Quantum Theory – From its Beginnings to its *Modern Foundations*

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Goal of Lectures:

Present a *new perspective* on Quantum Mechanics, solving some of the main conundrums that have pervaded this theory.

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"Eine neue wissenschaftliche Wahrheit pflegt sich nicht in der Weise durchzusetzen, dass ihre Gegner überzeugt werden und sich als belehrt erklären, sondern vielmehr dadurch, dass die Gegner allmählich aussterben und dass die heranwachsende Generation von vornherein mit der Wahrheit vetraut ist."
(Max Planck)

—

"We have to ask what it means." (Ken Wilson)

Acknowledgements

I warmly thank my last PhD student *Baptiste Schubnel* for joy- and fruitful collaboration over a fairly extended period, to *Philippe Blanchard* for his unwavering interest in our efforts and countless animated discussions, and to Jérémy Faupin, as well as a few other younger friends for some useful collaborations. I have profited from many lively discussions with numerous colleagues including *Detlef Dürr* and *Shelly Goldstein*, Gian Michele Graf, Klaus Hepp, Frank Laloë, Sandu Popescu, Daniel Wyler, and others. –
I am very grateful to *Heinz Siedentop* for inviting me to present these lectures within the TMP framework organized by LMU and TU Munich. Thanks to Heinz, I had to sit down and meditate, ever since I received this invitation, and think and write lecture notes, during almost two months,[1] in order to prepare these lectures. This activity helped me a lot to better understand various apects of Quantum Mechanics, *including my own ideas.*

[1] missing his workshop in Marseille

Tentative Table of Contents of Lectures

"He who is deficient in the art of selection may, by showing nothing but the truth, produce all the effects of the grossest falsehoods. It perpetually happens that one writer tells less truth than another, merely because he tells more 'truth'."
(T. Macauley, 'History', in 'Essays', Vol. 1, Sheldon, NY 1860)

Contents of Lectures – ctd.

Contents of Lectures – ctd.

Contents of Lectures – ctd.

Contents of Lectures – ctd.

Etc.

Specific topics to be addressed; basic convictions

- Why are physical theories never fully predictive?
  Why is quantum theory *intrinsically probabilistic*?
  What are *"events"* in quantum theory & how do we record them?
  How do *states* of physical systems evolve according to *QM*?
- What is the role of *(Space-)Time* in quantum theory, and why and how does quantum theory distinguish between *past* and *future*?
- What is the fundamental significance of *"locality"* and *Einstein causality* in quantum theory?
  Could it be that a consistent *"Quantum Theory of Events"* must necessarily be *relativistic*; could it be that it explains why space-time is *even-dimensional* and why it is *curved*?

Etc. ...

1. Against "interpretations" of physical theories – in favor of solid "foundations"; (the example of electro-magnetism before STR).
2. ... – in favor of *clear concepts and fundamental principles!*
3. Against denigration of precise mathematical tools.

Dedicated to the memory of Res Jost, Edward Nelson and Ernst Specker

1918-1990

1932-2014

1920-2011

1. Historical Introduction

"Wer sich mit einer Wissenschaft bekannt machen will, darf nicht nur nach den reifen Früchten greifen – er muss sich darum kümmern, wie und wo sie gewachsen sind." (J.C. Poggendorff)

"Anyone who is not shocked by quantum theory has not understood it." (N. Bohr)

"If someone tells you they understand quantum mechanics then all you've learned is that you're not a liar." (R.P. Feynman)

"Truth is ever to be found in the simplicity, and not in the multiplicity and confusion of things." (I. Newton)

"I cannot define the real problem; therefore I suspect there's no real problem; but I'm not sure there's no real problem." (R.P. Feynman)

"He who is deficient in the art of selection may, by showing nothing but the truth, produce all the effect of the grossest falsehood." (Henry Neele, 1828)

"C'est cette contradiction qui apparaît dans l'idée de quanta ou d'atomes d'energies ... qui a ôté à la science, à partir de 1900, la signification qu'elle avait eue au cours de quatre siècles sans qu'on ait pu lui en donner aucune autre." (Simone Weil)

"In our description of Nature the purpose is not to disclose the 'real essence' of the phenomena but only to track down, so far as it is possible, relations between the manifold aspects of our experience." (Niels Bohr)

"Philosophy is written in that great book which ever lies before our eyes – I mean the Universe – but we cannot understand it if we do not first learn the language and grasp the symbols, in which it is written. This book is written in the mathematical language." (Galileo Galilei)

Later the quotes about "atomism" (e.g., by Sommerfeld); see "The Quest for Laws and Structure".

QM is soon 100 years old. But there still is enormous confusion about its meaning. My goal is to eliminate some of this confusion!

Description of present situation: Sean Carroll.

“I think I can safely say that nobody really understands quantum mechanics,” observed the physicist and Nobel laureate Richard Feynman. That’s not surprising, as far as it goes. Science makes progress by confronting our lack of understanding, and quantum mechanics has a reputation for being especially mysterious.

What’s surprising is that physicists seem to be O.K. with not understanding the most important theory they have.

Quantum mechanics, assembled gradually by a group of brilliant minds over the first decades of the 20th century, is an incredibly successful theory. We need it to account for how atoms decay, why stars shine, how transistors and lasers work and, for that matter, why tables and chairs are solid rather than immediately collapsing onto the floor.

Scientists can *use* quantum mechanics with perfect confidence. But it’s a black box. We can set up a physical situation, and make predictions about what will happen next that are verified to spectacular accuracy. What we don’t do is claim to *understand* quantum mechanics. Physicists don’t understand their own theory any better than a typical smartphone user understands what’s going on inside the device.

There are two problems. One is that quantum mechanics, as it is enshrined in textbooks, seems to require separate rules for how quantum objects behave when we're not looking at them, and how they behave when they are being observed. When we're not looking, they exist in "superpositions" of different possibilities, such as being at any one of various locations in space. But when we look, they suddenly snap into just a single location, and that's where we see them. We can't predict exactly what that location will be; the best we can do is calculate the probability of different outcomes.

The whole thing is preposterous. Why are observations special? What counts as an "observation," anyway? When exactly does it happen? Does it need to be performed by a person? Is *consciousness* somehow involved in the basic rules of reality? Together these questions are known as the "measurement problem" of quantum theory.

The other problem is that we don't agree on what it is that quantum theory actually describes, even when we're not performing measurements. We describe a quantum object such as an electron in terms of a "wave function," which collects the superposition of all the possible measurement outcomes into a single mathematical object. When they're not being observed, wave functions evolve according to a famous equation written down by Erwin Schrödinger.

But what is the wave function? Is it a complete and comprehensive representation of the world? Or do we need additional physical quantities to fully capture reality, as Albert Einstein and others suspected? Or does the wave function have no direct connection with reality at all, merely characterizing our personal ignorance about what we will eventually measure in our experiments?

Until physicists definitively answer these questions, they can't really be said to understand quantum mechanics — thus Feynman's lament. Which is bad, because quantum mechanics is the most fundamental theory we have, sitting squarely at the center of every serious attempt to formulate deep laws of nature. If nobody understands quantum mechanics, nobody understands the universe.

You would naturally think, then, that understanding quantum mechanics would be the absolute highest priority among physicists worldwide. Investigating the foundations of quantum theory should be a glamour specialty within the field, attracting the brightest minds, highest salaries and most prestigious prizes. Physicists, you might imagine, would stop at nothing until they truly understood quantum mechanics.

The reality is exactly backward. Few modern physics departments have researchers working to understand the foundations of quantum theory. On the contrary, students who demonstrate an interest in the topic are gently but firmly — maybe not so gently — steered away, sometimes with an admonishment to "Shut up and calculate!" Professors who become interested might see their grant money drying up, as their colleagues bemoan that they have lost interest in serious work.

This has been the case since the 1930s, when physicists collectively decided that what mattered was not understanding quantum mechanics itself; what mattered was using a set of *ad hoc* quantum rules to construct models of particles and materials. The former enterprise came to be thought of as vaguely philosophical and disreputable. One is reminded of Aesop's fox, who decided that the grapes he couldn't reach were probably sour, and he didn't want them anyway. Physicists brought up in the modern system will look into your eyes and explain with all sincerity that they're not really interested in understanding how nature really works; they just want to successfully predict the outcomes of experiments.

This attitude can be traced to the dawn of modern quantum theory. In the 1920s there was a series of famous debates between Einstein and Niels Bohr, one of the founders of quantum theory. Einstein argued that contemporary versions of quantum theory didn't rise to the level of a complete physical theory, and that we should try to dig more deeply. But Bohr felt otherwise, insisting that everything was in fine shape. Much more academically collaborative and rhetorically persuasive than Einstein, Bohr scored a decisive victory, at least in the public-relations battle.

Not everyone was happy that Bohr's view prevailed, but these people typically found themselves shunned by or estranged from the field. In the 1950s the physicist David Bohm, egged on by Einstein, proposed an ingenious way of augmenting traditional quantum theory in order to solve the measurement problem. Werner Heisenberg, one of the pioneers of quantum mechanics, responded by labeling the theory "a superfluous ideological superstructure," and Bohm's former mentor Robert Oppenheimer huffed, "If we cannot disprove Bohm, then we must agree to ignore him."

Around the same time, a graduate student named Hugh Everett invented the “many-worlds” theory, another attempt to solve the measurement problem, only to be ridiculed by Bohr’s defenders. Everett didn’t even try to stay in academia, turning to defense analysis after he graduated.

A more recent solution to the measurement problem, proposed by the physicists Giancarlo Ghirardi, Alberto Rimini and Tulio Weber, is unknown to most physicists.

These ideas are not simply woolly-headed “interpretations” of quantum mechanics. They are legitimately distinct physical theories, with potentially new experimental consequences. But they have been neglected by most scientists. For years, the leading journal in physics had an explicit policy that papers on the foundations of quantum mechanics were to be rejected out of hand.

Of course there are an infinite number of questions that scientists could choose to worry about, and one must prioritize somehow. Over the course of the 20th century, physicists decided that it was more important to put quantum mechanics to work than to understand how it works. And to be fair, part of their rationale was that it was hard to actually see a way forward. What were the experiments one could do that might illuminate the measurement problem?

The situation might be changing, albeit gradually. The current generation of philosophers of physics takes quantum mechanics very seriously, and they have done crucially important work in bringing conceptual clarity to the field. Empirically minded physicists have realized that the phenomenon of measurement can be directly probed by sufficiently subtle experiments. And the advance of technology has brought questions about quantum computers and quantum information to the forefront of the field. Together, these trends might make it once again respectable to think about the foundations of quantum theory, as it briefly was in Einstein and Bohr’s day.

Meanwhile, it turns out that how reality works might actually matter. Our best attempts to understand fundamental physics have reached something of an impasse, stymied by a paucity of surprising new experimental results. Scientists discovered the Higgs boson in 2012, but that had been predicted in 1964. Gravitational waves were triumphantly observed in 2015, but they had been predicted a hundred years before. It's hard to make progress when the data just keep confirming the theories we have, rather than pointing toward new ones.

The problem is that, despite the success of our current theories at fitting the data, they can't be the final answer, because they are internally inconsistent. Gravity, in particular, doesn't fit into the framework of quantum mechanics like our other theories do. It's possible — maybe even perfectly reasonable — to imagine that our inability to understand quantum mechanics itself is standing in the way.

After almost a century of pretending that understanding quantum mechanics isn't a crucial task for physicists, we need to take this challenge seriously.

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Technology and the author of the forthcoming book "Something Deeply Hidden: Quantum Worlds and the Emergence of Spacetime," from which this essay is adapted.

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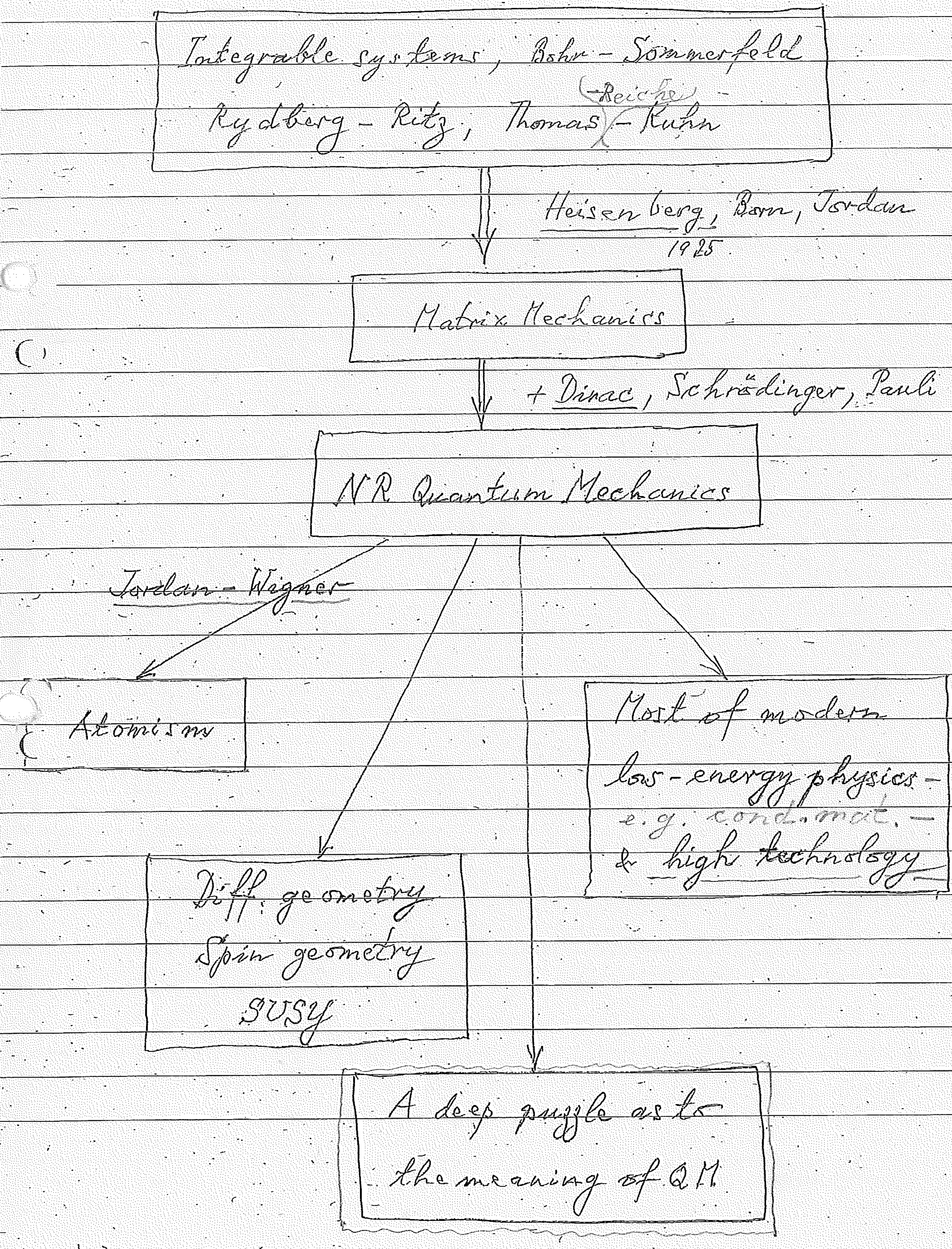
1. Introduction — general remarks



(1) Introduction: Notions & Concepts

Purpose of theor. physics:
Should enable "observers" to interpret what they learn about their "past" and use their knowledge of the "past" to make plausible predictions about their "future" – within a precise math. framework.

Interpretation & prediction of "histories" of events"

To fulfill this purpose, we must find out what "<u>events</u>" are & how to characterize them in terms of "invariant props.";

"<u>relations</u>" betw. "events" → "<u>histories</u>".

<u>Some basic notions</u>:

1) "<u>Observer</u>" who does exps., wants to interpret them & wants to make predictions about future.

<u>In theory</u>:

- Countably many observers with <u>relations</u>, e.g.,

$1 < 2$ (1 in the "<u>past</u>" of 2)

1×2; etc. (1 & 2 <u>space-like separated</u>)

Observers tend to <u>share</u> "events" in their past about which they can communicate. <u>Ambiguities</u> about <u>past events</u> should <u>not</u> arise; and: <u>One</u> theory should descr./predict what they have seen in their common past / are likely to see in the future.

