# Differential geometry of Lie algebroids for non-geometric string theory

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# this talk is based on

This talk is based on work done in collaboration with R. Blumenhagen, A. Deser and F. Rennecke:

- A bi-invariant Einstein-Hilbert action for the non-geometric string arXiv:1210.1591

- Non-geometric strings, symplectic gravity and differential geometry of Lie algebroids arXiv:1211.0030

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Related talks at this workshop by C. Hull, M. Larfors and P. Schupp.

## motivation

String theory is often studied in regimes where a geometric description is available.

# motivation

String theory is often studied in regimes where a geometric description is available.

But string theory also admits non-geometric backgrounds as solutions.

$$H_{abc} \xleftrightarrow{\;T_c\;} f_{ab}{}^{c} \xleftrightarrow{\;T_b\;} Q_a{}^{bc} \xleftrightarrow{\;T_a\;} R^{abc}$$

# motivation

$$H_{abc} \xleftrightarrow{\quad T_c \quad} f_{ab}{}^{c} \xleftrightarrow{\quad T_b \quad} Q_{a}{}^{bc} \xleftrightarrow{\quad T_a \quad} R^{abc}$$

$$\boxed{H_{abc}} \xleftrightarrow{\;\;T_c\;\;} f_{ab}{}^{c} \xleftrightarrow{\;\;T_b\;\;} Q_a{}^{bc} \xleftrightarrow{\;\;T_a\;\;} R^{abc}$$

Consider string theory compactified on a three-torus with *H*-flux:

- The geometry is characterized by $$ds^2 = dx^2 + dy^2 + dz^2 \,,$$ $$B_{yz} = Nx \,.$$
- The *H*-flux is determined by $$\frac{1}{(2\pi)^3}\int H = N \,.$$

# motivation :: f-flux background

$$H_{abc} \xleftrightarrow{\;T_c\;} \boxed{f_{ab}{}^{c}} \xleftrightarrow{\;T_b\;} Q_a{}^{bc} \xleftrightarrow{\;T_a\;} R^{abc}$$

After a T-duality in the $z$-direction, one arrives at a twisted torus:

- The geometry is characterized by
$$ds^2 = dx^2 + dy^2 + (dz + N x\, dy)^2\,,$$
$$B = 0\,.$$

- The geometric flux follows from
$$e^x = dx\,, \qquad e^y = dy\,, \qquad e^z = dz + N x\, dy\,,$$
$$\omega^z{}_{xy} = -N/2\,,$$
$$[e_x, e_y] = -N\, e_z\,.$$

Scherk, Schwarz - 1979
Kachru, Schulz, Tripathy, Trivedi - 2002

$$H_{abc} \quad \xleftrightarrow{\;T_c\;} \quad f_{ab}{}^{c} \quad \xleftrightarrow{\;T_b\;} \quad \boxed{Q_a{}^{bc}} \quad \xleftrightarrow{\;T_a\;} \quad R^{abc}$$

After a second T-duality in the $y$-direction, one arrives at a T-fold:

- The geometry is characterized by
$$ds^2 = dx^2 + \frac{1}{1+N^2x^2}\left(dy^2+dz^2\right),$$
$$B_{yz} = -\frac{N\,x}{1+N^2x^2}\,.$$

- The non-geometric $Q$-flux reads $\quad Q_x{}^{yz} = N\,.$

- The metric and $B$-field are well-defined locally, but not globally.
  Transition functions between local trivializations involve T-duality transformations, hence the name T-fold.

Hellermann, McGreevy, Williams - 2002
Dabholkar, Hull - 2002
Hull - 2004

# motivation :: r-flux background

$$H_{abc} \xleftrightarrow{T_c} f_{ab}{}^{c} \xleftrightarrow{T_b} Q_a{}^{bc} \xleftrightarrow{T_a} \boxed{R^{abc}}$$

After formally applying a third T-duality, one obtains an *R*-flux background:

- The metric and *B*-field are not even locally well-defined.
- The non-geometric *R*-flux is formally written as $R^{xyz} = N$ .
- It has been observed that this background gives rise to a non-associative structure.

Bouwknegt, Hannabuss, Mathai - 2004
Shelton, Taylor, Wecht - 2005
Ellwood, Hashimoto - 2006

... - 2010

# motivation :: generalized geometry I

An approach to study non-geometric fluxes is provided by generalized geometry.

# motivation :: generalized geometry I

An approach to study non-geometric fluxes is provided by generalized geometry.

- Consider a manifold *M* with generalized tangent bundle $TM \oplus T^*M$ and sections $X + \xi$ .

- On this bundle there is a natural *O(d,d)*-structure, and two abelian subgroups thereof are generated by

$$B\text{-transform} :: \qquad X + \xi \;\mapsto\; X + (\xi - \iota_X \omega)$$

$$\beta\text{-transform} :: \qquad X + \xi \;\mapsto\; (X + \beta^\sharp \xi) + \xi$$

- A generalized metric which encodes the metric *G* and a *B*-field reads

$$\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix},$$

and a particular set of corresponding vielbeins reads $(\mathcal{E}^a, \mathcal{E}_a) = (e^a, e_a - \iota_{e_a} B)$ .

Hitchin - 2002
Gualtieri - 2004
Graña, Minasian, Petrini, Waldram - 2008

## motivation :: generalized geometry II

Using the Courant bracket, the algebra for the vielbeins can be determined:

- For the basis $(\mathcal{E}^a, \mathcal{E}_a)$ one finds

$$\begin{aligned} \big[\mathcal{E}_a, \mathcal{E}_b\big] &= +f_{ab}{}^m \mathcal{E}_m - H_{abm}\mathcal{E}^m \,, \\ \big[\mathcal{E}_a, \mathcal{E}^b\big] &= \qquad\qquad\quad - f_{am}{}^b \mathcal{E}^m \,, \\ \big[\mathcal{E}^a, \mathcal{E}^b\big] &= \qquad\qquad\qquad\quad 0 \,. \end{aligned}$$

- But, after performing a $\beta$-transform on the vielbeins, one has

$$\begin{aligned} \big[\tilde{\mathcal{E}}_a, \tilde{\mathcal{E}}_b\big] &= \qquad\qquad\qquad f_{ab}{}^m \tilde{\mathcal{E}}_m \,, \\ \big[\tilde{\mathcal{E}}_a, \tilde{\mathcal{E}}^b\big] &= -f_{am}{}^b \tilde{\mathcal{E}}^m + Q_a{}^{bm} \tilde{\mathcal{E}}_m \,, \\ \big[\tilde{\mathcal{E}}^a, \tilde{\mathcal{E}}^b\big] &= +Q_m{}^{ab} \tilde{\mathcal{E}}^m + R^{abm} \tilde{\mathcal{E}}_m \,. \end{aligned}$$

The non-geometric fluxes are expressed in terms of a bi-vector $\beta$ as

$$Q_a{}^{bc} = \partial_a \beta^{bc} + 2 f_{am}{}^{[\underline{b}} \beta^{m\underline{c}]} \,, \qquad\qquad R^{abc} = 3\big(\beta^{[\underline{a}m}\partial_m \beta^{\underline{bc}]} + f_{mn}{}^{[\underline{a}} \beta^{\underline{b}m} \beta^{\underline{c}n]}\big) \,.$$

Grange, Schäfer-Nameki - 2006
Graña, Minasian, Petrini, Waldram - 2008
Halmagyi - 2009

# motivation :: double field theory

A further approach to study non-geometric fluxes is provided by double field theory.

A further approach to study non-geometric fluxes is provided by double field theory.

