Hyperconifold singularities and transitions

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Based on (**arXiv: 0911.0708, 1102.1428**) plus work in progress

Outline

Introduction/Overview

Toric Geometry and Mirror Symmetry

Topology and resolutions

Applications

Hyperconifold Transitions in String Theory

Summary

Hyperconifold Singularities

- The conifold, $\mathcal{C}$, is the simplest singularity of a Calabi–Yau threefold:

$$\mathcal{C} \;=\; \{\; y_1 y_2 - y_3 y_4 = 0 \mid (y_1, y_2, y_3, y_4) \in \mathbb{C}^4 \;\}$$

- We can take quotients $\mathcal{C}/G$, G a finite group of symmetries of $\mathcal{C}$.
- When only $\vec{0}$ is fixed, we get new *isolated* singularities – hyperconifolds.
- Interesting features:
  - Hyperconifolds occur naturally in *compact* CY3's.
  - Can be either deformed or resolved $\longrightarrow$ hyperconifold *transitions*
  - Mirror to ordinary conifolds

Multiply-Connected Calabi-Yau Threefolds

- CY3's X with $\pi_1(X) \neq 1$ are of particular interest:
  - Wilson line gauge symmetry breaking in heterotic models
  - Most manifolds with small Hodge numbers are in this class
  - One of two independent torsion subgroups of CY3 (co)homology
- Typically, $X = \tilde{X}/G$, where
  - $\tilde{X}$ is a complete intersection in a toric variety T, $\pi_1(\tilde{X}) = 1$
  - G acts on T; generic invariant $\tilde{X}$ misses the fixed points
- $\tilde{X}$ can often be deformed to intersect a unique G-fixed point

$$\Rightarrow G\text{-hyperconifold singularity on } X.$$

Note: Focus only on cyclic groups $\mathbb{Z}_n$.

Example

Consider $\tilde{X} \cong \begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Let $\{Y_i\}$ and $\{Z_m\}$ be coordinates on the two $\mathbb{P}^2$. Define a $\mathbb{Z}_3$ action:

$$Y_i \to \zeta^i Y_i \ , \ Z_m \to \zeta^m Z_m \quad \zeta = e^{2\pi \mathrm{i}/3}$$

Let $\tilde{X}$ be defined by an invariant polynomial. Then:

- $\tilde{X}$ is generically smooth.
- $\tilde{X}$ avoids the fixed points, so $X = \tilde{X}/\mathbb{Z}_3$ is smooth.
- $\pi_1(X) \cong \mathbb{Z}_3$

Example (Continued)

- Local coordinates: $y_1 = Y_1/Y_0, y_2 = Y_2/Y_0, z_1 = Z_1/Z_0, z_2 = Z_2/Z_0$

$$(y_1, y_2, z_1, z_2) \to (\zeta y_1, \zeta^2 y_2, \zeta z_1, \zeta^2 z_2)$$

- Expand invariant polynomial:

$$p = \alpha_0 + \alpha_1 y_1 y_2 + \alpha_2 y_1 z_2 + \alpha_3 y_2 z_1 + \alpha_4 z_1 z_2 + \dots$$

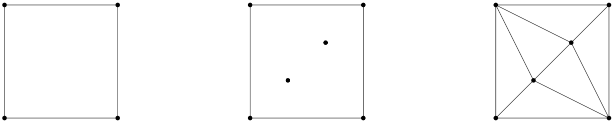
- The origin is the unique $\mathbb{Z}_3$-fixed point in this patch, $p(\vec{0}) = \alpha_0$.
- With $\alpha_0 = 0$, we get a conifold singularity at $\vec{0}$.

$X = \tilde{X}/\mathbb{Z}_3$ develops a $\mathbb{Z}_3$-hyperconifold singularity.

Resolution

Obvious question: can we resolve the singularity?

Toric geometry makes the analysis easy:



Conifold $\mathbb{Z}_3$-hyperconifold Resolution

This is manifestly crepant; one can check that it is projective.

