Nonlinear Fluid Dynamics from Gravity

Shiraz Minwalla

Department of Theoretical Physics
Tata Institute of Fundamental Research, Mumbai.

Fluid Gravity Correspondence, Munich, 2009

Introduction

- Consider any two derivative theory of gravity in $d+1$ dimensions, interacting with other fields of spins smaller than two. Assume that this theory admits an AdS_{d+1} solution.
- $SO(d-1,2)$ invariance sets all scalars to constants and other fields vanish to zero on this soln.
- The most general form of the action, expanded about this solution to first order in non metric fields is

$$S = \int \sqrt{g}\,(V_1(\phi_i) + V_2(\phi_i)R)$$

- $V_1(\phi_i)$ may be set to unity by a conformal redefinition of the metric (Einstein Frame). The requirement that AdS is a solution then sets V_2 to zero.

- Consequently Einstein's equations with a negative cosmological constant constitute a consistent truncation of a wide variety of gravitational theories. This implies a decoupled and universal dynamics for the stress tensor for every CFT that has a bulk supergravity dual description
- Moreover, the uncharged black brane solution - the dual to the field theory at finite temperature- lies in this universal sector.
- It follows, for example, that stress tensor correlators are universal in these theories at every temperature. Perhaps even more interestingly, every time dependent Lorentzian solution of Einstein's equations describes a dynamical process in each of these strongly coupled field theories

- It is clear that we would learn an enormous amount about strongly coupled large N field theory dynamics if we could somehow classify all solutions to Einstein's equations with a negative cosmological constant

$$R_{MN} - \frac{R}{2} g_{MN} = \frac{d(d-1)}{2} g_{MN} : \quad : M, N = 1 \ldots d+1$$

- In this talk we report some modest progress towards this goal; we uncover the structure of special classes of gravitational solutions. We will also review similar progress for the study of systems of equations that include selected fields apart from the metric.

Boosted Black Branes

$$R_{MN} - \frac{R}{2} g_{MN} = \frac{d(d-1)}{2} g_{MN} : \quad : M, N = 1 \ldots d+1$$

The black brane at temperature T and velocity u_μ are a d parameter set of exact solutions

$$ds^2 = \frac{dr^2}{r^2 f(r)} + r^2 \mathcal{P}_{\mu\nu} dx^\mu dx^\nu - r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu$$

$$f(r) = 1 - \left(\frac{4\pi T}{d\, r}\right)^d ; \quad \mathcal{P}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

Idea: Try to extend these to a much larger class of approximate solutions using the collective coordinate method, i.e. by permitting slow variations in u^μ and T.

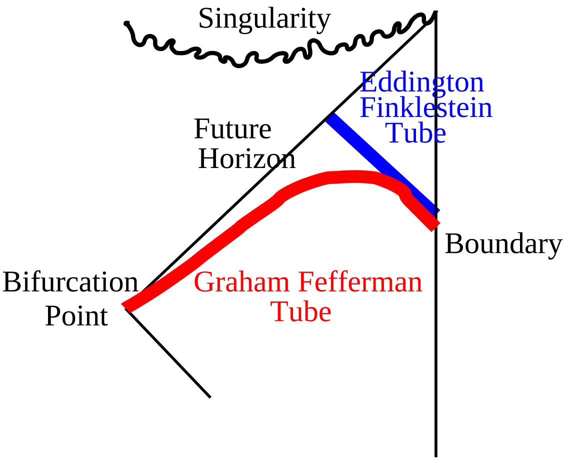
- That is we search for the bulk dual of slowly varying locally thermalized boundary fluid configurations. Suggests bulk solution tubewise approximated by black branes. But along which tubes?
- Naive guess: lines of constant x^μ in Schwarschild (Graham Fefferman) coordinates, i.e. metric approximately

$$ds^2 = \frac{dr^2}{r^2 f(r)} + r^2 \mathcal{P}_{\mu\nu}(x) dx^\mu(x) dx^\nu(x) - r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu$$

$$f(r) = 1 - \left(\frac{4\pi T(x)}{d\ r}\right)^d ; \quad \mathcal{P}_{\mu\nu} = g_{\mu\nu}(x) + u_\mu(x) u_\nu(x)$$

- Does not seem useful. Appears to be a bad starting point for perturbation theory. Also has several interpretative difficulties.