- Here, one first *doubles* the geometry

$$x^a \;\rightarrow\; x^A = (x^a, \tilde{x}_a)\,, \qquad\qquad \partial_a \;\rightarrow\; \partial_A = (\partial_a, \tilde{\partial}^a)\,.$$

- The (NS-NS sector of the) action can then be expressed as

$$\mathcal{S}_{\mathsf{DFT}} \sim \int dx\, d\tilde{x}\, e^{-2d} \left( \frac{1}{8} \mathcal{H}^{AB} (\partial_A \mathcal{H}^{CD})(\partial_B \mathcal{H}_{CD}) + \ldots \right)\,.$$

- This action is manifestly invariant under *O(d,d)*-transformations.

- Upon setting $\tilde{\partial}^a = 0$, one recovers the usual action.

Tseytlin - 1991
Siegel - 1993
Hull - 2004
Hull, Zwiebach - 2009
Hohm, Hull, Zwiebach - 2010
Hohm, Kwak - 2010
Jeon, Lee, Park - 2010 & 2011
Hohm, Zwiebach - 2011

To obtain an action for non-geometric fluxes, the following steps have been performed:

1. Consider the DFT action with generalized metric depending on *G* and *B*.
2. Perform an *O(d,d)*-transformation (T-duality transformation) and a field redefinition, to arrive at a DFT action depending on $(\tilde{g}, \tilde{\beta})$.
3. Set $\tilde{\partial}_a = 0$ and obtain an action for non-geometric fluxes.

$$\tilde{\mathcal{S}}_{\text{non-geometric}} = \int dx \sqrt{-|\tilde{g}|}\, e^{-2\tilde{\phi}}\, \tilde{\mathcal{L}}(\tilde{g}, \tilde{\beta}, \tilde{\phi})\,.$$

Alternatively, starting from the usual NS-NS Lagrangian a field redefinition has been employed to obtain a non-geometric action

$$G^{-1} = \tilde{g}^{-1} - \tilde{\beta}\,\tilde{g}\,\tilde{\beta}\,, \qquad\qquad B^{-1} = \tilde{\beta} - \tilde{g}^{-1}\tilde{\beta}^{-1}\tilde{g}^{-1}\,.$$

Andriot, Larfors, Lüst, Patalong - 2011
Andriot, Hohm, Larfors, Lüst, Patalong - 2012

As has been reviewed, for non-geometric fluxes a bi-vector $\beta$ plays an important role

$$Q_a{}^{bc} = \partial_a \beta^{bc}\,, \qquad\qquad \Theta^{abc} = 3\,\beta^{[\underline{a}m}\partial_m \beta^{\underline{bc}]}\,.$$

An action incorporating the bi-vector $\beta$ can be obtained as follows:

1. Introduce a mathematical framework for describing $\beta$
   - $\rightarrow$ Theory of Lie algebroids

2. Study diffeomorphisms and construct and invariant action
   - $\rightarrow$ Differential geometry

3. Relation to string theory
   - $\rightarrow$ Field redefinition à la Seiberg-Witten

4. Developments
   - $\rightarrow$ Extension to R-R and fermionic sectors
   - $\rightarrow$ Equations of motion and solutions

Blumenhagen, Deser, EP, Rennecke - 2012

# outline

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1. motivation
2. lie algebroids
3. differential geometry
4. string theory
5. solutions
6. conclusions

A natural mathematical framework to describe a bi-vector $\beta$ is given by Lie algebroids.

Hull - 2004
Halmagyi - 2008 & 2009
Berman, Perry - 2010
Blumenhagen, Deser, EP, Rennecke - 2012

# lie algebroids :: definition

Let $M$ be a manifold, and $E \to M$ a vector bundle with

bracket $\quad [\cdot,\cdot]_E : E \times E \to E\,,$

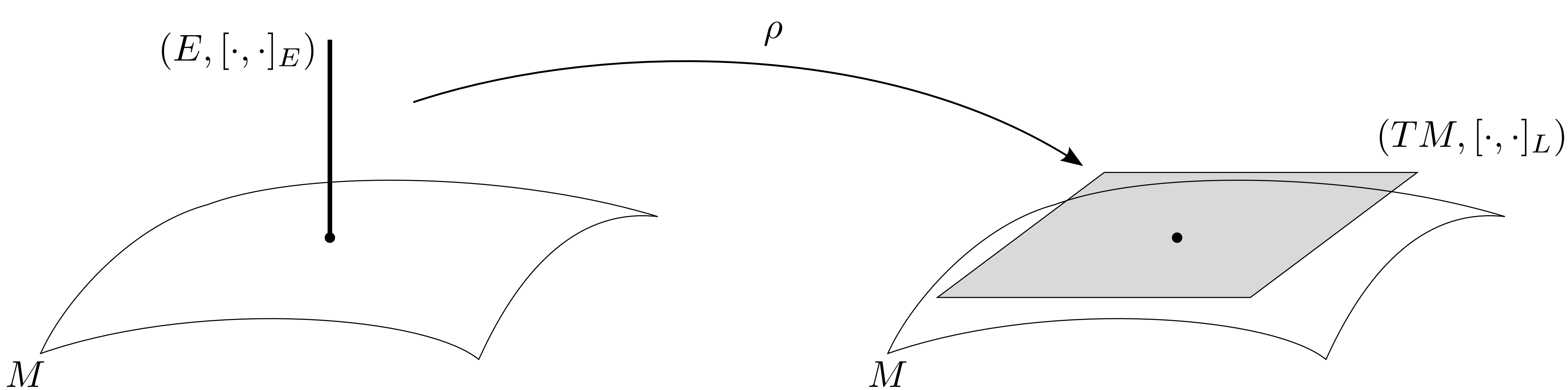
anchor map $\quad \rho : E \to TM\,.$

Then $(E, [\cdot,\cdot]_E, \rho)$ is called a Lie algebroid, if (for $s_i \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$)

homomorphism $\quad \rho([s_1, s_2]_E) = [\rho(s_1), \rho(s_2)]_L\,,$

Leibniz rule $\quad [s_1, f s_2]_E = f\,[s_1, s_2]_E + \rho(s_1)(f) s_2\,,$

Jacobi identity $\quad [s_1, [s_2, s_3]_E]_E = [[s_1, s_2]_E, s_3]_E + [s_2, [s_1, s_3]_E]_E\,.$



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| Jacobi identity | $[s_1, [s_2, s_3]_E]_E = [[s_1, s_2]_E, s_3]_E + [s_2, [s_1, s_3]_E]_E\,.$ |

There are two important properties of a Lie algebroid:

- The bracket on $E$ can be extended to a Gerstenhaber algebra on $\Gamma(\wedge^\star E)$.
- The space of dual sections $\Gamma(\wedge^\star E^*)$ is a graded differential algebra with respect to

$$
\begin{aligned}
(d_E\,\omega)(s_0, \ldots, s_k) = \;& \sum_{i=0}^{k} (-1)^i \rho(s_i)\left(\omega(s_0, \ldots, \hat{s}_i, \ldots, s_k)\right) \\
&+ \sum_{i<j} (-1)^{i+j} \omega\left([s_i, s_j]_E, s_0, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_k\right)\,.
\end{aligned}
$$

Let $M$ be a manifold, and $E \to M$ a vector bundle with

| | |
|---|---|
| bracket | $[\cdot,\cdot]_E : E \times E \to E$ , |
| anchor map | $\rho : E \to TM$ . |

Then $(E,[\cdot,\cdot]_E,\rho)$ is called a Lie algebroid, if (for $s_i \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$)

| | |
|---|---|
| homomorphism | $\rho([s_1,s_2]_E) = [\rho(s_1),\rho(s_2)]_L$ , |
| Leibnitz rule | $[s_1, f s_2]_E = f\,[s_1,s_2]_E + \rho(s_1)(f)s_2$ , |
| Jacobi identity | $[s_1,[s_2,s_3]_E]_E = [[s_1,s_2]_E,s_3]_E + [s_2,[s_1,s_3]_E]_E$ . |

---

The standard example for a Lie algebroid is $(TM,[\cdot,\cdot]_L,\rho=\mathrm{id})$:

- The bracket on $TM$ is the Lie bracket $[\cdot,\cdot]_L$ between vector fields.
- The extension to multi-vector fields gives the Schouten-Nijenhuis bracket $[\cdot,\cdot]_{SN}$ .
- The differential on $\Gamma(\wedge^\star T^*M)$ is the de Rham differential $d$ .