$\mathbb{Z}_3$-hyperconifold *transition* $X^{2,29} \to \hat{X}^{4,28}$

The exceptional set is simply-connected $\Rightarrow \pi_1(\hat{X}) \cong 1$

Outline

Classification

Context:

- Complete intersection CY3 $\tilde{X}$ in a toric variety
- $\mathbb{Z}_n$ acts linearly on homogeneous coordinates of ambient space
- Invariant $\tilde{X}$ is smooth and misses fixed points; $X = \tilde{X}/\mathbb{Z}_n$ is smooth

Near a fixed point, choose coordinates on which $\mathbb{Z}_n$ acts diagonally.

> **Conjecture:** *Locally, the system can be reduced to a single invariant polynomial p on a 4D slice, with non-degenerate quadratic piece.*

The coordinates (y_1, y_2, y_3, y_4) must each transform with a *primitive* n^{th} root of unity, and therefore pair up to make invariants

$$p = \alpha + y_1 y_4 - y_2 y_3 + \ldots$$

Classification — Toric Diagrams

$$\alpha \to 0 \;\Rightarrow\; p = y_1 y_4 - y_2 y_3 + \dots$$

The covering space thus develops a conifold singularity.

Toric coordinates

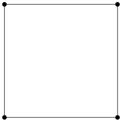
Let (t_1, t_2, t_3) parametrise the torus $\left(\mathbb{C}^*\right)^3$. Embedding:

$$y_1 = \frac{t_1}{t_3} \;,\; y_2 = t_2 \;,\; y_3 = \frac{t_1}{t_2} \;,\; y_4 = t_3$$

Work out the fan; a single cone generated by vertices

$$(1,0,0) \;,\; (1,1,0) \;,\; (1,0,1) \;,\; (1,1,1)$$

Toric diagram:



Classification — Toric Diagrams

$$p = y_1 y_4 - y_2 y_3 + \ldots$$

WLOG, assume $(y_1, y_2, y_3, y_4) \to (\zeta y_1, \zeta^k y_2, \zeta^{-k} y_3, \zeta^{-1} y_4)$, $\zeta := e^{2\pi \mathrm{i}/n}$, k relatively prime to n.[1] These actions are all subgroups of the torus:

$$t_1 \to t_1 \ , \ t_2 \to e^{2\pi \mathrm{i} k/n} t_2 \ , \ t_3 \to e^{-2\pi \mathrm{i}/n} t_3$$

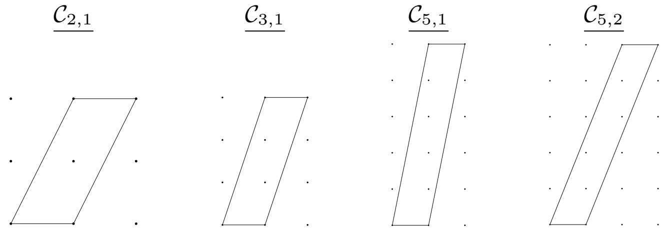
The quotient corresponds to refining the lattice; choosing a basis for the new lattice, the cone now has vertices at

$$(1,0,0) \ , \ (1,1,0) \ , \ (1,k,n) \ , \ (1,k+1,n)$$

(n,k)-hyperconifold: $\mathcal{C}_{n,k}$

[1] $k \sim \pm k^{\pm 1}$ by $y_2 \leftrightarrow y_3$, $(y_1, y_4) \leftrightarrow (y_2, y_3)$

Toric Diagrams — Examples



(Note: These all look nicer in coordinates where the diagram is 'nearly square'.)

Mirror Symmetry

Given a toric Calabi–Yau, its mirror is given by (Gross, **math/0012002**) :

$$\{F(u,v,y,z) := uv - f(y,z) = 0 \mid (u,v) \in \mathbb{C}^2 \ , \ (y,z) \in \left(\mathbb{C}^*\right)^2\} \ ,$$

where f is a Laurent polynomial, with Newton polygon the toric diagram.

Claim: *The mirror of any $\mathbb{Z}_n$-hyperconifold has n nodes (conifolds).*

Proof: Recalling the vertices, the mirror of $\mathcal{C}_{n,k}$ is given by

$$0 = F = uv - (1 + y + y^k z^n + y^{k+1} z^n) = uv - (1+y)(1 + y^k z^n)$$

Singularities occur when

$$F = dF = 0 \iff u = v = 0 \ , \ y = -1 \ , \ z^n = (-1)^{k+1} \ ,$$

hence there are n singular points. Easy to check Hessian is non-zero. □

Mirror Symmetry

Counter-example(s) to naïve expectation that conifold $\overset{\text{mirror}}{\longleftrightarrow}$ conifold.

