”Superconformal Albegra and Mathieu moonshine”

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♣ Elliptic genus of K3

Elliptic genus in string theory is expressed as

$$Z_{elliptic}(z;\tau) = Tr_{\mathcal{H}_L \times \mathcal{H}_R}(-1)^{F_L+F_R} e^{4\pi i z J^3_{L,0}} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{c}{24}}$$

and describes the topological invariants of the target manifold and counts the number of BPS states in the theory. Here L_0 denotes the zero mode of the Visasoro operators and F_L and F_R are left and right moving fermion numbers. In ellitpic genus the right moving sector is frozen to the supersymmetric ground states (BPS states) while in the left moving sector all the states in the Hilbert space $\mathcal{H}_L$ contribute.

Elliptic genus of K3 surface is known: EOTY

$$Z_{K3}(z;\tau) = 8\left[\left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)}\right)^2 + \left(\frac{\theta_3(z;\tau)}{\theta_3(0;\tau)}\right)^2 + \left(\frac{\theta_4(z;\tau)}{\theta_4(0;\tau)}\right)^2\right]$$

$$Z_{K3}(z=0)=24, \quad Z_{K3}(z=\frac{1}{2})=16+O(q),$$
$$Z_{K3}(z=\frac{1+\tau}{2})=2q^{-\frac{1}{2}}+O(q^{\frac{1}{2}})$$

Elliptic genus of a complex D-dimensional manifold is a Jacobi form of weight=0 and index=D/2. When D=2, space of Jacobi form is one-dimensional and given by the above formula.

String theory on K3 has an N=4 superconformal symmetry and its states fall into representations of N=4 superconformal algebra (SCA). N=4 SCA contains an affine $SU(2)_k$ symmetry and has a central charge $c = 6k$. $k = n$ case decsribes complex-$2n$ dimensional hyperKähler manifolds.

We would like to study the decomposition of the elliptic genus in terms of irreducible characters of N=4 SCA. In N=4 SCA, hightest-weight states $|h, I\rangle$ are charactered by

$$L_0|h,\ell\rangle = h|h,\ell\rangle, \qquad J_0^3|h,\ell\rangle = \ell|h,\ell\rangle$$

and the theory possesses two different type of representations, BPS and non-BPS representations. In the case of $k = 1$

there are representations (in Ramond sector)

$$\text{BPS rep.} \qquad h=\frac{1}{4}; \quad \ell=0,\ \frac{1}{2}$$
$$\text{non-BPS rep.} \qquad h>\frac{1}{4}; \qquad \ell=\frac{1}{2}$$

Character of a representation is given by

$$Tr_{\mathcal{R}}(-1)^F q^{L_0} e^{4\pi i z J_0^3}$$

Its index is given by the value at $z=0$, $Tr_{\mathcal{R}}(-1)^F q^{L_0}$. BPS representations have a non-vanishing index

$$\text{index (BPS, } \ell=0\text{)} = 1$$
$$\text{index (BPS, } \ell=\frac{1}{2}\text{)} = -2$$

Character function of $\ell = 0$ BPS representation has the form

$$ch^{\tilde{R}}_{k=1,h=\frac{1}{4},\ell=0}(z;\tau) = \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}\mu(z;\tau)$$

where

$$\mu(z;\tau) = \frac{-ie^{\pi iz}}{\theta_1(z;\tau)}\sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)}e^{2\pi inz}}{1-q^n e^{2\pi iz}}$$

On the other hand the character of non-BPS representations are given by

$$ch^{\tilde{R}}_{k=1,h>\frac{1}{4},\ell=\frac{1}{2}} = q^{h-\frac{3}{8}}\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

These have vanishing index

$$\text{index (non-BPS rep)} = 0$$

At the unitarity bound non-BPS representation splits into a sum of BPS representations

$$\lim_{h\to\frac{1}{4}} q^{h-\frac{3}{8}} \frac{\theta_1^2}{\eta^3} = ch^{\tilde{R}}_{k=1,h=\frac{1}{4},\ell=\frac{1}{2}} + 2ch^{\tilde{R}}_{k=1,h=\frac{1}{4},\ell=0}$$

Function $\mu(z;\tau)$ is a typical example of the so-called Mock theta functions (Lerch sum or Appell function). Mock theta functions look like theta functions but they have anomalous modular transformation laws and are difficult to handle. Recently there have been developments in understanding the nature of Mock theta functions initiated by Zwegers who has developed a way to improve their modular properties. We will adopt his method of handling Mock theta functions.