# lie algebroids :: example II

Let $M$ be a manifold, and $E \to M$ a vector bundle with

| | |
|---|---|
| bracket | $[\cdot,\cdot]_E : E \times E \to E\,,$ |
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Then $(E, [\cdot,\cdot]_E, \rho)$ is called a Lie algebroid, if (for $s_i \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$)

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| Leibnitz rule | $[s_1, f s_2]_E = f\,[s_1, s_2]_E + \rho(s_1)(f) s_2\,,$ |
| Jacobi identity | $[s_1, [s_2, s_3]_E]_E = [[s_1, s_2]_E, s_3]_E + [s_2, [s_1, s_3]_E]_E\,.$ |

---

For $(M, \beta)$ a Poisson manifold, a Lie algebroid is given by $(T^*M, [\cdot,\cdot]_K, \rho = \beta^\sharp)$.

- The anchor is characterized by: $\rho(e^a) = \beta^\sharp(e^a) = \beta^{ab} e_b\,.$
- The bracket is the Koszul bracket:
$$[\xi, \eta]_K = L_{\beta^\sharp(\xi)}\eta - \iota_{\beta^\sharp(\eta)}\, d\xi\,,$$
$$[e^a, e^b]_K = (\partial_c \beta^{ab}) e^c\,.$$
- The differential on $\Gamma(\wedge^\star TM)$ is: $d_\beta = [\beta, \cdot]_{SN}\,.$

A Lie derivative for a Lie algebroid can be defined as follows:

- action on functions $f \in \mathcal{C}^\infty(M)$: $\qquad \mathcal{L}_s f := s(f) := \rho(s)(f)\,,$
- action on sections $s \in \Gamma(E)$ : $\qquad \mathcal{L}_{s_0} s = [s_0, s]_E\,,$
- action on sections $\alpha \in \Gamma(E^*)$: $\qquad \mathcal{L}_{s_0}\alpha = \iota_{s_0} \circ d_E \alpha + d_E \circ \iota_{s_0}\alpha\,.$

A covariant derivative is a bilinear map $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ satisfying

$$\nabla_{f s_1} s_2 = f \nabla_{s_1} s_2\,, \qquad \nabla_{s_1} f s_2 = \rho(s_1)(f) s_2 + f \nabla_{s_1} s_2\,.$$

Curvature and torsion tensors can be defined as

$$R(s_a, s_b) s_c = \nabla_{s_a} \nabla_{s_b} s_c - \nabla_{s_b} \nabla_{s_a} s_c - \nabla_{[s_a, s_b]_E} s_c\,,$$

$$T(s_a, s_b) = \nabla_{s_a} s_b - \nabla_{s_b} s_a - [s_a, s_b]_E\,.$$

A metric on a Lie algebroid gives rise to a scalar product for sections in $E$

$$\langle s_a, s_b \rangle = g_{ab} \,.$$

The analogue of the Levi-Civita connection is obtained by requiring

- vanishing torsion $\qquad \mathring{\nabla}_{s_1} s_2 - \mathring{\nabla}_{s_2} s_1 = [s_1, s_2]_E \,,$
- metric compatibility $\qquad \rho(s_1)\langle s_2, s_3\rangle = \langle \mathring{\nabla}_{s_1} s_2, s_3 \rangle + \langle s_1, \mathring{\nabla}_{s_2} s_3 \rangle \,,$

and it is characterized by the Koszul formula

$$\begin{aligned} 2\big\langle \mathring{\nabla}_{s_1} s_2, s_3 \big\rangle &= s_1\big(\langle s_2, s_3\rangle\big) + s_2\big(\langle s_3, s_1\rangle\big) - s_3\big(\langle s_1, s_2\rangle\big) \\ &\quad - \langle s_1, [s_2, s_3]_E \rangle + \langle s_2, [s_3, s_1]_E \rangle + \langle s_3, [s_1, s_2]_E \rangle \,. \end{aligned}$$

# lie algebroids :: applications I

Recall that there is a Lie algebroid structure on $T^*M$ incorporating a bi-vector $\beta$

- given by $(T^*M, [\cdot,\cdot]_K, \rho = \beta^\sharp)$,
- defined in terms of the Koszul bracket,
- and with anchor $\beta^\sharp : T^*M \to TM$ .

The Jacobi identity for $(T^*M, [\cdot,\cdot]_K, \rho = \beta^\sharp)$

- is computed as (with $\eta, \chi, \zeta \in \Gamma(T^*M)$)

$$\mathrm{Jac}_K(\eta,\chi,\zeta) = d\big(\Theta(\eta,\chi,\zeta)\big) + \iota_{(\iota_\zeta \iota_\chi \Theta)} d\eta + \iota_{(\iota_\eta \iota_\zeta \Theta)} d\chi + \iota_{(\iota_\chi \iota_\eta \Theta)} d\zeta \ ,$$

- where the defect $\Theta$ is given by the *R*-flux

$$\Theta^{abc} = 3\,\beta^{[\underline{a}m} \partial_m \beta^{\underline{bc}]} \ .$$

- Thus, for non-vanishing *R*-flux this construction is only a quasi Lie algebroid ...

To obtain a proper Lie algebroid for non-vanishing *R*-flux $\Theta$, consider

- the *H*-twisted Koszul bracket defined by

$$[\xi,\eta]_K^H = [\xi,\eta]_K - \iota_{\beta^\sharp\eta}\iota_{\beta^\sharp\xi} H \, .$$

- The corresponding Jacobi identity reads

$$\mathrm{Jac}_K^H(\eta,\chi,\zeta) = d\big(\mathcal{R}(\eta,\chi,\zeta)\big) + \iota_{(\iota_\zeta\iota_\chi\mathcal{R})}d\eta + \iota_{(\iota_\eta\iota_\zeta\mathcal{R})}d\chi + \iota_{(\iota_\chi\iota_\eta\mathcal{R})}d\zeta \ ,$$

- with the defect given by

$$\mathcal{R}^{abc} = \Theta^{abc} - \beta^{am}\,\beta^{bn}\,\beta^{ck}\,H_{mnk} \, .$$

Therefore, a proper Lie algebroid $(T^*M,[\cdot,\cdot]_K^H,\beta^\sharp;\mathcal{R}=0)$ is obtained provided that

$$\Theta^{abc} = \beta^{am}\,\beta^{bn}\,\beta^{ck}\,H_{mnk} \, .$$

To summarize, a proper Lie algebroid structure on $T^*M$ incorporating a bi-vector $\beta$

- is given by $(T^*M, [\cdot,\cdot]_K^H, \beta^\sharp; \mathcal{R}=0)$,
- provided that the *R*-flux $\Theta^{abc}$ is related to the twist *H* as

$$\Theta^{abc} = \beta^{am}\,\beta^{bn}\,\beta^{ck}\,H_{mnk}\,.$$

- The metric and partial derivative will be denoted by $\hat{g}$ and $D^a = \beta^{ab}\partial_b$ .

One can develop a differential geometry calculus on $T^*M$ ,

- with Lie derivative, covariant derivative,
- curvature and torsion tensors,
- and Levi-Civita connection.