- Compact X with $\mathbb{Z}_n$-hyperconifold $\overset{\text{mirror}}{\longleftrightarrow}$ compact Y with n nodes
- Deformation of one is mirror to resolution of the other.
- Explicitly checked for some examples in (RD, 1102.1428)

Outline

Topology

The conifold $\mathcal{C}$ is a cone over $S^3 \times S^2$. Evslin & Kuperstein (hep-th/0702041) give parametrisation:

$$W := \begin{pmatrix} y_1 & y_2 \\ y_2 & y_4 \end{pmatrix} \ , \quad \mathcal{C} \ : \ \det W = 0$$

Base $S^3 \times S^2$ is $|y_1|^2 + |y_2|^2 + |y_3|^2 + |y_4|^2 = \mathrm{Tr}(W^\dagger W) = 1$. Write

$$W = X v v^\dagger \ ,$$

where

$$X \in SU(2) \ , \quad v \in \mathbb{P}^1 \ (||v|| = 1; \text{phase irrelevant}) \ .$$

Topology

Base $S^3 \times S^2$ is $\mathrm{Tr}(W^\dagger W) = 1$. Write

$$W = X v v^\dagger \ , \quad X \in SU(2) \ , \quad v \in \mathbb{C}^2, ||v|| = 1 \ .$$

Action for $\mathcal{C}_{n,k}$ is

$$(y_1, y_2, y_3, y_4) \to (\zeta y_1, \zeta^k y_2, \zeta^{-k} y_3, \zeta^{-1} y_4) \ , \ \zeta = e^{2\pi \mathrm{i}/n}$$

which is realised by

$$X \to \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-k} \end{pmatrix} X \begin{pmatrix} 1 & 0 \\ 0 & \zeta^{k-1} \end{pmatrix} \ , \quad v \to \begin{pmatrix} 1 & 0 \\ 0 & \zeta^{1-k} \end{pmatrix} v \ .$$

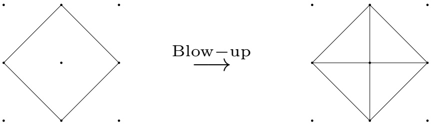
Vanishing 3-cycle is $S^3/\mathbb{Z}_n$; check that this action gives lens space $L(n,k)$.

(Lens spaces: k rel prime to n, and $k \sim \pm k^{\pm 1}$)

Resolutions

X_0 has a hyperconifold singularity; is there a smooth Calabi-Yau $\hat{X} \to X_0$?

- Blowing up the $\mathbb{Z}_2$-hyperconifold resolves it:



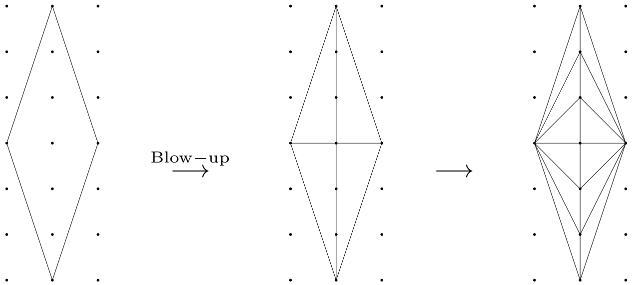
- Actually, blow-up commutes with quotient:
  - $\mathrm{Bl}_{\vec{0}}(\mathcal{C}) \;\cong\; \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1) \;\; ; \;\; \frac{\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)}{\mathbb{Z}_2} \;\cong\; \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,-2)$

- Similarly, blowing up any $\mathbb{Z}_{2m}$-hyperconifold gives an orbifold CY.

- These all have CY resolutions.

