Non-Abelian bosonization is used when we deal with models having internal symmetry higher than $U(1)$.

In fermion theories with symmetry group G the role of generators of group transformations is played by so-called current operators defined as

$$J^a = \sum_{n=1}^{k} R^+_{n\alpha} \tau^a_{\alpha\beta} R_{n\beta}$$

$$\bar{J}^a = \sum L^+_{n\alpha} \tau^a_{\alpha\beta} L_{n\beta}$$

$$\{\tau^a, \tau^b\} = i f^{abc} \tau^c$$

$$Tr\, \tau^a \tau^b = \frac{1}{2} \delta_{ab}$$

τ^a - generators of the corresponding algebra, for the $SU(2)$ group they are matrices of spin $S = 1/2$.

~~One~~ R and L are massless right and left-moving fermions.

One can show that $[J, \bar{J}] = 0$ and the currents satisfy the so-called Kac-Moody algebra.

Anomalous commutator.

Under the sign of correlation function

$$\langle\langle [A(x), B(y)] \rangle\rangle = \langle\langle A(\tau, x) B(0,y) - A(-\tau, x) B(0,y) \rangle\rangle \quad \tau \to +0$$

$$[J^a(x), J^b(y)] = k \lim_{\tau \to +0} \frac{1}{4\pi^2} \left(\frac{1}{(\tau + i(x-y))^2} - \frac{1}{(-\tau + i(x-y))^2} \right) \frac{\delta_{ab}}{2}$$

$$= \frac{k}{4\pi^2} \partial_x \left[\frac{1}{x-y-i0} - \frac{1}{x-y+i0} \right] \delta_{ab}/2$$

$$= \frac{ik}{4\pi} \delta'(x-y) \delta_{ab}$$

$$[J^a(x), J^b(y)] = i f^{abc} J^c(x) \delta(x-y) + \frac{ik}{4\pi} \delta'(x-y) \delta_{ab}$$

This a generalization of the angular momentum operators.

In a system of finite size L with periodic boundary conditions one can expand currents into Fourier series:

$$J(x) = \frac{1}{L} \sum e^{-\frac{2i\pi n x}{L}} J_n$$

$$[J^a_n, J^b_m] = i f^{abc} J^c_{n+m} + \frac{nk}{2} \delta_{n+m,0} \delta_{ab}$$

Then $[J^a_0, J^b_0] = i f^{abc} J^c_0$ compose a subalgebra of the Kac-Moody algebra which is $= G$.

The method of non-Abelian bosonization is based on conformal embedding which is actually similar to decomposition of 3D motion into radial and angular components:

$$\frac{m\vec{V}^2}{2} = \frac{m\dot{r}^2}{2} + \frac{\vec{L}^2}{2mr^2}$$

In a likewise fashion one decomposes

$$-i R^+_{n\alpha} \partial_x R_{n\alpha} = \pi :j_R^2: + \frac{2\pi}{K+N} :J^a_R J^a_R: + \frac{2\pi}{k+N} :\mathcal{F}^a_R \mathcal{F}^a_R:$$

$$j = \sum_{n,\alpha} R^+_{n\alpha} R_{n\alpha}$$

$$J^a \in SU(N), \quad \mathcal{F}^a \in SU(k).$$

The quadratic current Hamiltonian is called Sugawara one, in fact it is the Hamiltonian of the WZNW model.

$$H_R = \frac{2\pi}{L(N+k)} \left\{ J^a_0 J^a_0 + 2 \sum_{n>0} J^a_{-n} J_n \right\}$$

$$J^a_n |0\rangle = 0. \quad n>0$$

One can forget that currents are made of fermions and ~~continue~~ consider them as operators realizing reps. of the Kac-Moody algebra.

Is ~~mod~~ WZNW model tractable?

Yes, the Sugawara Hamiltonian can be diagonalized

Introduce vacuum states which satisfy:

$$J^a_{+n} |v\rangle = 0, \quad n > 0$$

$$\vec{J}_0^{\,2} |v\rangle = \mathbb{C} |v\rangle$$

quadratic Casimir.

For simplicity, I'll consider $G = SU(2)$

$$|v\rangle = |J,j\rangle$$

$$J_0^3 |J,j\rangle = j |J,j\rangle$$

$$\vec{J}^{\,2} |J,j\rangle = j(j+1) |J,j\rangle$$

$J^{a_1}_{-n_1} J^{a_2}_{-n_2} \dots J^{a_N}_{-n_N} |J,j\rangle$ are eigenstates

$$E_n = \frac{2\pi}{(k+2)L} [j(j+1)] + \text{integer} \times \frac{2\pi}{L}$$

In the thermodynamic limit we have a linear spectrum $\varepsilon \sim |q|$

Specific heat $\quad \dfrac{C_v}{T} = \dfrac{\pi^2}{3} \cdot \dfrac{3k}{k+2}$

Central charge

Central charge of the WZNW

$$C = \frac{k \dim G}{k + C_v}$$

For $SU(N)$, $C_v = N$, $\dim SU(N) = N^2 - 1$.

$$Nk = 1 + \frac{k(N^2-1)}{k+N} + \frac{N(k^2-1)}{N+k}$$

The embedding works.