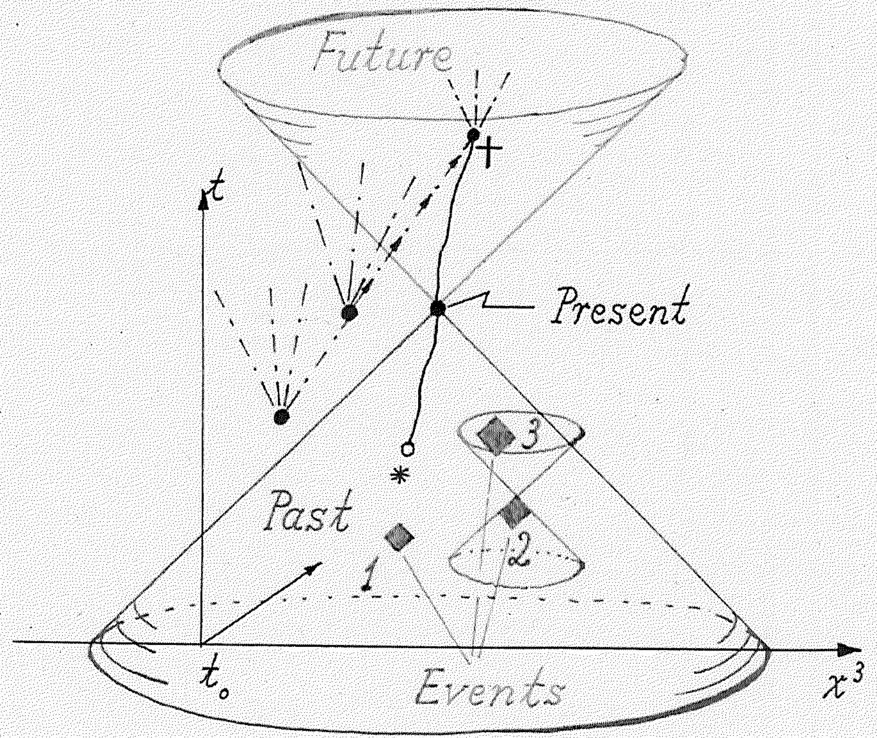
2) <u>"Events"</u>. Nature manifests itself to "obs." in "(causal) <u>time-ordered</u> <u>sequences</u> of <u>events</u>", or

"histories". "Events" have an obj. meaning indep. of obs.; their ordering may depend on obs., though.
Theory must talk about "possible events"; exp. about "observed events".

3) "Frequencies" (as opposed to "prob."; ↗ J. Bernoulli, M. Fierz, Bayes.)
Theory must enable us to predict frequencies of a certain class of histories (e.g. "consistent histories")..., as

2. What prevents theories from being (fully) predictive?

Space-time with an *event horizon*. (Observer sits at "Present"; is unaware of dangers lurking from outside his past light-cone; he might get killed at †. Events 1 & 2 are space-like separated; event 3 is in the future of 2)



t_0: time right after inflation $\rightarrow$ event horizon $\Rightarrow$ initial conditions not fully accessible!

Past = History of Events (Facts) / Future = Ensemble of Potentialities

This fundamental dichotomy should be retained in Quantum Mechanics!

dictable by any loc. obs. of finite life time, because such an obs. never knows enough about the past, (about "initial conditions")

Class. relat. physics <u>deterministic</u>, but not fully <u>predictive</u>.

<u>Order</u> of events 1 & 2 <u>dep.</u> on observer; order of 2 & 3 does <u>not</u>; (<u>is universal</u> ← <u>time-like separation of 2 & 3</u>).

Spacelike sep. events (1 & 2) can be <u>correlated</u> when <u>causally conn. to common precursor</u>!

After these general remarks, we comment on the revolutions in theor. physics in the 20th Century.

(3) Planck's Law for black-body radiation:

Spectral energy density, (Planck, Dec. 1900)

$$\rho(\nu, T) = \frac{8\pi}{c^3}\nu^2 \cdot \frac{h\nu}{e^{h\nu/kT} - 1} \qquad (*)$$

Limiting laws

(i) $h\nu \ll kT$ (exp. difficult!)

$$\rho(\nu, T) \approx \frac{8\pi\nu^2}{c^3} kT$$

(Rayleigh - Jeans, 1900)

(ii) $h\nu \gg kT$ (exp. easier!)

$$\rho(\nu, T) \approx \frac{8\pi\nu^2}{c^3} \cdot h\nu\, e^{-\frac{h\nu}{kT}}$$

(Wien 1896)

(New) constants of Nature appearing in (*):

c: speed of light

k: Boltzmann's constant

h: Planck's constant

Values:

$k = 1.381 \times 10^{-23}\ J/K$ (J: Joule, K: Kelvin)

$h = 6.625 \times 10^{-34}\ J\sec$

$$\begin{cases} N = \dfrac{R}{k} = 6.022 \times 10^{23}\ /\mathrm{Mol} \\ \\ e = \dfrac{F}{N} = 1.602 \times 10^{-19}\ \text{Coulombs} \\ \quad = 4.803 \times 10^{-10}\ \text{statcoulombs (cgs)} \end{cases}$$

Finally,

$l_P^2 := \dfrac{G_N \hbar}{c^3}$, G_N: Newton's gravit. const.

$l_P \simeq 1.6 \times 10^{-33}$ cm ; Planck length.

Revolutions in 20th Century Physics

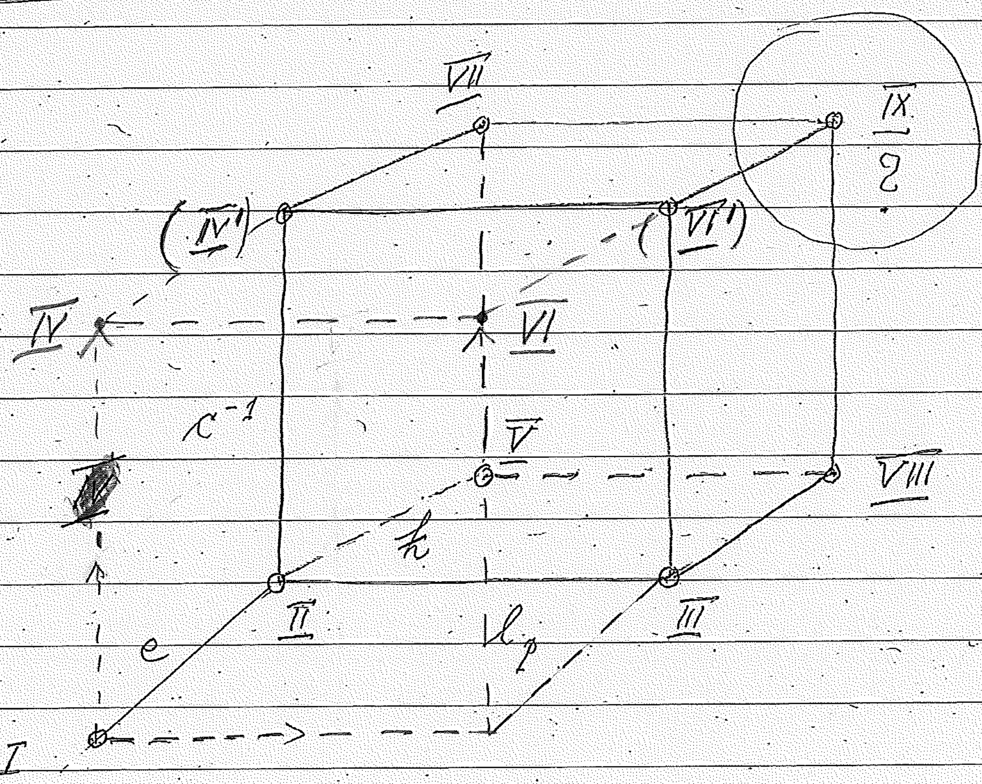
$c \longleftrightarrow$ special relativity

$e \longleftrightarrow$ atomism

$h \longleftrightarrow$ quantum theory

$l_p \longleftrightarrow$ Space, time and gravitation

Bronstein – Planck (Hyper) Cube



I: Mechanics of Continuous Media, Thermodynamics

II: Hamiltonian mech. and statistical mech. of point particles

III: Celestial mechanics; Newtonian space, absolute time, and gravitation

IV: Classical (special) relativistic field theory, electrodynamics

V: NR quantum mechanics of point particles

VI: General Relativity

VII: Relativistic QFT

VIII: NR Quantum Gravity

IX: "String - and M- theory"

The constants of Nature

$$c^{-1},\ e,\ h,\ l_P$$

can be interpreted as "<u>deformation parameters</u>";

(see Chapter 2).

<u>Examples.</u>

Galilei symmetry $\xrightarrow{c^{-1}}$ Poincaré symmetry

(Deformation of Lie algebras and -groups)

Hamiltonian mechanics $\xrightarrow{h}$ matrix mechanics

(Deformation of associative algebras)

Continuum mechanics $\xrightarrow{e}$ atomism

Representation categories of compact topol. groups

$\xrightarrow{\vartheta}$ braided C^* tensor categories

More details in Chapters 1 and 2.

We must require that the "new theories" (such as quantum mechanics) reproduce the "ancestor theories" (such as Hamiltonian mechanics) in appropriate <u>limiting regimes</u>.

<u>Remarks</u>

(a) The vision of physics sketched above remains to be completed! More understanding is desirable, e.g., concerning the <u>mathematics</u> of <u>limiting regimes</u>:

$$\left.\begin{array}{l} "\hbar \searrow 0": \text{ Classical limit} \\ \left.\begin{array}{l} "k \searrow 0" \\ \simeq "e \searrow 0" \end{array}\right\} \text{Mechanics of continuous media} \end{array}\right\} \simeq \underline{\text{Mean-Field Limits}}$$

$"c^{-1} \searrow 0"$: Non-relativistic (Newtonian) limit

(b) <u>Goal of lectures</u>:

Clarify to what extent we have understood the "new theory" corresponding to <u>$\hbar > 0$</u>, <u>$k > 0$</u>, i.e., <u>atomistic quantum theory</u>, and its relation to ancestor theories (Hamiltonian mechanics of point particles, Vlasov mechanics).

Examples of general consequences:

$c^{-1} > 0$, $\ell_P > 0$ $\longrightarrow$ Loss of predictability of future, dependence of observational data on state of the observer.

$k > 0$, $\hbar > 0$ $\longrightarrow$ Loss of determinism and of "realism" ($\nearrow$ S. Weil): physical quantities char. of a phys. system do, a priori, not have fixed values; probabilistic natural laws.

$\hbar > 0$, $c^{-1} > 0$, $\ell_P > 0$: Loss of the notion of space-time as a classical (smooth) Lorentzian manifold;

space-time is an "emergent structure" (as envisioned by Leibniz, Grothendieck and others - in contrast to Copernicus, Kepler, Newton, ...);

"relational geometry"

(4) What is the fundamental problem in (with) QM?

Conventional formulation of QM

In text books (and, originally, by Schrödinger), QM is formulated in terms of the following mathematical structure, $\mathcal{S}$:

$\mathcal{S}$

- a Hilbert space, $\mathcal{H}$, of pure state vectors;
- unitary propagator, U, describing time evolution of states;
- "Pictures" ("measurement bases"):

Configuration - space rep. (position measurements - Schrödinger)

Momentum - space rep. (momentum measurements - Pauli)

Energy rep. (energy (-difference) measurements - Heisenberg)

+ Transformations between different pictures (Dirac)

<u>Missing:</u> Dictionary between structure $\mathcal{S}$ and processes in Nature!

- In particular, we don't know what an "event" is and how it is described in quantum mechanics.
- Where does fundamentally probabilistic nature of QM come from? No mention of probabilities in $\mathcal{S}$; so, how do they enter the formalism?
- What is the meaning of "probabilities" in a theory of an evolving world where (exactly) situations do not repeat?

↓

$\boxed{\text{Copenhagen mumbo jumbo}}$

Next, we trace the history of the discovery of QM in more detail.

1.1. Planck's formula and Einstein's discovery of the photon.

We begin these lectures with a short recapitulation of the theory of black-body radiation culminating in Planck's formula for the spectral energy density of black-body radiation. I do not go into details about the experimental situation. The crucial experiments were carried out in Berlin by

Otto Lummer & Ernst Pringsheim

and by

Heinrich Rubens & Ferdinand Kurlbaum

in the year 1900.

At the end of 1900,* Max Planck extracted from the experimental data the formula

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* after a dinner with the Rubens' on which occasion Rubens showed Planck his data.

$$u(\omega, T) = \frac{\omega^2}{\pi^2 c^3} \cdot \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}, \qquad (1.1)$$

where ω is the circular frequency of radiation;

[or, in terms of the Hertzian frequency:

$$\rho(\nu, T) = 2\pi\, u(2\pi\nu, T) = \frac{8\pi\nu^2}{c^3} \cdot \frac{h\nu}{e^{h\nu/kT} - 1}$$

density of states

u : spectral energy density

T : absolute temperature

c : speed of light

h : Planck's constant

k : Boltzmann constant

Limiting laws:

(i) $\hbar\omega \ll kT$ (infrared regime)

$$u(\omega, T) \simeq \frac{\omega^2}{\pi^2 c^3} kT \qquad (1.2)$$

Rayleigh–Jeans, 1900

(ii) $\hbar\omega \gg kT$ (ultraviolet regime)

$$u(\omega,T) \simeq \frac{\hbar\omega^3}{\pi^2 c^3}\, e^{-\frac{\hbar\omega}{kT}} \qquad (1.3)$$

Wien, 1896

Wien's law reproduced the measurements of Paschen, Rubens and Wien in the ultraviolet regime. Measurements in the infrared regime were more challenging.]

The thermodynamics of black-body radiation.

From the second law of thermodynamics one concludes that

(i) Black-body radiation is homogeneous

(ii) —— " —— is isotropic

($\underbrace{\text{Black-body radiation}}_{\text{BBR}}$ = electromagnetic radiation in thermal equilibrium)

We recall the expression for the energy-momentum tensor of electromagnetic radiation:

$$T = (T_{\mu\nu}) = \left(\begin{array}{c|c} u & \vec{S} \\ \hline \vec{S}^T & P_{ij} \end{array} \right) \qquad (1.4)$$

$$u = \frac{1}{8\pi} \left(\vec{E}^2 + \vec{B}^2 \right) \quad \text{; energy density} \qquad (1.5)$$

$$\vec{S} = \frac{1}{4\pi} \vec{E} \wedge \vec{B} \quad \text{; Poynting vector}$$

$$P_{ij} = \frac{1}{4\pi} \left\{ \frac{1}{2} \left(\vec{E}^2 + \vec{B}^2 \right) \delta_{ij} - E_i E_j - B_i B_j \right\}$$

Stress tensor

$$=: p\, \delta_{ij} + P^{o}_{ij}, \qquad (1.6)$$

where $\mathrm{tr}\, P^{o}_{ij} \overset{!}{=} 0$, $\quad p = \frac{1}{3} \sum_{j=1}^{3} P_{jj}$: pressure

From (1.6),

$$3p = \sum_{j=1}^{3} P_{jj} = \frac{3}{8\pi} \left(\vec{E}^2 + \vec{B}^2 \right) - \frac{1}{4\pi} \left(\vec{E}^2 + \vec{B}^2 \right)$$

$$= u \qquad (1.7)$$

Then

$$U(T,V) = u(T) \cdot V = 3 V p(T), \qquad (1.8)$$

where V is the volume of the stove containing

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Stress tensor

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$$= u \qquad (1.7)$$

Then

$$U(T, V) = u(T) \cdot V = 3Vp(T), \qquad (1.8)$$

where V is the volume of the stove containing the radiation,

The first and second laws of thermodynamics for <u>BBR</u> are summarized in

$$dU = T\,dS - p\cdot dV, \quad \left(T\,dS = dQ^{\checkmark}\right) \quad (1.9)$$

$$\Rightarrow \quad dS = \frac{dU + p\cdot dV}{T} \underset{\substack{\uparrow \\ (1.7),(1.8)}}{=} \frac{3\,d(pV) + p\,dV}{T} \quad (1.10)$$

We view S as a function of T and V, i.e.,

$$dS = \frac{\partial S}{\partial V}\,dV + \frac{\partial S}{\partial T}\,dT$$

Then (1.10) yields

$$\frac{\partial S}{\partial V} = 4\,\frac{p}{T}, \quad \frac{\partial S}{\partial T} = 3\,\frac{V}{T}\,\frac{\partial p}{\partial T}$$

(p independent of V!) Since

$$\frac{\partial^2 S}{\partial T\,\partial V} = \frac{\partial^2 S}{\partial V\,\partial T},$$

we find

$$4\,\frac{\partial}{\partial T}\left(\frac{p}{T}\right) = \frac{3}{T}\,\frac{\partial p}{\partial T} \Longleftrightarrow \frac{1}{T}\,\frac{\partial p}{\partial T} = 4\,\frac{p}{T^2}$$

which implies that

$$p = \frac{\alpha}{3}\,T^4, \quad \alpha \text{ a constant},$$

and

$$\boxed{u = \alpha T^4,} \qquad (1.11)$$

which is the Stefan-Boltzmann law.

