Ancient solutions to Ricci flow

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Motivations

- Ricci flow equation
  \[ \frac{\partial}{\partial t} g(t) = -2 \text{Ric}(g(t)) \]

- An ancient solution \( g(t) \) exists on \(( -\infty, T_0 )\).

- Ancient solutions arise as the limit of a sequence of suitable blow-ups, via the compactness result of Hamilton, as the time approaches the singular time for Ricci flow on a manifold with a generic initial Riemannian metric.

- Classifications/properties on the ancient solutions provide useful information to understand the singularities.
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Non-collapsing

\( (M, g) \) is called \( \kappa \)-noncollapsed if for any ball \( B_x(r) \) satisfying 
\[
|R_m|_{y} \leq r^{-2}
\]
for \( y \in B_x(r) \), 
\[
V_x(r) \geq \kappa r^n.
\]

If the above only holds for \( r \leq a \), we say that \( (M, g) \) is \( \kappa \)-noncollapsed on the scale \( a \). Perelman proved that for \( (M, g(t)) \), any solution to Ricci flow on a compact manifold \( M \times [0, T] \), there exists \( \kappa \) such that \( (M, g(t)) \) is \( \kappa \)-noncollapsed on scale \( \sqrt{T} \).

This implies that all the ancient solutions arising from the blow-up limit is \( \kappa \)-noncollapsed on all scales.

Hence for the sake of understanding the singularities of Ricci flow on compact manifolds, we can assume that the ancient solutions are \( \kappa \)-noncollapsed.
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Why ancient solutions

Any nonflat ancient solution with bounded nonnegative curvature operator must satisfy
\[ \lim_{r \to \infty} V(B(o, r)) = 0. \]

Consequences:
1) Curvature estimates for the ancient solutions.
2) Compactness on the space of 3-dimensional $\kappa$-noncollapsed ancient solutions with bounded curvature.

We proved a similar result (2005) on ancient solutions to Kähler-Ricci flow on manifolds with nonnegative bisectional curvature.

Several geometric applications can be derived, including a proof to speculation of Yau on the relation between the curvature and volume of a Kähler manifold with nonnegative bisectional curvature.
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Classification results-I

Theorem (Hamilton)

The only solutions to Ricci flow on a surface which are complete with bounded curvature on an ancient time interval $-\infty < t < T$ and where the curvature $S$ satisfies

$$\limsup_{t \to -\infty} (T - t)|S| < \infty$$

are round sphere and the flat plane, and their quotients.
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Remarks

First $S \geq 0$ by the maximum principle. In fact either $S > 0$ or $S \equiv 0$.

The condition $\limsup_{t \to -\infty} (T - t) |S| < \infty$ is labeled Type I. (A singularity with such condition is also called a fast-forming singularity.)

The proof has two parts. 1) Compact case, prove that it must be the sphere quotient. 2) Rule out the noncompact, non-flat, type I ancient solutions on surfaces.
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  2) Rule out the noncompact, non-flat, type I ancient solutions on surfaces.
We first consider the compact case. Can we classify all compact ancient solutions? Note that ancient solutions on close surfaces can only resides on the sphere unless it is flat. Theorem (Daskalopoulos, Hamilton and Sesum) On a compact surfaces, there are only two ancient solutions: the round sphere and a sausage.
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Sausage: Fateev-Onofri-Zamolodchikov, Rosenau and King solution and Perelman’s 3-dim example

Let $R \times S^1(2)$ define $h = dx^2 + d\theta^2$. Then the solution $g(t)$ is $g(t) = u(t)h$, $u(t) = \sinh(-t) \cosh(x) + \cosh(t)$.

It can be extended to $S^2$, the two-point compactification of $R \times S^1$.

Perelman showed the existence of a 3-dimensional example of ancient solution. Moreover, the example is $\kappa$-noncollapsed with positive curvature.

The example is obtained by taking the limit of a sequence of solutions on $(-t_i, 0)$.
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A remark

Hamilton-Ivey estimate: If $g(t)$ exists on $(0, T)$ with $0 \leq t \leq T$ and $\lambda_1(R_m) < 0$, then (without assuming anything on the initial condition)

\[ S \geq |\lambda_1(R_m)| (\log |\lambda_1(R_m)| + \log(t) - 3). \]

This implies that any compact ancient solution of dimension 3 must have non-negative curvature (operator).

Concerning ancient solutions of Ricci flow, there also exist motivations from physics, as they describe trajectories of the renormalization group equations of certain asymptotically free local quantum field theories in the ultra-violet regime.
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Theorem (N, 2008)

Assume that \((M, g(t))\) is a closed type I, \(\kappa\)-non-collapsed (for some \(\kappa > 0\)) ancient solution to the Ricci flow with positive curvature operator. Then \((M, g(t))\) must be the quotients \(S^n\).

