Boost-invariant dynamics – near and far from equilibrium physics and AdS/CFT.

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Workshop on the Fluid-Gravity Correspondence (03/09/2009)

Based on 0805.3774 and 0906.4423 [hep-th]

Idea

- heavy ion collisions @ RHIC - strongly coupled quark-gluon plasma (QGP)
- fully dynamical process - need for a new tool
- idea: exchange

  QCD in favor of $\mathcal{N} = 4$ SYM

  and use the gravity dual
- there are differences
  - SUSY
  - conformal symmetry at the quantum level
  - no confinement...
- ... but not very important at high temperature

Motivation

- RHIC suggests that QGP behaves as an almost perfect fluid
- there has been an enormous progress in understanding

  QGP hydrodynamics with the AdS/CFT

- can the AdS/CFT be used to shed light on

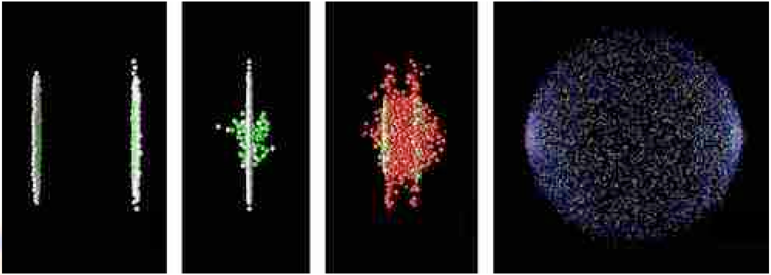
  far from equilibrium part of the QGP dynamics?

- maybe, but only at $\lambda \gg 1$!
- let's focus on

  the boost-invariant flow

  and use the AdS/CFT to grab some non-equilibrium physics.

Boost-invariant dynamics



- one-dimensional expansion along the collision axis x^1
- natural coordinates
  - proper time τ and rapidity y
  - $x^0 = \tau \cosh y$, $x^1 = \tau \sinh y$
- boost invariance (no rapidity dependence)

Gauge-gravity duality is an equivalence between

$\mathcal{N} = 4$ Supersymmetric Yang-Mills in $\mathbb{R}^{1,3}$

- strong coupling
- non-perturbative results
- gauge theory operators

Superstrings in curved $AdS_5 \times S^5$ 10D spacetime

- (super)gravity regime
- classical behavior
- supergravity fields

AdS/CFT dictionary relates
energy-momentum tensor of $\mathcal{N} = 4$ SYM to 5D **AdS metric**

Holographic reconstruction of spacetime

- AdS$_5$ metric in Fefferman-Graham gauge takes the form

$$ds^2 = m_{AB}\, dx^A dx^B = \frac{dz^2 + g_{\mu\nu}\, dx^\mu dx^\nu}{z^2}$$

where $z = 0$ corresponds to the boundary of AdS

- Einstein equations

$$\mathcal{G}_{AB} = \mathcal{R}_{AB} - \frac{1}{2}\mathcal{R}\cdot m_{AB} - 6\, m_{AB} = 0$$

can be solved near boundary given the boundary metric (here assumed to be $\mathbb{R}^{1,3}$) and any traceless and conserved $\langle T_{\mu\nu}\rangle$

$$g_{\mu\nu} = g^{(0)}_{\mu\nu}\left\{= \eta_{\mu\nu}\right\} + z^4 g^{(4)}_{\mu\nu}\left\{= \frac{2\pi^2}{N_c^2}\langle T_{\mu\nu}\rangle\right\} + g^{(6)}_{\mu\nu}\left(\langle T_{\alpha\beta}\rangle\right) z^6 + \ldots$$

- however most $\langle T_{\mu\nu}\rangle$ will lead to singularities in the bulk

Gravity dual to the boost-invariant flow

- the energy-momentum tensor is specified by $\epsilon(\tau)$

$$T^{\mu\nu} = \mathrm{diag}\left\{\epsilon(\tau), -\frac{1}{\tau^2}\epsilon(\tau) - \frac{1}{\tau}\epsilon'(\tau), \epsilon(\tau) + \frac{1}{2}\tau\epsilon'(\tau)_\perp\right\}$$