Since
$$\frac{\partial S}{\partial T} = 3\,\frac{V}{T}\,\frac{\partial p}{\partial T} = 4\alpha V T^2$$

we find that

$$S = \frac{4\alpha}{3} V T^3 + \sigma(V),$$

and, since $\frac{\partial S}{\partial V} = 4\,\frac{p}{T}$,

$$S = \frac{4\alpha}{3} V T^3 + \underbrace{S_0}_{=0,\ \text{according to Nernst's } 3^{rd} \text{ law.}} \qquad (1.12)$$

Calculation of density of states of BBR

Stove = cube with ideally conducting walls, sides of length L.

Then BBR consists of standing waves indexed by wave vectors, $\vec{k}$, in

$$K := \left\{ \frac{\pi}{L}\vec{n} \;\middle|\; n_i = 0, 1, 2, \ldots, \forall i,\ n_1 n_2 + n_2 n_3 + n_3 n_1 > 0 \right\} \qquad (1.13)$$

Since e.m. radiation is transversally polarized, there are two states (field oscillators) associated with every wave vector $\vec{k}$. The circular frequency ω of a field oscillator with wave vector $\vec{k}$ is

$$\omega = c\,|\vec{k}|. \qquad (1.14)$$

Let

$$N(\omega) := \#\text{ field oscillators with } |\vec{k}| \leq \frac{\omega}{c}.$$

Then (1.9) gives:

$$N(\omega, V) = 2 \cdot \frac{1}{8} \cdot \frac{4\pi}{3}\left(\frac{\omega}{c}\right)^3 \cdot \left(\frac{\pi}{L}\right)^{-3}$$

+ surface terms

Explanation:

1st factor, 2: 2 oscillators per wave vector

2nd —"—, $\frac{1}{8}$: K is an octant

3rd —"—, $\frac{4\pi}{3}\left(\frac{\omega}{c}\right)^3$: volume of 3D sphere of radius $\frac{\omega}{c}$

4th —"—, $\left(\frac{\pi}{L}\right)^{-3}$: $\left(\frac{\pi}{L}\right)^3$ is the volume in $\vec{k}$-space assoc. with every wave-vector in K

$$\Rightarrow \boxed{N(\omega, V) = \frac{1}{3\pi^2}\,\frac{\omega^3}{c^3} \cdot V} \quad \left(V = L^3\right) \quad (1.15)$$

The spectral density, $n(\omega, V)$, of field operators is thus given by

$$\left.\begin{aligned}\frac{dN(\omega,V)}{d\omega} =: n(\omega,V) = \frac{1}{\pi^2 c^3}\,\omega^2 V,\\ n(\omega) := n(\omega,V)/V.\end{aligned}\right\} \qquad (1.16)$$

The mean energy E of a field oscillator in thermal equilibrium can only depend on temperature T and it's angular frequency ω (by homogeneity and isotropy, 2nd law!):

$$E = E(T,\omega)$$

Thus

$$\boxed{u(\omega, T) = \frac{1}{\pi^2 c^3}\,\omega^2 \cdot E(\omega, T)} \qquad (1.17)$$

Formula (1.17) was first derived by Rayleigh (with a mistake of a factor of 8, corrected by Jeans) and then, independently, by Planck.

$$\boxed{\begin{aligned} &U^3 V = \text{const.} \\ &\Longrightarrow \quad 3U^2\,dU\,V + U^3\,dV = 0 \\ &\Longleftrightarrow \quad 3\,\frac{dU}{U} + \frac{dV}{V} = 0 \end{aligned}} \quad \Longrightarrow$$

Wien's displacement law:

Claim: $$\boxed{E(\omega, T) = \omega\, \mathcal{E}\left(\frac{\omega}{T}\right)}$$

Consider a cubical stove, and compress it adiabatically ($dQ^{\downarrow} = 0$) and quasi-statically. Then

$$dU = -p\,dV = -\frac{1}{3}\frac{U}{V}\,dV$$

$$\Leftrightarrow \quad 3\frac{dU}{U} + \frac{dV}{V} = 0$$

$$\Leftrightarrow \quad \boxed{U^3 V = \text{const.}} \tag{1.18}$$

The state of the stove after (infinitely slow) compression is still an equilibrium state.

$$\Rightarrow \quad U = \alpha T^4 V \qquad \text{(Stefan-Boltzmann)}$$

With (1.18) it follows that

$$\boxed{T^3 V = \text{const.}} \tag{1.19}$$

Consider a field oscillator with wave vector

$$\vec{k} = \frac{\pi}{L}\vec{n} \in K$$

Since $\omega = c|\vec{k}| = c\frac{\pi}{L}|\vec{n}|$, and

$\vec{n}$ remains constant under quasi-static ("adiabatic theorem"!), adiabatic compression, we have that

$$\boxed{\omega^3 \cdot V = \text{const.}} \tag{1.20}$$

Energy balance for quasi-static adiabatic compression:

$$dU = -p\,dV \qquad (dQ^{\ell} = 0), \tag{1.21}$$

with

$$V = V(t), \quad dV = \dot{V}(t)\,dt, \quad (t = \text{time})$$

By (1.19),

$$V\,dT = -\frac{1}{3} T\,dV, \tag{1.22}$$

and by (1.20),

$$V\,d\omega = -\frac{1}{3}\omega\,dV \tag{1.23}$$

Note: Only variable is time t, and $V(t)$ is fixed by the process, with $\dot{V}$ so small that thermal equilibrium is always maintained.

We specialize the energy balance to a shell in $\vec{k}$-space of width $\Delta|\vec{k}| = \frac{\Delta\omega}{c}$

about a value $|\vec{k}| = \frac{\omega}{c}$. The number of modes in this shell is given by

$$\Delta N(\omega, V) \simeq \frac{1}{\pi^2 c^4} \omega^2 V \Delta\omega, \qquad (1.24)$$

as follows from (1.15).

<u>Contribution of these modes to internal energy and pressure</u>:

$$\Delta U(\omega, T, V) = \frac{1}{\pi^2 c^3} \omega^2 E(\omega, T) V \frac{\Delta\omega}{c}$$

$$= u(\omega, T) V \frac{\Delta\omega}{c}$$

$$\Delta p(\omega, T) = \frac{1}{3} \frac{\Delta U(\omega, T, V)}{V} = \frac{1}{3} u(\omega, T) \frac{\Delta\omega}{c}$$

Then (1.21) yields

$$d\Delta U = -\Delta p \, dV, \qquad (1.25)$$

with

$$d\Delta U = du \, V \frac{\Delta\omega}{c} + u \, dV \frac{\Delta\omega}{c} + u V \frac{d\Delta\omega}{c}$$

$$\underset{(1.23)}{\uparrow}{=} du \, V \frac{\Delta\omega}{c} + \frac{2}{3} u \, dV \frac{\Delta\omega}{c} \qquad (1.26)$$

Thus, (1.25) yields:

$$du\,V + \frac{2}{3}u\,dV = -\frac{1}{3}u\,dV,$$

or

$$du\,V + u\,dV = 0. \qquad (1.27)$$

Next, $dV = \dot{V}\,dt$, t = time, and

$$du = \frac{\partial u}{\partial \omega}d\omega + \frac{\partial u}{\partial T}dT.$$

(1.22) and (1.23) then imply that

$$V\,du = -\frac{1}{3}\frac{\partial u}{\partial \omega}\omega\,dV - \frac{1}{3}\frac{\partial u}{\partial T}T\,dV, \qquad (1.28)$$

so that, with (1.27),

$$-\frac{1}{3}\frac{\partial u}{\partial \omega}\cdot\omega - \frac{1}{3}\frac{\partial u}{\partial T}\cdot T + u = 0, \qquad (1.29)$$

because $|\dot{V}| > 0$. This equation has the solution

$$u(\omega, T) = \frac{1}{\pi^2 c^3}\omega^3 \mathcal{F}\left(\frac{\omega}{T}\right), \qquad (1.30)$$

as the reader easily verifies. With (1.17) we find

$$E(\omega, T) = \omega\,\mathcal{F}\left(\frac{\omega}{T}\right), \qquad (1.31)$$

for the mean energy of a field oscillator of frequency ω at temperature T; which is <u>Wien's displacement law</u>!

<u>Dimension of $\hbar$:</u> $[\hbar] =$ action, because

$[E] =$ energy, $[\omega] = \frac{1}{\text{time}}$.

<u>Different choices for ε:</u>

$$\hbar\left(\frac{\omega}{T}\right) = k\,\frac{T}{\omega} \longrightarrow \text{Rayleigh-Jeans}$$

$$\hbar\left(\frac{\omega}{T}\right) = \hbar\, e^{-\frac{\hbar\omega}{kT}} \longrightarrow \text{Wien}$$

$$\hbar\left(\frac{\omega}{T}\right) = \hbar\,\frac{1}{e^{\hbar\omega/kT}-1} \longrightarrow \text{Planck}$$

Planck finds:

$$\boxed{\left.\begin{aligned} \hbar &= 1.04\ (1.05459) \times 10^{-34}\ \text{J.sec} \\ k &= 1.34\ \underbrace{(1.3807)}_{\text{present values}} \times 10^{-23}\ \text{J/K} \end{aligned}\right\}} \qquad (1.32)$$

The Rayleigh-Jeans law leads to an <u>ultraviolet catastrophe</u> when one integrates over all frequencies $\Longrightarrow$ <u>Classical physics fails!</u>

(Rayleigh-Jeans follows from classical electrodynamics for the free e.m. field and the <u>equipartition law</u> of classical stat. mech.!)

$\longrightarrow \exists$ new constant of nature $\hbar$ *)!

*) Why must there be a natural constant with the dimension of action?

(1) Gibbs: $Z = \frac{1}{n!} \int \frac{\prod_{i=1}^{n} d^3q_i \, d^3p_i}{h^{3n}} \, e^{-\beta H(p,q)} \longrightarrow$

$\boxed{\text{Hints to QM!}}$

(2) Planck's law

(3) TD of ideal gases (Sackur – Tetrode)

From the gas constant R and his value for k Planck finds the value of Avogadro's number (= Loschmidtsche Zahl):

$$N_A = \frac{R}{k} = 6.17\ (6.022) \times 10^{23} / \mathrm{Mol},$$

and the value of the elementary electric charge:

$$e = \frac{F}{N_A}, \quad F = \text{Faraday constant},$$

yielding:

$$e \simeq 1.6\ (1.60218) \times 10^{-19}\ C\ (\text{oulomb})$$

(Exp.: Robert A. Millikan)

Josephson constant:

$$K_J = \frac{2e}{h}$$

(from voltage oscillations in the Josephson effect)

von Klitzing constant:

$$R_K = \frac{h}{e^2}$$

("resistivity quantum" in IQHE)

$$\rightarrow \boxed{e = \frac{2}{R_K K_J}}$$

Feinstructure constant:

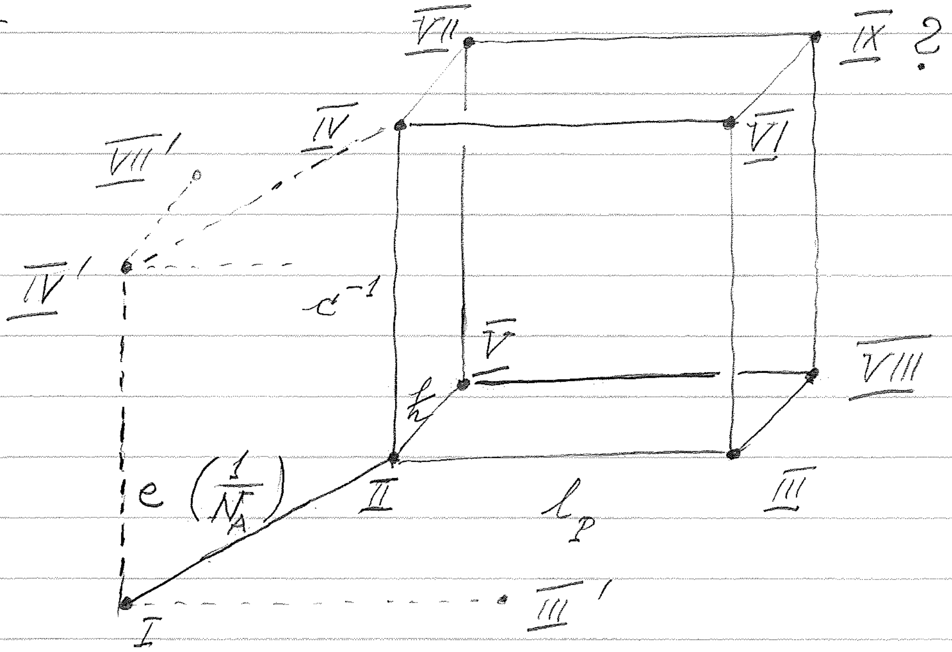
$$\alpha := \frac{e^2}{\hbar c} \simeq \frac{1}{137}$$

from measurements of anomalous magnetic moment of electron (↗ storage ring – Telegdi; Gabrielse's exp.: electron in ext. magn. field)

$$\alpha, e, c \longrightarrow \hbar$$

Nowadays, units are defined in terms of constants of Nature: $c, \hbar\,(e), \alpha, k, \dots$

[Bronstein – Planck (hyper-) cube:



I: Continuum mechanics, thermodynamics

IV': Electrodynamics, relativistic continuum mech. etc.

II: Class. mech. of point particles, class. stat. mech of point particles ("atomism")

III: Celestial mechanics, Newton's law of universal gravitation, with

$$\ell_P^2 := \frac{G_{Newton}\hbar}{c^3}$$

ℓ_P: Planck length,

G_{Newton}: Newton's gravitational constant

IV: Classical electrodynamics of charged point particles (→ problem of runaway solutions!)

V: Non-relativistic QM

VI: General Theory of Relativity

VII: Relativistic QFT

VIII: Non-relativistic "quantum gravity"

IX: "String theory, M-theory, ..."

$e, c^{-1}, \hbar, \ell_P$: "Deformation parameter"]

Einstein's discovery of the photon (1905)

Einstein notices that Rayleigh-Jeans is classical and untenable, and that Wien's law manifests the deviation from classical physics. He therefore begins his analysis of BBR with Wien's law.

Consider an ideal gas of particles, and let $S(U, N, V)$ be its entropy, with

U: internal energy, N: # of particles,

V: volume occupied by gas.

According to the first and second law of thermodynamics, for N fixed,

$$dS = \frac{1}{T}\left(dU + p\, dV\right) \qquad (1.33)$$

$$\Rightarrow \left(\frac{\partial S}{\partial V}\right)_{U,N} = \frac{p}{T} = \frac{Nk}{V}, \qquad (1.34)$$

where we have used the equation of state of an ideal gas

$$p \cdot V = n_{Mol} RT = NkT \qquad (1.35)$$

From (1.34):

$$S(U,V,N) = S(U,V_0,N) + k \ln \underbrace{\left(\frac{V}{V_0}\right)^N}_{\equiv W} \qquad (1.36)$$

$W := \left(\frac{V}{V_0}\right)^N$ is the probability that N in a box of volume V_0 equi-distributed, non-interacting, independent particles all occupy a smaller box of volume V. Thus,

$$S = S_0 + k \ln W,$$

<u>in accordance with a formula first proposed by Planck</u>, (& found on Boltzmann's grave stone).

Eq. (1.33) also implies that

$$\left(\frac{\partial S}{\partial U}\right)_{V,N} = \frac{1}{T} \qquad (1.37)$$

We apply (1.34), (1.37) to a single field oscillator of the e.m. field. Let $E(\omega, T)$ be the mean energy of such an oscillator, as before, and let s denote its entropy; frequency ω.