This provides a high dimension analogue of Hamilton’s surface result, at least for the compact case. The positivity can be replaced by 2-positivity of curvature operator (if we use the full result of B"ohm-Wilking), or complex sectional curvature if we use Brendle-Schoen’s result.

Question: Is there any other ancient solutions on spheres besides Perelman’s example and round metric?

Brendle-Huisken-Sinestrari (2011): If one assumes a pointwise pinching condition on the curvature the round metric is the only possibility.
Classification results-III

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Speculations on high dimensional ancient solutions

There exists a speculation: Compact type-II ancient solution at dimension there must be Perelman’s example or its quotient.

Ancient solutions on sphere with positive curvature (maybe assume additionally that it is non-collapsed?) should be rotationally symmetric since there exists enough time to make the solution become symmetric.

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Fateev’s three dimensional sausages

Anastz: 

\[ w(\tau,\theta) = a^2(\tau) - b^2(\tau)(x_1^2 + x_2^2 - x_3^2 - x_4^2) \]

where 

\[ a(\tau) = \lambda \sqrt{\cosh^2 \xi - k^2 \sinh^2 \xi + 1 \sinh \xi} \]

\[ b(\tau) = \lambda \sqrt{\cosh^2 \xi - k^2 \sinh^2 \xi - 1 \sinh \xi} \]

\[ c(\tau) = -\lambda kl \tanh \xi, \]

\[ d(\tau) = \lambda \sqrt{1 - k^2 \tanh^2 \xi - \cosh \xi \sinh \xi} \]

\[ u(\tau) = 2\lambda \coth \xi, \]

\[ \phi_1 \equiv x_1 dx_2 - x_2 dx_1, \]

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\[ ds^2_{\nu,k}(\tau) = \frac{1}{w(\tau, \theta)} (u(\tau)ds^2_{\text{Stan}} + 2d(\tau)(\phi_1^2 + \phi_2^2) + 4c(\tau)\phi_1\phi_2), \]

where \( w(\tau, \theta) = a^2(\tau) - b^2(\tau)(x_1^2 + x_2^2 - x_3^2 - x_4^2)^2 = a^2(\tau) - b^2(\tau) \cos^2 2\theta \) and \( a, b, c, d, u \) are functions of \( \tau \), \( \phi_1 \equiv x_1 dx_2 - x_2 dx_1, \phi_2 \equiv x_3 dx_4 - x_4 dx_3. \)
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\begin{align*}
a(\tau) &= \lambda \frac{\sqrt{\cosh^2 \xi - k^2 \sinh^2 \xi} + 1}{\sinh \xi}, \\
b(\tau) &= \lambda \frac{\sqrt{\cosh^2 \xi - k^2 \sinh^2 \xi} - 1}{\sinh \xi}, \\
c(\tau) &= -\lambda k \tanh \xi, (0.1) \\
d(\tau) &= \lambda \frac{\sqrt{1 - k^2 \tanh^2 \xi} - \cosh \xi}{\sinh \xi}, \\
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\end{align*}
Fateev’s three dimensional sausages

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\[ \text{ds}_{\nu, k}^2(\tau) = \frac{1}{w(\tau, \theta)} \left( u(\tau) \text{ds}_{\text{stan}}^2 + 2d(\tau)(\phi_1^2 + \phi_2^2) + 4c(\tau)\phi_1\phi_2 \right), \]

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Formulae

\[ \lambda = \nu^2 (1 - k^2) > 0 \text{ and } \nu, k \text{ are two parameters with } \nu > 0 \text{ and } k^2 < 1. \]

The new variable \( \xi \) is related to \( \tau \) via the equation:

\[ \nu \tau = \xi - k^2 \log \left( \frac{1 + k \tanh \xi}{1 - k \tanh \xi} \right). \] (0.2)

Obtained by solving ODE:

\[ \frac{du}{d\tau} = -\left( u + 2 \frac{dc}{du} \right)^2, \]

\[ \frac{d(uc)}{d\tau} = 0, \]

\[ \frac{d(ab)}{d\tau} = 0 \] (0.3)

under integrability conditions:

\[ (u + d)^2 = a^2 + c^2, \]

\[ d^2 = b^2 + c^2 \] (0.4)