It is possible to derive the following idenities

$$\begin{aligned} ch^{\tilde{R}}_{k=1,h=\frac{1}{4},\ell=0}(z;\tau) &= \left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)}\right)^2 + \mu_2(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} \\ &= \left(\frac{\theta_3(z;\tau)}{\theta_3(0;\tau)}\right)^2 + \mu_3(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} \\ &= \left(\frac{\theta_4(z;\tau)}{\theta_4(0;\tau)}\right)^2 + \mu_4(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} \end{aligned}$$

where

$$\mu_2(\tau)=\mu(z=\frac{1}{2};\tau),\ \mu_3(\tau)=\mu(z=\frac{1+\tau}{2};\tau),\ \mu_4(\tau)=\mu(z=\frac{\tau}{2};\tau)$$

$$\mu(z;\tau) = \frac{-ie^{\pi iz}}{\theta_1(z;\tau)}\sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)}e^{2\pi inz}}{1-q^n e^{2\pi iz}}$$

Then we can rewrite the ellitpic genus as

$$Z_{K3} = 24ch^{\tilde{R}}_{k=1,h=\frac{1}{4},\ell=0}(z;\tau) - 8\sum_{i=2}^{4}\mu_i(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

Using q-expansion of functions μ_i we find

$$8\left(\mu_2(\tau)+\mu_3(\tau)+\mu_4(\tau)\right) = \quad 2q^{-\frac{1}{8}} \; -2\sum_{n=1} A(n)q^{n-\frac{1}{8}}$$

$\Uparrow$

polar term

$A(n)$ $(n=1,2,\cdots)$ are positive integers.

At smaller values of n, Fourier coefficients $A(n)$ may be obtained by direct expansion. We find

n	1	2	3	4	5	6	7	8	$\cdots$
$A(n)$	45	231	770	2277	5796	13915	30843	65550	$\ldots$

Surprize: Dimensions of some irreducible reps. of Mathieu group M_{24} appear

dimensions : $\{ \; 45 \quad 231 \quad 770 \quad 990 \quad 1771 \quad 2024 \quad 2277$
$3312 \quad 3520 \quad 5313 \quad 5544 \quad 5796 \quad 10395 \quad \cdots \}$

$$A(6) = 13915 = 3520 + 10395,$$

$$A(7) = 30843 = 10395 + 5796 + 5544 + 5313 + 2024 + 1771$$

Mathieu moonshine? T.E.-Ooguri-Tachikawa

cf. Monsterous moonshine:

$$J(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

q-expansion coeffcients of J-function are decomposed into a sum of irred. reps. of the monster group.

$$196884 = 1 + 196883, \;\; 21493760 = 1 + 196883 + 21296876$$

Mukai: enumeration of eleven K3 surfaces with finite non-Abelian automorphism group. All these groups are sugb-groups of M_{23}.

Fantasy: Is it possible that these automorphism groups at iso-letd points in K3 moduli space are enhanced to M_{24} over the whole moduli space when one cosnider the elliptic genus?

On the other hand, using the method of Rademacher expansion adapted to the case of Mock theta functions (Bringmann-Ono) we can determine the asymptotic behavior of coefficients $A(n)$ as

$$A(n) \approx \frac{2}{\sqrt{8n-1}} e^{2\pi\sqrt{\frac{1}{2}(n-\frac{1}{8})}} \qquad \text{T.E.-Hikami}$$

Above exponent may be identified as the entropy of a baby Black Hole in string theory compactified on K3 with $Q_1 = 1, Q_5 = 1$, D_1 and D_5 branes.

