Ancient solutions to Ricci flow

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- Classifications/properties on the ancient solutions provide useful information to understand the singularities.

Non-collapsing

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- (M, g) is called κ-noncollapsed if for any ball $B_x(r)$ satisfying

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- If the above only holds for $r \leq a$, we say that (M, g) is κ-noncollpased on the scale a. Perelman proved that for $(M, g(t))$, any solution to Ricci flow on a compact manifold $M \times [0, T]$, there exists κ such that $(M, g(t))$ is κ-noncollapsed on scale $\sqrt{T}$.

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- This implies that all the ancient solutions arising from the blow-up limit is κ-noncollapsed on all scales.
- Hence for the sake of understanding the singularities of Ricci flow on compact manifolds, we can assume that the ancient solutions are κ-noncollapsed.

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- We proved a similar result (2005) on ancient solutions to Kähler-Ricci flow on manifolds with nonnegative bisectional curvature.
- Several geometric applications can be derived, including a proof to speculation of Yau on the relation between the curvature and volume of a Kähler manifold with nonnegative bisectional curvature.

Classification results-I

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Theorem (Hamilton)

The only solutions to Ricci flow on a surface which are complete with bounded curvature on an ancient time interval $-\infty < t < T$ *and where the curvature* S *satisfies*

$$\limsup_{t \to -\infty} (T - t)|S| < \infty$$

are round sphere and the flat plane, and their quotients.

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2) Rule out the noncompact, non-flat, type I ancient solutions on surfaces.

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- Note that ancient solutions on close surfaces can only resides on the sphere unless it is flat.
- Theorem (Daskalopoulos, Hamilton and Sesum)

  On a compact surfaces, there are only two ancient solutions: the round sphere and a sausage.

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$$g(t) = u(t)h, \quad u = \frac{\sinh(-t)}{\cosh x + \cosh t}.$$

  It can be extended to S^2, the two-point compactification of $R \times S^1$.

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- The example is obtained by taking the limit of a sequence of solutions on $(-t_i, 0)$.

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- Concerning ancient solutions of Ricci flow, there also exist motivations from physics, as they describe trajectories of the renormalization group equations of certain asymptotically free local quantum field theories in the ultra-violet regime.

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  Assume that $(M, g(t))$ *is a closed type I,* κ*-non-collapsed (for some* $\kappa > 0$*) ancient solution to the Ricci flow with positive curvature operator. Then* $(M, g(t))$ *must be the quotients* $\mathbb{S}^n$*.*

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  - This provide a high dimension analogue of Hamilton's surface result, at least for the compact case. The positivity can be replaced by 2-positivity of curvature operator (if we use the full result of Böhm-Wilking), or complex sectional curvature if we use Brendle-Schoen's result.

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  - Question: Is there any other ancient solutions on spheres besides Perelman's example and round metric?
  - Brendle-Huisken-Sinestrari (2011): If one assumes a pointwise pinching condition on the curvature the round metric is the only possibility.

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- One may be able to remove the non-collapsed assumption in the previous high dimensional classification result.

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- Anastz:

$$\mathrm{d}s^2_{\nu,k}(\tau) = \frac{1}{w(\tau,\theta)}\left(u(\tau)\mathrm{d}s^2_{\mathrm{stan}} + 2d(\tau)(\phi_1^2+\phi_2^2) + 4c(\tau)\phi_1\phi_2\right),$$

where $w(\tau,\theta) = a^2(\tau) - b^2(\tau)(x_1^2+x_2^2-x_3^2-x_4^2)^2 = a^2(\tau) - b^2(\tau)\cos^2 2\theta$ and a, b, c, d, u are functions of τ, $\phi_1 \doteqdot x_1\mathrm{d}x_2 - x_2\mathrm{d}x_1, \phi_2 \doteqdot x_3\mathrm{d}x_4 - x_4\mathrm{d}x_3$.