The reduction is computational, but it is remarkable that the anastz works. The example is neither homogeneous nor rotationally symmetric.
where $\lambda = \frac{\nu}{2(1-k^2)} > 0$ and $\nu$ and $k$ are two parameters with $\nu > 0$ and $k^2 < 1$. The new variable $\xi$ is related to $\tau$ via the equation:

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- Obtained by solving ODE:

\[
\frac{du}{d\tau} = -(u + 2d)^2, \quad \frac{d(u \, c)}{d\tau} = 0, \quad \frac{d(a \, b)}{d\tau} = 0 \tag{0.3}
\]
where $\lambda = \frac{\nu}{2(1-k^2)} > 0$ and $\nu$ and $k$ are two parameters with $\nu > 0$ and $k^2 < 1$. The new variable $\xi$ is related to $\tau$ via the equation:

$$\nu \tau = \xi - \frac{k}{2} \log \left( \frac{1 + k \tanh \xi}{1 - k \tanh \xi} \right). \quad (0.2)$$

Obtained by solving ODE:

$$\frac{du}{d\tau} = -(u + 2d)^2, \quad \frac{du}{d\tau} c = 0, \quad \frac{da}{d\tau} b = 0 \quad (0.3)$$

under integrability conditions:

$$(u + d)^2 = a^2 + c^2, \quad d^2 = b^2 + c^2 \quad (0.4)$$
where $\lambda = \frac{\nu}{2(1-k^2)} > 0$ and $\nu$ and $k$ are two parameters with $\nu > 0$ and $k^2 < 1$. The new variable $\xi$ is related to $\tau$ via the equation:

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under integrability conditions:

$$(u + d)^2 = a^2 + c^2, \quad d^2 = b^2 + c^2 \quad (0.4)$$

The reduction is computational, but it is remarkable that the ansatz works. The example is neither homogeneous nor rotationally symmetric.
Special cases-1

\[ d s^2 \nu(\tau) = \omega(\tau, \theta) \left( a(\tau) + b(\tau) \right) d s^2 \tan^2 b(\tau) (\phi_1^2 + \phi_2^2) \].

\[ d a d \tau = - a(a - b), \quad (0.5) \]

\[ d b d \tau = b(a - b) (0.6) \]

Note that this system has a simple first integral \( ab = \text{constant} \). It is worthwhile to mention that this solution has pinched sectional curvature with pinching constant tends to zero as \( \tau \to \infty \). Since its curvature operator has three different eigenvalues, it can not be rotationally symmetric.
Special cases-1

- $k = 0$:

\[
ds^2_{\nu}(\tau) = \frac{1}{w(\tau, \theta)} \left( (a(\tau) + b(\tau))ds^2_{\text{stan}} - 2b(\tau)(\phi_1^2 + \phi_2^2) \right).
\]
Special cases-1

$k = 0$:

\[
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\]

\[
\begin{align*}
\frac{da}{d\tau} & = -a(a - b), \quad (0.5) \\
\frac{db}{d\tau} & = b(a - b) \quad (0.6)
\end{align*}
\]
Special cases-I

- $k = 0$:

$$ds^2_{\nu}(\tau) = \frac{1}{w(\tau, \theta)} ((a(\tau) + b(\tau))ds^2_{\text{stan}} - 2b(\tau)(\phi_1^2 + \phi_2^2)).$$

- 

$$\frac{da}{d\tau} = -a(a - b), \quad (0.5)$$

$$\frac{db}{d\tau} = b(a - b) \quad (0.6)$$

- Note that this system has a simple first integral $a b =$ constant. It is worthwhile to mention that this solution has pinched sectional curvature with pinching constant tends to zero as $\tau \to \infty$. Since its curvature operator has three different eigenvalues, it can not be rotationally symmetric.
Special cases- I

- \( k = 0 \):

\[
\begin{aligned}
\frac{ds^2(\tau)}{\nu} &= \frac{1}{w(\tau, \theta)} \left( (a(\tau) + b(\tau))ds^2_{stan} - 2b(\tau)(\phi_1^2 + \phi_2^2) \right).
\end{aligned}
\]

- \[
\begin{aligned}
\frac{da}{d\tau} &= -a(a - b), \quad (0.5) \\
\frac{db}{d\tau} &= b(a - b) \quad (0.6)
\end{aligned}
\]

- Note that this system has a simple first integral \( ab = \text{constant} \). It is worthwhile to mention that this solution has pinched sectional curvature with pinching constant tends to zero as \( \tau \to \infty \). Since its curvature operator has three different eigenvalues, it can not be rotationally symmetric.
A tricky limit: let \( k \to 1 \) and \( \nu \to 0 \), but in the manner that
\[ 2\nu = \Omega \]
d\( s \) \( d\Omega(\tau) = \sinh 2\xi \Omega \left( d\theta^2 + \cos^2\theta (1 - \tanh^2 2\xi \cos^2\theta) \right) \]
d\( \chi \)
\[ -2 \sin^2\theta \cos^2\theta \tanh 2\xi \]
d\( \chi \)

