simeq S^-$	$S = S^+ \oplus S^-$
$J \simeq J^+ \simeq J^-$	$J = J^+ \oplus J^-$

Torsion-free compatible (spin) connection [Hilmann '09; Grana & Orsi '11]

- Connection can then act on spinor ($\in S^{\pm}$) by

$$
\begin{aligned}
D_a &= \mathrm{e}^{\Delta}\Big(\nabla_a + \tfrac{1}{2}\left(\tfrac{7-d}{d-1}\right)(\partial_b\Delta)\gamma_a{}^b \pm \tfrac{1}{2}\tfrac{1}{4!}F_{ab_1b_2b_3}\gamma^{b_1b_2b_3} \\
&\qquad\qquad - \tfrac{1}{2}\tfrac{1}{7!}\tilde{F}_{ab_1\dots b_6}\gamma^{b_1\dots b_6} + \not{C}_a\Big) \\
D^{a_1a_2} &= \mathrm{e}^{\Delta}\left(\tfrac{1}{4}\tfrac{2!}{4!}F^{a_1a_2}{}_{b_1b_2}\gamma^{b_1b_2} \pm \tfrac{3}{(d-1)(d-2)}(\partial_b\Delta)\gamma^{a_1a_2b} + \not{C}^{a_1a_2}\right) \\
D^{a_1\dots a_5} &= \mathrm{e}^{\Delta}\left(\tfrac{1}{4}\tfrac{5!}{7!}\tilde{F}^{a_1\dots a_5}{}_{b_1b_2}\gamma^{b_1b_2} + \not{C}^{a_1\dots a_5}\right) \\
D^{a,a_1\dots a_7} &= \mathrm{e}^{\Delta}\left(\not{C}^{a,a_1\dots a_7}\right)
\end{aligned}
$$

Unique operators

For torsion-free compatible D, and $\varepsilon \in S$, $\psi \in J$

$$D \times_J \varepsilon \qquad D \times_S \varepsilon$$
$$D \times_J \psi \qquad D \times_S \psi$$

are uniquely defined operators, independent of the choice of D

SUSY and fermion equations

- SUGRA theory contains fermions ψ_m and ρ
- These can be promoted to $\tilde{H}_d$ objects $\psi \in J$ and $\rho \in S$
- Their SUSY variations are

$$\delta\psi = (D \times_J \varepsilon)$$
$$\delta\rho = (D \times_S \varepsilon)$$

- Their equations of motion are

$$(D \times_J \psi) + (D \times_J \rho) = 0$$
$$(D \times_S \psi) + (D \times_S \rho) = 0$$

Curvatures and bosonic equations

- Closure of SUSY algebra $\Rightarrow$ tensors R^0 and R

$$D \times_J (D \times_J \varepsilon) + D \times_J (D \times_S \varepsilon) = R^0 \cdot \varepsilon,$$
$$D \times_S (D \times_J \varepsilon) + D \times_S (D \times_S \varepsilon) = R\, \varepsilon,$$

- R^0 and R are the 2 parts of Ricci tensor R_{AB}
- R_{AB} lives in the representation $h^\perp$
- I.e. R_{AB} has same degrees of freedom as bosonic fields

Alternative Form of Curvature

- Can also write the curvature as

$$(D \wedge D) \times_J \varepsilon + (D \times_{N^*} D) \times_J \varepsilon = R^0 \cdot \varepsilon,$$
$$(D \wedge D) \times_S \varepsilon + (D \times_{N^*} D) \times_S \varepsilon = R\varepsilon.$$

- This guarantees that 2nd partial derivatives of ε vanish since

$$\partial \wedge \partial = 0 \qquad \partial \times_{N^*} \partial = 0$$

$d = 7$ in $SU(8)$ indices

- Under $SU(8)$ have $E \sim \mathbf{56} \to \mathbf{28} + \bar{\mathbf{28}}$ so that

$$(V^A) \to (V^{[\alpha\beta]}, \bar{V}_{[\alpha\beta]})$$

- Spin representations:

$$S \sim \mathbf{8} + \bar{\mathbf{8}} \qquad J \sim \mathbf{56} + \bar{\mathbf{56}}$$

$$\varepsilon \to (\varepsilon^{\alpha}, \bar{\varepsilon}_{\alpha}) \qquad \psi \to (\psi^{[\alpha\beta\gamma]}, \bar{\psi}_{[\alpha\beta\gamma]})$$

- Generalised metric $G(V, V') = \frac{1}{2}(V^{\alpha\beta}\bar{V}'_{\alpha\beta} + \bar{V}_{\alpha\beta}V'^{\alpha\beta})$

Unique operators in $SU(8)$ indices [de Wit & Nicolai '86; Hilmann '09]

- Unique derivatives are

$$D^{[\alpha\beta}{}_{\varepsilon}{}^{\gamma]} \qquad \bar{D}_{\alpha\beta}\varepsilon^{\beta}$$

$$\epsilon_{[\alpha\beta\gamma]\theta_1\theta_2\theta_3\theta_4\theta_5} D^{\theta_1\theta_2}\psi^{\theta_3\theta_4\theta_5} \qquad \bar{D}_{\beta\gamma}\psi^{\alpha\beta\gamma}$$

- SUSY variations of fermions

$$\delta\psi^{\alpha\beta\gamma} = D^{[\alpha\beta}{}_{\varepsilon}{}^{\gamma]} \qquad \delta\rho_\alpha = -\bar{D}_{\alpha\beta}\varepsilon^{\beta}$$