♣ Twisted Elliptic Genus

Dimension of the representation equals the trace of the identity element: we may identify

$$A(n) = Tr_{V_n} \; 1$$

$$V_1 = 45 + 45^*, \;\; V_2 = 231 + 231^*, \;\; V_3 = 770 + 770^*, \cdots$$

We may consider the trace of other group elements in M_{24}

$$A_g(n) = Tr_{V_n} \; g, \qquad g \in M_{24}$$

$Tr \; g$ depends only on the conjugacy class of g. There exists 26 conjugacy classes $\{g\}$ in M_{24} and also 26 irreducible representations $\{R\}$. We have the character table given by

$$\chi_R{}^g = Tr_R\, g$$

1A	2A	3A	5A	4B	7A	7B	8A	6A	11A	15A	15B	14A	14B	23A	23B	12B	6B	4C	3B
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
23	7	5	3	3	2	2	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1
252	28	9	2	4	0	0	0	1	-1	-1	-1	0	0	-1	-1	0	0	0	0
253	13	10	3	1	1	1	-1	-2	0	0	0	-1	-1	0	0	1	1	1	1
1771	-21	16	1	-5	0	0	-1	0	0	1	1	0	0	0	0	-1	-1	-1	7
3520	64	10	0	0	-1	-1	0	-2	0	0	0	1	1	1	1	0	0	0	-8
45	-3	0	0	1	e_7^+	e_7^-	-1	0	1	0	0	$-e_7^+$	$-e_7^-$	-1	-1	1	-1	1	3
$\overline{45}$	-3	0	0	1	e_7^-	e_7^+	-1	0	1	0	0	$-e_7^-$	$-e_7^+$	-1	-1	1	-1	1	3
990	-18	0	0	2	e_7^+	e_7^-	0	0	0	0	0	e_7^+	e_7^-	1	1	1	-1	-2	3
$\overline{990}$	-18	0	0	2	e_7^-	e_7^+	0	0	0	0	0	e_7^-	e_7^+	1	1	1	-1	-2	3
1035	-21	0	0	3	$2\,e_7^+$	$2\,e_7^-$	-1	0	1	0	0	0	0	0	0	-1	1	-1	-3
$\overline{1035}$	-21	0	0	3	$2\,e_7^-$	$2\,e_7^+$	-1	0	1	0	0	0	0	0	0	-1	1	-1	-3
$1035'$	27	0	0	-1	-1	-1	1	0	1	0	0	-1	-1	0	0	0	2	3	6
231	7	-3	1	-1	0	0	-1	1	0	e_{15}^+	e_{15}^-	0	0	1	1	0	0	3	0
$\overline{231}$	7	-3	1	-1	0	0	-1	1	0	e_{15}^-	e_{15}^+	0	0	1	1	0	0	3	0
770	-14	5	0	-2	0	0	0	1	0	0	0	0	0	e_{23}^+	e_{23}^-	1	1	-2	-7
$\overline{770}$	-14	5	0	-2	0	0	0	1	0	0	0	0	0	e_{23}^-	e_{23}^+	1	1	-2	-7
483	35	6	-2	3	0	0	-1	2	-1	1	1	0	0	0	0	0	0	3	0
1265	49	5	0	1	-2	-2	1	1	0	0	0	0	0	0	0	0	0	-3	8
2024	8	-1	-1	0	1	1	0	-1	0	-1	-1	1	1	0	0	0	0	0	8
2277	21	0	-3	1	2	2	-1	0	0	0	0	0	0	0	0	0	2	-3	6
3312	48	0	-3	0	1	1	0	0	1	0	0	-1	-1	0	0	0	-2	0	-6
5313	49	-15	3	-3	0	0	-1	1	0	0	0	0	0	0	0	0	0	-3	0
5796	-28	-9	1	4	0	0	0	-1	-1	1	1	0	0	0	0	0	0	0	0
5544	-56	9	-1	0	0	0	0	1	0	-1	-1	0	0	1	1	0	0	0	0
10395	-21	0	0	-1	0	0	1	0	0	0	0	0	0	-1	-1	0	0	3	0

Here we have used $e_p^{\pm} = \frac{1}{2}\left(\pm\sqrt{-p} - 1\right)$.

There are two types of conjugacy classes in M_{24}, type I and type II.

Conjugacy class of type I fixes at least one element out of 24 and thus they arise from the conjugacy classes of M_{23}.

On the other hand conjugacy class of type II does not have a fixed point and is intrinsically M_{24}.