Here \( \Omega_\tau = \xi + 2 \sinh 2\xi \). This is a type-I solution. The curvature operator has three distinct eigenvalues at a generic point.
A tricky limit: let $k \to 1$ and $\nu \to 0$, but in the manner that

$$
\frac{2\nu}{1 - k^2} = \Omega
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$$\frac{2\nu}{1 - k^2} = \Omega$$

$$ds^2_\Omega(\tau) = \frac{\sinh 2\xi}{\Omega} (d\theta^2 + \cos^2 \theta(1 - \tanh^2 \xi \cos^2 \theta)d\chi_1^2$$

$$+ \sin^2 \theta(1 - \tanh^2 \xi \sin^2 \theta)d\chi_2^2$$

$$- 2 \sin^2 \theta \cos^2 \theta \tanh^2 \xi d\chi_1 d\chi_2)$$

$$= \frac{\sinh 2\xi}{\Omega} (ds^2_{\text{stan}} - \tanh^2 \xi (\phi_1^2 + \phi_2^2 + 2\phi_1\phi_2)) .$$

Here $\Omega_{\tau} = \xi + 2 \sinh 2\xi$. This is a type-I solution. The curvature operator has three distinct eigenvalues at a generic point.
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d_{\Omega}(\tau) = \frac{\sinh 2\xi}{\Omega} \left( d\theta^2 + \cos^2 \theta (1 - \tanh^2 \xi \cos^2 \theta) d\chi_1^2 \\
+ \sin^2 \theta (1 - \tanh^2 \xi \sin^2 \theta) d\chi_2^2 \\
- 2 \sin^2 \theta \cos^2 \theta \tanh^2 \xi d\chi_1 d\chi_2 \right) \\
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\[
ds^2_\Omega(\tau) = \frac{\sinh 2\xi}{\Omega} \left( d\theta^2 + \cos^2 \theta (1 - \tanh^2 \xi \cos^2 \theta) d\chi^2_1 \\
+ \sin^2 \theta (1 - \tanh^2 \xi \sin^2 \theta) d\chi^2_2 \\
- 2 \sin^2 \theta \cos^2 \theta \tanh^2 \xi d\chi_1 d\chi_2 \right)
\]

\[
= \frac{\sinh 2\xi}{\Omega} \left( ds^2_{\text{stan}} - \tanh^2 \xi (\phi^2_1 + \phi^2_2 + 2\phi_1 \phi_2) \right).
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This is a type-I solution. The curvature operator has three distinct eigenvalues at a generic point.
Special cases-II

- A tricky limit: let \( k \to 1 \) and \( \nu \to 0 \), but in the manner that

\[
\frac{2\nu}{1 - k^2} = \Omega
\]

- This is a type-I solution. The curvature operator has three distinct eigenvalues at a generic point.

\[
ds^2_{\Omega}(\tau) = \frac{\sinh 2\xi}{\Omega} \left( d\theta^2 + \cos^2 \theta (1 - \tanh^2 \xi \cos^2 \theta) d\chi_1^2 + \sin^2 \theta (1 - \tanh^2 \xi \sin^2 \theta) d\chi_2^2 - 2 \sin^2 \theta \cos^2 \theta \tanh^2 \xi d\chi_1 d\chi_2 \right) = \frac{\sinh 2\xi}{\Omega} \left( ds^2_{\text{stan}} - \tanh^2 \xi (\phi_1^2 + \phi_2^2 + 2\phi_1\phi_2) \right).
\]

Here \( \Omega_{\tau} = \xi + \frac{2\sinh 2\xi}{2} \).
Now introduce the following change of variables:

\[ \Theta = 2 \theta, \quad \Phi = \chi_1 + \chi_2, \quad \Psi = \chi_1 - \chi_2. \]

Introduce the 1-forms

\[ \psi_1 = \sin \Phi \, d\Theta - \sin \Theta \cos \Phi \, d\Psi, \]
\[ \psi_2 = -\cos \Phi \, d\Theta - \sin \Theta \sin \Phi \, d\Psi, \]
\[ \psi_3 = -d\Phi - \cos \Theta \, d\Psi. \]

Direct calculation shows that

\[ ds^2 = \sinh 2\xi \Omega (\psi_2 \psi_1 + \psi_2 \psi_2) + 2 \tanh \xi \Omega \psi_2 \psi_3. \]