One simple example how non-Ab. bosonization can be used.

$$\hat{H} = \int dx \left\{ -i R^+_{n\alpha} \partial_x R_{n\alpha} + i L^+_{n\alpha} \partial_x L_{n\alpha} + g R^+_{n\alpha} R_{n\beta} L^+_{m\beta} L_{m\alpha} \right\}$$

$$-\psi^+_{n\alpha} \frac{\partial^2}{\partial x^2} \psi_{n\alpha} - \mu \psi^+ \psi - g \psi^+_{n\alpha} \psi_{m\alpha} \psi^+_{m\beta} \psi_{n\beta}$$

$$\psi = e^{-ik_F x} R + e^{ik_F x} L$$

$$+ g \underbrace{J^a_R J^a_L}_{SU_k(2)} - g \underbrace{\mathcal{F}^a_R \mathcal{F}^a_L}_{SU_2(k)}$$

$$\underbrace{\frac{dg_s}{d\Lambda} = 2g_s^2}_{\text{grows}}, \qquad \underbrace{\frac{dg_f}{d\Lambda} = k g_f^2}_{\text{decreases}}$$

$$H = H_{U(1)} + H^{(0)}_{SU_k(2)} + H^{(0)}_{SU_2(k)} + g J^a_R J^a_L - g \mathcal{F}^a_R \mathcal{F}^a_L$$

$$\underline{H_0[SU_k(2)] + g J^a_R J^a_L}$$ integrable

Scattering theory has no well defined asymptotic states.

Particles remain entangled

The sector with $g>0$ acquires a gap.
Some operators acquire vacuum averages
This enhances response ~~of~~ to certain perturbations.

How to characterize WZNW theory?

1. This theory is critical. (1+1) one.
2. Critical (1+1) - theories ~~have~~ possess conformal symmetry.

This means that on infinite plane their correlation functions ~~are power law~~ decay as power law.

~~on~~

It also means that holomorphic and antiholomorphic sectors ($z=\tau+ix$, $\bar{z}=\tau-ix$) are separated

$$\langle\langle \mathcal{O}_1(1)\ldots\mathcal{O}_N(N)\rangle\rangle = \sum C_{ij}\,\mathcal{F}_i(z_1,\ldots z_N)\,\bar{\mathcal{F}}_j(\bar{z}_1,\ldots \bar{z}_N)$$

$$\mathcal{O}_\Delta(\xi) = \mathcal{O}_\Delta[z(\xi)]\left(\frac{dz}{d\xi}\right)^\Delta$$

There are operator = primary fields transforming under holomorphic transformations as

Eigen States $\leftrightarrow$ operators

What can we do with conformal embedding?

One example.

$U(1) \times SU(2) \times SU(2)$

$$-i R^{+}_{n\alpha} \partial_x R_{n\alpha} + i L^{+}_{n\alpha} \partial_x L_{n\alpha} + V$$

$n = 1, 2 \; ; \quad \alpha = 1, 2$

$$V = g_s \left(R^{+}_{n\alpha} \vec{\sigma}_{\alpha\beta} R_{n\beta} \right) \left(L^{+}_{m\gamma} \vec{\sigma}_{\gamma\delta} L_{m\delta} \right)$$

$$+ g_o \left(R^{+}_{n\alpha} \vec{\tau}_{nm} R_{m\alpha} \right) \left(L^{+}_{\bar{n}\beta} \vec{\tau}_{\bar{n}\bar{m}} L_{\bar{m}\beta} \right)$$

$$+ g_{so} \left(R^{+} \tau^{a} \otimes \sigma^{b} R \right) \left(L^{+} \tau^{a} \otimes \sigma^{b} L \right)$$

$$\downarrow$$

$$g_s \left(\vec{\chi}_R \cdot \vec{\chi}_L \right)^2 + g_o \left(\vec{\xi}_R \cdot \vec{\xi}_L \right)^2$$

$$+ g_{so} \left(\vec{\chi}_R \cdot \vec{\chi}_L \right) \left(\vec{\xi}_R \cdot \vec{\xi}_L \right)$$

$g_s > 0, \quad g_o < 0$ is the most interesting <u>case</u>