Penrose diagram



- Causality suggests the use of tubes centered around ingoing null geodesics. In particular we try

$$ds^2 = g^{(0)}_{MN} dx^M dx^N = -2u_\mu(x) dx^\mu dr + r^2 \mathcal{P}_{\mu\nu}(x) dx^\mu dx^\nu \\ - r^2 f(r, T(x)) u_\mu(x) u_\nu(x) dx^\mu dx^\nu$$

- Metric generally regular but not solution to Einstein's equations. However solves equations for constant $u^\mu, T, g_{\mu\nu}$. Consequntly appropriate starting point for a perturbative soln of equations in the parameter $\epsilon(x)$.

Perturbation Theory: Redn to ODEs

- That is we set

$$g_{MN} = g^{(0)}_{MN}(\epsilon x) + \epsilon g^{(1)}_{MN}(\epsilon x) + \epsilon^2 g^{(2)}_{MN}(\epsilon x) \ldots$$

  and attempt to solve for $g^{(n)}_{MN}$ order by order in ϵ.
- Perturbation expansion surprisingly simple to implement. Nonlinear partial differential equation $\rightarrow$ 15 ordinary differential equations, in the variable r at each order and each boundary point. Skipping all details we summarize the results.

- This perturbation theory can be implemented at n^{th} order provided the velocities and temperatures obey an integrability condition of the form $\nabla_\mu T^{\mu\nu}_{n-1} = 0$. Here $T^{\mu\nu}_{n-1}$ is a traceless tensor of $n-1$ order in derivatives. The explicit form of the function $T^{\mu\nu}_{n-1}$ is determined by the perturbative analysis at order $n-1$.
- When this condition is fulfilled the solution is unique provided we insist it has a regular future horizon.
- Naive Graham Fefferman counting: $\frac{d(d+1)}{2} - 1$ parameter solution. Rougly parameterized by fluctuation fields $g^n_{\mu\nu}$. However $\frac{d(d-1)}{2} - 1$ of these modes - the tensor sector - fixed by the requirement of regularity.
- Remaining solutions parameterized by d velocities and temperatures. Closed dynamical system named fluid dynamics.

- Schematic form of 2nd order stress tensor:

$$T_{\mu\nu} = aT^d(g_{\mu\nu} + du_\mu u_\nu) + bT^{d-1}\sigma_{\mu\nu} + T^{d-2}\sum_{i=1}^{5} c_i S^i_{\mu\nu}$$

- a is a thermodynamic parameter. b is related to the viscosity: we find $\eta/s = 1/(4\pi)$. c_i coefficients of the five traceless symmetric Weyl covariant two derivative tensors are second order transport coefficients. Values disagree with the predictions of the Israel Stewart formalism.
- Recall that results universal. Should yield correct order of magnitude estimate of transport coefficients in any strongly coupled CFT.

Event Horizons

- Our solutions are singular at $r = 0$. Quite remarkably it is possible to demonstrate that under certain conditions these solutions have event horizons. The event horizon manifold $r = r(x)$, may explicitly be determined order by order in the derivative expansion. This horizon shields the $r = 0$ singularity from the boundary.
- Need some knowledge of the long time behaviour of the solution. Sufficient, though far from necessary, to assume fluid flows that reduce to constant temperature and velocity at late times. Not very strong assumption. Probably true of all finite fluctuations about uniform motion in $d \geq 2$.