All these examples are included in two papers by Fateev (1995 and 1996). The last type I example was later also found by X. Cao and Sallof-Coste independently.
Now introduce the following change of variables:

\[ \Theta = 2\theta, \quad \Phi = \frac{\chi_1 + \chi_2}{2}, \quad \Psi = \frac{\chi_1 - \chi_2}{2}. \]

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\end{align*}
\]

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\[
\begin{align*}
ds_{\Omega}^2 &= \frac{\sinh 2\xi}{\Omega} \left( \psi_1^2 + \psi_2^2 \right) + \frac{2 \tanh \xi}{\Omega} \psi_3^2.
\end{align*}
\]
Type-I example

Now introduce the following change of variables:

\[ \Theta = 2\theta, \quad \Phi = \frac{\chi_1 + \chi_2}{2}, \quad \Psi = \frac{\chi_1 - \chi_2}{2}. \]

Introduce the 1-forms

\[ \psi_1 = \sin \Phi \, d\Theta - \sin \Theta \cos \Phi \, d\Psi, \]
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Direct calculation shows that

\[ ds^2_\Omega = \frac{\sinh 2\xi}{\Omega} (\psi_1^2 + \psi_2^2) + \frac{2 \tanh \xi}{\Omega} \psi_3^2. \]

All these examples are included in two papers by Fateev (1995 and 1996). The last type I example was later also found by X. Cao and Sallof-Coste independently.
Solutions on principle bundles

Viewing $S^3$ as the total space of the Hopf fibration over $\mathbb{C}P^1$, it is easy to check that $\psi_1 + \psi_2 = d\Theta + \sin^2\Theta d\Psi$ corresponds to the metric on the base manifold $\mathbb{C}P^1$.

Hence, \{\psi_1, \psi_2\} form a moving frame of the base manifold $\mathbb{C}P^1$.

Also $d\psi_3 = -\psi_1 \wedge \psi_2$, which is the $-1$ multiple of the Kähler form. Hence $\psi_3$ can be viewed as a connection 1-form on the total space;
Viewing $S^3$ as the total space of the Hopf fibration over $\mathbb{C}P^1$, it is easy to check that

$$\psi_1^2 + \psi_2^2 = d\Theta^2 + \sin^2 \Theta d\psi^2$$

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Theorem
Besides the rotationally symmetric type-II ancient solution constructed by Perelman, on $S^15$, there are at least five nontrivial (non-Einstein) type-I ancient solutions to Ricci flow. The first one is collapsed with positive curvature operator, which converges to the round metric as the time approaches to the singularity. The second and the third ones are non-collapsed, with positive sectional curvature, each 'connecting' one of the two known nonstandard Einstein metrics (at $t=−∞$) to the round metric as the time approaches to the singularity. The fourth one 'starts' with (at $t=−∞$) a nonstandard Einstein metric and collapses the fiber sphere $S^3$ in the generalized Hopf fibration $S^3 \to S^15 \to H\mathbb{P}^3$ as the time approaches to the singularity. The fifth ancient solution 'starts' with another nonstandard Einstein metric and collapses the fiber sphere $S^7$ in the generalized Hopf fibration $S^7 \to S^15 \to S^8$ as the time approaches to the singularity.
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Einstein metrics

It is joint work with Ioannis Bakas and S.-L. Kong. Also works on general odd dimensional spheres and $\mathbb{C}P^2_m + 1$. J.-P. Bourguignon and H. Karcher, G. R. Jensen and W. Ziller have constructions on the nonstandard Einstein metrics on the generalized Hopf fibrations. Further constructions were done by M. Wang-Ziller, etc.

The Einstein metrics amount to solving an algebraic/quadratic equation. Our result is a dynamic version of the earlier results for Einstein metrics. Our dynamic version requires solving a nonlinear ODE system with Einstein metrics as its equilibrium. The key is to find the first integral which links with the previous Einstein metric construction.
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- Our dynamic version requires solving a nonlinear ODE system with Einstein metrics as its equilibrium. The key is to find the first integral which links with the previous Einstein metric construction.
Stability consequences

It was initiated by Cao-Hamilton-Ilmanen to investigate the stability of Einstein metrics and solitons (via the entropy functional). Our examples show that the non-canonical Einstein metrics are unstable fixed point of Ricci flow. This particularly applies to non-standard Einstein metrics on odd dimensional spheres and complex projective spaces.
Stability consequences