- this suggests the metric Ansatz for the gravity dual

$$\mathrm{d}s^2 = \frac{-e^{a(\tau,z)}\mathrm{d}\tau^2 + \tau^2 e^{b(\tau,z)}\mathrm{d}y^2 + e^{c(\tau,z)}\mathrm{d}\mathbf{x}_\perp^2 + \mathrm{d}z^2}{z^2}$$

- Einstein equations

$$\mathcal{G}_{AB} = \mathcal{R}_{AB} - \frac{1}{2}\mathcal{R}\cdot m_{AB} - 6\, m_{AB} = 0$$

cannot be solved exactly ($\rightarrow$ numerics)

- however there are two regimes

$$\tau \gg 1 \text{ or } \tau \approx 0$$

where analytic calculations can be done

$\tau \gg 1$ regime – hydrodynamics

Reconstructing the space-time near the boundary – redux

- generic $\epsilon(\tau)$ will **NOT** lead to **smooth geometry**
- this cannot be seen within the Fefferman-Graham expansion
- let's focus then on proper time

$$\tau \gg 1 \text{ or } \tau \approx 0$$

and solve Einstein eqns **exactly in z** but **approximately in τ**

- to begin with let's assume that for $\tau \to \infty$

$$\epsilon(\tau) \sim \frac{1}{\tau^s}$$

and analyze the structure of the $z = 0$ expansion to resum it

- at this level there are no constraints* on s

(* positivity of energy density in any frame forces $0 < s < 4$)

Large times and the scaling variable [hep-th/0512162]

- resummation involves choosing at each order in z

$$a(\tau, z) = -\epsilon(\tau) \cdot z^4 + \left\{ -\frac{\epsilon'(\tau)}{4\tau} - \frac{\epsilon''(\tau)}{12} \right\} \cdot z^6 + \ldots$$

the leading (at $\tau \gg 1$) contribution given the energy density

$$\epsilon(\tau) \sim \frac{1}{\tau^s}$$

- this amounts to introduction of scaling variable $v = z/\tau^{s/4}$
- Einstein equations reduce then to a set of solvable ODEs for

$$a(\tau, z) = a_0\left(z/\tau^{s/4}\right), \ldots \text{for } \tau \to \infty$$

- $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$ evaluated on the scaling solution is singular for $s \neq 4/3$ (that scaling corresponds to perfect fluid hydro)

$\tau \to \infty$ metric and the fluid/gravity correspondence

- $\tau \to \infty$ metric in Fefferman-Graham coordinates looks like

$$ds^2 = \frac{1}{z^2}\left\{-\frac{\left(1-\frac{1}{3}z^4\tau^{-4/3}\right)^2}{1+\frac{1}{3}z^4\tau^{-4/3}}d\tau^2 + \left(1+\frac{1}{3}z^4\tau^{-4/3}\right)\left(\tau^2 dy^2 + dx_\perp^2\right) + dz^2\right\}$$

- it looks like a boosted and dilated black brane

$$ds^2 = \frac{1}{z^2}\left\{-\frac{\left(1-z^4\lambda^4\right)^2}{1+z^4\lambda^4}u_\mu u_\nu dx^\mu dx^\nu + \left(1+z^4\lambda^4\right)\left(\eta_{\mu\nu}+u_\mu u_\nu\right)dx^\mu dx^\nu + dz^2\right\}$$

  with boost and dilation parameters being $u^\mu = 1 \cdot [\partial_\tau]^\mu$ and $\lambda \sim T \sim \tau^{-1/3}$

- at the same time it describes perfect fluid hydrodynamics of boost-invariant plasma $\epsilon(\tau) \sim 1/\tau^{4/3}$

- this is of course the key observation of the fluid/gravity duality

Basics

- long-wavelength effective theory
- vast reduction of # degrees of freedom
  - velocity $\mathrm{u}^{\mu}(x)$ constrained by $\mathrm{u}^{\mu}\,\mathrm{u}_{\mu} = -1$
  - temperature $\mathrm{T}(x)$
- slow changes $\rightarrow$ gradient expansion
- expansion parameter $\frac{1}{\mathrm{L}\cdot\mathrm{T}}$
  (T is temperature, L is characteristic length-scale)