For each conjugacy class we want to construct a twisted genus (analogue of Thompson series in monstrous moonshine)

$$Z_g = \sum_{n=1}^{\infty} Tr_{V_n}\, g \times q^n$$

For instance,

$$Z_{2A} = -6q + 14q^2 - 28q^3 + 42q^4 - 56q^5 + 86q^6 + \cdots$$

and has the right modular property ($Z_{2A} \in \Gamma_0(2)$).

Twisted genus is decomposed into massless and massive parts

$$Z_g(\tau, z) = \chi_g \, ch^{\tilde{R}}_{h=\frac{1}{4}, I=0} - A_g(\tau) \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3}$$

Here χ_g is the Euler number assigned to the class g

g	1A	2A	3A	5A	4B	7A	8A	6A	11A	15A	14A	23A	typeII
χ_g	24	8	6	4	4	3	2	2	2	1	1	1	0

χ_g vanishes for type II classes.

conjugacy class	cycle shape	
1A	1^{24}	()
2A	$1^8 \cdot 2^8$	(1,8)(2,12)(4,15)(5,7)(9,22)(11,18)(14,19)(23,24)
3A	$1^6 \cdot 3^6$	(3,18,20)(4,22,24)(5,19,17)(6,11,8)(7,15,10)(9,12,14)
5A	$1^4 \cdot 5^4$	(2,21,13,16,23)(3,5,15,22,14)(4,12,20,17,7)(9,18,19,10,24)
4B	$1^4 \cdot 2^2 \cdot 4^4$	(1,17,21,9)(2,13,24,15)(3,23)(4,14,5,8)(6,16)(12,18,20,22)
7A	$1^3 \cdot 7^3$	(1,17,5,21,24,10,6)(2,12,13,9,4,23,20)(3,8,22,7,18,14,19)
7B	$1^3 \cdot 7^3$	(1,21,6,5,10,17,24)(2,9,20,13,23,12,4)(3,7,19,22,14,8,18)
8A	$1^2 \cdot 2^1 \cdot 4^1 \cdot 8^2$	(1,13,17,24,21,15,9,2)(3,16,23,6)(4,22,14,12,5,18,8,20)(7,11)
6A	$1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$	(1,8)(2,24,11,12,23,18)(3,20,10)(4,15)(5,19,9,7,14,22)(6,16,13)
11A	$1^2 \cdot 11^2$	(1,3,10,4,14,15,5,24,13,17,18)(2,21,23,9,20,19,6,12,16,11,22)
15A	$1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$	(2,13,23,21,16)(3,7,9,5,4,18,15,12,19,22,20,10,14,17,24)(6,8,11)
15B	$1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$	(2,23,16,13,21)(3,12,24,15,17,18,14,4,10,5,20,9,22,7,19)(6,8,11)
14A	$1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$	(1,12,17,13,5,9,21,4,24,23,10,20,6,2)(3,18,8,14,22,19,7)(11,15)
14B	$1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$	(1,13,21,23,6,12,5,4,10,2,17,9,24,20)(3,14,7,8,19,18,22)(11,15)
23A	$1^1 \cdot 23^1$	(1,7,6,24,14,4,16,12,20,9,11,5,15,10,19,18,23,17,3,2,8,22,21)
23B	$1^1 \cdot 23^1$	(1,4,11,18,8,6,12,15,17,21,14,9,19,2,7,16,5,23,22,24,20,10,3)
12B	12^2	(1,12,24,23,10,8,18,6,3,21,2,7)(4,9,11,15,13,16,20,5,22,17,14,19)
6B	6^4	(1,24,10,18,3,2)(4,11,13,20,22,14)(5,17,19,9,15,16)(6,21,7,12,23,8)
4C	4^6	(1,23,18,21)(2,12,10,6)(3,7,24,8)(4,15,20,17)(5,14,9,13)(11,16,22,19)
3B	3^8	(1,10,3)(2,24,18)(4,13,22)(5,19,15)(6,7,23)(8,21,12)(9,16,17)(11,20,14)
2B	2^{12}	(1,8)(2,10)(3,20)(4,22)(5,17)(6,11)(7,15)(9,13)(12,14)(16,18)(19,23)(21,24)
10A	$2^2 \cdot 10^2$	(1,8)(2,18,21,19,13,10,16,24,23,9)(3,4,5,12,15,20,22,17,14,7)(6,11)
21A	$3^1 \cdot 21^1$	(1,3,9,15,5,12,2,13,20,23,17,4,14,10,21,22,19,6,7,11,16)(8,18,24)
21B	$3^1 \cdot 21^1$	(1,12,17,22,16,5,23,21,11,15,20,10,7,9,13,14,6,3,2,4,19)(8,24,18)
4A	$2^4 \cdot 4^4$	(1,4,8,15)(2,9,12,22)(3,6)(5,24,7,23)(10,13)(11,14,18,19)(16,20)(17,21)
12A	$2^1 \cdot 4^1 \cdot 6^1 \cdot 12^1$	(1,15,8,4)(2,19,24,9,11,7,12,14,23,22,18,5)(3,13,20,6,10,16)(17,21)