- It was initiated by Cao-Hamilton-Ilmanen to investigate the stability of Einstein metrics and solitons (via the entropy functional). Our examples show that the non-canonical Einstein metrics are unstable fixed point of Ricci flow. This particularly applies to non-standard Einstein metrics on odd dimensional spheres and complex projective spaces.
The previous result is a combination of the special cases of three general theorems. The first one is on $U(1)$-bundles over a Kähler-Einstein manifold. Let $(M, g)$ be a Kähler-Einstein manifold with positive Chern class and let $P$ be a $U(1)$ principle bundle over $M$ with a connection $1$-form $\theta$ such that its curvature is a non-zero multiple of the Kähler form. There exist positive functions $a$ and $b$ on $(0, \infty)$ (depending on a parameter $\Lambda$) such that

$$\tilde{g} = a \langle \cdot, \cdot \rangle_g + b \pi^* g$$

is an ancient solution to Ricci flow on the total space $P^n$ ($n = 2m + 1$). Moreover, the solution is of type-I and collapsed. It has positive curvature operator when $(M, g)$ is $(C^P_m, c g_{FS})$, where $g_{FS}$ is the Fubini-Study metric and $c > 0$ is a constant.
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The first one is on $U(1)$-bundles over a Kähler-Einstein manifold

**Theorem**

Let $(M, g)$ be a Kähler-Einstein manifold with positive Chern class and let $P$ be a $U(1)$ principle bundle over $M$ with a connection 1-form $\theta$ such that its curvature is a non-zero multiple of the Kähler form. There exist positive functions $a_\Lambda(\tau)$ and $b_\Lambda(\tau)$ on $(0, \infty)$ (depending on a parameter $\Lambda$) such that $\tilde{g}_{a,b} = a \langle \cdot, \cdot \rangle_g + b \pi^* g$ is an ancient solution to Ricci flow on the total space $P^n$ ($n = 2m + 1$). Moreover, the solution is of type-I and collapsed. It has positive curvature operator when $(M, g)$ is $(\mathbb{CP}^m, c g_{FS})$, where $g_{FS}$ is the Fubini-Study metric and $c > 0$ is a constant.
The second case is on the principle SU(2) bundles on quaternion-Kähler manifolds. Again, using the variation of the connection metrics. Let \( y = \frac{a}{b} \), the slope. First if \( p > 2\sqrt{(2m + 3)q^2} \), where \( p \) is Einstein constant (for the Einstein metric on \( M \)) and \( q \) is below, there are Einstein metrics \( \tilde{g}_e_1 \) and \( \tilde{g}_e_2 \) on the total space.

**Theorem**

Assume that \((M, g)\) is a quaternion-Kähler manifold with Einstein constant \( p > 0 \). Let \( P \) be the associated SU(2)-principle bundle with connection \( A \) satisfying a compatibility condition \( (F = q(w_1 + w_2 + w_3)) \). Then, there exists a type-I ancient solution \( \tilde{g}_{a_1, b_1} \) to Ricci flow on the total space \( P \) with \( r_2 < y(\tau) < r_1 \), which flows (after re-normalization) the Einstein metric \( \tilde{g}_e_2 \) into the Einstein metric \( \tilde{g}_e_1 \) as \( t \) increases from \( -\infty \) to some \( t_0 \). There also exists a type-I ancient solution \( \tilde{g}_{\tilde{a}, \tilde{b}} \) to Ricci flow on the total space \( P \) with \( r_2 > y(\tau) > 0 \) which flows the Einstein metric \( \tilde{g}_e_2 \) from \( t = -\infty \) into a singularity, at time \( t_0 \) when it collapses the SU(2) fiber.
The second case is on the principle $SU(2)$ bundles on quaternion-Kähler manifolds. Again, using the variation of the connection metrics. Let $y = a/b$, the slope. First if $p > 2\sqrt{(2m + 3)q^2}$, where $p$ is Einstein constant (for the Einstein metric on $M$) and $q$ is below, there are Einstein metrics $\tilde{g}_{e_1}$ and $\tilde{g}_{e_2}$ on the total space.
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**Theorem**

Assume that $(M, g)$ is a quaternion-Kähler manifold with Einstein constant $p > 0$. Let $P$ be the associated $SU(2)$-principle bundle with connection $A$ satisfying a compatibility condition $(F_A = q(w_1 i + w_2 j + w_3 k))$. Then, there exists a type-I ancient solution $\tilde{g}_{a,b}$ to Ricci flow on the total space $P$ with $r_2 < y(\tau) < r_1$, which flows (after re-normalization) the Einstein metric $\tilde{g}_{e_2}$ into the Einstein metric $\tilde{g}_{e_1}$ as $t$ increases from $-\infty$ to some $t_0$. There also exists a type-I ancient solution $\tilde{g}_{\tilde{a},\tilde{b}}$ to Ricci flow on the total space $P$ with $r_2 > y(\tau) > 0$ which flows the Einstein metric $\tilde{g}_{e_2}$ from $t = -\infty$ into a singularity, at time $t_0$. 
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The third formulation is via the Riemannian submersion structure:

