$SU(5) \times U(1)$ F-Theory models from Toric Elliptic Fibrations

Volker Braun

Dublin Institute for Advanced Studies (DIAS)
starting May 1: Oxford University

21. March 2013

Introduction

Definition (F-Theory)

Defines a (real) $(12 - 2d)$-dimensional effective field theory after compactification on elliptically fibered $2d$-dimensional Calabi-Yau variety.

$$\begin{array}{ccc} T^2 & \longrightarrow & Y^{2d} \\ & & \downarrow \pi \\ & & B^{2d-2} \end{array}$$

Gauge group, matter, and Yukawa couplings localized at different dimensions:

- $\dim_{\mathbb{C}} Y = 1$: IIB in 10-d
- $\dim_{\mathbb{C}} Y = 2$: Degenerate (Kodaira) fibers $\Rightarrow$ Gauge group
- $\dim_{\mathbb{C}} Y = 3$: Discriminant components intersect $\Rightarrow$ Matter
- $\dim_{\mathbb{C}} Y = 4$: Matter curves intersect $\Rightarrow$ Yukawa couplings, flux.

Elliptic Curves

First, look at $\dim Y = 1$.

- Can write down CY 1-fold explicitly: $Y = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$
- But not in higher dimenison, better use embedding in 2-d ambient space
- For example, cubic hypersurface in $\mathbb{P}^2$
- Can always be written in Weierstrass form

$$y^2 = x^3 + ax + b$$

- Or, more generally, a (crepant resolution of a singular Fano) toric surface

16 Reflexive Polygons

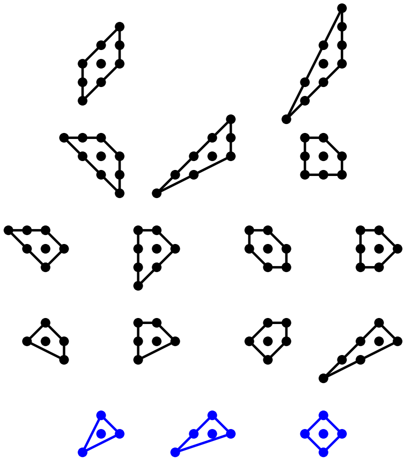
Definition (Reflexive)

A lattice polytope ∇ is called reflexive if its dual Δ is also a lattice polytope.

Note: Larger $\nabla \Leftrightarrow$ smaller Δ.

The blue polygons:

- minimal with respect to removing a vertex (blow-down).
- dual is maximal with respect to inclusion.



$\mathbb{P}^2$ $\quad$ $\mathbb{P}^2[1,1,2]$ $\quad$ $\mathbb{P}^1 \times \mathbb{P}^1$

Normal Form of a Cubic

Cubic surface:

$$\sum_{i,j,k} a_{ijk} u^i v^j w^k = 0, \qquad [u : v : w] \in \mathbb{P}^2$$

The undergrad method:

- Find a flex
- Translate flex to $[0 : 1 : 0]$
- ...

Picking a point (= zero-section) necessary, what if its not a flex?

Better solution: Artin, Rodriguez-Villegas, Tate

- Switch to the Jacobian $\mathrm{Pic}^0(E)$
- Weierstrass parameters a, b = polynomial in a_{ijk}.

Weierstrass Form

How to go from this:

$$P(u, v, w) = \sum_{i+j+k=3} a_{ijk} u^i v^j w^k = 0$$

to this: $y^2 = x^3 + fxz^4 + gz^6$ (the Weierstrass form)?

SL_3-rotation of $[u : v : w]$ should not change f, g.

The Ternary Cubic

A cubic in three variables

$$P(u,v,w) = \sum_{i+j+k=3} a_{ijk} u^i v^j w^k = 0, \qquad [u:v:w] \in \mathbb{P}^2$$

has

- two invariants S, T, and
- four covariants $P(u,v,w)$, $H(u,v,w)$, $\Theta(u,v,w)$, and $J(u,v,w)$

satisfying the syzygy

$$\begin{aligned} J^2 = & 4\Theta^3 + TP^2\Theta^2 + \Theta\big(-4S^3P^4 + 2STP^3H - 72S^2P^2H^2 \\ & - 18TPH^3 + 108SH^4 \big) - 16S^4P^5H - 11S^2TP^4H^2 \\ & - 4T^2P^3H^3 + 54STP^2H^4 - 432S^2PH^5 - 27TH^6 \end{aligned}$$

Weierstrass Form From Invariants

For $P = 0$, the syzygy is

$$J^2 = 4\Theta^3 + 108\Theta S H^4 - 27 T H^6$$

so up to some rescaling: $y = J$, $x = \Theta$, $z = H$, $f = S$, and $g = T$.