Gradient expansion

- definition of the energy-momentum tensor

$$T^{\mu\nu} = \epsilon\cdot\mathrm{u}^{\mu}\mathrm{u}^{\nu} + \mathrm{p}\cdot\Delta^{\mu\nu} - \eta\cdot\left(\Delta^{\mu\lambda}\nabla_{\lambda}\mathrm{u}^{\nu} + \Delta^{\nu\lambda}\nabla_{\lambda}\mathrm{u}^{\mu} - \frac{2}{3}\Delta^{\mu\nu}\nabla^{\lambda}\mathrm{u}_{\lambda}\right) + \ldots$$

- EOMs $\nabla_{\mu} T^{\mu\nu} = 0$ + equation of state (e.g. $\epsilon = 3p$)

Perfect hydrodynamics

- in conformal boost invariant hydrodynamics

$$\epsilon(\tau) \sim \mathrm{T}(\tau)^4, \quad \mathrm{u}^\mu = 1 \cdot [\partial_\tau]^\mu, \quad \eta_{\mu\nu} = \mathrm{diag}\left\{-1, \tau^2, 1, 1\right\}$$

- perfect hydro ($\nabla_\mu T^{\mu\nu} = 0$ for $T^{\mu\nu} = \epsilon \cdot \mathrm{u}^\mu \mathrm{u}^\nu + \mathrm{p} \cdot \Delta^{\mu\nu}$) gives

$$\partial_\tau \epsilon(\tau) = -\frac{\epsilon(\tau) + p(\tau)}{\tau}$$

- which together with $\epsilon = 3p$ leads to $\epsilon \sim \frac{1}{\tau^{4/3}}$

Gradient expansion

- *remainder:* in hydro the expansion parameter is $\frac{1}{\mathrm{L}\cdot\mathrm{T}}$
- in this setting $\mathrm{T} \sim \tau^{-1/3}$, $\mathrm{L}^{-1} \sim \nabla \mathrm{u} = \tau^{-1}$, so $\frac{1}{\mathrm{L}\cdot\mathrm{T}} \sim \frac{1}{\tau^{2/3}}$
- one should expect the general structure of $\epsilon(\tau)$ of the form

$$\epsilon(\tau) \sim \frac{1}{\tau^{4/3}} \left\{\#_0 + \frac{1}{\tau^{2/3}} \#_1 + \frac{1}{\tau^{4/3}} \#_2 + \ldots\right\}$$

Reminder:

$$ds^2 = \frac{-e^{a(\tau,z)}d\tau^2 + \tau^2 e^{b(\tau,z)}dy^2 + e^{c(\tau,z)}dx_\perp^2 + dz^2}{z^2}$$

Gravitational gradient expansion:

$$a(\tau,z) = a_0\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{2/3}}a_1\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{4/3}}a_2\left(\frac{z}{\tau^{1/3}}\right) + \ldots$$

$$b(\tau,z) = b_0\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{2/3}}b_1\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{4/3}}b_2\left(\frac{z}{\tau^{1/3}}\right) + \ldots$$

$$c(\tau,z) = c_0\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{2/3}}c_1\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{4/3}}c_2\left(\frac{z}{\tau^{1/3}}\right) + \ldots$$

$$\mathcal{R}^2(\tau,z) = \mathcal{R}_0^2\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{2/3}}\mathcal{R}_1^2\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{4/3}}\mathcal{R}_2^2\left(\frac{z}{\tau^{1/3}}\right) + \ldots$$

This is AdS counterpart of hydrodynamics

$$\epsilon(\tau) = \left(\frac{N_c^2}{2\pi^2}\right)\frac{1}{\tau^{4/3}}\left\{1 - 2\eta_0\frac{1}{\tau^{2/3}} + \left[\frac{3}{2}\eta_0^2 - \frac{2}{3}(\eta_0\tau_\Pi^0 - \lambda_1^0)\right]\frac{1}{\tau^{4/3}} + \cdots\right\}$$

Fefferman-Graham vs Eddington-Finkelstein

- in [hep-th/0703243] it was found that

$$\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} = \mathrm{REGULAR} + \frac{1}{\tau^2}\left\{\# \cdot \log\left(3^{1/4} - v\right) + \ldots\right\} + \ldots$$