Proposition

Let \( \pi : (M, g) \rightarrow (B, \tilde{g}) \) be a Riemannian submersion with totally geodesic fiber. Let \( g = \hat{g} + \tilde{g} \) be the metric decomposition. Suppose that the metrics on \( M \), \( B \) and on the fiber are all Einstein with 

\[
\text{Ric}(\hat{g}) = \hat{\lambda} \hat{g}, \\
\text{Ric}(\tilde{g}) = \tilde{\lambda} \tilde{g}, \\
\text{Ric}(g) = \lambda g.
\]

(0.7)

Let \( \tilde{g}_{a, b}(\tau) = a(\tau) \hat{g} + b(\tau) \tilde{g} \). Then, \( \tilde{g}_{a, b} \) solving the Ricci flow equation is equivalent to 

\[
d a d \tau = 2\hat{\lambda} + 2(\lambda - \hat{\lambda}) a^2 b^2 ,
\]

(0.8)

\[
d b d \tau = 2\tilde{\lambda} - 2(\tilde{\lambda} - \lambda) a b.
\]

(0.9)
The third formulation is via the Riemannian submersion structure:

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\text{Ric}(\hat{g}) = \hat{\lambda} \hat{g}, \\
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\]

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Let \( \tilde{g}_{ab}(\tau) = a(\tau) \hat{g} + b(\tau) \check{g} \). Then, \( \tilde{g}_{ab} \) solving the Ricci flow equation is equivalent to

\[
d_a d_\tau = 2\hat{\lambda} + 2(\lambda - \hat{\lambda}) a^2 b^2, \\
d_b d_\tau = 2\check{\lambda} - 2(\check{\lambda} - \lambda) a b.
\]

(0.8) (0.9)
The third formulation is via the Riemannian submersion structure:

**Proposition**

Let $\pi : (M, g) \rightarrow (B, \breve{g})$ be a Riemannian submersion with totally geodesic fiber. Let $g = \hat{g} + \breve{g}$ be the metric decomposition. Suppose that the metrics on $M$, $B$ and on the fiber are all Einstein with

$$\operatorname{Ric}(\hat{g}) = \hat{\lambda} \hat{g}, \quad \operatorname{Ric}(\breve{g}) = \breve{\lambda} \breve{g}, \quad \operatorname{Ric}(g) = \lambda g. \quad (0.7)$$

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$$\frac{da}{d\tau} = 2\hat{\lambda} + 2(\lambda - \hat{\lambda}) \frac{a^2}{b^2}, \quad (0.8)$$

$$\frac{db}{d\tau} = 2\breve{\lambda} - 2(\breve{\lambda} - \lambda) \frac{a}{b}. \quad (0.9)$$
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\]
Theorem

Assume that $\Lambda_1 \approx \hat{\lambda} \approx \lambda - \hat{\lambda} \neq 1$ and $\hat{\lambda} > 0$. There exists an ancient solution $d_s^2(\tau)$ to Ricci flow on the total space $P$ with the slope $y$ between $\Lambda_1$ and 1. If $\Lambda_1 < 1$, it flows (after re-normalization) the Einstein metric $d_s^2 e^{2(1/2\Lambda_1)}$ into the Einstein metric $d_s^2 e^{1(1/2\lambda)}$ as $t$ increases from $-\infty$ to some $t_0$. If $\Lambda_1 > 1$, it flows $d_s^2 e^{1(1/2\lambda)}$ into $d_s^2 e^{2(1/2\Lambda_1)}$ as $t$ increases from $-\infty$ to some $t'$. Both solutions are of type-I.

This is complementary to the case of $U(1)$ bundle over Kähler-Einstein manifolds since $\hat{\lambda} = 0$ there.

$\Lambda_1 = 1$ is a degenerate case in the sense that there exist only one Einstein metric in the family.

The key to the proof of the existence is the monotonicity of the slope $y = a/b$ is monotone non-increasing in $\tau$. 