Twisted genera for all conjugacy classes have been obtained.

They reproduce correct lower-order expansion coefficients and are invariant under the Hecke subgroup $\Gamma_0(N)$

$$\Gamma_0(N)=\{\begin{pmatrix} a & b \\ c & d \end{pmatrix},\ ad-bc=1,\ c\equiv 0,\ mod\, N\}$$

N denotes the order of the element g.

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From the study of K3 surface with Z_p (p=2,3,5,7) symmetry, for instance, twisted genera of classes pA (p=2,3,5,7) are known

A.Sen

$$Z_{pA}(z;\tau) = \frac{2}{p+1}\phi_{0,1}(z;\tau) + \frac{2p}{p+1}\phi_2^{(p)}(\tau)\phi_{-2,1}(z;\tau)$$

where

$$\phi_{0,1}(z;\tau) = \frac{1}{2}Z_{K3}(z;\tau), \quad \phi_{-2,1}(z;\tau) = -\frac{\theta_1(z;\tau)^2}{\eta(\tau)^6}$$

are the basis of Jacobi forms with index=1 and

$$\begin{aligned}\phi_2^{(p)}(\tau) &= \frac{24}{p-1}q\partial_q \log\left(\frac{\eta(p\tau)}{\eta(\tau)}\right), \\ &= \frac{24}{p-1}\sum_{k=1}\sigma_1(k)(q^k - pq^{pk})\end{aligned}$$

is an element of $\Gamma_0(p)$.

In the case of type II twisted genera are modular forms of $\Gamma_0(N)$ with a multiplier system (invariant up to a phase). They are given in terms of quotients of eta functions.

$$
\begin{aligned}
Z_{2B}(z;\tau) &= 2\frac{\eta(\tau)^8}{\eta(2\tau)^4}\phi_{-2,1}(z;\tau), \\
Z_{3B}(z;\tau) &= 2\frac{\eta(\tau)^6}{\eta(3\tau)^2}\phi_{-2,1}(z;\tau), \\
Z_{4A}(z;\tau) &= 2\frac{\eta(2\tau)^8}{\eta(4\tau)^4}\phi_{-2,1}(z;\tau), \\
Z_{4C}(z;\tau) &= 2\frac{\eta(\tau)^4\eta(2\tau)^2}{\eta(4\tau)^2}\phi_{-2,1}(z;\tau) \\
&\cdots
\end{aligned}
$$

etc. Thus we have a complete list of the twisted genera for 26 conjugacy classes. Making use of them we can uniquely decompose the coefficients of K3 elliptic genus into irreducible representations of M_{24} at arbitrary level.