For example, the Fermat cubic $P = u^3 + v^3 + w^3$:

$$\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2[2,3,1],$$
$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} -u^3v^3 - u^3w^3 - v^3w^3 \\ \frac{1}{2}\left(u^6v^3 - u^3v^6 - u^6w^3 + v^6w^3 + u^3w^6 - v^3w^6\right) \\ uvw \end{pmatrix}$$

Degenerate Fibers

The Weierstrass for an elliptically fibered K3:

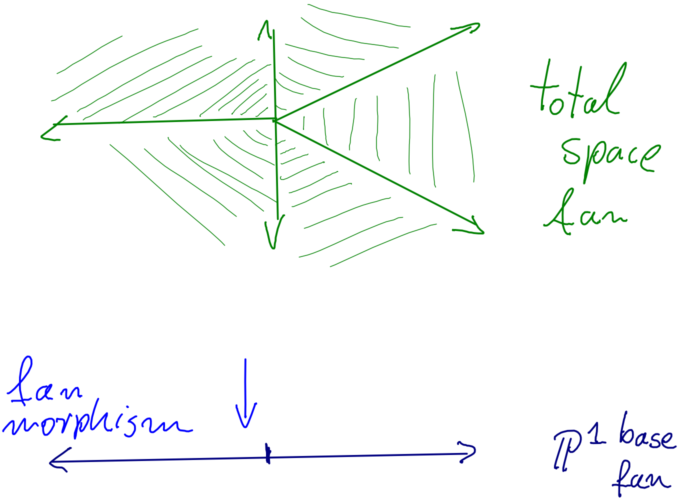
$$y^2 = x^3 + a(t)x + b(t)$$

where t is a coordinate on the base $\mathbb{P}^1$.

- Discriminant is $\delta = 4a^3 + 27b^2 = 0$
- Non-Abelian gauge group G determined by degree of vanishing of (a, b, δ) at the discriminant. [Tate]
- Number of $U(1)$-factors = Mordell-Weil rank

$$\operatorname{rank} MW(Y) + \operatorname{rank}(G) = h^{11}(Y) - h^{11}(B) - 1$$

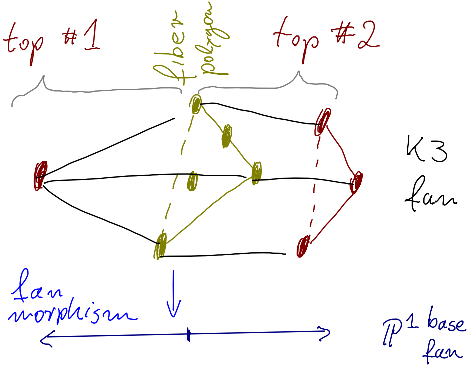
Toric Fibrations



Toric fibration of toric varieties equivalent to *fan morphism*

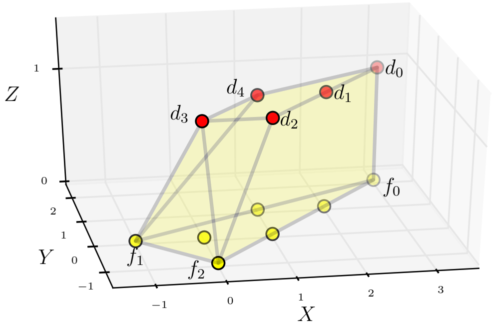
Tops and Bottoms

- K3 as hypersurface in 3-d toric variety from 3-d reflexive polygon
- Fiber = kernel of fan morphism = preimage of origin
- Fiber is one of the 16 reflexive polygons



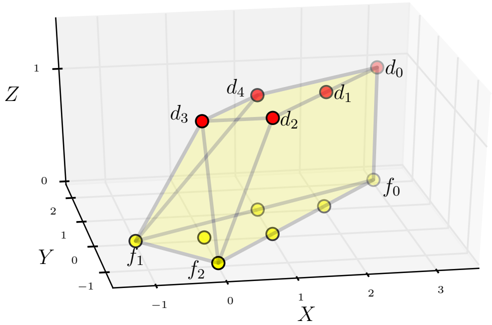
- Fiber cuts 3-d polytope in two halves (=*tops*)
- Non-fiber vertices and edges of top form extended Dynkin diagram of gauge group [Candelas]