Then (1.37) (for one harmonic oscillator in heat bath) $\Rightarrow$

$$\frac{\partial s}{\partial E} = \frac{1}{T} \tag{1.38}$$

Wien's law tells us that

$$\frac{1}{T} = -\frac{k}{\hbar\omega} \ln\left(\frac{E}{\hbar\omega}\right) \tag{1.39}$$

Plugging (1.39) into (1.38) and integrating in E yields

$$s(E) = -k\frac{E}{\hbar\omega}\left(\ln\left(\frac{E}{\hbar\omega}\right) - 1\right) + s_0 \tag{1.40}$$

Third law of TD: $s_0 = 0$

Eq. (1.40) yields a formula for the entropy of all field oscillators whose frequencies belong to $[\omega, \omega + \Delta\omega]$:

$$\Delta S(U, V, N_\omega) = s(E(\omega, T))\, n(\omega, V)\, \Delta\omega, \tag{1.41}$$

where $n(\omega, V)$ as in (1.16), with

$$U = E(\omega, T) \cdot n(\omega, V)\, \Delta\omega$$

Expressing E by U and n in (1.40) and plugging the result into (1.41) yields:

$$\Delta S(U,V,N_\omega) = -k\frac{U}{\hbar\omega}\left[\ln\left(\frac{U}{\frac{\omega^2}{\pi^2 c^3}V\Delta\omega}\right)-1\right]$$

hence

$$\Delta S(U,V,N_\omega) = \Delta S(U,V_0,N_\omega) + k\frac{U}{\hbar\omega}\ln\left(\frac{V}{V_0}\right)$$

$$= \Delta S(U,V_0,N_\omega) + k\ln\left(\frac{V}{V_0}\right)^{\frac{U}{\hbar\omega}} \quad (1.42)$$

Comparing (1.42) with (1.36) leads Einstein to the following interpretation: "Monochromatische Strahlung geringer Dichte (innerhalb des Gültigkeitsbereichs der Wienschen Strahlungsformel) verhält sich in wärmetheoretischer Beziehung so, wie wenn sie aus von einander unabhängigen Energiequanten der Grösse $\hbar\omega$ bestünde." $\longrightarrow$ Birth of the idea of the photon.

<u>Application</u>: Theory of the <u>photo-electric effect</u>; <u>Stokes' phenomenon</u>; (1905)

Back to (1.38): $\quad \dfrac{\partial s}{\partial E} = \dfrac{1}{T}$

$$\Rightarrow \quad \frac{\partial^2 s}{\partial E^2} = -\frac{1}{T^2}\frac{\partial T}{\partial E} = -\frac{1}{T^2 \frac{\partial E}{\partial T}} \qquad (1.43)$$

Using Rayleigh-Jeans, we then find that

$$\frac{1}{k}\frac{\partial^2 s}{\partial E^2} = -\frac{1}{E^2}; \qquad (1.44)$$

while Wien yields

$$\frac{1}{k}\frac{\partial^2 s}{\partial E^2} = -\frac{1}{\hbar\omega E} \qquad (1.45)$$

Planck interpolated between (1.44) and (1.45):

$$\frac{1}{k}\frac{\partial^2 s}{\partial E^2} = -\frac{1}{E(\hbar\omega + E)}$$

$$\overset{(1.43)}{=} -\frac{1}{T^2 \frac{\partial E}{\partial T}}$$

← Diff. eq. for E!

Integration yields Planck's formula for $E(\omega, T)$!

In 1906/1907 Einstein reproduces Planck's formula by applying the Boltzmann distribution

$$\left.\begin{array}{l} p_n = \dfrac{1}{Z_\beta} e^{-\beta E_n}, \\ \text{with} \quad \beta = (kT)^{-1}, \; E_n = n \cdot \hbar\omega \end{array}\right\} \qquad (1.46)$$

to the field oscillators. He then proposes to apply the same formula to the vibrational modes of crystals, which yields an expression for the specific heat of crystals and will inspire Nernst to discover the third law of thermodynamics.

1.2. Quantum Theory and Probability

We recall that

$$u(\omega, T) = \frac{\omega^2}{\pi^2 c^3} E(\omega, T) \tag{1.47}$$

In 1917, Einstein gives a new derivation of Planck's formula for $E(\omega, T)$ based on the following considerations:

Consider a stove filled with e.m. radiation and with material objects (atoms) with the following properties:

(1) Atoms can only occupy states with discrete energies $E_1 < E_2 < \cdots < E_k < \cdots$

(2) # of atoms, N_k, in state labelled by k is given by Boltzmann's distribution

$$N_k = A\, e^{-\beta E_k}$$

(3) Atoms can decay (spontaneously) from state labelled by k to state of lower energy labelled by $l < k$ by emitting a photon of energy $\hbar\omega = E_k - E_l$ (energy conservation!) The number, $\dot{N}^s_{kl}$, of such transitions per unit of time (sec) is given by

$$\dot{N}^s_{kl} = N_k A_{kl} \qquad (1.48)$$

(4) The number of transitions, $\dot{N}_{kl}$, from state with energy E_l to an excited state with energy $E_k > E_l$ induced by absorption of a photon with energy $\hbar\omega = E_k - E_l$ from BBR

is given by

$$\dot{N}_{\ell k} = u(\omega, T)\, N_\ell\, B_{\ell k} \qquad (1.49)$$

and the number, $\dot{N}^{i}_{k\ell}$, of induced transitions from states with energy E_k to states with energy $E_\ell < E_k$ accompanied by the emission of a photon of energy $\hbar\omega = E_k - E_\ell$ is given by

$$\dot{N}^{i}_{k\ell} = u(\omega, T)\, N_k\, B_{k\ell}\,, \qquad (1.50)$$

where $A_{k\ell}$, $B_{\ell k}$ and $B_{k\ell}$ are "transition rates" determined by a quantum theory of atoms.

(5) Condition for equilibrium

$$\dot{N}_{\ell k} = \dot{N}^{s}_{k\ell} + \dot{N}^{i}_{k\ell}\,. \qquad (1.51)$$

Hence, with (1.48) – (1.50)

$$u(\omega, T)\, N_\ell\, B_{\ell k} = u(\omega, T)\, N_k\, B_{k\ell} + N_k A_{k\ell}\,,$$

with $\omega = \dfrac{E_k - E_\ell}{\hbar}$.

Since, by (2), $N_k = A e^{-\beta E_k}$, we find

$$u(\omega,T)\, e^{\beta\hbar\omega} B_{\ell k} = u(\omega,T)\, B_{k\ell} + A_{k\ell}$$

and hence

$$u(\omega,T) = \frac{A_{k\ell}}{e^{\beta\hbar\omega} B_{\ell k} - B_{k\ell}} \qquad (1.52)$$

In the limiting regime where $\beta\hbar\omega \ll 1$, expression (1.52) for $u(\omega,T)$ must approach the law of Rayleigh and Jeans:

$$u(\omega,T) = \frac{A_{k\ell}}{e^{\beta\hbar\omega} B_{\ell k} - B_{k\ell}} \simeq \frac{A_{k\ell}}{(B_{\ell k} - B_{k\ell}) + \beta\hbar\omega\, B_{\ell k}}$$

$$\overset{!}{=} \frac{\omega^2}{\pi c^3}\, \beta^{-1}$$

Assuming that energy differences $E_k - E_\ell$, $k > \ell$, are "dense" in the positive real axis we conclude that

(i) $B_{\ell k} = B_{k\ell}$ (!), and

(ii) $$\frac{A_{k\ell}}{B_{k\ell}} = \frac{\hbar\omega^3}{\pi c^3} \qquad (1.53)$$

Plugging (i) and (ii) into (1.52), one finds Planck's law.

The coefficients $A_{k\ell}$ and $B_{k\ell}$ are called "Einstein's A- and B-coefficients", and (1.53) is called "Einstein's relation".

Morale:

- Emission and absorption of photons by material objects (atoms) is statistical, governed by rate equations; besides induced emission, there is also spontaneous emission.
- Induced emission → idea of lasers, STED-(stimulated emission depletion) microscopy (by Stefan Hell), ...,

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1.3. Bose Gases and Bose-Einstein Condensation

The first example of a system for which a phase transition accompanied by a broken symmetry has been exhibited, using the methods

of equilibrium statistical mechanics, is the ideal Bose gas. In 1925, Albert Einstein proved that this system exhibits what is called Bose-Einstein condensation, as the temperature is lowered at fixed, positive density. This phenomenon turned out to be accompanied by the spontaneous breaking of a $U(1)$ symmetry; (gauge trsfs. of the 1^{st} kind).

1.1 Bose-Einstein condensation in the ideal Bose gas

We consider an ideal gas of non-relativistic atoms confined to a cubical region Λ in phys. space $\mathbb{E}^3$ at a positive temperature T and density $\rho > 0$. (It is assumed that the system is in a state of thermal equilibrium.)

For simplicity, we impose periodic b.c. on the $\underbrace{\text{wave functions}}$ of the atoms, and we describe

← de Broglie, 1924 (thesis!)

the system in the grand-canonical ensemble. (!)

[The one-atom Hamiltonian is given by

$$h_\Lambda := -\frac{\hbar^2}{2m}\Delta_\Lambda \qquad (1) \quad (1.54)$$

with periodic b.c. at $\partial\Lambda$. The eigenfunctions of h_Λ are

$$\psi_{\vec{k},\sigma} = u_\sigma \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{L^3}}, \qquad (2) \quad (1.55)$$

where $\{u_\sigma\}_{\sigma=1}^{S}$ is a CONS in $\mathbb{C}^{2S+1}$, S is the spin of an atom, and] L is the length of an edge of Λ; states of an atom labelled by

de Broglie ⟶ $$\boxed{\vec{k} = \frac{2\pi}{L}\vec{n}, \quad \vec{n} \in \mathbb{Z}^3;} \qquad (3) \quad (1.56)$$

with energy (=Eigenvalue of h_Λ on $\psi_{\vec{k},\sigma}$) given by

$$\boxed{\varepsilon_{\vec{k}} = \frac{\hbar^2 \vec{k}^2}{2m}, \quad \text{indep. of } \sigma.}$$

[Hilbert space of system: Fock space

$$\mathcal{F}_\Lambda := \bigoplus_{n=0}^{\infty} L^2(\Lambda, d^3x)^{\otimes_s n} \quad]$$

$\Leftrightarrow$ A one-particle state corresp. to a mode $\vec{k}$, as in (1.56) can be occupied by $n = 0, 1, 2, \dots$ atoms; (Bose statistics!) (1.57)

[Hamiltonian

$$H_\Lambda := \bigoplus_{n=0}^{\infty} H_\Lambda^{(n)} \,,$$

$$H_\Lambda^{(0)} = 0, \quad H_\Lambda^{(1)} = h_\Lambda, \quad H_\Lambda^{(n)} = \sum_{j=1}^{n} h_{\Lambda, j}, \dots$$

N: particle # operator

μ: chemical potential

[Mention 2nd quantization; a^*-, a-ops.!]

Grand partition function:

$$\Xi_\Lambda(\beta, \mu) = \mathrm{Tr}_{\mathcal{F}_\Lambda}\left(e^{-\beta(H_\Lambda - \mu N)}\right)$$

$$\underset{(1.57)}{=} \prod_{\vec{k} \in \frac{2\pi}{L}\mathbb{Z}^3} \left(1 - e^{-\beta(\varepsilon_{\vec{k}} - \mu)}\right)^{-(2S+1)} \qquad \cancel{(4)}\ (1.58)$$

summing geometric series.

$\Rightarrow$ Pressure:

$$p_\Lambda(\beta, \mu) = \frac{1}{\beta|\Lambda|} \log \Xi_\Lambda(\beta, \mu)$$

$$- \frac{2S+1}{\beta(2\pi)^3} \sum_{\vec{k}} \left(\frac{2\pi}{L}\right)^3 \log\left(1 - z e^{-\beta \varepsilon_{\vec{k}}}\right), \qquad \cancel{(5)}\ (1.59)$$

where $z = e^{\beta\mu}$ (fugacity).

R.S. is <u>Riemann-sum approximation</u> to

$$-\frac{2S+1}{\beta(2\pi)^3}\int d^3k \log\left(1-z e^{-\beta \hbar^2 \vec{k}^2/2m}\right)$$

As $L\to\infty$ (TD limit), we find that (polar coos.)

$$p(\beta,\mu) = -\frac{2S+1}{\beta}\frac{1}{2\pi^2\hbar^3}\left(\frac{2m}{\beta}\right)^{3/2}\int_0^\infty \log\left(1-ze^{-x^2}\right)x^2\,dx$$

$$= \frac{2}{3}\frac{2S+1}{\beta}\frac{1}{2\pi^2\hbar^3}\left(\frac{2m}{\beta}\right)^{3/2}\sum_{n=1}^\infty\int_0^\infty dx\, x^4\left(ze^{-x^2}\right)^n$$

$$= \frac{2}{3}\frac{2S+1}{\beta}\frac{1}{2\pi^2\hbar^3}\left(\frac{2m}{\beta}\right)^{3/2}\frac{3\sqrt{\pi}}{8}\left(\sum_{n=1}^\infty\frac{z^n}{n^{5/2}}\right)$$

<u>Def.</u> (ξ-function)

$$\xi_r(z) := \sum_{n=1}^\infty \frac{z^n}{n^r} \tag{1.60}$$

Then

$$\boxed{p(\beta,\mu) = \frac{2S+1}{\beta}\frac{1}{\lambda^3}\xi_{5/2}(z),} \qquad (7)\ (1.61)$$

where

$$\lambda := \left(\frac{2\pi\hbar^2}{mkT}\right)^{1/2}$$ (<u>thermal wave length</u>)

$$\left(\beta = \frac{1}{kT}\right)$$

From (4) and (5):

Particle density: $$\rho = \frac{\partial p}{\partial \mu} = z\,\frac{\partial(\beta p)}{\partial z} = \frac{2S+1}{\lambda^3}\,\zeta_{3/2}(z), \qquad (8)\ (1.62)$$

energy density: $$u = \lim_{L\to\infty} \frac{U_\Lambda}{|\Lambda|} = \frac{\langle H_\Lambda\rangle_{\beta,\mu}}{L^3}$$

$$= -\frac{\partial(\beta p)}{\partial \beta} \underset{\uparrow}{=} \frac{3}{2}\,p \qquad (9)\ (1.63)$$

(1.61) ~~(7)~~, def. of λ.

<u>Classical limit</u>: $\rho\lambda^3 \ll 1$, (i.e., $|z| \ll 1$, or T large; see (6), (7)). Then

$$p(\beta,\mu) \simeq kT\,(2S+1)\,\lambda^{-3} z,$$

$$\rho \simeq (2S+1)\,\lambda^{-3} z, \qquad (10)\ (1.64)$$

i.e.,

$$\underbrace{p(\beta,\mu) \simeq kT\rho}_{\text{(ideal gas)}}, \quad u = \frac{3}{2}kT\rho.$$

For ρ large, $\boxed{p(\beta,\mu) < p_{cl.}(\beta,\mu)\,!}$

- <u>BEC</u>

Eq. (59) meaningful, as long as

$$z\, e^{-\beta \min \varepsilon_{\vec{k}}} = z < 1,$$

i.e.,

$$\mu < \varepsilon_{\vec{k}=0} = 0. \qquad \cancel{(11)}\ (1.65)$$

For $\mu < 0$, we have $\cancel{(7)}$ (1.61), $\cancel{(8)}$ (1.62) and $\cancel{(9)}$ (1.63)!

Let us consider low temperatures at fixed ρ:

By (6), $\zeta_r(z)$ well def. for $|z| < 1$, monotone increasing for $z \in [0,1)$. For $r > 1$,

$$\zeta_r(z=1) < \infty, \qquad (12)$$

$$\zeta_{3/2}(1) \simeq 2.612, \quad \zeta_{5/2}(1) \simeq 1.342. \qquad (1.66)$$

By (7),

$$p(\beta,\mu) \underset{\mu \nearrow 0}{\nearrow} p^*(\beta) = kT\, \frac{2S+1}{\lambda^3}\, \zeta_{5/2}(1) \qquad (13)\ (1.67)$$

$$\rho(\beta,\mu) \underset{\mu \nearrow 0}{\nearrow} \rho_c(T) = \frac{2S+1}{\lambda^3}\, \zeta_{3/2}(1) \qquad (14)\ (1.68)$$

$$\boxed{\begin{array}{l} \Longrightarrow \text{ Fixing } \rho > 0,\ \exists \text{ crit. temperature} \\ T_c = T_c(\rho) \text{ such that } z \nearrow 1, \text{ as } T \searrow T_c! \end{array}} \qquad (1.69)$$

$$z\, e^{-\beta \min \varepsilon_{\vec{k}}} = z < 1,$$

i.e.,

$$\mu < \varepsilon_{\vec{k}=0} = 0. \qquad \cancel{(11)} \quad (1.65)$$

For $\mu < 0$, we have ~~(7)~~ (1.61), ~~(8)~~ (1.62) and ~~(9)~~ (1.63)!