n	1A	2A	3A	5A	4B	7A	8A	6A	11A	15A	14A	23A	12B	6B	4C	3B
1	90	-6	0	0	2	-1	-2	0	2	0	1	-2	2	-2	2	6
2	462	14	-6	2	-2	0	-2	2	0	-1	0	2	0	0	6	0
3	1540	-28	10	0	-4	0	0	2	0	0	0	-1	2	2	-4	-14
4	4554	42	0	-6	2	4	-2	0	0	0	0	0	0	4	-6	12
5	11592	-56	-18	2	8	0	0	-2	-2	2	0	0	0	0	0	0
6	27830	86	20	0	-2	-2	2	-4	0	0	2	0	0	0	6	-16
7	61686	-138	0	6	-10	2	-2	0	-2	0	2	0	-2	-2	-2	30
8	131100	188	-30	0	4	-3	0	2	2	0	-1	0	0	0	-12	0
9	265650	-238	42	-10	10	0	-2	2	0	2	0	0	-2	6	10	-42
10	521136	336	0	6	-8	0	-4	0	0	0	0	2	-2	2	16	42
11	988770	-478	-60	0	-14	6	2	-4	2	0	-2	0	0	0	-6	0
12	1830248	616	62	8	8	0	0	-2	2	2	0	0	2	-6	-16	-70
13	3303630	-786	0	0	22	-6	2	0	0	0	-2	2	0	-4	6	84
14	5844762	1050	-90	-18	-6	0	2	6	0	0	0	2	0	0	18	0
15	10139734	-1386	118	4	-26	-4	-2	6	0	-2	0	0	2	2	-10	-110
16	17301060	1764	0	0	12	0	0	0	-4	0	0	0	2	6	-28	126
17	29051484	-2212	-156	14	28	0	-4	-4	0	-1	0	0	0	0	12	0
18	48106430	2814	170	0	-18	8	-2	-6	-2	0	0	-2	2	-6	38	-166
19	78599556	-3612	0	-24	-36	0	0	0	2	0	0	0	-2	-6	-20	210
20	126894174	4510	-228	14	14	-6	-2	4	0	2	2	0	0	0	-42	0
21	202537080	-5544	270	0	48	4	4	6	-2	0	0	0	-2	6	16	-282
22	319927608	6936	0	18	-16	-7	4	0	0	0	-1	0	0	4	48	300
23	500376870	-8666	-360	0	-58	0	-2	-8	4	0	0	2	0	0	-18	0
24	775492564	10612	400	-36	28	0	0	-8	0	0	0	0	0	-8	-60	-392
25	1191453912	-12936	0	12	64	12	-4	0	0	0	0	0	2	-10	32	462
26	1815754710	15862	-510	0	-34	0	-6	10	0	0	0	-1	0	0	78	0
27	2745870180	-19420	600	30	-76	-10	4	8	-2	0	-2	0	0	8	-36	-600
28	4122417420	23532	0	0	36	2	0	0	0	0	-2	0	0	12	-84	660
29	6146311620	-28348	-762	-50	100	-6	4	-10	-2	-2	2	0	0	0	36	0
30	9104078592	34272	828	22	-40	0	4	-12	4	-2	0	0	0	-8	96	-840
31	13401053820	-41412	0	0	-116	0	-4	0	0	0	0	-2	-2	-10	-44	966
32	19609321554	49618	-1062	34	50	18	2	10	-2	-2	2	0	0	0	-126	0
33	28530824630	-59178	1220	0	126	0	-6	12	0	0	0	2	-4	12	62	-1204
34	41286761478	70758	0	-72	-66	-10	-6	0	6	0	2	0	0	12	150	1332
35	59435554926	-84530	-1518	26	-154	6	2	-14	0	2	2	0	0	0	-66	0
36	85137361430	100310	1670	0	70	-12	-2	-10	0	0	0	0	-2	-18	-170	-1666

♣ Mathieu moonshine

Orthogonality relation of characters:

$$\sum_g n_g \, \chi^g_{R'} \, \bar{\chi}^g_R = |G| \delta_{RR'}$$

n_g is the number of elements in the conjugacy class g and $|G|$ denotes the order of the group. Let $c_R(n)$ be he multiplicity of representation R in the decompostion of K3 elliptic genus at level n. We then have

$$\sum_R c_R(n) \chi^g_R = A_g(n)$$

Then using the orthogonality relation we find

$$\sum_g \frac{1}{|G|} n_g \bar{\chi}_R^{\,g} A_g(n) = c_R(n)$$

We have checked that the multiplicities $c_R(n)$ are all positive integers upto $n = 1000$ and this gives a very strong evidence for Mathieu moonshine conjecture.

T.Gannon now has a mathematical proof of Mathieu moonshine (to appear).

