Bosonization.

~~Let me remind you what is the standard~~ What is bosonization and what is the logic behind it?

Bosonization is a reformulation of one theory (a fermionic one) in terms of another to make the solution easier.

Such reformulation may not necessarily ~~to~~ immediately lead to a solution, it may refer us to a model which solution can be obtained by other means (Bethe ansatz etc.).

The standard (Abelian) bosonisation establishes a correspondence between ~~a~~ theories of spinless relativistic (1+1)-d fermions and a scalar bosonic field:

Hamiltonian formulation	Lagrangian formulation
Operators	Path integral.
Commutators	Time derivatives

$$H_F = \int dx \left[-i R^\dagger \partial_x R + i L^\dagger \partial_x L \right]$$

$$= \sum_{|k|<\Lambda} k \left(R_k^\dagger R_k - L_k^\dagger L_k \right)$$

$$H_B = \int dx \left[\frac{1}{2} \hat{\Pi}^2 + \frac{1}{2} (\partial_x \phi)^2 \right]$$

$$[\hat{\pi}(x), \hat{\phi}(y)] = -i \delta(x-y)$$

Dual field $\quad \partial_x \hat{\theta} = \hat{\pi}$

$$H_B = \frac{1}{2} \int dx \left[(\partial_x \hat{\theta})^2 + (\partial_x \hat{\phi})^2 \right]$$

Lagrangian formulation (Gucliden)

$$L_F = \int dx \left[R^\dagger (\partial_\tau - i\partial_x) R + L^\dagger (\partial_\tau + i \partial_x) L \right]$$

$$L_B = \frac{1}{2} \int \left[(\partial_\tau \phi)^2 + (\partial_x \phi)^2 \right] dx \quad \backsim$$

$$\frac{1}{2} \int \left[(\partial_x \theta)^2 + (\partial_x \phi)^2 + 2i \partial_\tau \theta \, \partial_x \phi \right] dx$$

This form is convenient when one has to use ϕ and θ simultaneously

Theorem.

~~Every~~ 1. The Hilbert spaces of bosonic and fermionic theories are isomorphic.

2. Every local fermionic operator can be constructed bosonic ones

Operator dictionary

$$\frac{1}{2\pi a} e^{i\sqrt{4\pi}\,\phi} = i R^{+} L$$

$$\frac{1}{\sqrt{\pi}} \partial_x \phi = \,:R^{+}R + L^{+}L:$$

$$\frac{1}{\sqrt{\pi}} \hat{\Pi} = R^{+}R - L^{+}L$$

$$\frac{1}{\sqrt{2\pi a}} \exp\left\{ i\sqrt{\pi}\left[\int_{-\infty}^{x} \hat{\Pi}(y)\,dy + \phi(x) \right]\right\} = R$$

$$\frac{1}{\sqrt{2\pi a}} \exp\left\{ +i\sqrt{\pi}\left[\int^{x} \hat{\Pi}(y)\,dy - \phi(x) \right]\right\} = L$$

$$\frac{1}{2\pi a} e^{i\sqrt{4\pi}\,\theta} = i\,\cancel{L}R$$

What can we do with it?

Examples:

$$-iR^+\partial_x R + iL^+\partial_x L + g\,R^+R\,L^+L$$

$$\downarrow$$

$$\frac{1}{2}\left(1+ g/\pi\right)(\partial_\mu \phi)^2$$

$$\phi \to \beta\phi\,, \qquad \beta^2 = \frac{1}{1+g/\pi}$$

$$O_{CDW} = iR^+L \sim e^{i\sqrt{4\pi}\,\phi} \to e^{i\sqrt{4\pi}\,\beta\phi}$$

$$\langle e^{i\gamma\phi(\tau,x)}\, e^{-i\gamma\phi(0,0)} \rangle = \left(\frac{a_0^2}{\tau^2+x^2}\right)^d$$

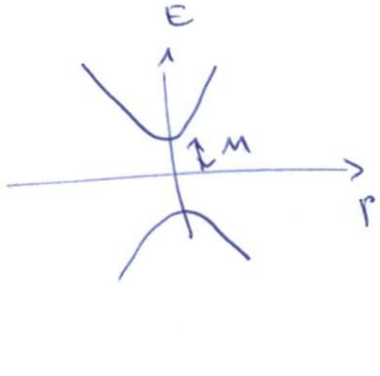
$$d = \gamma^2/4\pi = \beta^2 = \frac{1}{1+g/\pi}$$