Let us consider low temperatures at fixed ρ:

By (6), $\zeta_r(z)$ well def. for $|z| < 1$, monotone increasing for $z \in [0,1)$. For $r > 1$,

$$\zeta_r(z=1) < \infty, \qquad (12) \quad (1.66)$$

$\zeta_{3/2}(1) \simeq 2.612$, $\zeta_{5/2}(1) \simeq 1.342$.

By (7),

$$p(\beta,\mu) \nearrow_{\mu \nearrow 0} p^*(\beta) = kT\, \frac{2S+1}{\lambda^3}\, \zeta_{5/2}(1) \qquad (13) \quad (1.67)$$

$$\rho(\beta,\mu) \nearrow_{\mu \nearrow 0} \rho_c(T) = \frac{2S+1}{\lambda^3}\, \zeta_{3/2}(1) \qquad (14) \quad (1.68)$$

$\Longrightarrow$ Fixing $\rho > 0$, $\exists$ crit. temperature $T_c = T_c(\rho)$ such that $z \nearrow 1$, as $T \searrow T_c$! (1.69)

$$z\, e^{-\beta \min \varepsilon_{\vec{k}}} = z < 1,$$

i.e.,

$$\mu < \varepsilon_{\vec{k}=0} = 0. \qquad \cancel{(11)}\ (1.65)$$

For $\mu < 0$, we have $\cancel{(7)}$ (1.61), $\cancel{(8)}$ (1.62) and $\cancel{(9)}$ (1.63)!

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$$\boxed{\begin{array}{l} \Rightarrow \text{Fixing } \rho > 0,\ \exists \text{ crit. temperature} \\ T_c = T_c(\rho) \text{ such that } z \nearrow 1, \text{ as } T \searrow T_c! \end{array}} \qquad (1.69)$$

By (14), Eq. for T_c given by

$$\rho \simeq \underbrace{2.612}_{\zeta_{3/2}(1)} (2S+1) \left(\frac{m k T_c}{2\pi \hbar^2} \right)^{3/2}, \quad \text{i.e.,}$$

$$T_c \simeq \frac{2\pi \hbar^2}{m k} \left(\frac{\rho}{2.612(2S+1)} \right)^{2/3} \qquad \cancel{(15)} \; (1.70)$$

What happens for $T < T_c$?

When $z \nearrow 1$, the Riemann sum in (5) cannot be replaced by the integral

$$\int d^3k \log\left(1 - z\, e^{-\beta \varepsilon_{\vec{k}}}\right),$$

anymore, as $L \to \infty$. The occupation #, $n_{\vec{k}}$, of state with wave vector $\vec{k}$ [$\leftrightarrow$ orbital $\psi_{\vec{k},\sigma}$] has expectation value (see (4))

$$\langle n_{\vec{k}} \rangle_{\beta,\mu} = \frac{2S+1}{z^{-1} e^{\beta \varepsilon_{\vec{k}}} - 1}, \qquad \cancel{(16)} \; (1.71)$$

which diverges, for $\vec{k} = 0$, as $z \nearrow 1$!

For $z \lesssim 1$ and $L < \infty$,

$$\underset{|\vec{k}| \neq 0}{\langle n_{\vec{k}} \rangle_{\beta,\mu}} \ll \langle n_{\vec{k}=0} \rangle_{\beta,\mu}$$

(1.71) (pg 1.35)

By ~~(16)~~,

$$\beta\mu = -\ln\left(1 + \frac{1}{\langle n_{\vec{0}} \rangle_{\beta,\mu}}\right) \simeq -\frac{1}{\langle n_{\vec{0}} \rangle_{\beta\mu}}$$

Thus, for $L < \infty$, $\langle n_{\vec{0}} \rangle_{\beta\mu} \propto \rho L^3$ is "macroscopically occupied", when $T < T_c(\rho)$, with $\langle n_{\vec{0}} \rangle_{\beta,\mu} \propto N = \rho L^3$. In the TD limit, at fixed T but variable ρ

$$\frac{\langle n_{\vec{0}} \rangle_{\beta,\mu}}{L^3} \xrightarrow[L\to\infty]{} \rho - \rho_c > 0, \qquad (1.72)$$

for $\rho > \rho_c(T)$, (with $\rho_c(T)$ given by ~~(14)~~ (1.68) (pg. 1.34); after $T_c \to T$). Thus, we find for the occupation density $\rho_L(\vec{k}) := \frac{\langle n_{\vec{k}} \rangle_{\beta,\mu}}{L^3}$ in the thermodynamic limit, at $z = 1$:

$$\rho(\vec{k}) := \lim_{L\to\infty} \rho_L(\vec{k}) = \frac{2S+1}{e^{\beta\hbar^2\vec{k}^2/2m} - 1} + (\rho - \rho_c)\,\delta^{(3)}(\vec{k}) \qquad \cancel{(18)}\ (1.73)$$

and hence

$$\rho = \int \rho(\vec{k})\, d^3k = \underbrace{(2S+1)\lambda^{-3}\zeta_{3/2}(1)}_{\rho_c = \rho_c(T)} + \underbrace{\rho_{\vec{k}=0}}_{=\rho-\rho_c}$$

With $\lambda = \left(\dfrac{2\pi\hbar^2}{mkT}\right)^{1/2}$, it follows that,

for $T < T_c = T_c(\rho)$, (as in ~~(15)~~ (1.70)!),

$$\rho_{\vec{k}=0} = \rho\left(1 - \left(\frac{T}{T_c}\right)^{3/2}\right) \qquad (19)\ (1.74)$$

Note: 0 - mode does not contribute to $p(\beta,\mu)$,

nor to u!

$$u = \frac{3}{2}p = \frac{3}{2}(2S+1)kT\lambda^{-3}\zeta_{5/2}(z)$$

$$\underset{z\nearrow 1}{=} \frac{3}{2}kT\rho_c\frac{\zeta_{5/2}(1)}{\zeta_{3/2}(1)}$$

$$\simeq \left(\frac{3}{2}\times 0.513\right)\cdot kT\rho\left(\frac{T}{T_c}\right)^{3/2} \qquad (20)\ (1.75)$$

$$\left.\begin{aligned} \Longrightarrow c_v &\propto T^{3/2} \quad (T \text{ small}) \\ &\simeq \frac{3}{2}k \quad (T \text{ large}) \end{aligned}\right\} \longrightarrow \text{3rd Law valid.}$$

$\dfrac{dc_v}{dT}$ has discontinuity at $T = T_c$!

Symmetry breaking:

Creation- & annihilation ops.

$$\frac{1}{|\Lambda|^2}\int d^3x\, d^3y\, \left\langle a^*(x)\, a(y)\right\rangle^{(\Lambda)}_{\beta,\mu=0}$$

$$\underset{L\to\infty}{\longrightarrow}\ \rho-\rho_c>0\,,\quad \text{for } T<T_c$$

(at fixed $\rho>0$). Thus, LRO!

$\Rightarrow$ For symmetry breaking b.c.,

$$\left\langle a^{\#}(x)\right\rangle_{\beta,\text{b.c.}} \neq 0 \qquad (22)\ (1.77)$$

$\Rightarrow$ $\boxed{\text{Symmetry } a^{\#}(x)\longrightarrow e^{\pm i\theta}a^{\#}(x),\ \theta\in[0,2\pi) \text{ spontaneously broken!}}$

Bose gases in 1 and 2 dimensions

The function $\rho(\vec{k}) := \lim_{L\to\infty} \rho_L(\vec{k}) = \frac{\langle n_{\vec{k}} \rangle_{\beta,\mu}}{L^3}$

$$= \frac{2S+1}{e^{\beta \hbar^2 \vec{k}^2/2m} - 1} \qquad (\vec{k} \neq 0)$$

has a singularity at $\vec{k} = 0$ that, in $d = 1, 2$, is non-integrable.

$$\rho(\beta,\mu) \propto T \zeta_1(z) \underset{z \nearrow 1}{\nearrow} \infty \qquad \begin{matrix}(23)\\(1.78)\end{matrix}$$

Eq. (23) has soln. $z = z(\rho) < 1$, for arb. large ρ;

$\Rightarrow \rho - \rho_c = 0$, $\forall \rho$, i.e., no BEC, and

$$\rho(\vec{k}) = \frac{2S+1}{z(\rho)^{-1} e^{\beta \hbar^2 \vec{k}^2/2m} - 1}, \quad \forall \vec{k},$$

with $z(\rho) < 1$, i.e., $\rho(0) < \infty$!

Aus: "Aspekte der frühen Quantenmechanik", Diplomarbeit von Stefan Schneider, ETHZ, Sept. 2007.

~~2~~ Die Entdeckung der Quantenmechanik – Heisenbergs Matrizenmechanik

In den Jahren 1923-1925 zeichnete sich immer mehr ab, dass eine Änderung der bisherigen Methoden in der Behandlung der Atomstruktur und der Spektrallinien unabdingbar war. Vor allem in den beiden Zentren Göttingen (Born) und Kopenhagen (Bohr) machte sich langsam Unbehagen über die fehlerhafte Beschreibung von Mehrkörpersystemen innerhalb der alten Quantentheorie breit.

Der Ausweg aus dieser misslichen Lage wurde von WERNER HEISENBERG, in seiner Arbeit *Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen* [Hei25] gefunden. Heisenberg war 23 Jahre alt, als er die Arbeit schrieb.

1.4 ~~2.1~~ Heisenbergs Umdeutung

In der Einleitung seiner Arbeit drückt Heisenberg die konzeptuellen Probleme der alten Quantentheorie folgendermassen aus:

> Bekanntlich läßt sich gegen die formalen Regeln, die allgemein in der Quantentheorie zur Berechnung beobachtbarer Größen (z.B. der Energie im Wasserstoffatom) benutzt werden, der schwerwiegende Einwand erheben, daß jene Rechenregeln als wesentlichen Bestandteil Beziehungen enthalten zwischen Größen, die scheinbar prinzipiell nicht beobachtet werden können (wie z.B. Ort, Umlaufszeit des Elektrons), daß also jenen Regeln offenbar jedes anschauliche physikalische Fundament mangelt, wenn man nicht immer noch an der Hoffnung festhalten will, daß jene bis jetzt unbeobachtbare Größen später vielleicht experimentell zugänglich gemacht werden könnten.

Weiter spricht er die Bereiche an, in denen die alte Quantentheorie versagt (anomaler Zeeman-Effekt, Unmöglichkeit der Behandlung von Atomen mit mehreren Elektronen). Die Notwendigkeit einer neuen Quantentheorie – der Quantenmechanik – welche nicht mehr auf der alten Idee der quantenmechanisch erlaubten klassischen Bahnen basiert, wird deutlich:

> [Es scheint] geratener, jene Hoffnung auf eine Beobachtung der bisher unbeobachtbaren Größen (wie Lage, Umlaufszeit des Elektrons) ganz aufzugeben, gleichzeitig also einzuräumen, daß die teilweise Übereinstimmung der genannten Quantenregeln mit der Erfahrung mehr oder weniger zufällig sei, und zu versuchen, eine der klassischen Mechanik analoge quantentheoretische Mechanik auszubilden, in welcher nur Beziehungen zwischen beobachtbaren Größen vorkommen..

Heisenberg liess sich in der Entwicklung seiner Theorie von folgenden Prinzipien leiten:

1. Es muss eine quantentheoretische Mechanik gefunden werden, in welcher nur Beziehungen zwischen beobachtbaren Grössen vorkommen. Dies wurde schon von Born in [Bor24] in ähnlicher Weise verlangt.

Puzzling features of quantum mechanics originate in its combination with atomism.

1.4. Heisenberg's discovery of matrix mechanics.

Heisenberg used the following input:

(i) Rydberg – Ritz combination principle; (1905/08)

(ii) "Quantization" of integrable Hamiltonian systems, according to the rules of Bohr – Sommerfeld – Epstein (Schwarzschild) – Einstein; (1913 – 1917)

(iii) Thomas – Kuhn sum rule.

Heisenberg's paper bears the title:

"Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen", Zeitschrift für Physik 33, 879 – 893 (1925)

Spectroscopy, the interactions between the e.m. field and matter, plays a fundamental rôle.

That is the case for (i):

(i) Rydberg-Ritz combination principle

Already known was Balmer's formula for the emission spectrum of hydrogen, (and extensions due to Lyman, Paschen, Brackett, and others; see J. Balmer, Ann. Physik 25, 80 (1885)):

$$\omega_{n_i n_f} = \underbrace{Z^2 R}_{\substack{\uparrow\\ \text{Rydberg const.}}}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right), \quad n_f < n_i \qquad (1.79)$$

Z : atom number (Ze: charge of nucleus/ion)

(Rydberg and) Ritz have generalized this formula as follows:

For every species of atoms, there exists a series of energies, $\{E_n\}_{n=1}^{\infty}$ ("Termschema", or "Grotrian diagram") such that the circular frequencies of spectral lines are given by

$$\omega \equiv \omega_{nm} = \frac{1}{\hbar}(E_n - E_m) \qquad (1.80)$$

for a photon emission accompanying a transition from state $|n\rangle$ to state $|m\rangle$, with $E_n > E_m$.

Certain pairs (n, m) are "forbidden", i.e., corresp. ω_{nm} not observed ("selection rules").

If ω_{nm} and ω_{mk}, as given by (1.80), are observed then

$$\omega_{nk} = \omega_{nm} + \omega_{mk} \qquad (1.81)$$

is observed, too.

It is found experimentally that the "spectral terms" (energies), E_n, of a fixed species of atoms depend on external magnetic and electric fields (Zeeman- and Stark effect).

Within classical mechanics and electrodynamics, the Rydberg-Ritz combination principle cannot be explained. (A mathematically rigorous explanation had to wait $\approx$ 100 years: Bach-JF-Pizzo-Sigal.) We note, however, that, for $E_n \to \infty$, with $|E_n - E_m|$ bounded, the predictions of the Rydberg-Ritz principle agree with classical radiation theory ("correspondence principle" of Bohr).

(ii) <u>"Quantization" of integrable Hamiltonian systems.</u>

Consider a mechanical system, S, with phase space Γ. Locally one can always introduce
$\nwarrow$ Ex.: T^*M, $\mathbb{R}^{2f}$, S^2, ...
Darboux coordinates, $q_1, \dots, q_f, p_1, \dots, p_f$, with

$$\left.\begin{aligned} &\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \forall i,j, \\ \text{and} \\ &\{q_i, p_j\} = -\delta_{ij}, \quad \forall i,j, \end{aligned}\right\} \quad (1.82)$$

$\{\cdot,\cdot\}$: Poisson bracket on $C^\infty(\Gamma)$ (= alg. of "observables")

S is said to be integrable if $\exists$ f functions, $G_1, \dots, G_f$ in $C^\infty(\Gamma)$ in <u>involution</u>, i.e., satisfying

$$\{G_i, G_j\} = 0, \quad \forall i,j, \qquad (1.83)$$

$\{dG_1, \dots, dG_f\}$ <u>linearly</u> <u>indep.</u> everywhere on Γ, which are <u>constants of motion</u>, i.e.,

$$\{H, G_i\} = 0, \quad i = 1, \dots, f, \qquad (1.84)$$

where H is the Hamiltonian of S.

A famous <u>theorem</u>, due to Liouville, Jacobi, Arnol'd and Jost, says that if S is <u>integrable</u>, with

$$\Gamma = \mathbb{R}^{2f} \qquad (1.85)$$

then Γ is foliated by f-<u>dimensional tori</u>, $\mathbb{T}^f = \mathbb{T}^f(\underline{W})$, that are invariant under the mechanical flow generated by H (and the flows generated by the G_i's). Every torus in the foliation

$$\{\mathbb{T}^f(\underline{W}) \mid \underline{W} = (W_1, \dots, W_f) \in \mathbb{R}^f\} \qquad (1.86)$$

can be parametrized by f <u>angle variables</u>

$$\underline{\varphi} = (\varphi_1, \dots, \varphi_f) \qquad (1.87)$$

where $\varphi_i \in [0, 2\pi]$ parametrizes the i^{th} non-contractible cycle (circle) of $\mathbb{T}^f(\underline{W})$.

Since the functions G_i are constants on every $\mathbb{T}^f(\underline{W})$, with $\{dG_1, \dots, dG_f\}$ lin. indep., the parameters $\underline{W}$ are functions of $\underline{G} = \{G_1, \dots, G_f\}$, and conversely. Actually, one can choose the functions $\underline{W}$ in such a way that

$$\{W_i, W_j\} = 0 \ , \ \{W_i, \varphi_j\} = \delta_{ij} \ , \qquad (1.88)$$

with $\{\varphi_i, \varphi_j\} = 0$, $\forall\, i, j$.