♣ Borcherds product and lift of a Jacobi form

Borcherds lift:

Go back to K3 ellitpic genus

$$Z_{K3}(\tau;z) \equiv Z_{1A}(\tau;z) = 2\phi_{0,1}(\tau;z)$$

and consider its "second quantized" version DVV

$$\mathcal{M}(\Omega) = \sum_{m}^{\infty} Z_{K3}^{[m]}(\tau,z)p^m = \exp\left(\sum_{m=1}^{\infty} T_m(Z_{K3}(\tau,z))p^m\right)$$

where $Z_{K3}^{[m]}$ denotes the elliptic genus of a m-th symmetric product of $K3$ and T_m is the Hecke transformation

$$T_m(Z_{K3}(\tau;z)) = m^{-1} \sum_{\substack{ad=m \\ b=0..d-1}} Z_{K3}(\frac{a\tau+b}{d}\tau, az)$$

This sum is written into an infinite product form

$$\mathcal{M}(\Omega) = \prod_{\substack{m=1 \\ n=0, r\in Z}} (1-p^m q^n y^r)^{-c_{1A}(nm,r)}$$

Here $c_{1A}(n,r)$ are expansion coefficients of Z_{K3}

$$Z_{1A}(\tau;z) = \sum_{n,\ell} c_{1A}(n,r) q^n y^r$$

By symmetrizing in p, q we can construct a Siegel modular form

$$\Phi(\Omega) = \prod_{\substack{n\geq 0, m\geq 0\\ r\in \mathbb{Z}}} (1 - p^m q^n y^r)^{c_{1A}(nm,r)} \tag{1}$$

(when $n = m = 0, r < 0$). It is well-known that this is the wt=10 Igusa form. One also finds a "Hodge anomaly" term

$$\Phi(\Omega)\mathcal{M}(\Omega) = p\,\eta^{24}(\tau, z)\phi_{-2,1}(\tau, z)$$

Additive lift:

On the other hand class 1A has a cycle shape 1^{24} and one may introduce a wt=10 Jacobi form

$$\eta_{1A}(\tau;z) = \eta(\tau)^{24}\phi_{-2,1}(\tau;z)$$

We consider the horizontal lift

$$\Phi(\Omega) = \sum_{m\geq 1} T_m(\eta_{1A}(\tau;z))p^m \tag{2}$$

Then the above sum (2) becomes also a Siegel modular form of wt=10. It is known that these two Siegel forms in fact agree.

Borcherds lift (1)= additive lift (2)

Hence we obtain the correspondence of Jacobi forms;

$Z_{1A} \Longleftrightarrow \eta_{1A} = \eta^{24} \times \phi_{-2,1}$ which maps a twisted genus to an eta product.

The above identiy implies an infinite number of relations

$$Z_{1A} = -T_2(\eta_{1A})/\eta_{1A},$$
$${Z_{1A}}^2/2 - T_2(Z_{1A}) = T_3(\eta_{1A})/\eta_{1A}, \cdots$$

It is possible to consider "twisted" version of the above correspondence such as

$$Z_{2A} \Longleftrightarrow \eta_{2A} = \eta(\tau)^8\eta(2\tau)^8 \times \phi_{-2,1}$$

Note that class 2A has a cycle shape $1^8 2^8$. Relevance of cycle shape and eta product is very well-known. Mason, McKay,,,

The pairing between twisted K3 genus and eta product holds for classes 2A,3A,4B,5A,8A. Gritsenko-Nikulin,Sen,Gritsenko-Clery,Dabholkar-Nampuri,Govindarajan,,,

We studied the correspondence in detail and found that the following relation holds for all type I conjugacy classes

$$Z_g = -T_2(\eta_g)/\eta_g, \ \cdots \tag{3}$$

(Above formula becomes modified in the case of classes 11A,14A, 15A, 23A when the Jacobi form η_g has a vanishing or negative weight.)