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The ancient solutions on the total space

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**Theorem**
Assume that \( \Lambda_1 \doteq \frac{\hat{\lambda}}{\hat{\lambda} - \hat{\lambda}} \neq 1 \) and \( \hat{\lambda} > 0 \). There exists an ancient solution \( ds^2(\tau) \) to Ricci flow on the total space \( P \) with the slope \( y \) between \( \Lambda_1 \) and 1. If \( \Lambda_1 < 1 \), it flows (after re-normalization) the Einstein metric \( ds_{e_2}^2(\frac{1}{2\Lambda_1}) \) into the Einstein metric \( ds_{e_1}^2(\frac{1}{2\lambda}) \) as \( t \) increases from \( -\infty \) to some \( t_0 \). If \( \Lambda_1 > 1 \), it flows \( ds_{e_1}^2(\frac{1}{2\lambda}) \) into \( ds_{e_2}^2(\frac{1}{2\Lambda_1}) \) as \( t \) increases from \( -\infty \) to some \( t'_0 \). Both solutions are of type-I.
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This is complementary to the case of $U(1)$ bundle over Kähler-Einstein manifolds since $\hat{\lambda} = 0$ there.
The ancient solutions on the total space

**Theorem**
Assume that $\Lambda_1 \neq 1$ and $\dot{\lambda} > 0$. There exists an ancient solution $ds^2(\tau)$ to Ricci flow on the total space $P$ with the slope $y$ between $\Lambda_1$ and 1. If $\Lambda_1 < 1$, it flows (after re-normalization) the Einstein metric $ds^2_{e_2}(\frac{1}{2\Lambda_1})$ into the Einstein metric $ds^2_{e_1}(\frac{1}{2\lambda})$ as $t$ increases from $-\infty$ to some $t_0$. If $\Lambda_1 > 1$, it flows $ds^2_{e_1}(\frac{1}{2\lambda})$ into $ds^2_{e_2}(\frac{1}{2\Lambda_1})$ as $t$ increases from $-\infty$ to some $t'_0$. Both solutions are of type-I.

- This is complementary to the case of $U(1)$ bundle over Kähler-Einstein manifolds since $\dot{\lambda} = 0$ there.
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An first integral

\[ |1 - ab| \left| \lambda \hat{\lambda} - 2\hat{\lambda} \right| |\hat{\lambda} - \hat{\lambda} - ab| \left| \hat{\lambda} - \hat{\lambda} - 2\hat{\lambda} \right| + \hat{\lambda}\lambda(\hat{\lambda} - 2\hat{\lambda})(\hat{\lambda} - \hat{\lambda}) b - 1 = \Lambda, \quad (0.10) \]

where \( \Lambda \geq 0 \) is a constant.
Another key to the proof is that there exists a first integral to the ODE system.
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\[ 1 - \frac{a}{b} \left| \frac{\lambda}{\lambda - \hat{\lambda}} \right| \frac{\hat{\lambda}}{\hat{\lambda} - \hat{\lambda}} - \frac{a}{b} \left| - \frac{\hat{\chi}^2 - 2\hat{\lambda} \hat{\chi} + \hat{\lambda} \lambda}{(\hat{\lambda} - 2\hat{\lambda})(\hat{\lambda} - \hat{\lambda})} \right| ^{b^{-1}} = \Lambda, \quad (0.10) \]

where \( \Lambda \geq 0 \) is a constant.
Another solution which collapse the fiber
Another solution which collapse the fiber

- **Corollary**

  Assume that $\hat{\lambda} > 0$. If $\Lambda_1 < 1$, there exists an ancient solution $ds^2(\tau)$ to Ricci flow on the total space $P$, such that it exists for $t \in (-\infty, t_0)$ and with $y(\tau) < \Lambda_1$. Moreover, it is of type-I and as $t \to t_0$, $ds^2(\tau)$ collapses into $\hat{b} \hat{g}$. 
### Corollary

Assume that \( \hat{\lambda} > 0 \). If \( \Lambda_1 < 1 \), there exists an ancient solution \( ds^2(\tau) \) to Ricci flow on the total space \( P \), such that it exists for \( t \in (\neg \infty, t_0) \) and with \( y(\tau) < \Lambda_1 \). Moreover, it is of type-I and as \( t \to t_0 \), \( ds^2(\tau) \) collapses into \( b \hat{g} \).

The proof again rely on the monotonicity of the slope \( y \) together with the first integral. But now it is non-decreasing in \( \tau \).
Another solution which collapse the fiber

- **Corollary**

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- The proof again rely on the monotonicity of the slope \( y \) together with the first integral. But now it is non-decreasing in \( \tau \).

- This shows that \( S^3 \times \mathbb{R}^{4n} \) and \( S^7 \times \mathbb{R}^8 \) can be the singularity models of Ricci flow on spheres.