The variables $W_1, \dots, W_f$ are called <u>action variables</u>. The Hamilton function H of S is a function of the $\underline{W}$'s alone. The mechanical flow on $\mathbb{T}^f(\underline{W})$ is typically <u>not</u> periodic, but <u>quasi-periodic</u>, which is a <u>problem</u> for Bohr-Sommerfeld quantization.

Einstein had, to say the least, an intuitive understanding of the above. He proposed the following generalization of Bohr-Sommerfeld-Epstein:

Quantum-mechanically, the "allowed orbits" of S lie on tori $\mathbb{T}^f(\underline{W})$ for which

$$W_k = \nu_k h \,, \quad \nu_k \in \mathbb{Z} \,, \qquad (1.89)$$

(or $\nu_k \in \mathbb{Z}_+$ if $W_k \geq 0$), for all $k = 1, \dots, f$.

The corresponding "energy levels" are given by

$$E_{\underline{\nu}} = H(\underline{W} = \underline{\nu} h) \,, \qquad (1.90)$$

$\underline{\nu} = (\nu_1, \dots, \nu_f) \in \mathbb{Z}^f$.

Einstein points out that, according to Poincaré, the 3-body Coulomb problem is not integrable; which explains why already the helium atom could not be understood using these ideas.

Chronology of the year 1925.

January: Pauli discovers his exclusion principle and "electron spin"; (application to anomalous Zeeman effect, periodic table of elements).

Spring: W. Thomas, F. Reiche and W. Kuhn discover a sum rule for the intensity of spectral lines (to be explained below).

July: W. Heisenberg discovers the essence of matrix mechanics.

September: Born and Jordan find the proper mathematical formulation of Heisenberg's discovery.

November: Independently of Born and Jordan, Dirac develops the "definitive" form of quantum mechanics from Heisenberg's ideas; including "transformation theory"

Later in November: Born, Heisenberg and Jordan, indep. of Dirac, present the "definitive" form of quantum mechanics ("Dreimännerarbeit").

1926: Pauli determines hydrogen spectrum, using matrix mechanics.

1926: E. Schrödinger: "Wave mechanics"

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1926: Pauli determines hydrogen spectrum, using matrix mechanics.

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Interpretation of Heisenberg's fundamental ideas

We start by considering an integrable Hamiltonian system, S, with f degrees of freedom. We use angle- and action variables $(\underline{\varphi}, \underline{W})$, as introduced above.

Let x be an "observable" of S, i.e., a real-valued smooth function on $\Gamma = \mathbb{R}^{2f}$. In the following we imagine that S consists of electrically charged particles, and x is a component of the total electric dipole moment of S.

The function $x = x(\underline{\varphi}, \underline{W})$ is a periodic function of the angle variables $\underline{\varphi} = (\varphi_1, \dots, \varphi_f)$, hence can be expanded in a Fourier series

$$x(\underline{\varphi}, \underline{W}) = \sum_{\underline{n} \in \mathbb{Z}^f} \hat{x}_{\underline{n}}(\underline{W})\, e^{i \underline{n} \cdot \underline{\varphi}} \qquad (1.91)$$

The time evolution of x can be derived from the one of the angle variables:

$$\dot{\varphi}_j = \frac{\partial H(\underline{W})}{\partial W_j} =: \omega_j(\underline{W}) \left(= \text{const.}\right) \quad (1.92)$$

$$\dot{W}_j = -\frac{\partial H(\underline{W})}{\partial \varphi_j} = 0 \quad (1.93)$$

Solution: $\underline{W} = \text{const.}, \quad \underline{\varphi} = \underline{\varphi}_0 + \underline{\omega} \cdot t \quad (1.94)$

Thus:

$$x(t) = \sum_{\underline{n} \in \mathbb{Z}^f} \hat{x}_{\underline{n}}(\underline{W})\, e^{i(\varphi_0^{(\underline{n})} + \omega^{(\underline{n})} \cdot t)}, \quad (1.95)$$

where

$$\varphi_0^{(\underline{n})} = \underline{n} \cdot \underline{\varphi}_0, \quad \omega^{(\underline{n})} = \underline{n} \cdot \underline{\omega} = \underline{n} \cdot \underline{\nabla} H \quad (1.96)$$

$\Rightarrow$ $x(t)$ is a quasi-periodic function of time t with frequencies, $\omega^{(\underline{n})}$, as in (1.96).

If x is a component of the electric dipole moment of S, then Maxwell's equations imply that e.m. radiation with frequencies $\omega^{(\underline{n})}$, $\underline{n} \in \mathbb{Z}^f$, is emitted by S, provided the recoil of the e.m. field on the mechanical motion can be neglected. (This was first shown theoretically and experimentally by H. Hertz.)

For every fixed $\underline{W}$, the frequencies $\omega^{(\underline{n})} = \omega^{(\underline{n})}(\underline{W})$ define a representation of the abelian group $\mathbb{Z}^f$:

$$\omega^{(\underline{n})} + \omega^{(\underline{m})} = \omega^{(\underline{n}+\underline{m})}, \quad \underline{n}, \underline{m} \in \mathbb{Z}^f.$$

→ With a frequency, ω, all its harmonics, $n\omega$, $n \in \mathbb{Z}$, are emitted, too!

This contradicts the Rydberg-Ritz combination principle. For, the frequencies of light emitted by a physical system represent a groupoid, $\mathcal{G}$, rather than an (abelian) group! $\mathcal{G}$ consists of pairs, $(\underline{\mu}, \underline{\nu})$, of so-called "quantum numbers", $\underline{\mu}$, labelling "stationary states" of S. If $(\underline{\mu}, \underline{\nu}) \in \mathcal{G}$ and $(\underline{\nu}, \underline{\lambda}) \in \mathcal{G}$ then

$$(\underline{\mu}, \underline{\nu}) \circ (\underline{\nu}, \underline{\lambda}) = (\underline{\mu}, \underline{\lambda}) \in \mathcal{G} \qquad (1.97)$$

(The composition $(\underline{\mu}, \underline{\nu}) \circ (\underline{\kappa}, \underline{\lambda})$ is defined only if $\underline{\nu} = \underline{\kappa}$ and is then given by (1.97).)

If $\omega_{\underline{\mu}\underline{\nu}}$ is the frequency of the photon emitted by S when a transition from state $\underline{\mu}$ to state $\underline{\nu}$ occurs then, according to Rydberg and Ritz,

$$\omega_{\underline{\mu}\underline{\lambda}} = \omega_{\underline{\mu}\underline{\nu}} + \omega_{\underline{\nu}\underline{\lambda}}, \qquad (1.98)$$

for $(\underline{\mu}, \underline{\nu}), (\underline{\nu}, \underline{\lambda})$ in G, with

$$\omega_{\underline{\mu}\underline{\nu}} = \frac{1}{\hbar}(E_{\underline{\mu}} - E_{\underline{\nu}}) \qquad (1.99)$$

Bohr, Sommerfeld, Einstein, ... interpret $E_{\underline{\mu}}$ as the <u>energy</u> of state $\underline{\mu}$ and (1.99) as an expression of <u>energy</u> <u>conservation</u>.

If, <u>classically</u>, the system S is integrable then one may suspect that

$$E_{\underline{\nu}} = H(\underline{W}_{\underline{\nu}}), \text{ where } \underline{W}_{\underline{\nu}} = \underline{\nu} \cdot h, \ \underline{\nu} \in \mathbb{Z}^f, \qquad (1.100)$$

see (1.89).

The ansatz (1.98)–(1.100) yields the correct "<u>Grotrian diagram</u>" (Termschema) for the spectral lines of hydrogen and of harmonic oscillators.

Heisenberg's ansatz of July 1925:
Quantum-mechanically, all "observables", such as x, only depend on pairs $(\underline{\mu}, \underline{\nu}) \in \mathcal{G}$ of allowed transitions between stationary states ("allowed trajectories") of S. The Fourier coefficients, $\hat{x}_{\underline{n}}(\underline{W} = \underline{\mu}h)$, must be replaced by what Heisenberg called "schemes"

$$\hat{x} = (x_{\underline{\mu}\underline{\nu}})_{(\underline{\mu},\underline{\nu}) \in \mathcal{G}}, \left(\mathcal{G} = \mathbb{Z}^f \times \mathbb{Z}^f\right), \quad (1.101)$$

subsequently recognized by Born and Jordan as (infinite) matrices.

Bohr's "correspondence principle" then suggests that

$$x_{\underline{\mu}\,\underline{\mu}+\underline{n}} \simeq \hat{x}_{\underline{n}}(\underline{W} = \underline{\mu}h), \quad (1.102)$$

provided $0 < |\underline{n}| \ll |\underline{\mu}|$. Since, classically, x is real, we have that

$$\hat{x}_{-\underline{n}}(\underline{W}) = \hat{x}_{\underline{n}}(\underline{W})^*,$$

where * denotes complex conjugation. With (1.102), this suggests that

$$x^*_{\underline{\mu}\,\underline{\nu}} = x_{\underline{\nu}\,\underline{\mu}} \qquad (1.103)$$

(hermiticity).

If $x^{(1)}$ and $x^{(2)}$ are two classical observables of S then $x^{(1)} \cdot x^{(2)}$ is an observable, too, and

$$\widehat{(x^{(1)} \cdot x^{(2)})}_{\underline{n}}(\underline{W}) = \sum_{\underline{m} \in \mathbb{Z}^f} \hat{x}^{(1)}_{\underline{n}-\underline{m}}(\underline{W})\, \hat{x}^{(2)}_{\underline{m}}(\underline{W})$$

Heisenberg proposes to replace this convolution product on the group $\mathbb{Z}^f$ by

$$(x^{(1)} * x^{(2)})_{\underline{\mu}\,\underline{\nu}} = \sum_{\underline{\lambda}} x^{(1)}_{\underline{\mu}\,\underline{\lambda}} \cdot x^{(2)}_{\underline{\lambda}\,\underline{\nu}}, \qquad (1.104)$$

which is the <u>matrix product</u> of $x^{(1)}$ and $x^{(2)}$. Setting $\underline{\lambda} = \underline{\mu} + \underline{n}$ and cutting off the sum over $\underline{n}$ in (1.104) in such a way that $|\underline{n}| \ll |\underline{\mu}|$, $|\underline{\mu} + \underline{n} - \underline{\nu}| \ll |\underline{\mu}|$, and using (1.102), we see that (1.104) reproduces the convolution product in this limiting regime.

Combining the Rydberg - Ritz combination principle (1.98) with the frequency condition (1.99) and Hertz' theory of dipole radiation, one arrives at the idea that if x is a component of the electric dipole moment of S then

$$x_{\underline{\mu}\underline{\nu}}(t) = x_{\underline{\mu}\underline{\nu}}\, e^{i\omega_{\underline{\mu}\underline{\nu}} t}$$

$$= e^{iE_{\underline{\mu}} t/\hbar}\, x_{\underline{\mu}\underline{\nu}}\, e^{-iE_{\underline{\nu}} t/\hbar} \qquad (1.105)$$

If, quantum mechanically, the Hamilton function H of S is re-interpreted as a "diagonal scheme" (diagonal matrix), $(H_{\underline{\mu}\underline{\nu}})$, given according to (1.100) by

$$H_{\underline{\mu}\underline{\nu}} = E_{\underline{\mu}}\, \delta_{\underline{\mu}\underline{\nu}} \qquad (1.106)$$

one finds that

$$x_{\mu\nu}(t) = \left[e^{iHt/\hbar} * x * e^{-iHt/\hbar} \right]_{\mu\nu}, \quad (1.107)$$

which is now postulated for general observables of S.

Heisenberg now proposes to ban the Bohr-Sommerfeld picture of "allowed (mechanical) orbits", as expressed in (1.89), (1.90), (1.100), because the classical orbits are not observable (following the philosophy of the young Einstein). He finds a replacement for the Bohr-Sommerfeld conditions in the sum rule of Thomas-Reiche and Kuhn.

Let us consider an atom with N electrons of charge $-e$. Then

$x = -Ne$ x component of center of mass position of electron configuration

We set

$$X := -\frac{x}{Ne}, \qquad (1.108)$$

M = total mass of N electrons,

$$P := M\dot{X}, \qquad (1.109)$$

which is the center of mass momentum canonically conjugate to X. The Thomas-Reiche-Kuhn sum rule amounts to

$$[X,P]_{\underline{\mu}\underline{\mu}} := (X * P)_{\underline{\mu}\mu} - (P * X)_{\mu\underline{\mu}}$$

$$\overset{!}{=} i\hbar \qquad (1.110)$$

We propose to present heuristic reasoning for the validity of (1.110): Using (1.103) (hermiticity), the matrix product (1.104) and formula (1.105) for the time dependence of $x_{\underline{\mu}\underline{\nu}}(t)$, we find, after replacing P in (1.110) by $M\dot{X}$, that

$$[X,P]_{\underline{\mu}\underline{\mu}} = (X * M\dot{X})_{\underline{\mu}\underline{\mu}} - (M\dot{X} * X)_{\underline{\mu}\underline{\mu}}$$

$$= 2iM \sum_{\underline{\nu}} |X_{\underline{\mu}\underline{\nu}}|^2 \omega_{\underline{\nu}\underline{\mu}} \qquad (1.111)$$

In the region of quantum numbers where

$$0 < |\underline{\mu} - \underline{\nu}| \ll |\underline{\mu}|$$

we can use Bohr's correspondence principle:

$$\left.\begin{aligned} & X_{\underline{\mu}\,\underline{\nu}} \simeq \hat{X}_{\underline{\mu}-\underline{\nu}}(\underline{W} = h\underline{\mu}), \quad \omega_{\underline{\nu}\,\underline{\mu}} \simeq (\underline{\nu} - \underline{\mu}) \cdot \underline{\omega}^{(\underline{\mu})}, \\ & \text{where} \\ & \qquad \underline{\omega}^{(\underline{\mu})} = (\underline{\nabla} H)(\underline{W} = h\underline{\mu}). \end{aligned}\right\} \quad (1.112)$$

If these equations were <u>exact</u> then

$$[X, P]_{\underline{\mu}\,\underline{\mu}} = 0,$$

as was to be expected. For, if (1.112) were exact then

$$[X, P]_{\underline{\mu}\,\underline{\mu}} = \left(\widehat{X \cdot M\dot{X}}\right)_{\underline{n}=0} - \left(\widehat{M\dot{X} \cdot X}\right)_{\underline{n}=0},$$

where "$\cdot$" is the ordinary product of functions on phase space, which is commutative. The Thomas-Reiche-Kuhn sum rule (1.111) captures the deviation of the matrix product "$*$" from a commutative product.

We now calculate the R.S. in (1.111) heuristically for the hydrogen atom (or for the harmonic oscillator considered by Heisenberg).