- this strange logarithmic singularity is not present in Eddington-Finkelstein coordinates

$$\mathrm{d}s^2 = 2\mathrm{d}r\mathrm{d}\tilde{\tau} - r^2\tilde{A}\left(\tilde{\tau}, r\right)\mathrm{d}\tilde{\tau}^2 + (1 + r\tilde{\tau})^2\, e^{\tilde{b}(\tilde{\tau},r)}\mathrm{d}y^2 + r^2 e^{\tilde{b}(\tilde{\tau},r)}\mathrm{d}x_\perp^2$$

- it turns out that there is a singular coordinate transformation

$$\tilde{\tau}\left(\tau, z\right) = \tau\left\{1 + \frac{1}{\tau^{2/3}}\tilde{\tau}_1\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{4/3}}\tilde{\tau}_2\left(\frac{z}{\tau^{1/3}}\right) + \ldots\right\}$$

$$r\left(\tau, z\right) = \frac{1}{z}\left\{1 + \frac{1}{\tau^{2/3}}r_1\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{4/3}}r_2\left(\frac{z}{\tau^{1/3}}\right) + \ldots\right\}$$

given order by order in $\tau^{-2/3}$ (see 0805.3774 [hep-th])

$\tau \approx 0$ regime – dynamics far from equilibrium

Scaling variable doesn't work @ $\tau \approx 0$

- let's start with $\epsilon(\tau) \sim \frac{1}{\tau^s}$ and solve $\mathcal{G}_{AB}$ near the boundary

$$a(\tau, z) = \tilde{a}_0(\tau) z^4 + \tilde{a}_2(\tau) z^6 + \tilde{a}_4(\tau) z^8 + \dots$$

- for $\tau \gg 1$ certain terms dominate at each z^{4+2k} and picking them gives

$$a(\tau, z) = f\left(\frac{z}{\tau^{s/4}}\right)$$

- for $\tau \approx 0$ other terms dominate leading to

$$a(\tau, z) = \frac{z^4}{\tau^s} \cdot \tilde{f}\left(\frac{z}{\tau}\right)$$

- `arXiv:0705.1234` argued that s has to be 0 in this case
- this is wrong, since each term in this scaling expansion is

multiplied by s, thus vanishes identically

Initial conditions and early times expansion of $\epsilon(\tau)$

- warp factors can be solved near the boundary given $\epsilon(\tau)$

$$a(\tau, z) = -\epsilon(\tau) \cdot z^4 + \left\{ -\frac{\epsilon'(\tau)}{4\tau} - \frac{\epsilon''(\tau)}{12} \right\} \cdot z^6 + \ldots$$

- for $\epsilon(\tau) = \epsilon_0 + \epsilon_1 \tau + \epsilon_2 \tau^2 + \epsilon_3 \tau^3 + \epsilon_4 \tau^4 + \epsilon_5 \tau^5 + \ldots$
  all ϵ_{2k+1} must vanish, otherwise $a(0, z) \to \infty$
- setting τ to zero in $a(\tau, z)$ for

$$\epsilon(\tau) = \epsilon_0 + \epsilon_2 \tau^2 + \epsilon_4 \tau^4 + \ldots$$

gives

$$a(0, z) = a_0(z) = \epsilon_0 \cdot z^4 + \frac{2}{3}\epsilon_2 \cdot z^6 + \left(\frac{\epsilon_4}{2} - \frac{\epsilon_0^2}{6} \right) \cdot z^8 + \ldots$$

- it defines map between initial profiles in the bulk and $\epsilon(\tau)$

Geometry @ $\tau \approx 0$

- warp factors a, b and $c\,(\tau, z)$ have $\tau \approx 0$ expansion

$$a\,(\tau, z) = a_0\,(z) + \tau^2 a_2\,(z) + \tau^4 a_4\,(z) \dots$$

- both $\mathcal{G}_{\tau z}$ and $\mathcal{G}_{zz}$ at $\tau = 0$ are constraints equations
- $\mathcal{G}_{\tau z}$ forces $b_0\,(z) = a_0\,(z)$ whereas $\mathcal{G}_{zz}$ takes the form

$$v'\,(z) + w'\,(z) + 2\,z\left\{v\,(z)^2 + w\,(z)^2\right\} = 0$$

where $v\,(z) = \frac{1}{4z} a_0'\,(z)$ and $w\,(z) = \frac{1}{4z} c_0'\,(z)$

- this equation does not have any regular solution

$$\int_0^\infty (v'+w')\,\mathrm{d}z + 2\int_0^\infty (v^2+w^2) z\,\mathrm{d}z = \int_0^\infty (v^2+w^2) z\,\mathrm{d}z = 0$$

what is then the allowed set of initial data?