♣ Recent Developments

Umbral moonshine: Cheng, Duncan and Harvey

Consider a series of Jacobi forms with index $m = k + 1$ $(m =$

$\mathbf{2, 3, 4, 5, 7)}$

$$\boldsymbol{Z(m=2) = 8 \times [X + Y + Z],}$$

$$\boldsymbol{Z(m=3) = 4[XY + YZ + ZX],}$$

$$\boldsymbol{Z(m=4) = 8XYZ,}$$

$$\boldsymbol{Z(m=5) = 4[X^2YZ + \cdots] - 2[X^2Y^2 + \cdots]}$$

$$\boldsymbol{Z(m=7) = -4[X^3Y^3 + \cdots] + 4[X^3Y^2Z + \cdots] \\ -8X^2Y^2Z^2}$$

where $\boldsymbol{X \equiv \frac{\theta_2(z)^2}{\theta_2(0)^2}, Y \equiv \frac{\theta_3(z)^2}{\theta_3(0)^2}, Z \equiv \frac{\theta_4(z)^2}{\theta_4(0)^2}}$

These Jacobi forms are characterized by their q^0 term

$$Z(m) \approx 2y + (\frac{24}{m-1} - 4) + 2y^{-1} \tag{4}$$

It turns out that the expansion of the above Jacobi forms in terms of $\mathcal{N} = 4$ characters all exhibit moonshine phenomena, with the group M_{24} for $m = 2$ and M_{12} for $m = 3$ etc.

Note:

At $m = 3$, for instance, there exist two Jacobi forms with index 2

$$J_1 = X^2 + Y^2 + Z^2, \quad J_2 = XY + YZ + ZX$$

It is known that the identity operator in NS sector is contained in J_1. The elliptic genus of symmetric product $K3^{[2]}$, for instance, is given by

$$48J_1 + 60J_2.$$

It is somewhat awkward to consider $Z(m = 3) = 4J_2$ which does not contain the identity operator. Thus $Z(m = 3)$ may not possess well-defined geometrical significance. The same comment applies to all cases $m \geq 3$.

We point out Umbral series still appears to give a natural extension of original Mathieu moonshine. Let us consider an infinite product

$$\mathcal{M}_k = \prod (1 - p^m q^n y^r)^{-kc(nm,r)} \tag{5}$$

Here $c(n, r)$ are the expansion coefficients of $Z(m = k+1)$. Umbral condition (4) implies $c(-1) = 2$, $c(0) = 24/k - 4$. Thus we have a Hodge anomaly

$$\prod \left((1 - yq^n)^2 (1 - q^n)^{24/k-4} (1 - y^{-1} q^n)^2 \right)^k$$
$$= (\eta^{24/k} \phi_{-2,1})^k = \eta^{24} {\phi_{-2,1}}^k$$

Thus it seems reasonable to consider a Jacobi form

$$\eta(m) = \eta^{24} {\phi_{-2,1}}^{m-1}$$

By computing the Hecke transformation of $\eta(m)$ we find

$$Z(m) = -\frac{1}{(m-1)}\frac{T_2(\eta(m))}{\eta(m)}$$

for $m = 3, 4, 5$ and an additive correction term of $1/2 \cdot \eta(m = 7)$ for $m = 7$.

Idetification of target manifold is unclear in Umbral moonshine. Revelevant algebra is either $\mathcal{N} = 4$ (hyperKäler) or $\mathcal{N} = 2$ (CY). We have studied the expansion of $Z(m = 3)$ in terms of characters of $\mathcal{N} = 2$ representations.

N=2 moonshine

T.E. and Hikami

$$
\begin{aligned}
Z(m=3) &= \mathrm{massless}(N=2, Q=0) \\
&+ \sum_n F_1(n)\,\mathrm{massive}(N=2, Q=\pm 1) \\
&+ \sum_n F_2(n)\,\mathrm{massive}(N=2, Q=\pm 2)
\end{aligned}
$$

F_1, F_2 are decomposed into sums of representations of group $SL_2(11)$.

Summary

- There is a strong evidence for Mathieu mooonshine phenomenon for K3 surface.

- It is beyond classical geometry and no fundamental explanations so far.
- Individual K3 surfaces (with its holomorphic structure) do not possess symmetry under (subgroups of) M_{24}. Gaberdiel-Hohenegger-Volpato. Rather the symmetry should act on the BPS states or topololgical sector of the theory and this seems a very sublte situation.
- Umbral moonshine gives a natural generalization of Mathieu moonshine although its geometrical significance is somewhat obscure.
- We may use $\mathcal{N}=2$ algebra instead of $\mathcal{N}=4$ and find moonshine phenomenon.