The Grotrian diagrams (Termschema) of hydrogen and of the harmonic oscillator were known, including selection rules. This can be used to determine corrections to the formula

$$\omega_{\nu\mu} = (\nu - \mu) \cdot \omega^{(\mu)}$$

If, for hydrogen, we only take into account circular orbits and the principal quantum number, n, with angular momentum $L = \hbar n$, (or if n denotes the "occupation number" of a harmonic oscillator: $\oint p\,dq = hn$), and if we denote the corresponding mechanical orbit by n then we have that

$$X_{n\,n+1} = \hat{X}_1(n+1) = \xi_{n+1}$$

$$X_{n\,n-1} \underset{(1.103)}{=} X^*_{n-1\,n} = \underbrace{X_1(n)}_{\text{real}} = \xi_n \qquad (1.113)$$

where ξ_n is the radius of the Bohr orbit with principal quantum number n. According to Bohr's

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where ξ_n is the radius of the Bohr orbit with principal quantum number n. According to Bohr's model of a hydrogen-like ion with a nucleus of charge Ze

$$|\xi_n|^2 \simeq \frac{1}{2}\frac{a_n^2}{Z^2}, \quad a_n = n^2 a_1, \quad a_1 = \frac{\hbar^2}{e^2 M}, \qquad (1.114)$$

M = mass of electron, a_1 = Bohr radius of hydrogen atom. Furthermore,

$$\omega_{n\,n+k} = \frac{Z^2}{\hbar} \mathrm{Ryd}\left[\frac{1}{n^2} - \frac{1}{(n+k)^2}\right]$$
$$= \frac{2Z^2}{\hbar} \mathrm{Ryd}\frac{k}{n^3} - 3\frac{Z^2}{\hbar}\mathrm{Ryd}\frac{k^2}{n^4} + \cdots, \qquad (1.115)$$

with $k = \pm 1$, and

$$\mathrm{Ryd} = \frac{M}{2}\frac{e^4}{\hbar^2} \qquad (1.116)$$

We now plug (1.113) – (1.116) into (1.111), with $\mu \mapsto n$, $\nu \mapsto n \pm 1$ ($k = \pm 1$!), and find that

$$[X, P]_{nn} = [X, M\dot{X}]_{nn} = 2iM\left\{|X_{n\,n+1}|^2 \omega_{n+1\,n} + |X_{n\,n-1}|^2 \omega_{n-1\,n}\right\}$$

$$\overset{(1.115)}{=} 2iM\Omega\left\{\xi_{n+1}^2 \frac{1}{n^3} - \frac{3}{2}\xi_{n+1}^2\frac{1}{n^4} + \xi_n^2\left(-\frac{1}{n^3}\right) - \frac{3}{2}\xi_n^2\frac{1}{n^4}\right\}, \qquad (1.117)$$

where

$$\Omega = \frac{2Z^2}{\hbar} Ryd = \frac{Z^2}{\hbar^3} M e^4$$

Next,

$$\xi_{n+1}^2 - \xi_n^2 \overset{(1.114)}{\downarrow}{=} \frac{\hbar^4}{2e^4 M^2 Z^2}\left[(n+1)^4 - n^4\right]$$

$$= 2\frac{\hbar^4}{e^4 M^2 Z^2} n^3 + L.O. \qquad (1.118)$$

With (1.117) it follows that

$$[X, P]_{nn} = iM \frac{Z^2}{\hbar^3} M e^4 \frac{\hbar^4}{e^4 M^2 Z^2}\{4-3\}$$

$$= i\hbar, \qquad (1.119)$$

which is (1.110)!

Next, we treat the harmonic oscillator.

For a harmonic oscillator with frequency ω and mass M we have

$$x(t) = \xi \cos(\omega t + \theta), \text{ for some } \theta \in [0, 2\pi), \quad (1.120)$$

hence

$$\dot{x}(t) = -\omega \xi \sin(\omega t + \theta);$$

the energy of this orbit is therefore

$$E(\xi) = \frac{M}{2}\dot{x}(t)^2 + \frac{M\omega^2}{2} x(t)^2 = \frac{M\omega^2}{2}\xi^2, \quad (1.121)$$

with $f = M\omega^2$ the "spring constant". The energies of stationary states ("terms") of a harmonic oscillator of frequency ω are, according to Planck,

$$E(\xi_n) \equiv E_n = \hbar\omega \cdot n, \qquad (1.122)$$

$n = 0, 1, 2, \cdots$; (zero-point energy neglected).

Thus,

$$\xi_n = \sqrt{\frac{2\hbar n}{M\omega}} \qquad (1.123)$$

Next,

$$\left.\begin{aligned} x_{n\,n+1} &\simeq \hat{x}_1(\xi_n) = \frac{1}{2}\xi_n , \\ x_{n\,n-1} &\simeq \hat{x}_{-1}(\xi_{n-1}) = \frac{1}{2}\xi_{n-1} , \end{aligned}\right\} \qquad (1.124)$$

and

$$\left.\begin{aligned} \omega_{n+1\,n} &= \omega(n+1-n) = \omega = \\ &= \omega_{n\,n-1} = -\omega_{n-1\,n} \end{aligned}\right\} \qquad (1.125)$$

Thus

$$[x, p]_{nn} = [x, M\dot{x}]_{nn}$$

$$= 2iM\left\{ |x_{n\,n+1}|^2 \omega_{n+1\,n} + |x_{n\,n-1}|^2 \omega_{n-1\,n} \right\}$$

(1.125), (1.124)

$$\overset{\downarrow}{=} 2iM\omega \frac{1}{4}\left\{\xi_n^2 - \xi_{n-1}^2\right\}$$

$$= \frac{iM\omega}{2}\,\frac{2\hbar}{M\omega}\left\{n-(n-1)\right\}$$

$$= i\hbar, \qquad (1.126)$$

which is (1.110)!

In September 1925, Born and Jordan interpret Heisenberg's "schemes", $(x_{\mu\nu})$, as infinite matrices and develop the mathematical foundations of matrix mechanics: Hermitian matrices are associated with real-valued functions on classical phase space. Born and Jordan argue that, for Hamilton functions, H, of the form

$$H(P,X) = \frac{P^2}{2M} + V(X), \qquad (1.127)$$

the quantum-mechanical equations of motion of matrices (see (1.107)) must have the same form as the classical equations of motion of the corresponding functions on phase space. Thus,

$$\dot{X} \overset{(1.107)}{=} \frac{i}{\hbar}[H, X] \overset{!}{=} \frac{P}{M},$$

and

$$\dot{P} \overset{(1.107)}{=} \frac{i}{\hbar}[H, P] \overset{!}{=} -\frac{\partial V}{\partial X}(X),$$

for H as in (1.127), (with X and P now interpreted as infinite matrices). It follows that

$$\frac{i}{\hbar}[H,[X,P]] = [\dot{X}, P] + [X, \dot{P}]$$
$$= \left[\frac{P}{M}, P\right] - \left[X, \frac{\partial V}{\partial X}(X)\right] = 0.$$

But

$$\frac{i}{\hbar}[H,[X,P]]_{\underline{\mu}\underline{\nu}} = i(\omega_{\underline{\mu}} - \omega_{\underline{\nu}})[X,P]_{\underline{\mu}\underline{\nu}}$$
$$= i\,\omega_{\underline{\mu}\underline{\nu}}\,[X,P]_{\underline{\mu}\underline{\nu}}$$

Assuming that $\omega_{\underline{\mu}} \neq \omega_{\underline{\nu}}$, for $\underline{\mu} \neq \underline{\nu}$, it follows that

$$[X,P]_{\underline{\mu}\underline{\nu}} = 0, \quad \text{for } \underline{\mu} \neq \underline{\nu}. \qquad (1.128)$$

Together with (1.110), it follows that

$$\boxed{[X,P] = i\hbar\,\mathbb{1},} \qquad (1.129)$$

which are the <u>canonical commutation relations</u>.

<u>October 1925:</u> Goudsmit and Uhlenbeck interpret Pauli's binary quantum number ($s = \pm\frac{\hbar}{2}$) as "electron spin" and propose a linear relation between the magnetic moment of the electron and its spin, (which gave rise to some confusion eventually resolved by Dirac: Dirac equation $\rightarrow g = 2$!)

<u>Beginning of November 1925:</u> Independently of Born and Jordan, <u>Dirac</u> develops the definitive form of non-relativistic quantum mechanics, starting from Heisenberg's ansatz:

C		QM
Symplectic geometry of phase spaces	$\rightarrow$	"Quantum geometry" (Born & Jordan)
real function on phase space	$\rightarrow$	hermitian operator (square matrix)
Poisson brackets $\{F, G\}$	$\rightarrow$	$\frac{i}{\hbar}[F, G]$ (1.30) (<u>ordering problems!</u>)
$\dot{F} = \{H, F\}$	$\rightarrow$	$\dot{F} = \frac{i}{\hbar}[H, F]$

Transformation theory:

Assigning an operator $\mathbb{F}$ to a function F on phase space is equivalent to assigning an operator $T\mathbb{F}T^{-1}$, $\forall F$, with T indep. of F. If one insists that, with $\mathbb{F}$, $T\mathbb{F}T^{-1}$ must also be hermitian then

T must be unitary!

Middle of November 1925:

"Dreimännerarbeit" (Born-Heisenberg-Jordan)

They introduce Hilbert space, discover continuous spectra, develop q.m. perturbation theory and the quantum theory of angular momentum based on the commutation relations

$$[L_x, L_y] = \frac{\hbar}{i} L_z + \text{cyclic} \qquad (1.131)$$

$\hookrightarrow$ representation theory of $SU(2) = \widetilde{SO(3)}$!

They quantize the vibrating string (∞ many degrees of freedom, ..., Planck's formula).

1.5. Dirac's discovery of the path integral

"The Lagrangian in Quantum Mechanics" (Nov. 1932)

Dirac recalls that QM has been built on foundations of an analogy with Hamiltonian mechanics, employing canonically conjugate variables, such as $\underset{\sim}{q} = (q_1, \dots, q_f)$ and $\underset{\sim}{p} = (p_1, \dots, p_f)$ (Darboux coordinates in Γ), and replacing Poisson brackets, $\{\cdot,\cdot\}$, by $\frac{i}{\hbar}[\cdot,\cdot]$.

Question: Could QM be built on foundations of an analogy with Lagrangian mechanics?

$$(\underset{\sim}{q}, \underset{\sim}{p}) \longmapsto (\underset{\sim}{q}, \dot{\underset{\sim}{q}}) \qquad (1.132)$$

Hamiltonian mech.	Lagrangian mech.
Hamiltonian, H, generates time evolution after singling out a particular time variable → "non-relativistic" formalism.	Eqs. of motion = stationary points of an action functional, which is a "relativistic invariant" → Easy generalization to RELATIVISTIC theories → basis for QFT.

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What corresponds in Quantum Theory to the Lagrangian method of classical mechanics?

Dirac proposes to take over to quantum theory the <u>Ideas</u>, rather than the <u>equations</u>, of classical Lagrangian theory.

(i) <u>Lagrangian theory and contact transformations.</u>

Is there something like "<u>quantum contact transformations</u>"?

Let $q := \underline{q} = (q_1, \dots, q_f)$, $p := \underline{p} = (p_1, \dots, p_f)$

Symplectic transformation on $\Gamma \, (= \mathbb{R}^{2f})$:

$$(q, p) \longmapsto (Q, P), \qquad (1.133)$$

leaving invariant symplectic 2-form, ω.

Let us suppose that (q, Q) are <u>independent</u> coordinates, (i.e., (dq, dQ) linearly independent). Then $\exists$ a <u>generating function</u>, $S(q, Q)$, for (1.133) on Γ, with:

$$p_k = \frac{\partial S}{\partial q_k}\ , \quad P_k = -\frac{\partial S}{\partial Q_k}\ , \quad k=1,\dots,f. \qquad (1.134)$$

In Quantum theory, we consider two "Schrödinger representations",

R_1 : $\hat{q}_1, \dots, \hat{q}_f$ diagonal,

R_2 : $\hat{Q}_1, \dots, \hat{Q}_f$ diagonal,
with $\hat{F}$ operator corresp. to function F on Γ.
If ψ is a state vector in the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^f, dq) \simeq L^2(\mathbb{R}^f, dQ)$$

then $\psi_1(q)$ is the wave function corresponding to ψ in rep. R_1, while $\psi_2(Q)$ corresponds to ψ in R_2. There is then a unitary transformation, T, intertwining the two equivalent representations R_1 and R_2:

$$\psi_1(q) = \int dQ\, T(q,Q)\, \psi_2(Q) \qquad (1.135)$$

T is the "quantum analogue" of $e^{iS(q,Q)/\hbar}$.

Let A be some "quantum observable", i.e., a self-adjoint operator on $\mathcal{H}$. We can represent A as an integral operator in two ways:

$$A \Longleftrightarrow A_1(q', q'') \qquad \text{in rep. } R_1$$

$$A \Longleftrightarrow A_2(Q', Q'') \qquad \text{in rep. } R_2.$$

We define

$$\left.\begin{aligned} A_{12}(q', Q') &:= \int dq'' \, A_1(q', q'') \, T(q'', Q') \\ &= \int dQ'' \, T(q', Q'') \, A_2(Q'', Q'). \end{aligned}\right\} \quad (1.136)$$

Then

$$\begin{aligned} \hat{q}_r(q', Q') &= \int dq'' \, \hat{q}_r(q', q'') \, T(q'', Q') \\ &= \int dq'' \, q'_r \, \delta(q' - q'') \, T(q'', Q') \\ &= q'_r \, T(q', Q'), \end{aligned} \qquad (1.137)$$

and

$$\begin{aligned} \hat{p}_r(q', Q') &= \int dq'' \, \hat{p}_r(q', q'') \, T(q'', Q') \\ &= \int dq'' \, \frac{\hbar}{i} \frac{\partial}{\partial q'_r} \delta(q' - q'') \, T(q'', Q') \\ &= -i\hbar \frac{\partial}{\partial q'_r} T(q', Q') \end{aligned} \qquad (1.138)$$

Using the second equation in (1.136), we also find that

$$\hat{Q}_r(q', Q') = Q'_r \, T(q', Q') \qquad (1.139)$$

and $$\hat{P}_r(q';Q') = i\hbar \frac{\partial}{\partial Q'_r} T(q';Q') \qquad (1.140)$$

$\uparrow$

change of sign from integration by parts.

It also follows from (1.136) and (1.139) that

$$[f(\hat{q})\cdot g(\hat{Q})](q',Q')$$

$$= \int dQ''\left(\int dq''\, f(\hat{q})(q',q'')\, T(q'',Q'')\right) g(\hat{Q})(Q'',Q')$$

$$= f(q')\, g(Q')\, T(q';Q'), \quad \text{etc.} \qquad (1.141)$$

If F is a "well-ordered" function of the operators $\hat{q}$ and $\hat{Q}$ of the form

$$F(\hat{q},\hat{Q}) = \sum_l f_l(\hat{q})\, g_l(\hat{Q}) \qquad (1.142)$$

then

$$F(\hat{q},\hat{Q})(q',Q') = F(q',Q')\, T(q',Q') \qquad (1.143)$$

as is easily checked.

If we write $T(q',Q') = e^{iU(q',Q')/\hbar}$ in Eqs. (1.138) — (1.143), we find that

$$\hat{p}_r(q',Q') \overset{(1.138)}{=} \frac{\partial U(q',Q')}{\partial q'_r}\, T(q',Q')$$

$$\overset{(1.143)}{=} \left(\frac{\partial U}{\partial q_r}\right)(\hat{q},\hat{Q})(q',Q'),$$

i.e.,

$$\left.\begin{aligned}\hat{p}_r &= \left(\frac{\partial U}{\partial q_r}\right)(\hat{q},\hat{Q})\\ &\text{and, likewise,}\\ \hat{P}_r &= -\left(\frac{\partial U}{\partial Q_r}\right)(\hat{q},\hat{Q})\end{aligned}\right\} \quad (1.144)$$

Thus, $U(\hat{q},\hat{Q})$ is the <u>quantum analogue</u> of the generating function, $S(q,Q)$, of (1.133).

(ii) <u>The Lagrangian and the action principle</u>

The classical flow map of time evolution, i.e.,

$$\phi_{t,T} : (q_T, p_T) \to (q_t, p_t), \quad (1.145)$$

where t and T are two times, and $\{(q_s, p_s) \mid s \in \mathbb{R}\}$ is a solution of the equations of motion, is a symplectic transformation on phase space Γ. If q_T and q_t are independent variables then $\exists$ generating function, $S(q_T, q_t)$ for (1.145) such that (1.134) holds, with $q := q_T$, $Q := q_t$.

A solution $\{q_s \mid T \leq s \leq t\}$ of the Euler-Lagrange equations of Lagrangian mechanics is uniquely determined by $q_{s=T} = q_T$ and $q_{s=t} = q_t$, (assuming that $t-T$, $|q_t - q_T|$ are "small"; note that $\mathbb{R}^f$ is contractible.)

Claim:

$$S(q, Q) = \int_T^t ds\, L(q_s, \dot{q}_s) \Big|_{q_T = q,\, q_t = Q}, \qquad (1.146)$$

where L is the Lagrangian of the mechanical system, and $\{q_s \mid T \leq s \leq t\}$ is a solution of the Euler-Lagrange equations

$$\frac{d}{ds} p_s = \frac{\partial L}{\partial q_s}, \quad \text{with } p_s := \frac{\partial L}{\partial \dot{q}_s}. \qquad (1.147)$$

Proof:

$$\frac{\partial S}{\partial q_t}(q \equiv q_t, Q \equiv q_T)$$

$$= \int_T^t ds \left[\underbrace{\frac{\partial L}{\partial q_s}}_{= \dot{p}_s, \text{ by } (1.147)} \cdot \frac{\partial q_s}{\partial q_t} + \underbrace{\frac{\partial L}{\partial \dot{q}_s}}_{= p_s, \text{ by def.}} \cdot \frac{\partial \dot{q}_s}{\partial q_t} \right]$$

$$\overset{(1.147)}{=} \int_T^t ds \left[\dot{p}_s \frac{\partial q_s}{\partial q_t} + p_s \frac{\partial \dot{q}_s}{\partial q_t} \right]$$

$$= \int_T^t ds \frac{d}{ds} \left[p_s \cdot \frac{\partial q_s}{\partial q_t} \right]$$

$$= p_t \cdot \underbrace{\frac{\partial q_t}{\partial q_t}}_{= \mathbb{1}} - p_T \cdot \underbrace{\frac{\partial q_T}{\partial q_t}}_{= 0}$$

$$= p_t \equiv p$$

i.e., $\frac{\partial S}{\partial q} = p$; likewise: $\frac{\partial S}{\partial Q} = -P \equiv -p_T$.