# outline

1. motivation
2. lie algebroids
3. differential geometry
4. string theory
5. solutions
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# diffeomorphisms :: definitions

For the Lie algebroid on $T^*M$, two different Lie derivates appear:

- The Lie derivative based on the Lie bracket $L_X$ for $X \in \Gamma(TM)$.
- The Lie derivative based on the Koszul bracket $\hat{\mathcal{L}}_\xi$ for $\xi \in \Gamma(T^*M)$.

Both can be used to describe and define (infinitesimal) diffeomorphisms ...

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Both can be used to describe and define (infinitesimal) diffeomorphisms ...

Definitions:

- An object $T \in \Gamma\big((\otimes^r TM) \otimes (\otimes^s T^*M)\big)$ is called a tensor, if it behaves under diffeomorphisms as

$$\delta_X T^{a_1\ldots a_r}{}_{b_1\ldots b_s} = (L_X T)^{a_1\ldots a_r}{}_{b_1\ldots b_s} \,.$$

- A tensor $T \in \Gamma\big((\otimes^r TM) \otimes (\otimes^s T^*M)\big)$ is called a $\beta$ - tensor, if it behaves as under $\beta$-diffeomorphisms as

$$\hat{\delta}_\xi T^{a_1\ldots a_r}{}_{b_1\ldots b_s} = \big(\hat{\mathcal{L}}_\xi T\big)^{a_1\ldots a_r}{}_{b_1\ldots b_s} \,.$$

For usual diffeomorphisms,

- the transformation behavior of a scalar *f* implies

$$\delta_X f = X(f) = L_X f \qquad \longrightarrow \qquad \delta_X df = L_X df\,.$$

- The metric $\hat{g}$ is a tensor, that is

$$\delta_X \hat{g} = L_X \hat{g}\,.$$

- If the bi-vector is a tensor, it implies for the *R*-flux $\Theta^{abc} = 3\,\beta^{[a|m}\partial_m\beta^{|bc]}$ that

$$\delta_X \beta = L_X \beta \qquad \longrightarrow \qquad \delta_X \Theta = L_X \Theta\,.$$

For $\beta$-diffeomorphisms,

- a scalar *f* transforms as $\hat{\delta}_\xi f = \hat{\mathcal{L}}_\xi f = \xi_a D^a f$, where $D^a = \beta^{ab}\partial_b$.
- Requiring that the *partial* derivative of a scalar is a $\beta$-tensor implies

$$\hat{\delta}_\xi \big(D^a f\big) = \big(\hat{\mathcal{L}}_\xi\, Df\big)^a + \big(\hat{\delta}_\xi\, \beta^{ab} - \Theta^{abm}\xi_m\big)\,\partial_b f \overset{!}{=} \big(\hat{\mathcal{L}}_\xi\, Df\big)^a$$

$$\longrightarrow \qquad \hat{\delta}_\xi\, \beta^{ab} = \Theta^{abm}\,\xi_m = \hat{\mathcal{L}}_\xi\, \beta + \beta^{am}\beta^{bn}\big(\partial_m\xi_n - \partial_n\xi_m\big)$$

$$\longrightarrow \qquad \hat{\delta}_\xi \Theta = \hat{\mathcal{L}}_\xi \Theta\,.$$

The algebra of infinitesimal $\beta$-transformations does not close (with $\eta, \xi_{1,2} \in \Gamma(T^*M)$)

$$\left[\hat{\delta}_{\xi_1}, \hat{\delta}_{\xi_2}\right]\eta = \hat{\delta}_{[\xi_1,\xi_2]_K}\eta + \iota_{(\iota_{\xi_1}\iota_{\xi_2}\Theta)}d\eta - d\big(\Theta(\xi_1,\xi_2,\eta)\big)\,,$$

where the defect is given by the *R*-flux $\Theta$ .

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where the defect is given by the *R*-flux $\Theta$ .

However, the combined algebra of standard and $\beta$-diffeomorphisms does close

$$\begin{aligned}
\big[\delta_{X_1}, \delta_{X_2}\big] &= \delta_{[X_1,X_2]_L}\,, \\
\big[\hat{\delta}_{\xi_1}, \delta_{X_1}\big] &= \delta_{(\hat{\mathcal{L}}_{\xi_1}X_1)}\,, \\
\big[\hat{\delta}_{\xi_1}, \hat{\delta}_{\xi_2}\big] &= \hat{\delta}_{[\xi_1,\xi_2]_K} + \delta_{(\iota_{\xi_1}\iota_{\xi_2}\Theta)}\,.
\end{aligned}$$

Since two different Lie derivates appear for the Lie algebroid on $T^*M$,

- one can describe infinitesimal diffeomorphisms by $\delta_X = L_X$,
- and a new type of $\beta$-diffeomorphisms by $\hat{\delta}_\xi = \hat{\mathcal{L}}_\xi$.

The infinitesimal $\beta$-transformations of the metric and bi-vector read

$$\hat{\delta}_\xi \, \hat{g}^{ab} = (\hat{\mathcal{L}}_\xi \, \hat{g})^{ab} \,, \qquad \hat{\delta}_\xi \, \beta^{ab} = (\hat{\mathcal{L}}_\xi \beta)^{ab} + \beta^{am} \beta^{bn} \big( \partial_m \xi_n - \partial_n \xi_m \big) \,.$$

The behavior under standard and $\beta$-diffeomorphisms can be summarized as

| | metric $\hat{g}$ | bi-vector $\beta$ | derivative $Df$ | $R$-flux $\Theta$ |
|---|---|---|---|---|
| tensor | ✓ | ✓ | ✓ | ✓ |
| $\beta$-tensor | ✓ | | ✓ | ✓ |

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# bi-invariant geometry :: technical remarks

For a basis $\{e^a\} \in \Gamma(T^*M)$, the $H$-twisted Koszul bracket evaluates to

$$\big[e^a, e^b\big]^H_K = \big(\partial_c \beta^{ab} - \beta^{am}\,\beta^{bn}\,H_{mnc}\big)\,e^c = \mathcal{Q}_c{}^{ab}\,e^c\,.$$

Due to the anomalous transformation behavior of $\beta$, the $H$-twisted Koszul bracket of $\beta$-tensors is a $\beta$-tensor

$$\hat{\delta}_\xi[\eta,\chi]^H_K = \hat{\mathcal{L}}_\xi[\eta,\chi]^H_K\,.$$

Thus, objects construct via this bracket are $\beta$-tensors.

For the Lie algebroid $(T^*M, [\cdot,\cdot]^H_K, \beta^\sharp; \mathcal{R}=0)$ ,

- the Leibnitz rule of the covariant derivative reads $(\xi,\eta \in \Gamma(T^*M),\ f\in \mathcal{C}^\infty(M))$

$$\begin{aligned}\hat\nabla_\xi(f\,\eta) &= f\,\hat\nabla_\xi\eta + \big((\beta^\sharp\xi)f\big)\,\eta \\ &= f\,\hat\nabla_\xi\eta + \xi_m(D^m f)\,\eta\,.\end{aligned}$$

- Connection coefficients for a basis $\{e^a\}\in\Gamma(T^*M)$ are defined as

$$\hat\nabla_{e^a} e^b \equiv \hat\nabla^a\, e^b = \hat\Gamma_c{}^{ab}\, e^c\,,$$

- which for the components of a one-form and a vector field implies

$$\begin{aligned}\hat\nabla^a\eta_b &= D^a\eta_b + \hat\Gamma_b{}^{am}\,\eta_m\,,\\ \hat\nabla^a X^b &= D^a X^b - \hat\Gamma_m{}^{ab}\,X^m\,.\end{aligned}$$

In order for $\hat\nabla$ to be a tensor and a $\beta$-tensor, $\hat\Gamma_c{}^{ab}$ has to transform anomalously

$$\begin{aligned}\big(\delta_X - L_X\big)\,\hat\Gamma_c{}^{ab} &= -D^a(\partial_c X^b)\,,\\ \big(\hat\delta_\xi - \hat{\mathcal{L}}_\xi\big)\ \ \hat\Gamma_c{}^{ab} &= +D^a(D^b\xi_c - \xi_m\,\mathcal{Q}_c{}^{mb})\,.\end{aligned}$$

The torsion operator for the present Lie algebroid

- takes the form $\hat{T}(\xi,\eta) = \hat{\nabla}_\xi\,\eta - \hat{\nabla}_\eta\,\xi - [\xi,\eta]^H_K\,,$
- which in components reads $\hat{T}_c{}^{ab} = \iota_{e_c}\hat{T}(e^a,e^b) = \hat{\Gamma}_c{}^{ab} - \hat{\Gamma}_c{}^{ba} - \mathcal{Q}_c{}^{ab}\,.$
- It is a tensor with respect to standard and $\beta$-diffeomorphisms.