A $SU(5)$ Top



- $d_0, \ldots, d_4$ form $SU(5)$ extended Dynkin diagram
- Correspond to irreducible toric surfaces in the fiber over torus fixed point
- Hypersurface cuts out I_5 Kodaira fiber

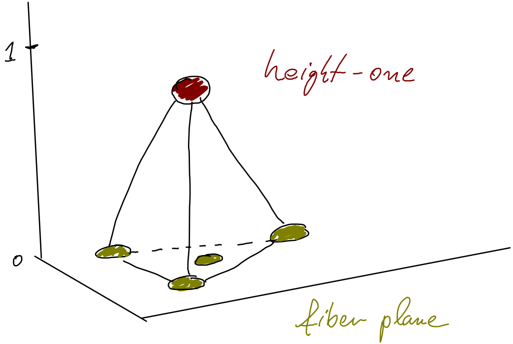
Toric Sections



- Base of top = fiber polygon
- Base vertices whose two adjacent points are a lattice basis are toric sections
- Here: single toric section $f_0 = 0$

Trivial Top

- For each fiber polygon there is the trivial top with a single point at height 1.



- This means that the fiber over the torus fixed point in the base has only a single irreducible component.
- Cartesian products and bundles are all built with trivial tops.

Matter Charges in $SU(5) \times U(1)$ Models

- Try to impose constraints on $SU(5)$ GUT couplings by additional $U(1)$
- Open question: Which $U(1)$ charges can the different $SU(5)$-reps acquire?
- Really question about elliptic CY 3-folds
- We constructed and analyzed a relatively complicated example

VB-Grimm-Keitel

The Toric Data for the Calabi-Yau Threefold X

Point $n_z \in \nabla \cap N$				Coordinate z	Divisor $V(z)$
-1	-1	-1	-1	h_0	$\hat{H}_0$
0	0	0	1	h_1	$\hat{H}_1$
-2	-1	1	0	d_0	$\hat{D}_0$
-1	0	1	0	d_1	$\hat{D}_1$
0	0	1	0	d_2	$\hat{D}_2$
0	-1	1	0	d_3	$\hat{D}_3$
-1	-1	1	0	d_4	$\hat{D}_4$
-1	0	0	0	f_0	$\hat{F}_0$
0	1	0	0	f_1	$\hat{F}_1$
1	0	0	0	f_2	$\hat{F}_2$
-1	-1	0	0	f_3	$\hat{F}_3$

The fan morphism is the projection on the last two coordinates.

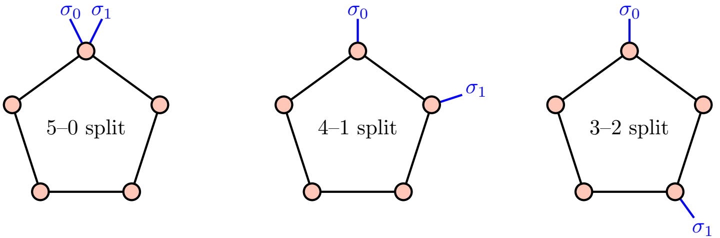
Mordell-Weil Group

- The Hodge numbers are $h^{11}(X) = 7$ and $h^{21}(X) = 63$
- Therefore rank $MW(X) = 1$
- But only one toric section $\sigma_0 = \{f_0 = 0\}$
- What is the generator of MW? Using intersection theory, we guessed

$$[\sigma_1] = [\hat{F}_1 - \hat{F}_0 - \hat{D}_0 - \hat{D}_3 - \hat{D}_4 + \hat{H}_0].$$

- To verify the guess, compute $H^0\big(X, \mathcal{O}(\sigma_1)\big) = 1.$

Orientations of I_5 and Two Sections



5–0 split: The $SU(5)$ singlets have minimal $U(1)$ charge one.

4–1 split: The $SU(5)$ singlets have $U(1)$ charges in $5\mathbb{Z}$. The $\underline{\mathbf{5}}$ of $SU(5)$ (fundamental representation) have $U(1)$ charge $2, 3$ mod 5. The $\underline{\mathbf{10}}$ (antisymmetric representation) have $U(1)$ charges $1, 4$ mod 5.