Event Horizon in the derivative expansion

- The event horizon of the dual bulk geometry is the unique null manifold that reduces to the event horizon $r = \frac{4\pi T}{d} = \frac{1}{b}$ of the dual uniform black brane at late times.
- It turns out to be simple to construct this event horizon manifold in the derivative expansion: explicitly

$$r_H = \frac{1}{b} + b\left(\lambda_1 \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \lambda_2 \omega_{\alpha\beta}\omega^{\alpha\beta} + \lambda_3 \mathcal{R}\right) + \ldots$$

$$\lambda_1 = \frac{2(d^2+d-4)}{d^2(d-1)(d-2)} - \frac{K_2(1)}{d(d-1)}$$

$$\lambda_2 = -\frac{d+2}{2d(d-2)} \quad \text{and} \quad \lambda_3 = -\frac{1}{d(d-1)(d-2)}$$

- We can put our control over the event horizon to practical use. Recall that a d dimensional event horizon is generated by a $d-1$ dimensional family of null geodesics. Let α^i $i = 1 \ldots (d-1)$ label these geodesics. Let λ be any future directed coordiante along the geodesics.
- The line element on the event horizon takes the form

$$ds^2 = g^{eh}_{ij} d\alpha^i d\alpha^j$$

  Define the area $d-1$ form as $a = \sqrt{g^{eh}} d\alpha^1 \wedge d\alpha^2 \ldots d\alpha^{d-1}$.
- Now $da = \phi d\lambda \wedge d\alpha^1 \ldots d\alpha^{d-1}$ The classic area increase theorem of black hole physics implies the assertion that $\phi \geq 0$.

- Consider the pullback of a to the boundary using the map generated by the radial ingoing null geodesics described above. The boundary hodge dual of pullback of this $d-1$ form is a current whose divergence may be shown to be non negative.
- Consequently fluid dynamics dual to gravity is equipped with a local current whose divergence is always non negative, and which agrees with the thermodynamic entropy current in equilibrium. This 'entropy current' is a local 'Boltzman H' function whose non negative divergence rigorously estabilishes the locally irreversable nature of the dual fluid flows.

Entropy Current at second order

Explicitly this entropy current is given to second order by

$$4\,G_{d+1}\,b^{d-1}\,J^{\mu}_{S} = [\,1 + b^2\,(\,A_1\,\sigma^{\alpha\beta}\,\sigma_{\alpha\beta} + A_2\,\omega^{\alpha\beta}\,\omega_{\alpha\beta} + A_3\,\mathcal{R}\,)\,]\,u^{\mu}$$
$$+\, b^2\,[\,B_1\,\mathcal{D}_{\lambda}\,\sigma^{\mu\lambda} + B_2\,\mathcal{D}_{\lambda}\,\omega^{\mu\lambda}\,]$$

where

$$A_1 = \frac{2}{d^2}(d+2) - \frac{K_1(1)d + K_2(1)}{d}\,, \qquad A_2 = -\frac{1}{2d}, \qquad B_2 = \frac{1}{d-2}$$
$$B_1 = -2A_3 = \frac{2}{d\,(d-2)}$$

Conceptual Issues

- While our solutions all have regular future event horizons, they are in general singular on the past horizon. This is simply a reflection of the fact that fluid dynamics is anti dissipative in the past. Nontrivial solutions generally diverge at early times.
- Consider a fluid in a lab, (possibly violently) stirred out of its equilibrium state at $t = 0$, and then allowed to relax back to equilibrium. This relaxation process is well described by fluid dynamics. The unforced fluid relaxation may formally be continued back to early times. This continuation is, of course, unphysical in the context of the actual experiment.

- The thought experiment of the previous page must clearly have an everywhere nonsingular gravitational dual. Moreover, it must reduce, at late times, to the solutions described earlier in this talk, if our interpretation of those solutions is correct.
- In simple enough situations one may directly check that this is indeed the case. Perturbative description of thermalization at weak amplitude. Sets fluid dynamical initial conditions along null ingiong geodesics. Subsequent motion fluid dynamical.
- This gives an ‘explaination’ for our tubes running along null ingoing geodesics. The explaination is basically causality.