Allowed initial conditions

- contraint equation

$$v'(z) + w'(z) + 2\,z\left\{v(z)^2 + w(z)^2\right\} = 0$$

can be solved using $v_+ = -w - v$ and $v_- = w - v$

$$v_-\left(u = z^2\right) = \sqrt{2v_+'(u) - v_+(u)^2}$$

- the regularity of $\mathcal{R}_{ABCD}\mathcal{R}^{ABCD}$ @ $\tau = 0$ fixes $v_+(u)$ to be

$$v_+(u) = \frac{2\epsilon_0\,u_0}{3}\cdot\frac{u^3}{u_0 - u}f(u)$$

where $f(0) = 1$, $f(u_0) = \frac{3}{2u_0^4\epsilon_0}$ and otherwise just regular

- the space of allowed initial data is parametrized by

all $v_+(u)$ satisfying above conditions

Resummation of the energy density

- energy density power series @ $\tau = 0$

$$\epsilon(\tau) = \epsilon_0 + \epsilon_2 \tau^2 + \ldots + \epsilon_{2N_{cut}} \tau^{2N_{cut}} + \ldots$$

has a finite radius of convergence and thus

a resummation is needed

- presumably the simplest can be given by Pade approximation

$$\epsilon_{\mathrm{approx}}(\tau)^3 = \frac{\epsilon_U^{(0)} + \epsilon_U^{(2)} \tau^2 + \ldots + \epsilon_U^{(N_{cut}-2)} \tau^{N_{cut}-2}}{\epsilon_D^{(0)} + \epsilon_D^{(2)} \tau^2 + \ldots + \epsilon_D^{(N_{cut}-2)} \tau^{N_{cut}+2}}$$

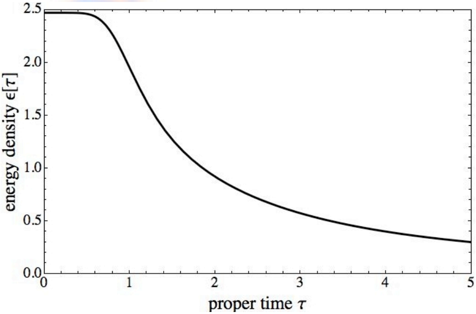
which uses the uniqueness of the asymptotic behavior

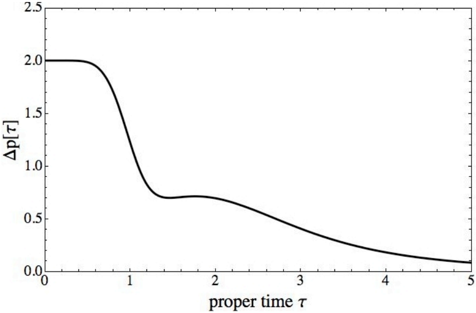
$$\epsilon \sim \frac{1}{\tau^{4/3}}$$

- nice example of allowed initial profile is given by

$$v_+(u) = \frac{\pi}{2}\tan\left(\frac{\pi}{2}u\right) - \frac{\pi}{2}\tanh\left(\frac{\pi}{2}u\right)$$

leading to the following $\epsilon(\tau)$ and $\Delta p(\tau) = 1 - \frac{p_{\|}(\tau)}{p_{\perp}(\tau)}$





Results:

- AdS/CFT is indispensable not only near equilibrium
- early time dynamics is not governed by the scaling limit
- gravity dual at $\tau = 0$ sets $\epsilon(\tau) = \epsilon_0 + \epsilon_2 \cdot \tau^2 + \ldots$
- simple resummation recovers reach dynamics

Open questions:

- numerics starting from some initial data
- towards colliding shock-waves