3–2 split: As 4–1 but fundamentals have charges $1, 4$ mod 5 and antisymmetrics have $2, 3$ mod 5.

$U(1)$ Charges

- The example is of the 4–1 split type, easy intersection theory computation.
- This fixes the $U(1)$ charge mod 5, but what are the actual $U(1)$ charges?
- The 6-d hypermultiplets come from vanishing curves on the discriminant.
- Their $U(1)$ charge is the intersection

$$U(1)\text{-charge}(C) = C \cap S(\sigma_1) = C \cap \sigma_1 - C \cap \sigma_0 + \sum_{1 \le a,b \le 4} (C \cap \hat{D}_a) \begin{pmatrix} \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{pmatrix}_{ab} (\sigma_1 \cap C_b)$$

Park-Morrison

Codimension-Two Fibers

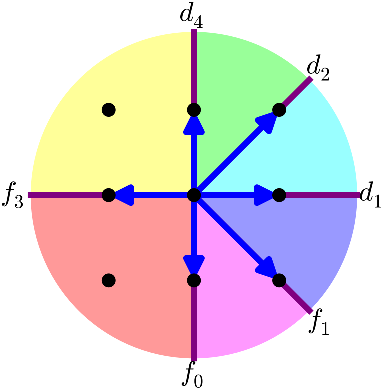
Need to identify the curves stuck over codimension-two fibers, for example where I_5 degenerates into an I_6.

- Very explicit: compute location of codimension-two fiber and plug into hypersurface equation.
- Here: projection map

$$\pi : \; [h_0 : h_1 : d_0 : \ldots : d_4 : f_0 : \ldots : f_3] \mapsto [h_0 : h_1 : d_0 d_1 d_2 d_3 d_4]$$

- For example, look at the $d_0 = 0$ toric fiber component over the point $[h_0 : h_1 : 0]$

The $d_0 = 0$ Toric Fiber Component



Point n_z		Coord. z
1	0	d_1
1	1	d_2
0	1	d_4
−1	0	f_3
1	−1	f_1
0	−1	f_0

It is embedded as

$$i_0 : [d_1 : d_2 : d_4 : f_0 : f_1 : f_3] \mapsto [h_0 : h_1 : 0 : d_1 : d_2 : 1 : d_4 : f_0 : f_1 : 1 : f_3]$$

Plugging into the Hypersurface Equation

1. Over a generic point $[h_0 : h_1 : 0]$, get

$$p(h_0, h_1, 0, d_1, d_2, 1, d_4, f_0, f_1, 1, f_3) = \beta_0 d_1 d_2^2 d_4 f_1 + \beta_1 d_1 d_2 f_0 f_1^2 + \beta_2 d_2 d_4 f_3 + \beta_3 f_0 f_1 f_3$$

2. at 2 distinct codimension-two fibers the coefficient β_2 vanishes and the polynomial factorizes as

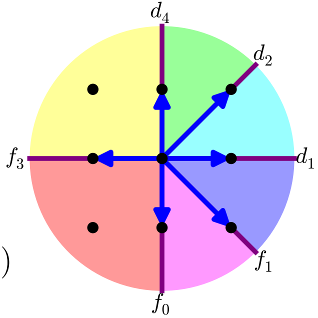
$$p(h_0, h_1, 0, d_1, d_2, 1, d_4, f_0, f_1, 1, f_3) = f_1 \times \left(\beta_0 d_1 d_2^2 d_4 + \beta_1 d_1 d_2 f_0 f_1 + \beta_3 f_0 f_3\right)$$

3. at 3 distinct codimension-two fibers the hypersurface equation factors as

$$p(h_0, h_1, 0, d_1, d_2, 1, d_4, f_0, f_1, 1, f_3) = (\beta'_0 d_1 d_2 f_1 + \beta'_1 f_3) \times (\beta'_2 d_2 d_4 + \beta'_3 f_0 f_1)$$

Codimension-Two Fiber Components

Previous slide: The $d_0 = 0$ node of the extended Dynkin diagram splits in two different ways.