The Levi-Civita connection is obtained by requiring

- metric compatibility $(\beta^\sharp\xi)\,\hat{g}(\eta,\chi) = \hat{g}\big(\hat{\nabla}_\xi\eta,\chi\big) + \hat{g}\big(\eta,\hat{\nabla}_\xi\chi\big)\,,$
- vanishing torsion $\mathcal{Q}_c{}^{ab} = \hat{\Gamma}_c{}^{ab} - \hat{\Gamma}_c{}^{ba}\,.$
- Employing the Koszul formula, the connection coefficients are computed as

$$\hat{\Gamma}_c{}^{ab} = \frac{1}{2}\,\hat{g}_{cm}\big(D^a\hat{g}^{bm} + D^b\hat{g}^{am} - D^m\hat{g}^{ab}\big) - \hat{g}_{cm}\,\hat{g}^{(a|n}\,\mathcal{Q}_n{}^{|b)m} + \frac{1}{2}\,\mathcal{Q}_c{}^{ab}\,.$$

The connection coefficients have the correct anomalous transformation behavior.

The curvature operator for the present Lie algebroid

- takes the form $$\hat{R}(\eta,\chi)\xi = \big[\hat{\nabla}_\eta, \hat{\nabla}_\chi\big]\xi - \hat{\nabla}_{[\eta,\chi]^H_K}\,\xi\,,$$
- which in components reads $$\hat{R}_a{}^{bcd} = 2\big(D^{[c}\hat{\Gamma}_a{}^{d]b} + \hat{\Gamma}_a{}^{[c|m}\,\hat{\Gamma}_m{}^{|d]b}\big) - \hat{\Gamma}_a{}^{mb}\,\mathcal{Q}_m{}^{cd}\,.$$
- It is a tensor with respect to standard and $\beta$-diffeomorphisms.

The curvature tensor satisfies (for the Levi-Civita connection)

$$\hat{R}^{abcd} = -\hat{R}^{bacd} = -\hat{R}^{abdc} = \hat{R}^{cdab}\,,$$

$$0 = \hat{R}^{abcd} + \hat{R}^{adbc} + \hat{R}^{acdb}\,,$$

$$0 = \hat{\nabla}^m\hat{R}^{abcd} + \hat{\nabla}^d\hat{R}^{abmc} + \hat{\nabla}^c\hat{R}^{abdm}\,.$$

The Ricci tensor and scalar both behave as tensors and $\beta$-tensors

$$\hat{R}^{ab} = \hat{R}_m{}^{amb}\,, \qquad \hat{R} = \hat{g}_{ab}\hat{R}^{ab}\,.$$

The transformation behavior of quantities discussed above can be summarized as:

| | metric $\hat{g}$ | bi-vector $\beta$ | derivative $Df$ | $R$-flux $\Theta$ | Ricci scalar $\hat{R}$ |
|---|---|---|---|---|---|
| tensor | ✓ | ✓ | ✓ | ✓ | ✓ |
| $\beta$-tensor | ✓ | | ✓ | ✓ | ✓ |

$$\delta_X \left(\sqrt{-|\hat{g}|}\,\hat{\mathcal{L}}\right) = \partial_m\Big( \sqrt{-|\hat{g}|}\,\hat{\mathcal{L}}\,X^m\Big) \qquad -2\,\sqrt{-|\hat{g}|}\,\hat{\mathcal{L}}\,(\partial_m X^m)\,,$$

$$\hat{\delta}_\xi \left(\sqrt{-|\hat{g}|}\,\hat{\mathcal{L}}\right) = \partial_m\Big( \sqrt{-|\hat{g}|}\,\hat{\mathcal{L}}\,\xi_n\Big)\beta^{nm} - \sqrt{-|\hat{g}|}\,\hat{\mathcal{L}}\,(\partial_n \beta^{mn})\xi_m\,.$$

# bi-invariant geometry :: invariant action I

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|---|---|---|---|---|---|
| tensor | ✓ | ✓ | ✓ | ✓ | ✓ |
| $\beta$-tensor | ✓ | | ✓ | ✓ | ✓ |

The following Lagrangian then behaves as a scalar under standard & $\beta$-diffeomorphisms

$$\hat{\mathcal{L}} = e^{-2\phi} \left( \hat{R} - \frac{1}{12} \Theta^{abc} \, \Theta_{abc} + 4 \hat{g}_{ab} \, D^a \phi D^b \phi \right) .$$

To construct an invariant action, the measure has to transform appropriately:

$$\begin{aligned}
\delta_X \left( \sqrt{-|\hat{g}|} \, \hat{\mathcal{L}} \right) &= \partial_m \left( \sqrt{-|\hat{g}|} \, \hat{\mathcal{L}} \, X^m \right) \quad -2 \sqrt{-|\hat{g}|} \, \hat{\mathcal{L}} \, (\partial_m X^m) \, , \\
\hat{\delta}_\xi \left( \sqrt{-|\hat{g}|} \, \hat{\mathcal{L}} \right) &= \partial_m \left( \sqrt{-|\hat{g}|} \, \hat{\mathcal{L}} \, \xi_n \right) \beta^{nm} - \sqrt{-|\hat{g}|} \, \hat{\mathcal{L}} \, (\partial_n \beta^{mn}) \xi_m \, .
\end{aligned}$$

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$$\hat{\mathcal{L}} = e^{-2\phi}\left(\hat{R} - \frac{1}{12}\Theta^{abc}\,\Theta_{abc} + 4\hat{g}_{ab}\,D^a\phi D^b\phi\right) .$$

To construct an invariant action, the measure has to transform appropriately:

$$\delta_X\left(\sqrt{-|\hat{g}|}\,\left|\beta^{-1}\right|\hat{\mathcal{L}}\right) = \partial_m\left(\sqrt{-|\hat{g}|}\,\left|\beta^{-1}\right|\hat{\mathcal{L}}\,X^m\right),$$

$$\hat{\delta}_\xi\;\left(\sqrt{-|\hat{g}|}\,\left|\beta^{-1}\right|\hat{\mathcal{L}}\right) = \partial_m\left(\sqrt{-|\hat{g}|}\,\left|\beta^{-1}\right|\hat{\mathcal{L}}\,\xi_n\beta^{nm}\right).$$

Combining these findings, one arrives at the bi-invariant action

$$\boxed{\hat{S} = \frac{1}{2\kappa^2}\int d^n x\,\sqrt{-|\hat{g}|}\,\left|\beta^{-1}\right|\,e^{-2\phi}\Big(\hat{R} - \frac{1}{12}\Theta^{abc}\,\Theta_{abc} + 4\,\hat{g}_{ab}\,D^a\phi D^b\phi\Big)\,.}$$