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a(\tau) &= \lambda\frac{\sqrt{\cosh^2\xi - k^2\sinh^2\xi}+1}{\sinh\xi}, \\
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- where $\lambda = \frac{\nu}{2(1-k^2)} > 0$ and ν and k are two parameters with $\nu > 0$ and $k^2 < 1$. The new variable ξ is related to τ via the equation:
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- Obtained by solving ODE:

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- The reduction is computational, but it is remarkable that the anastz works. The example is neither homogeneous nor rotationally symmetric.

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- $k = 0$:

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- Now introduce the following change of variables:

$$\Theta = 2\theta, \quad \Phi = \frac{\chi_1 + \chi_2}{2}, \quad \Psi = \frac{\chi_1 - \chi_2}{2}.$$

Introduce the 1-forms

$$\begin{aligned}
\psi_1 &= \sin\Phi\,\mathrm{d}\Theta - \sin\Theta\cos\Phi\,\mathrm{d}\Psi, \\
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- All these examples are included in two papers by Fateev (1995 and 1996). The last type I example was later also found by X. Cao and Sallof-Coste independently.

Solutions on principle bundles

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- Viewing $\mathbb{S}^3$ as the total space of the Hopf fibration over $\mathbb{C}P^1$, it is easy to check that

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- Theorem

  Besides the rotationally symmetric type-II ancient solution constructed by Perelman, on $\mathbb{S}^{15}$, there are at least five nontrivial (non-Einstein) type-I ancient solutions to Ricci flow. The first one is collapsed with positive curvature operator, which converges to the round metric as the time approaches to the singularity. The second and the third ones are non-collapsed, with positive sectional curvature, each 'connecting' one of the two known nonstandard Einstein metrics (at $t = -\infty$) to the round metric as the time approaches to the singularity. The fourth one 'starts' with (at $t = -\infty$) a nonstandard Einstein metric and collapses the fiber sphere $\mathbb{S}^3$ in the generalized Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^{15} \to \mathbb{H}P^3$ as the time approaches to the singularity. The fifth ancient solution 'starts' with another nonstandard Einstein metric and collapses the fiber sphere $\mathbb{S}^7$ in the generalized Hopf fibration $\mathbb{S}^7 \to \mathbb{S}^{15} \to \mathbb{S}^8$ as the time approaches to the singularity.

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- The Einstein metrics amount to solving an algebraic/quadratic equation. Our result is a dynamic version of the earlier results for Einstein metrics.
- Our dynamic version requires solving a nonlinear ODE system with Einstein metrics as its equilibrium. The key is to find the first integral which links with the previous Einstein metric construction.

Stability conseqeunces

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- It was initiated by Cao-Hamilton-Ilmanen to investigate the stability of Einstein metrics and solitons (via the entropy functional). Our examples show that the non-canonical Einstein metrics are unstable fixed point of Ricci flow. This particularly applies to non-standard Einstein metrics on odd dimensional spheres and complex projective spaces.

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- Theorem

  Let (M, g) be a Kähler-Einstein manifold with positive Chern class and let P be a $U(1)$ principle bundle over M with a connection 1-form θ such that its curvature is a non-zero multiple of the Kähler form. There exist positive functions $a_\Lambda(\tau)$ and $b_\Lambda(\tau)$ on $(0, \infty)$ (depending on a parameter Λ) such that $\widetilde{g}_{a,b} = a\,\langle\cdot,\cdot\rangle_{\mathfrak{g}} + b\,\pi^* g$ is an ancient solution to Ricci flow on the total space P^n $(n = 2m+1)$. Moreover, the solution is of type-I and collapsed. It has positive curvature operator when (M, g) is $(\mathbb{C}P^m, c\, g_{FS})$, where g_{FS} is the Fubini-Study metric and $c > 0$ is a constant.