1. The pull-back of the Calabi-Yau to the $d_0 = 0$ fiber component is

$$i_0^*(Y) = V(p) = V(f_0)+V(f_1)+V(f_3)$$

1. Over 2 points the fiber component decomposes as

$$i_0^*(Y) = V(p) = \big[V(f_1)\big] + \big[V(f_0) + V(f_3)\big],$$

2. and over 3 points the fiber component decomposes as

$$i_0^*(Y) = V(p) = \big[V(f_0) + V(f_1)\big] + \big[V(f_3)\big].$$

Intersection Numbers of Fibers and Sections

The pull-back of the sections is

$$
\begin{aligned}
i_0^*(\sigma_0) =& V(f_0), \\
i_0^*(\sigma_1) =& V(f_3) - V(f_0).
\end{aligned}
$$

I_6 component	$\bar{C}_0$	$\bar{C}_1$	$\bar{C}_2$	$\bar{C}_3$	$\bar{C}_4$	$\bar{C}_5$
Realization	$V(f_0) + V(f_3)$	$V(f_1)$	C_1	C_2	C_3	C_4
$\cap\sigma_0$	0	1	0	0	0	0
$\cap\sigma_1$	1	-1	0	0	1	0

I_6 component	$\bar{C}_0$	$\bar{C}_1$	$\bar{C}_2$	$\bar{C}_3$	$\bar{C}_4$	$\bar{C}_5$
Realization	$V(f_3)$	$V(f_0) + V(f_1)$	C_1	C_2	C_3	C_4
$\cap\sigma_0$	1	0	0	0	0	0
$\cap\sigma_1$	-1	1	0	0	1	0

$U(1)$-Charges

- The intersection numbers of the stuck curves determine the $U(1)$ charges of the $SU(5)$ matter rep that contains the hyper.
- In the above example, these are $\underline{\mathbf{5}}$ of $SU(5)$
- Plugging into the formula:

$$U(1)\text{-charge}(2\times\underline{\mathbf{5}}) = 1-0+\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{8}{5}$$

- By analogous computation, find complete matter spectrum:

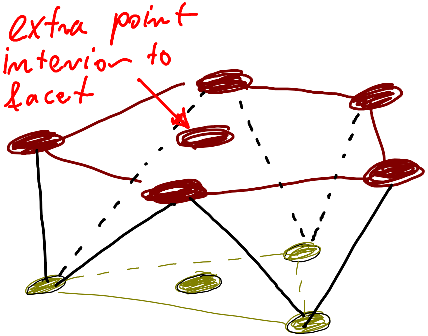
$$2 \times \underline{\mathbf{5}}_8 + 3 \times \underline{\mathbf{5}}_7 + 6 \times \underline{\mathbf{5}}_3 + 8 \times \underline{\mathbf{5}}_2 + 3 \times \underline{\mathbf{10}}_1$$

Flat Fibrations

- Want fibrations where all fibers are one-dimensional (otherwise, have tensionless strings)
- Starting with codimension-two fibers (CY 3-fold), the dimension of the fiber can jump up.
- A fibration where all fibers are of the same dimension is called flat.
- Note: flat in the sense of homological algebra, not in the geometric sense.
- As we go up in dimension, this gets more and more restrictive.

Non-Flat Tops

Consider a top with an integral point in the interior of the pentagon facet at height one.



- For an elliptic K3 built from this top, the corresponding divisor is interior to a facet.
- The following are equivalent:
  - Integral point p_i interior to a facet
  - Toric divisor $V(z_i)$ missed by the Calabi-Yau hypersurface
  - Toric divisor $V(z_i)$ such that the restriction of the Calabi-Yau equation is constant.

Non-Flat Top in Calabi-Yau Threefold

If we use this top in a Calabi-Yau threefold:

- The toric fiber is now fibred over the one-dimensional discriminant.
- The hypersurface equation is still constant in the fiber direction on the toric fiber component coresponding to the facet interior point of the top.
- But the facet interior point of the top is not in a facet of the 4-d polytope, so this constant varies along the discriminant.
- Hence, must be zero somewhere.
- There, the whole 2-dimensional toric fiber is part of the Calabi-Yau hypersurface.

The threefold fibration cannot be flat.

Tops in Higher Dimensions

- There are more/different things that can go wrong.
- They do not only depend on the top, but also its embedding in the polytope.
- Most 4-d toric hypersurfaces are not flat elliptic fibrations. WIP
- Extends to complete intersections as well. WIP