For the Lie algebroid $(T^*M, [\cdot,\cdot]^H_K, \beta^\sharp; \mathcal{R}=0)$, a corresponding differential geometry

- characterized by a Levi-Civita connection

$$\hat{\Gamma}_c{}^{ab} = \frac{1}{2}\,\hat{g}_{cm}\big(D^a\hat{g}^{bm} + D^b\hat{g}^{am} - D^m\hat{g}^{ab}\big) - \hat{g}_{cm}\,\hat{g}^{(a|n}\,\mathcal{Q}_n{}^{|b)m} + \frac{1}{2}\,\mathcal{Q}_c{}^{ab}\,,$$

- as well as a curvature tensor have been determined

$$\hat{R}_a{}^{bcd} = 2\big(D^{[c}\hat{\Gamma}_a{}^{d]b} + \hat{\Gamma}_a{}^{[c|m}\,\hat{\Gamma}_m{}^{|d]b}\big) - \hat{\Gamma}_a{}^{mb}\,\mathcal{Q}_m{}^{cd}\,.$$

A bi-invariant action has been constructed

$$\hat{S} = \frac{1}{2\kappa^2}\int d^nx\,\sqrt{-|\hat{g}|}\,\big|\beta^{-1}\big|\,e^{-2\phi}\Big(\hat{R} - \frac{1}{12}\Theta^{abc}\,\Theta_{abc} + 4\,\hat{g}_{ab}\,D^a\phi D^b\phi\Big)\,.$$

# outline

1. motivation
2. lie algebroids
3. differential geometry
4. string theory
5. solutions
6. conclusions

The following notation will be employed from now on:

- standard geometric frame $(G, B)$,
- non-geometric frame $(\hat{g}, \hat{\beta})$ .

To connect to string theory, consider a Seiberg-Witten field redefinition

$$\hat{g}^{ab} = \hat{\beta}^{am}\,\hat{\beta}^{bn}\,G_{mn}\,, \qquad\qquad \hat{\beta}^{ab} = (B^{-1})^{ab}\,.$$

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$$\hat{g}^{ab} = \hat{\beta}^{am}\,\hat{\beta}^{bn}\,G_{mn}\,, \qquad\qquad \hat{\beta}^{ab} = (B^{-1})^{ab}\,.$$

This relation between the frames $(G, B)$ and $(\hat{g}, \hat{\beta})$ then implies that

- the metric and bi-vector are tensors.
- The field identification $\hat{\Theta}^{abc} = \hat{\beta}^{am}\hat{\beta}^{bn}\hat{\beta}^{ck}\,H_{mnk}$ for a proper Lie algebroid is automatically satisfied.
- The $\beta$-transformations correspond to gauge transformations.

Given the field redefinition mentioned above, the *H*- and *R*-flux can be related as

$$
\begin{aligned}
H_{abc} &= 3\,\partial_{[a} B_{bc]} \\
&= -3\,B_{[b|m}\,(\partial_{|a|}\hat{\beta}^{mn})\,B_{n|c]} \\
&= 3\,B_{[a|k|}\,B_{b|m|}\,B_{c]n}\,D^{k}\hat{\beta}^{mn} \\
&= B_{ak}\,B_{bm}\,B_{cn}\,\hat{\Theta}^{mnk}\ ,
\end{aligned}
$$

which then implies $\hat{\Theta}^{abc} = \hat{\beta}^{am}\hat{\beta}^{bn}\hat{\beta}^{ck}\,H_{mnk}\,.$

Therefore, for the Lie algebroid $(T^*M, [\cdot,\cdot]^H_K, \beta^\sharp; \mathcal{R}=0)$ the Jacobi identity is satisfied.

The $B$-field behaves under gauge transformations in the following way

$$\delta_\xi^{\mathsf{gauge}} B_{ab} = \partial_a \xi_b - \partial_b \xi_a \,.$$

Using the field redefinitions given above, this implies

$$\begin{aligned}\delta_\xi^{\mathsf{gauge}}\, \hat{g}^{ab} &= 2\, \hat{g}^{(a|m}\, \hat{\beta}^{|b)n} \big(\partial_m \xi_n - \partial_n \xi_m\big)\,, \\ \delta_\xi^{\mathsf{gauge}}\, \hat{\beta}^{ab} &= \hat{\beta}^{am}\, \hat{\beta}^{bn} \big(\partial_m \xi_n - \partial_n \xi_m\big)\,.\end{aligned}$$

With $L$ the ordinary Lie derivative and $\hat{\mathcal{L}}$ the one based on the Koszul bracket, one has

$$\begin{aligned}\delta_\xi^{\mathsf{gauge}}\, \hat{g}^{ab} &= (L_{\hat{\beta}\sharp\xi}\, \hat{g})^{ab} - (\hat{\mathcal{L}}_\xi\, \hat{g})^{ab}\,, \\ \delta_\xi^{\mathsf{gauge}}\, \hat{\beta}^{ab} &= (L_{\hat{\beta}\sharp\xi}\, \hat{\beta})^{ab} - \big[(\hat{\mathcal{L}}_\xi \hat{\beta})^{ab} + \hat{\beta}^{am}\, \hat{\beta}^{bn} \big(\partial_m \xi_n - \partial_n \xi_m\big)\big]\,.\end{aligned}$$

This agrees with the previously introduced $\beta$-diffeomorphisms

$$\begin{aligned}\hat{\delta}_\xi\, \hat{g}^{ab} &= (\hat{\mathcal{L}}_\xi\, \hat{g})^{ab}\,, \\ \hat{\delta}_\xi\, \hat{\beta}^{ab} &= (\hat{\mathcal{L}}_\xi \hat{\beta})^{ab} + \hat{\beta}^{am}\, \hat{\beta}^{bn} \big(\partial_m \xi_n - \partial_n \xi_m\big)\,.\end{aligned}$$

The action in the non-geometric frame reads

$$\hat{S} = \frac{1}{2\kappa^2} \int d^n x \sqrt{-|\hat{g}|}\, \big|\hat{\beta}^{-1}\big|\, e^{-2\phi} \Big( \hat{R} - \tfrac{1}{12}\, \hat{\Theta}^{abc}\, \hat{\Theta}_{abc} + 4\, \hat{g}_{ab}\, D^a \phi D^b \phi \Big) \,.$$

Employing the field redefinitions

$$\begin{aligned} \hat{g}^{ab} &= \hat{\beta}^{am}\, \hat{\beta}^{bn}\, G_{mn} \,, \\ \hat{\beta}^{ab} &= (B^{-1})^{ab} \,, \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} \sqrt{-|G|} &= \sqrt{-|\hat{g}|}\, \big|\hat{\beta}^{-1}\big| \,, \\ R^d{}_{cab} &= -\hat{\beta}^{dq}\, \hat{\beta}_{cp}\, \hat{\beta}_{am}\, \hat{\beta}_{bn}\, \hat{R}_q{}^{pmn} \,, \\ H_{abc} &= \hat{\beta}_{am}\, \hat{\beta}_{bn}\, \hat{\beta}_{cp}\, \hat{\Theta}^{mnp} \,, \\ \partial_a \phi &= \hat{\beta}_{am} D^m \phi \,, \end{aligned}$$

one arrives at the gravity part of the string theory action

$$S = \frac{1}{2\kappa^2} \int d^n x \sqrt{-|G|}\, e^{-2\phi} \Big( R - \tfrac{1}{12} H_{abc} H^{abc} + 4 G^{ab}\, \partial_a \phi\, \partial_b \phi \Big) \,.$$

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The string theory action in the $(G,B)$-frame receives higher-order $\alpha'$-corrections.