- Fermion equations of motion

$$-\frac{1}{12}\,\epsilon_{[\alpha\beta\gamma]\theta_1\theta_2\theta_3\theta_4\theta_5} D^{\theta_1\theta_2}\psi^{\theta_3\theta_4\theta_5} + 2\bar{D}_{[\alpha\beta}\rho_{\gamma]} = 0$$

$$D^{\alpha\beta}\rho_\beta - \frac{1}{2}\bar{D}_{\beta\gamma}\psi^{\alpha\beta\gamma} = 0$$

Curvature in $SU(8)$ indices [de Wit & Nicolai '86; Hilmann '09]

- Curvature

$$-\tfrac{1}{12}\,\epsilon_{\alpha\beta\gamma\delta\delta'\epsilon\epsilon'\theta}D^{\delta\delta'}D^{\epsilon\epsilon'}\varepsilon^{\theta} + 2\bar{D}_{[\alpha\beta}\bar{D}_{\gamma]\delta}\varepsilon^{\delta} = R^{0}_{[\alpha\beta\gamma\delta]}\varepsilon^{\delta}$$

$$D^{\alpha\beta}\bar{D}_{\beta\gamma}\varepsilon^{\gamma} - \tfrac{1}{2}\bar{D}_{\beta\gamma}D^{[\alpha\beta}\varepsilon^{\gamma]} = R\varepsilon^{\alpha}$$

- $(R^{0}_{[\alpha\beta\gamma\delta]}, \bar{R}^{0\,[\alpha\beta\gamma\delta]}) \in \mathbf{35} + \bar{\mathbf{35}}$ (Complex self-duality condition)

- Scalar curvature comes out as

$$R \propto \mathrm{e}^{2\Delta}(\mathcal{R} - 6\nabla^2\Delta - 12(\partial\Delta)^2 - \tfrac{1}{2}\tfrac{1}{4!}F^2 - \tfrac{1}{2}\tfrac{1}{7!}\tilde{F}^2)$$

- Bosonic action

$$S_B \propto \int \mathrm{vol}_G\, R$$

Supergravity equations: summary

- Bosons:

$$S = \int \mathrm{vol}_G \, R \qquad \Rightarrow \qquad R_{AB} = 0$$

- SUSY

$$\begin{aligned} \delta\psi &= (D \times_J \varepsilon) \\ \delta\rho &= (D \times_S \varepsilon) \\ \delta G &= (\varepsilon \times_{h^\perp} \psi) \end{aligned}$$

- Fermion equations

$$\begin{aligned} (D \times_J \psi) + (D \times_J \rho) &= 0 \\ (D \times_S \psi) + (D \times_S \rho) &= 0 \end{aligned}$$

- All defined uniquely by the $\tilde{H}_d$ structure on E

Universal Construction of Fermions [Damour, Kleinschmidt & Nicolai '06]

- Above used decomposition of 11d spinors into $Spin(d)$
- Alternatively can embed

$$Spin(10-d,1) \times \tilde{H}_d \to \mathrm{Cliff}(10,1;\mathbb{R})$$

and act directly on 32 cpt $Spin(10,1)$ spinors

- Again: 2 inequivalent embeddings/representations

$$\{\Gamma^{mn}, \pm\Gamma^{mnp}, \Gamma^{m_1\dots m_6}\} \text{ on } \hat{S}^{\pm}$$

- Find $\epsilon \in \hat{S}^-$, $\rho \in \hat{S}^+$ and $\psi_m \in \hat{J}^-$
- Provides dimension independent expressions for fermions

Type II theories with RR flux [Hull '07; Grana, Louis, Sim, Waldram '09]

- Can do same thing with $GL(d-1) \subset E_{d(d)} \times \mathbb{R}^+$
- Two inequivalent embeddings $GL(d-1) \subset E_{d(d)} \times \mathbb{R}^+$
  $\rightarrow$ results in IIA and IIB
- Decompositions related by $Pin(d-1, d-1)$ transformation

Compactifications

- Globally defined spinors $\{\varepsilon\}$ on M_{int}
  $\rightarrow$ SUSY in effective theory

- E.g. N=1 in 4d $\Leftrightarrow$ $SU(7)$ structure on M_7
  N=2 in 4d $\Leftrightarrow$ $SU(6)$ structure on M_7

- For identity structure case (maximal SUSY)

Embedding tensor $\leftrightarrow$ Generalised torsion

[Aldazabal, Grana, Marqus & Rosabal '13]

Supersymmetric Backgrounds

- Clear that $D_M \varepsilon = 0 \Rightarrow$ SUSY
- In fact SUSY $\Rightarrow \exists$ torsion-free D s.t. $D_M \varepsilon = 0$
- Analogue of special holonomy for D
- SUSY background $=$ Torsion-free generalised G-structure

Conclusions

- $E_{d(d)} \times \mathbb{R}^+$ generalised geometrical description of supergravity
- Bosonic sector $\leftrightarrow$ Einstein gravity in generalised geometry
- Obtain all equations with local $\tilde{H}_d$ symmetry

Further extensions?

- What happens for $d \geq 8$?
  - Need to understand (non-linear) dual gravity
- Superalgebra?
  - $T = 0$ gives closure of SUSY algebra
- Higher derivative corrections?
- Other supergravities?
- Non-geometric backgrounds?
- Maths: Interesting new kind of algebroids?

The End

- Thanks for your attention!