- Several generalizations of the simplest story described above have been studied. Some generalizations deal with augmentations of Einstein gravity to, e.g. the Einstein Maxwell or Einstein dilaton systems.
- Other generalizations deal with nonconformal fluid dynamics - FD in gravitational systems whose vacuum is a geometry other than AdS spaces.
- We take these up in turn.

Non Conformal Hydrodynamics

- The gravitational dual of the hydrodynamical description of the theory on the world volume of Dp branes at finite temperature has also been worked out
- This turns out to be easier than expected for a remarkable reason. It turns out that every Dp backgroud admits a consistent truncation to the metric/dilaton sector. Further this metric dilaton system may formally be regarded as the dimensional reduction of Einstein gravity with a negative cosmological constant in a fictitious higher dimension.
- For the case of the fundamental string and the D4 branes this follows by descent from M theory. Howeve the observation is always true. This allows the immediate determination of the relevant hydrodynamical behaviours using the well understood *AdS* results.

- Consider a bulk Lagrangian describing gravity interacting with other fields that admits AdS space as a solution. We can construct long wavelength fluid like solutions in which these other fields are turned on. Gain: more solutions, wider dynamical behaviour. Price: reduced universality
- Let us first consider the addition of bulk gauge fields to the story. Every additional gauge field is implies a new local conserved boundary charge. It is possible to generalize our analysis to this situation, by perturbing around locally charged black brane backgrounds.

- The end result is a map between the solutions of charged fluid dynamics and those of the long distance Einstein Maxwell system. In $d = 4$ has also been done in the presence of a background magnetic field.
- The procedure yields expressions for the stress tensor and charge currents as a function of local temperatures, velocities and chemical potentials.
- We find a surprise here even at first order in the derivative expansion. In addition to the usual diffusive currents, in $d = 4$ we find a term in the charge current proportional to $\epsilon_{\mu\nu\rho\sigma}\omega^{\nu\rho}u^{\sigma}$. This is important because this term was ignored by Landau and Lifshitz and perhaps all authors subsequently.

- It turns out that this new term (which violates parity but not CP) arises only if the bulk dual has a Chern Simons term in addition to the Maxwell term for gauge fields.
- Landau Lifshitz ignored this term apparently because they thought that it would be impossible to construct a point wise increasing entropy current for fluid equations with such a term.
- However the gravitational construction is automatically equiped with a satisfactory local entropy current, so that cannot be correct. The story here has been clarrfied only very recntly. Anomaly and possible experimental consequences.

Dilaton Forcing

- Let us now study the Einstein-Dilaton rather than Einstein Maxwell system. In this case no additonal conserved charges. However varying boundary dilaton field forces stress tensor and produces Lagrangian vev
- Perturbation theory easily implemented. Solutions are in one to one correspondence with the forced Navier Stokes equations

$$\nabla_\mu T^{\mu\nu} = -\frac{(\pi T)^3}{16\pi G_5}\nabla^\nu \phi (u.\partial)\phi + \ldots$$

Simple Solutions

- A simple class of solutions to these equations are given by the dilaton chosen as a slowly varying function of time. If the fluid is initially at rest, it stays at rest but slowly heats up according to

$$\frac{dT}{dt} = \frac{(\dot{\phi})^2}{12\pi}.$$

The dual bulk solution has a dilaton pulses falling into the black hole, and at leading order is the Vaidya solution. Note that varying - whether increasing or decreasing - the dilaton heats up the gauge theory. Consistent with entropy. Speculations about the continuation to weak coupling.

Universality of second order transport coefficients

- As we have seen above, gravity completely and unambiguously determines the expansion of the universal fluid dynamical stress tensor. So it trivially gives universal results for various ratios of these coefficients.
- However once we move away from the universal sector - e.g. by turning on Maxwell fields and scalars - the results for arbitrary components of the fluid stress tensor are no longer universal. However it turns out that some ratios continue to be universal under arbitrary deformations of this sort. Remenicent of η/s?

Other issues

- Finite N effects
- Turbulence in gravity
- Generalization to confining theories (domain walls)
- Solutions that are everywhere regular
- A path integral to generate long distance correlators?
- More about equilibration