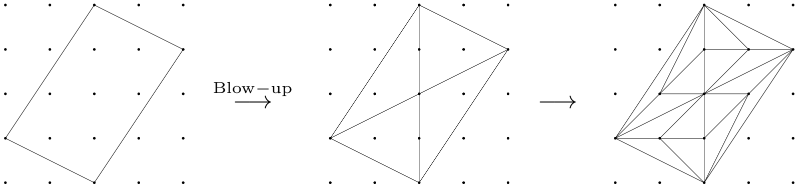
Example

Resolving $\mathcal{C}_{6,1}$:



Example

Resolving $\mathcal{C}_{8,3}$:



Resolving 'Odd' Hyperconifolds

What about $\mathbb{Z}_{2m+1}$-hyperconifolds? Need a more general approach.

Assume:

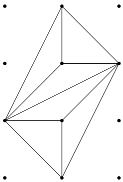
- X_0 has a single $\mathbb{Z}_n$-hyperconifold singularity at a point p.
- $Cl(X_0)$ has a basis of divisors which do not intersect p.
- $\pi : \hat{X} \to X_0$ is some (analytic) resolution map, with exceptional set E.

Let ω_0 be a Kähler form on X_0. Then $\pi^*\omega_0$ integrates to zero on all sub-varieties of E. So if we can find a 'local Kähler form' ω_L, built out of divisors contained in E, $\pi^*\omega_0 + \epsilon\omega_L$ will be a Kähler form for small $\epsilon > 0$.

So a Kähler resolution depends on the existence of a 'local Kähler form'.

Example: Resolving $\mathcal{C}_{3,1}$

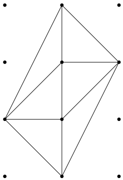
Two resolutions of $\mathcal{C}_{3,1}$; let $t_1 D_1 + t_2 D_2$ be putative local Kähler class

Case 1: 

$$D_1 \cdot C_1 = -3 \ , \ D_2 \cdot C_1 = 0 \qquad t_1 < 0$$

$$D_1 \cdot C_2 = 0 \ , \ D_2 \cdot C_2 = -3 \qquad t_2 < 0$$

$$D_1 \cdot C_3 = D_2 \cdot C_3 = 1 \qquad t_1 > 0 \ , \ t_2 > 0$$

Case 2: 

$$D_1 \cdot C_1 = -2 \ , \ D_2 \cdot C_1 = 1 \qquad -2t_1 + t_2 > 0$$

$$D_1 \cdot C_2 = 1 \ , \ D_2 \cdot C_2 = -2 \qquad t_1 - 2t_2 > 0$$

$$D_1 \cdot C_3 = D_2 \cdot C_3 = -1 \qquad -t_1 - t_2 > 0$$

So case 2 gives a Kähler resolution (e.g. $t_1 = t_2 < 0$); case 1 does not.

Hodge Numbers

Resolving a $\mathbb{Z}_n$-hyperconifold introduces $n-1$ new divisors

$$\Rightarrow \delta h^{1,1} = n-1$$

Asking that the CY hits a fixed point is one complex structure condition[2]

$$\Rightarrow \delta h^{2,1} = -1$$

These imply $\delta\chi = 2n$, which agrees with the toric calculation.

$$\underline{\mathbb{Z}_n\text{-hyperconifold transition: } \delta(h^{1,1}, h^{2,1}) = (n-1, -1)}$$

It is easy to calculate new intersection numbers from the toric diagrams.

[2] Not clear that this is always true, but true in examples.

Fundamental Group

Let $X = \tilde{X}/\mathbb{Z}_n$, and consider a $\mathbb{Z}_n$-hyperconifold transition:

$$X \overset{\text{def.}}{\leftrightsquigarrow} X_0 \overset{\text{res.}}{\leftarrow} \hat{X}$$

Topologically, this is a surgery:

- $\pi_1(X) \cong \mathbb{Z}_n$
- Delete a lens space $L(n,k)$; $\pi_1(L(n,k)) \cong \mathbb{Z}_n$
- Replace $L(n,k)$ with a simply-connected space
- $\implies \pi_1(\hat{X}) \cong 1$

Formally, this is a simple application of van Kampen's theorem.

Fundamental Group

More generally, suppose $X = \tilde{X}/G$ for some group G.