The reverse process: fermionization

$$\frac{1}{2}(\partial_\mu\phi)^2 - \frac{M}{\pi a}\cos\sqrt{4\pi}\,\phi$$

$$\downarrow$$

$$iM\left(R^+L - L^+R\right)$$



Massive Dirac fermions

Two coupled SC wires

$$\frac{1}{2}(\partial_\mu \phi)^2 - \frac{M_1}{\pi a}\cos\sqrt{4\pi}\,\phi - \frac{M_2}{\pi a}\cos\sqrt{4\pi}\,\theta$$

$$- i M_1 (R^+ L - L^+ R) - i M_2 (LR - R^+ L^+)$$

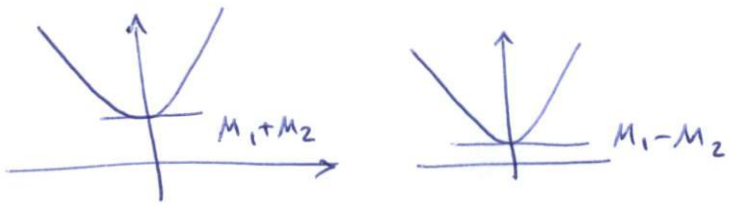
$$R = (\chi_R + i\eta_R)/\sqrt{2}, \quad R^+ = (\chi_R - i\eta_R)/\sqrt{2}$$

$$L \; -$$

$$\frac{1}{2}\chi_R(\partial_\tau + i\partial_x)\chi_R + \frac{1}{2}\chi_L(\partial_\tau - i\partial_x)\chi_L$$

$$\frac{1}{2}\eta_R(\partial_\tau + i\partial_x)\eta_R + \frac{1}{2}\eta_L(\partial_\tau - i\partial_x)\eta_L$$

$$+ i(M_1 + M_2)\chi_R\chi_L + i(M_1 - M_2)\eta_R\eta_L$$



~~What~~

Transverse field Ising model.

$M_1 > M_2$ $\quad \langle e^{\pm i\sqrt{\pi}\phi} \rangle \neq 0$

$M_1 < M_2$ $\quad \langle e^{\pm i\sqrt{\pi}\theta} \rangle \neq 0$

Ising model.

$$:\cos\sqrt{\pi}\,\phi: \;=\; \frac{\sqrt{\pi}}{(ma_0)^{1/4}}\,\mu_1\mu_2$$

$$:\sin\sqrt{\pi}\,\phi: \qquad \sigma_1\sigma_2$$

Itsykson & Zuber (1977)

$$:\cos\sqrt{\pi}\,\theta: \qquad \sigma_1\mu_2$$

$$:\sin\sqrt{\pi}\,\theta: \qquad \mu_1\sigma_2 .$$

$$m = J - 2h$$

$$-\sum\left(J\,\sigma^x_n\sigma^x_{n+1} + h\,\sigma^z_n\right)$$

$$\mu^x_{n+1/2} = \prod^{n}\sigma^z_j$$

$$\mu^z_{n+1/2} = \sigma^x_n\sigma^x_{n+1}$$

$$-\sum\left(h\,\mu^x_{n-1/2}\mu^x_{n+1/2} + J\,\mu^z_{n+1/2}\right)$$

The Gaussian model belongs to the class of ~~ex.~~ $(1+1)$-d critical theories.
They all have common features related to conformal invariance.

Linear excitation spectrum

Right and left-moving degrees of freedom are separated:

Correlation functions are sums of products of analytic functions of

$z = \tau + ix$, $\bar{z} = \tau - ix$.

$$\langle\langle O_1(1) \dots O_N(N) \rangle\rangle = \sum C_{ij} \mathcal{F}^{(i)}(z_1, \dots z_N) \mathcal{F}^{(j)}(\bar{z}_1, \dots \bar{z}_N)$$

Two-point functions on ∞ plane are power law

$$\langle O_\Delta O_\Delta^+ \rangle = \frac{1}{z_{12}^{2\Delta} \bar{z}_{12}^{2\bar{\Delta}}}$$

Operators are characterized by conformal dimensions $(\Delta, \bar{\Delta})$

Only o/p with the same $(\Delta, \bar{\Delta})$ have $\neq 0$ 2-point funct.

Locality: $\Delta - \bar{\Delta} =$ integer bosons
½ integer (fermions)

$$A_\Delta(\xi) = A_\Delta[z(\xi)] \left(\frac{dz}{d\xi}\right)^\Delta$$

There are fields, called primary one, which are transformed as

In the Gaussian theory these are $e^{i\beta\phi}$
$\Delta = \beta^2/8\pi$

$$\langle\langle A_\Delta(z,\bar{z}) A^+ \rangle\rangle = \left[\frac{\pi/L}{\sinh\frac{\pi}{L}(\tau+ix)}\right]^{2\Delta} \times \left[\frac{\pi/L}{\sinh\frac{\pi}{L}(\tau-ix)}\right]^{2\bar{\Delta}}$$