These can be expressed in terms of the building blocks $R_{abcd}$, $H_{abc}$ and $\partial_a\phi$.

The translations of such blocks to the non-geometric frame is known, thus

$$\hat{S}^{(1)} = \frac{1}{2\kappa^2}\,\frac{\alpha'}{4}\int d^{26}x\,\sqrt{-|\hat{g}|}\,\big|\hat{\beta}^{-1}\big|\,e^{-2\phi}\Big(\hat{R}^{abcd}\,\hat{R}_{abcd} - \tfrac{1}{2}\hat{R}^{abcd}\,\hat{\Theta}_{abm}\hat{\Theta}_{cd}{}^{m}$$
$$+\tfrac{1}{24}\hat{\Theta}_{abc}\,\hat{\Theta}^{a}{}_{mn}\,\hat{\Theta}^{bm}{}_{p}\,\hat{\Theta}^{cnp} - \tfrac{1}{8}(\hat{\Theta}_{amn}\hat{\Theta}_{b}{}^{mn})(\hat{\Theta}^{a}{}_{pq}\hat{\Theta}^{bpq})\Big)\,.$$

Metsaev, Tseytlin - 1987
Hull, Townsend - 1988

Note the following:

1. If $C_{a_1 \ldots a_n}$ is invariant under *B*-field gauge transformations in the *(G,B)*-frame, then
$$\hat{C}^{a_1 \ldots a_n} = \hat{\beta}^{a_1 b_1} \ldots \hat{\beta}^{a_n b_n} C_{b_1 \ldots b_n}$$
behaves as a $\beta$-tensor.

2. If $\hat{C}^{a_1 \ldots a_n}$ is a $\beta$-tensor, then also
$$\hat{F}^{a_1 \ldots a_{n+1}} = \hat{\nabla}^{[a_1} \hat{C}^{a_2 \ldots a_{n+1}]}$$
behaves as a $\beta$-tensor.

3. One can verify that $\hat{F}^{a_1 \ldots a_{n+1}}$ is invariant under
$$\delta_\Lambda \hat{C}^{a_1 \ldots a_n} = \hat{\nabla}^{[a_1} \Lambda^{a_2 \ldots a_n]} \,.$$

Therefore, the $\hat{C}^{a_1 \ldots a_n}$ can be considered as the analogues of the R-R gauge potentials.

To obtain an action for the gauge potentials $\hat{C}_1$ and $\hat{C}_3$,

- define the generalized field strengths

$$\hat{\mathcal{F}}_2 = \hat{F}_2 \; , \qquad \hat{\mathcal{F}}_4 = \hat{F}_4 - \hat{\Theta} \wedge \hat{C}_1 \; ,$$

- which are invariant under the gauge transformations

$$\delta_{\Lambda_{(0)}} \hat{C}^a = \hat{\nabla}^a \Lambda_{(0)} \; , \qquad \delta_{\Lambda_{(2)}} \hat{C}^{a_1 a_2 a_3} = \hat{\nabla}^{[a_1} \Lambda_{(2)}^{a_2 a_3]} \; ,$$
$$\delta_{\Lambda_{(0)}} \hat{C}^{a_1 a_2 a_3} = -\Lambda_{(0)} \, \hat{\Theta}^{a_1 a_2 a_3} \; .$$

- The action related to type IIA string theory via the above field redefinition reads

$$\hat{S}^{\text{R-R}}_{\text{IIA}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-|\hat{g}|} \, |\hat{\beta}^{-1}| \left( -\tfrac{1}{2} |\hat{\mathcal{F}}_2|^2 - \tfrac{1}{2} |\hat{\mathcal{F}}_4|^2 \right) ,$$

- which is invariant under gauge transformations and ($\beta$-)diffeomorphisms.

Similarly, the Chern-Simons action is found as

$$\hat{S}^{\text{CS}}_{\text{IIA}} = \frac{1}{4\,\kappa_{10}^2} \, \frac{1}{3!\,4!\,3!} \int d^{10}x \, |\hat{\beta}^{-1}| \, \epsilon_{b_1 \ldots b_{10}} \, \hat{\Theta}^{b_1 b_2 b_3} \, \hat{F}^{b_4 b_5 b_6 b_7}_{(4)} \, \hat{C}^{b_8 b_9 b_{10}}_{(3)} \; .$$

The Lagrangian for the dilatino in the standard frame reads

$$\mathcal{L}^{\lambda}_{\mathsf{IIA}} = \overline{\lambda}\gamma^{\alpha}e_{\alpha}{}^{a}\left(\partial_a - \tfrac{i}{4}\omega_{a\,\beta\gamma}\gamma^{\beta\gamma}\right)\lambda\,.$$

The notation is as follows:

$$\lambda = \hat{\lambda}\,,$$

$$e_{\alpha}{}^{a}\,e_{\beta}{}^{b}\,G_{ab} = \eta_{\alpha\beta}\,, \qquad \eta^{\alpha\beta} = \hat{e}^{\alpha}{}_{a}\,\hat{e}^{\beta}{}_{b}\,\hat{g}^{ab}\,,$$

$$e_{c}{}^{\alpha}\,e_{\beta}{}^{b}\,\Gamma^{c}{}_{ab} + e_{c}{}^{\alpha}\,\partial_a e_{\beta}{}^{c} = \omega_{a}{}^{\alpha}{}_{\beta}\,, \qquad \hat{\omega}^{a}{}_{\alpha}{}^{\beta} = \hat{e}_{\alpha}{}^{b}\,\hat{e}^{\beta}{}_{c}\,\hat{\Gamma}_{b}{}^{ac} + \hat{e}_{\alpha}{}^{b}D^{a}\hat{e}^{\beta}{}_{b}\,.$$

The Lagrangian in the non-geometric frame then becomes

$$\hat{\mathcal{L}}^{\lambda}_{\mathsf{IIA}} = \overline{\hat{\lambda}}\gamma_{\alpha}\hat{e}^{\alpha}{}_{a}\left(D^{a} - \tfrac{i}{4}\hat{\omega}^{a}{}_{\beta\gamma}\gamma^{\beta\gamma}\right)\hat{\lambda}\,.$$

The Lagrangian for the gravitino in the standard frame reads

$$\mathcal{L}^{\Psi}_{\mathsf{IIA}} = \overline{\Psi}_a \, \gamma^{\alpha\beta\gamma} e_\alpha{}^a e_\beta{}^b e_\gamma{}^c \left(\nabla_b - \tfrac{i}{4}\,\omega_{b\,\delta\epsilon}\,\gamma^{\delta\epsilon}\right) \Psi_c \,.$$

With the field redefinition

$$\hat{\Psi}^a = \hat{\beta}^{ab}\,\hat{\Psi}_b \,,$$

the Lagrangian in the non-geometric frame becomes

$$\hat{\mathcal{L}}^{\Psi}_{\mathsf{IIA}} = \overline{\hat{\Psi}}^a \, \gamma_{\alpha\beta\gamma} \hat{e}^\alpha{}_a \hat{e}^\beta{}_b \hat{e}^\gamma{}_c \left(\hat{\nabla}^b - \tfrac{i}{4}\,\hat{\omega}^b{}_{\delta\epsilon}\,\gamma^{\delta\epsilon}\right) \hat{\Psi}^c \,.$$

Note that the appearance of the covariant derivative is crucial.