Construction II

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- Theorem

Assume that (M, g) is a quaternion-Kähler manifold with Einstein constant $p > 0$. Let P be the associated SU(2)*-principle bundle with connection A satisfying a compatibility condition $(F_A = q(w_1 i + w_2 j + w_3 k))$. Then, there exists a type-I ancient solution $\widetilde{g}_{a,b}$ to Ricci flow on the total space P with $r_2 < y(\tau) < r_1$, which flows (after re-normalization) the Einstein metric $\widetilde{g}_{e_2}$ into the Einstein metric $\widetilde{g}_{e_1}$ as t increases from $-\infty$ to some t_0. There also exists a type-I ancient solution $\widetilde{g}_{\tilde{a},\tilde{b}}$ to Ricci flow on the total space P with $r_2 > y(\tau) > 0$ which flows the Einstein metric $\widetilde{g}_{e_2}$ from $t = -\infty$ into a singularity at time t_0*

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$$\mathrm{Ric}(\hat{g}) = \hat{\lambda}\hat{g}, \qquad \mathrm{Ric}(\check{g}) = \check{\lambda}\check{g}, \qquad \mathrm{Ric}(g) = \lambda g. \tag{0.7}$$

Let $\widetilde{g}_{a,b}(\tau) = a(\tau)\hat{g} + b(\tau)\check{g}$. Then, $\widetilde{g}_{a,b}$ solving the Ricci flow equation is equivalent to

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = 2\hat{\lambda} + 2(\lambda - \hat{\lambda})\frac{a^2}{b^2}, \tag{0.8}$$

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  Assume that $\Lambda_1 \doteqdot \frac{\hat{\lambda}}{\check{\lambda}-\hat{\lambda}} \neq 1$ *and* $\hat{\lambda} > 0$. *There exists an ancient solution* $\mathrm{d}s^2(\tau)$ *to Ricci flow on the total space* P *with the slope* y *between* Λ_1 *and* 1. *If* $\Lambda_1 < 1$, *it flows (after re-normalization) the Einstein metric* $\mathrm{d}s^2_{e_2}(\frac{1}{2\Lambda_1})$ *into the Einstein metric* $\mathrm{d}s^2_{e_1}(\frac{1}{2\lambda})$ *as* t *increases from* $-\infty$ *to some* t_0. *If* $\Lambda_1 > 1$, *it flows* $\mathrm{d}s^2_{e_1}(\frac{1}{2\lambda})$ *into* $\mathrm{d}s^2_{e_2}(\frac{1}{2\Lambda_1})$ *as* t *increases from* $-\infty$ *to some* t_0'. *Both solutions are of type-I.*

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  - This is complementary to the case of $U(1)$ bundle over Kähler-Einstein manifolds since $\hat{\lambda} = 0$ there.
  - $\Lambda_1 = 1$ is a degenerate case in the sense that there exist only one Einstein metric in the family.
  - The key to the proof of the existence is the monotonicity of the slope $y = \frac{a}{b}$ is monotone non-increasing in τ.

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- $$\left|1-\frac{a}{b}\right|^{\frac{\lambda}{\check{\lambda}-2\hat{\lambda}}}\left|\frac{\hat{\lambda}}{\check{\lambda}-\hat{\lambda}}-\frac{a}{b}\right|^{-\frac{\check{\lambda}^2-2\hat{\lambda}\check{\lambda}+\hat{\lambda}\lambda}{(\check{\lambda}-2\hat{\lambda})(\check{\lambda}-\hat{\lambda})}}b^{-1}=\Lambda, \qquad (0.10)$$

  where $\Lambda \geq 0$ *is a constant.*

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- Corollary

  Assume that $\hat{\lambda} > 0$*. If* $\Lambda_1 < 1$*, there exists an ancient solution* $\mathrm{d}s^2(\tau)$ *to Ricci flow on the total space* P*, such that it exists for* $t \in (-\infty, t_0)$ *and with* $y(\tau) < \Lambda_1$*. Moreover, it is of type-I and as* $t \to t_0$*,* $\mathrm{d}s^2(\tau)$ *collapses into* $b\,\check{g}$*.*

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  - This shows that $\mathbb{S}^3 \times \mathbb{R}^{4n}$ and $\mathbb{S}^7 \times \mathbb{R}^8$ can be the singularity models of Ricci flow on spheres.