- $\mathbb{Z}_n \cong H \leq G$ develops a fixed point $\to \frac{|G|}{|H|}$ fixed points by symmetry
- X develops a single $\mathbb{Z}_n$-hyperconifold singularity
- Resolution $\hat{X}$, $\pi_1(\hat{X}) \cong G/H^G$, where H^G is normal closure of H in G

Outline

Constructing New Calabi–Yau Threefolds

- Hundreds of known spaces with $\pi_1 \neq 1$
- Most admit multiple hyperconifold transitions, e.g. (RD, 1102.1428)

$$\begin{matrix}\mathbb{P}^2 \\ \mathbb{P}^2\end{matrix}\begin{bmatrix}3 \\ 3\end{bmatrix}_{/\mathbb{Z}_3\times\mathbb{Z}_3} \cong X^{2,11} \overset{\mathbb{Z}_3}{\rightsquigarrow} X^{4,10} \overset{\mathbb{Z}_3}{\rightsquigarrow} X^{6,9} \overset{\mathbb{Z}_3}{\rightsquigarrow} X^{8,8}$$

  The last three spaces have $\pi_1 \cong \mathbb{Z}_3$; $X^{4,10}$ and $X^{8,8}$ were unknown.

- Previously unknown fundamental groups:
  - (Braun, 1003.3235) : S_3 does not act freely on any known CY3...
  - ...but $\mathrm{Dic}_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ does: $X^{1,4} = \tilde{X}/\mathrm{Dic}_3$.
  - Dic_3 has $\mathbb{Z}_2$ as a normal subgroup; $\mathrm{Dic}_3/\mathbb{Z}_2 \cong S_3$
  - $X^{1,4} \overset{\mathbb{Z}_2}{\rightsquigarrow} X^{2,3}$, with $\pi_1(X^{2,3}) \cong S_3$ (RD, 1103.3156)
- Simple way to get 'local cycles' (for swiss cheese models, etc.)

Connectivity of Moduli Spaces

- There is speculation that all CY3's are connected by transitions.
- Conifold transitions do not change π_1 (nor do flops).
- Perhaps all connected by conifold + hyperconifold (+ flop) transitions?

Outline

A Reminder: Conifold Transitions

Type IIB on a CY3 (Strominger, hep-th/9504090; Greene, Morrison, Strominger, hep-th/9504145) :

- Vanishing $S^3 \Rightarrow$ massless $4D$ hypermultiplet from wrapped $D3$-branes.
- Charged under $U(1)$ associated to dual cycle; D-term prevents a VEV
- Multiple cycles in same homology class $\Rightarrow$ D-flat directions
- Matches mathematical criterion for CY resolution of nodal variety
- Higgs branch VEV(s) are the new Kähler parameters

Hyperconifold transitions

$X = \tilde{X}/\mathbb{Z}_n$

- X moduli are a subspace of $\tilde{X}$ moduli
- $\mathbb{Z}_n$-hyperconifold on $X \leftrightarrow$ single conifold on $\tilde{X}$
- Upstairs, D-term, one hypermultiplet $\Rightarrow$ no resolution
- Downstairs:
  - $D3$-worldvolume is $L(n,k) \Rightarrow$ Wilson lines $\Rightarrow n$ ground states
  - So theory on X has n massless hypermultiplets.[3]
  - Same D-term $\Rightarrow n-1$ Higgs branch hypermultiplets $\leftrightarrow$ new Kähler parameters

Again, there is a nice match between mathematics and physics.

[3]Quotienting renormalises the charge by $1/\sqrt{n}$, so one-loop corrections are the same downstairs as upstairs.

Outline

Summary

$\mathbb{Z}_n$-Hyperconifold Singularities

- Isolated singularities; cyclic quotients of the conifold $\mathcal{C}$
- One-to-one correspondence with vanishing lens spaces $L(n, k)$
- Arise naturally in *compact* Calabi–Yau threefolds, when a free group action develops a fixed point
- Can always be resolved, unlike conifold singularities

Summary

$\mathbb{Z}_n$-Hyperconifold Transitions

- Potential to yield hundreds of new Calabi–Yau threefolds.
- Are mirror to familiar conifold transitions
- Have a nice Type IIB description similar to conifold transitions