Using the field redefinition, a relation between actions has been established

$$\hat{S} = \frac{1}{2\kappa^2} \int d^n x \, \sqrt{-|\hat{g}|} \, \big|\hat{\beta}^{-1}\big| \, e^{-2\phi} \Big( \hat{R} - \tfrac{1}{12} \, \hat{\Theta}^{abc} \, \hat{\Theta}_{abc} + 4 \, \hat{g}_{ab} \, D^a \phi D^b \phi \Big)$$

$$\updownarrow$$

$$S = \frac{1}{2\kappa^2} \int d^n x \, \sqrt{-|G|} \, e^{-2\phi} \Big( R - \tfrac{1}{12} H_{abc} H^{abc} + 4 G^{ab} \, \partial_a \phi \, \partial_b \phi \Big)$$

Employing the same principle, actions have been derived

- for higher-order $\alpha'$-corrections,
- for the remaining bosonic terms in the type IIA action, and
- for the fermionic terms in the type IIA theory.

Using the field redefinition, a relation between actions has been established

$$\hat{S} = \frac{1}{2\kappa^2} \int d^n x \sqrt{-|\hat{g}|} \, \left|\hat{\beta}^{-1}\right| e^{-2\phi} \Big( \hat{R} - \tfrac{1}{12} \hat{\Theta}^{abc} \hat{\Theta}_{abc} + 4 \hat{g}_{ab} D^a \phi D^b \phi \Big)$$

$$\updownarrow$$

$$S = \frac{1}{2\kappa^2} \int d^n x \sqrt{-|G|} e^{-2\phi} \Big( R - \tfrac{1}{12} H_{abc} H^{abc} + 4 G^{ab} \partial_a \phi \partial_b \phi \Big)$$

Employing the same principle, actions have been derived

- for higher-order $\alpha'$-corrections,
- for the remaining bosonic terms in the type IIA action, and
- for the fermionic terms in the type IIA theory.

Contact with the action in the non-geometric $(\tilde{g}, \tilde{\beta})$-frame obtained via DFT is made via

$$\hat{g} = \tilde{g} - \tilde{g} \, \tilde{\beta}^{-1} \, \tilde{g} \, \tilde{\beta}^{-1} \, \tilde{g} \,, \qquad \hat{\beta} = \tilde{\beta} - \tilde{g} \, \tilde{\beta}^{-1} \tilde{g} \,.$$

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The equations of motion determined from the non-geometric action

$$\hat{S} = \frac{1}{2\kappa^2} \int d^n x \sqrt{-|\hat{g}|} \, |\hat{\beta}^{-1}| \, e^{-2\phi} \Big( \hat{R} - \tfrac{1}{12} \hat{\Theta}^{abc} \hat{\Theta}_{abc} + 4 \hat{g}_{ab} D^a \phi D^b \phi \Big) ,$$

can be expressed as follows

$$\begin{aligned}
0 &= -\frac{1}{2} \hat{g}_{ab} \hat{\nabla}^a \hat{\nabla}^b \phi + \hat{g}_{ab} \hat{\nabla}^a \phi \hat{\nabla}^b \phi - \frac{1}{24} \hat{\Theta}^{abc} \hat{\Theta}_{abc} , \\
0 &= \hat{R}^{ab} + 2 \hat{\nabla}^a \hat{\nabla}^b \phi - \frac{1}{4} \hat{\Theta}^{amn} \hat{\Theta}^b{}_{mn} , \\
0 &= \frac{1}{2} \hat{\nabla}^m \hat{\Theta}_{mab} - (\hat{\nabla}^m \phi) \hat{\Theta}_{mab} .
\end{aligned}$$

These field equations are of the same form as the ones in the geometric frame.

Consider $\mathbb{R}^4$ with a metric and bi-vector field given by

$$\hat{g}^{ab} = \delta^{ab}\,, \qquad \hat{\beta} = \begin{pmatrix} 0 & +\epsilon^{-1}(1+|x_4|) & 0 & 0 \\ -\epsilon^{-1}(1+|x_4|) & 0 & 0 & 0 \\ 0 & 0 & 0 & +\mathsf{sign}(x_4)\,\epsilon\,\theta \\ 0 & 0 & -\mathsf{sign}(x_4)\,\epsilon\,\theta & 0 \end{pmatrix}.$$

The resulting non-geometric quantities read

$$\hat{\mathcal{Q}}_1{}^{31} = -\hat{\mathcal{Q}}_1{}^{13} = \hat{\mathcal{Q}}_2{}^{32} = -\hat{\mathcal{Q}}_2{}^{23} = \frac{\theta\,\epsilon}{1+|x_4|}\,, \qquad Q_4{}^{12} = -Q_4{}^{21} = \frac{\mathsf{sign}(x_4)}{\epsilon}\,,$$

$$\hat{R}^{11} = \hat{R}^{22} = \frac{3}{4}\,\hat{R}^{33} = -3\,\frac{(\theta\,\epsilon)^2}{(1+|x_4|)^2}\,, \qquad \hat{\Theta}^{123} = \theta\,.$$

The equations of motion are satisfied (up to first order in the flux) in the limit $\epsilon \to 0$, i.e.

$$\hat{R}^{ab} \xrightarrow{\epsilon\to 0} 0\,, \qquad \hat{\mathcal{Q}}_c{}^{ab} \xrightarrow{\epsilon\to 0} 0\,, \qquad \hat{\Theta}^{123} = \theta\,.$$

In the geometric frame, (compact) Calabi-Yau manifolds can be characterized by

a Kähler form $\omega = \frac{i}{2} G_{a\bar{b}}\, dz^a \wedge d\bar{z}^{\bar{b}}$ satisfying $d\omega = 0\,,$

and by $R_{ab} = 0\,.$

For $H_{abc} = 0$ and $\phi = \text{const.}$, these are solutions to the equations of motion.

After the field redefinition, one obtains a non-geometric Calabi-Yau manifold given by

a two-vector $W = \frac{i}{2} \hat{g}^{a\bar{b}}\, \partial_{z_a} \wedge \partial_{\bar{z}_b}$ satisfying $d^H_\beta W = 0\,,$

and by $\hat{R}^{ab} = 0\,.$

For the corresponding $\hat{\Theta}^{abc} = 0$ and $\phi = \text{const.}$, this is a non-geometric solution.

# outline

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6. **conclusions**

# conclusions :: part I

1. As reviewed, non-geometric fluxes are expressed in terms of a bi-vector $\beta$ as

$$Q_a{}^{bc} = \partial_a \beta^{bc}\,, \qquad\qquad \Theta^{abc} = 3\,\beta^{[\underline{a}m}\partial_m\beta^{\underline{bc}]}\,.$$

2. A mathematical framework to describe a bi-vector is
   - the theory of Lie algebroids (generalization of the Lie bracket on $TM$).
   - A construction suitable for non-vanishing *R*-flux is $(T^*M, [\cdot,\cdot]^H_K, \beta^\sharp; \mathcal{R}=0)$.

3. The differential geometry calculus for Lie algebroids was used to construct an action

$$\hat{S} = \frac{1}{2\kappa^2}\int d^nx\,\sqrt{-|\hat{g}|}\,\left|\hat{\beta}^{-1}\right|\,e^{-2\phi}\Big(\hat{R} - \tfrac{1}{12}\,\hat{\Theta}^{abc}\,\hat{\Theta}_{abc} + 4\,\hat{g}_{ab}\,D^a\phi D^b\phi\Big)\,,$$

which is manifestly bi-invariant under standard and $\beta$-diffeomorphisms.

4. Motivated by the Seiberg-Witten map, a field redefinition
   - relating string theory and the above action has been obtained

$$\hat{g}^{ab} = \hat{\beta}^{am}\,\hat{\beta}^{bn}\,G_{mn}\,, \qquad\qquad \hat{\beta}^{ab} = (B^{-1})^{ab}\,,$$

   - which fits naturally into the Lie-algebroid construction.

5. Using the field redefinitions,
   - actions for the non-geometric R-R and fermionic sectors have been derived.
   - The equations of motion and some solutions have been discussed.