Thus,

$$T(q, Q) = e^{iU(q,Q)/\hbar} \leftrightarrow e^{i\int_T^t ds\, L(q_s, \dot{q}_s)/\hbar};$$

in particular,

$$T(q_{t+dt}, q_t) \simeq e^{iL(q_t, \dot{q}_t)\, dt/\hbar} \qquad (1.148)$$

We subdivide a general time interval $[T, t]$ into a large number of <u>very short</u> sub-intervals

$$[T,t] = [T, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_{m-1}, t_m] \cup [t_m, t]$$

in such a way that we can use (1.148) on every sub-interval. Let

$$\varphi(t,T) := \int_T^t ds\, L(q_s, \dot q_s)/\hbar .$$

Then

$$e^{iS(q_t,q_T)/\hbar} \overset{(1.146)}{=} e^{i\varphi(t,T)} =$$

$$= e^{i\varphi(t,t_m)} e^{i\varphi(t_m,t_{m-1})} \cdots e^{i\varphi(t_2,t_1)} e^{i\varphi(t_1,T)} \tag{1.149}$$

Quantum mechanically,

$$T(q_t, q_T) = \int T(q_t, q_{t_m}) T(q_{t_m}, q_{t_{m-1}}) \cdots T(q_{t_2}, q_{t_1}) \times$$

$$\times T(q_{t_1}, q_T) \prod_{k=1}^{m} dq_{t_k} \tag{1.150}$$

Using the "approximation" (1.148) in every small sub-interval, we find that

$$T(q_t, q_T) = \lim_{m\to\infty} \text{cst.} \int e^{iL(q_t,\dot q_t)[t-t_m]/\hbar} \times$$

$$\times e^{iL(q_{t_m},\dot q_{t_m})[t_m - t_{m-1}]/\hbar} \cdots e^{iL(q_{t_1},\dot q_{t_1})[t_1-T]/\hbar} \prod_{k=1}^{m} dq_{t_k}$$

$$= \int \mathcal{D}q\; e^{i\int_T^t ds\, L(q_s,\dot q_s)} , \tag{1.151}$$

where
$$\text{"}\mathcal{D}q := Z_{t,T} \prod_{s=T}^{t} dq_s\text{"}; \qquad (1.152)$$
$Z_{t,T}$: normalization const.

Dirac interprets $|T(q_t, q_T)|^2$ as the "transition probability" from state $|\hat{q} = q_T\rangle$ to state $|\hat{q} = q_t\rangle$ in time $t-T$. Thus

$$T(q_t = q, q_T = Q) = \left(e^{-iH(t-T)/\hbar}\right)(q, Q) \qquad (1.153)$$

With (1.151), one finds that

$$\boxed{\left(e^{-iHt/\hbar}\right)(q, Q) = \int_{\substack{q_0 = Q \\ q_t = q}} \mathcal{D}q \, e^{i\int_0^t ds \, L(q_s, \dot{q}_s)}} \qquad (1.154)$$

Dirac then sketches how to generalize this to relativistic QFT, proposing to start from a Lagrangian formulation of QFT and calculating transition amplitudes from one space-like surface to a future space-like surface.

(iii) Feynman's contribution to the path integral

Feynman attempts to justify (1.148) for systems with Lagrangian

$$L(q,\dot{q}) = \frac{m}{2}\dot{q}^2 - V(q) \qquad (1.155)$$

After canonical quantization, the Hamiltonian corresponding to (1.154) is given by

$$H = -\frac{\hbar^2}{2m}\Delta + V \qquad (1.156)$$

Next,

$$\left(e^{i\frac{\hbar^2}{2m}\Delta\cdot\tau/\hbar}\right)(q,Q)$$

$$= \int d^f p \; e^{-i\frac{p^2}{2}\cdot\frac{\hbar\tau}{m}} \; e^{ip\cdot(q-Q)}$$

$$\pi := p\sqrt{\frac{\hbar\tau}{m}} \Rightarrow d^f p = \left(\frac{m}{\hbar\tau}\right)^{f/2} d^f\pi$$

$$\Rightarrow \left(e^{i\frac{\hbar^2}{2m}\Delta\cdot\tau/\hbar}\right)(q,Q)$$

$$= \left(\frac{m}{\hbar\tau}\right)^{f/2} \underbrace{\int d^f\pi \; e^{-i\frac{\pi^2}{2}} \; e^{i\sqrt{\frac{m}{\hbar\tau}}\pi\cdot(q-Q)}}$$

$$= \left(\frac{2\pi m}{i\hbar\tau}\right)^{f/2} e^{+i\frac{m}{2}\frac{(q-Q)^2}{\tau}/\hbar} \qquad (1.157)$$

For $\tau > 0$ very small (i.e., $\overline{H}\tau \ll \hbar$),

$$e^{-iH\tau/\hbar} \simeq e^{\frac{i\hbar}{2m}\Delta\cdot\tau} e^{-iV\tau/\hbar},$$

so that

$$\left(e^{-iHdt/\hbar}\right)(q_{t+dt}, q_t)$$
$$\simeq \left(\frac{2\pi m}{i\hbar\, dt}\right)^{f/2} e^{i\frac{m}{2}\left(\frac{q_{t+dt}-q_t}{dt}\right)^2 dt/\hbar} \times$$
$$\times\, e^{-iV(q_t)dt/\hbar} \qquad (1.158)$$

Using that $\frac{q_{t+dt}-q_t}{dt} \simeq \dot{q}_t^{\,2}$ and inserting (1.158) into (1.150), using (1.153), we find again (1.151) after choosing the normalization constant $Z_{t,T}$ appropriately.

Of course, the arguments of Dirac and Feynman are <u>purely formal</u>, (as Dirac clearly realizes). $\rightarrow$ Cures!

<u>Cure 1:</u> Pass to imaginary time $t \to it$; then the path integral is converted into a

Wiener integral, and (1.154) becomes the so-called Feynman-Kac formula.

Cure 2: Use the path integral formally to derive an algebra of "local propagators", $T \equiv T(t, T')$, corresponding to time-dependent Lagrangians and encoding the canonical commutation relations; then do pure algebra. (See Buchholz-Fredenhagen! Their program is a rather straightforward "consolidation" of ideas of E.C.G. Stückelberg, combined with a property of quasi-invariance of the path-space measure under translations in path space, previously applied to Euclidian field theory by this lecturer.)

(iv) Dynamical C^*-algebras, à la Stückelberg and Buchholz – Fredenhagen

Consider the propagator

$$U(t,T;V)_{q,Q} := T(q_t = q, q_T = Q)$$

$$= \int_{\substack{q_t = q \\ q_T = Q}} \mathcal{D}q \; e^{i\int_T^t ds\, L_s(q_s, \dot{q}_s)}, \qquad (1.159)$$

where

$$L_s(q_s, \dot{q}_s) = \frac{m}{2}\dot{q}_s^2 - V_s(q_s), \qquad (1.160)$$

where $V_s(q)$ is a possibly time-dependent potential, and $q = (q_1, \dots, q_f) \in \mathbb{R}^f$.

We consider a translation in path space,

$$\mathcal{C}_{[T,t]} := \{ q_s \in C([T,t], \mathbb{R}^f), -\infty \le T < t \le \infty \} :$$

$$\left.\begin{array}{l} q_s \longmapsto q_s + q_s^{(0)} =: \tilde{q}_s, \\ \text{where} \\ \quad q_T^{(0)} = q_t^{(0)} = 0, \end{array}\right\} \qquad (1.161)$$

i.e., $q_s^{(0)}$ is a loop in $\mathbb{R}^f$ attached to the origin; space of such loops: $\mathcal{C}^{(0)}_{[T,t]}$. Formally, see (1.152),

$$\mathcal{D}q = \mathcal{D}\tilde{q}. \qquad (1.162)$$

By (1.161), (1.162), the R.S. of (1.159) does <u>not</u> change under the change of variables (1.161).

For L_s as in (1.160), we have that

$$L_s(q_s + q_s^{(0)}, \dot{q}_s + \dot{q}_s^{(0)}) = \frac{m}{2}\left(\dot{q}_s^2 + 2\dot{q}_s\dot{q}_s^{(0)} + \dot{q}^{(0)2}\right) - V_s(q_s + q_s^{(0)})$$

$$= \frac{m}{2}\dot{q}_s^2 + m\dot{q}_s \cdot \dot{q}_s^{(0)} + \frac{m}{2}\dot{q}_s^{(0)2} - V_s(q_s + q_s^{(0)})$$

$$= L_s(q_s, \dot{q}_s) + \delta L_s^{q_s^{(0)}}(q_s, \dot{q}_s), \qquad (1.163)$$

where

$$\int_T^t ds\, \delta L_s^{q_s^{(0)}}(q_s, \dot{q}_s) = \int_T^t ds \Big[\frac{m}{2}\dot{q}_s^{(0)2} - m\ddot{q}_s^{(0)} \cdot q_s + \underbrace{V_s(q_s) - V_s(q_s + q_s^{(0)})}\Big]$$

$$=: \int_T^t ds \left[\delta \mathring{\mathcal{L}}^{\dot{q}^{(0)}} - \delta V^{q^{(0)}}\right] \qquad (1.164)$$

The condition that the action

$$A(q, Q) = \int_T^t ds\, L_s(q_s, \dot{q}_s)\Big|_{q_T = Q}^{q_t = q} \qquad (1.165)$$

be <u>stationary</u> under the "variation" (1.161) implies the classical Euler–Lagrange eqs. of motion!

We set

$$V_s(q) := V(q) + W_s(q), \qquad (1.166)$$

where W_s has support in (T, t). For a fixed potential V (constant in time), we define

superfluous

$$S(W) := e^{itH_0/\hbar}\, U(t, T; V+W)\, e^{-iTH_0/\hbar}, \qquad (1.167)$$

with U as in (1.159). Using (1.159) and (1.162), we find that

$$\boxed{S(W) = S(W + \delta W^{q^{(0)}} - \delta \mathcal{L}^{q^{(0)}}),} \qquad (1.168)$$

where $\delta W_s^{q^{(0)}}(q) = W_s(q + q_s^{(0)}) - W_s(q)$. (1.169)

If $W = W^{(1)} + W^{(2)}$, with

$$\operatorname{supp} W^{(i)} = [s_-^{(i)}, s_+^{(i)}],\ i = 1, 2, \text{ and } s_-^{(1)} \geq s_+^{(2)}, \qquad (1.170)$$

i.e. "$W^{(1)}$ is in the future of $W^{(2)}$" $(W^{(1)} \succ W^{(2)})$, then

$$S(W^{(1)})\, S(W^{(2)}) = S(W^{(1)} + W^{(2)}). \qquad (1.171)$$

The operators $S(W)$ are unitary; thus

$$S(W)^{-1} := S(W)^* \qquad (1.172)$$

exists.

Choose an arbitrary $W^{(3)}$ with $\operatorname{supp} W^{(3)} \subseteq [T, t]$, and let $W^{(1)}, W^{(2)}$ be as above. We split

$$W^{(3)} = W_+^{(3)} + W_-^{(3)}$$

in such a way that $W^{(1)} + W_+^{(3)}$ is in the future of $W^{(2)} + W_-^{(3)}$, ... Then, by (1.171) (used 4 times)

$$S(W^{(1)} + W^{(3)})\, S(W^{(3)})^{-1}\, S(W^{(2)} + W^{(3)})$$

$$\overset{(1.171)}{=} S(W^{(1)} + W_+^{(3)})\, S(W_-^{(3)})\; S(W^{(3)})^{-1}\, S(W_+^{(3)})\, S(W^{(2)} + W_-^{(3)})$$

$$\overset{(1.171)}{=} S(W^{(1)} + W_+^{(3)})\, S(W_-^{(3)})\; S(W_-^{(3)})^{-1}\, S(W_+^{(3)})^{-1}\, S(W_+^{(3)})$$
$$S(W^{(2)} + W_-^{(3)})$$

$$= S(W^{(1)} + W_+^{(3)})\, S(W^{(2)} + W_-^{(3)})$$

$$\overset{(1.171)}{=} S(W^{(1)} + W^{(2)} + W^{(3)}), \quad \text{i.e.,}$$

$$\boxed{\begin{array}{l} S(W^{(1)} + W^{(3)})\, S(W^{(3)})^{-1}\, S(W^{(2)} + W^{(3)}) = S(W^{(1)} + W^{(2)} + W^{(3)}) \\ \text{if} \qquad W^{(1)} \succ W^{(2)}. \end{array}} \tag{1.173}$$

Identities (1.168) – (1.173) were first used by <u>E.C.G. Stückelberg</u> de Breidenbach in relativistic QFT in order to derive covariant perturbation theory.

<u>Definition.</u> We let $T=-\infty$, $t=\infty$, and we define $\mathcal{G}_L$ to be the <u>free group</u> generated by abstract operators $S(W)$, $W \in \mathcal{W} := \{W \mid \text{supp}\, W_s \text{ compact},$[1] $W_s \in C^\infty(\mathbb{R}^f), \forall s\}$, <u>modulo the relations</u>

$$\begin{cases} (\mathrm{I}) \quad S(W) = S\left(W + \delta W^{q^{(0)}} - \delta \mathcal{L}^{q^{(0)}}\right), \\ \qquad \text{for all } q^{(0)} \in \mathcal{C}^{(0)},\ W \in \mathcal{W}, \qquad (1.174) \\ (\mathrm{II}) \quad S(W^{(1)}+W^{(2)}+W^{(3)}) = S(W^{(1)}+W^{(3)})\, S(W^{(3)})^{-1} S(W^{(2)}+W^{(3)}), \end{cases}$$

$\forall\, W^{(1)}, W^{(2)}, W^{(3)}$ in $\mathcal{W}$, with $W^{(1)} \succ W^{(2)}$;

(and related identities for $S(W)^{-1}$).

(I) encodes <u>dynamical information</u>: Behaviour of $S(W)$ under <u>shifts</u> of trajectories q_s by <u>loops</u> $q_s^{(0)}$, given a Lagrangian L.

(II) encodes the causal properties of the theory.

We will see that $\mathcal{G}_L$ is a <u>non-abelian group</u>. It has a <u>center</u>: If $W_\hbar(q) =: \hbar = \text{const.}$, for all q, then $\text{supp}\, W_\hbar = \phi$; hence

[1] $\delta W^{q^{(0)}} = 0$, $\forall$ loops q^{0} with support in $(\text{supp}\, W_s)^c$

$$S(W)S(W_{t_\hbar}) = S(W+W_{t_\hbar}) = S(W_{t_\hbar})S(W),$$

$\forall\, W \in \mathcal{C}$; $\left(S(W_{t_\hbar}) = e^{i t_\hbar/\hbar}\mathbb{1}\right.$ in a factorial rep. of $\mathcal{G}_L$).

The group $\mathcal{G}_L$ gives rise to a group algebra

$$\mathcal{A}(\mathcal{G}_L) = \left\{ \sum z_j S(W_j) \,\middle|\, z_j \in \mathbb{C},\ W_j \in \mathcal{C},\ \forall j \right\} \qquad (1.175)$$

We define an unphysical, but faithful state, ω, on $\mathcal{A}(\mathcal{G}_L)$ by setting

$$\omega(\mathbb{1}) = 1, \quad \omega(g) = 0,\ \forall g \in \mathcal{G}_L \setminus \mathbb{T}^1, \qquad (1.176)$$

where $\quad \mathbb{T}^1 = \{ e^{i\theta}\mathbb{1} \mid 0 \leq \theta < 2\pi \}$.

Then $\quad \omega\left(\left(\sum_i z_i S(W_i)\right)^* \left(\sum_j z_j S(W_j)\right)\right)$

$$= \sum_{i,j} \bar{z}_i z_j\ \omega\left(S(W_i)^{-1} S(W_j)\right) = \sum_j |z_j|^2 > 0,$$

assuming that $S(W_i) \neq e^{i\theta} S(W_j)$; $(z_1, z_2, \dots) \neq 0$.

For $A \in \mathcal{A}(\mathcal{G}_L)$, we define

$$\|A\| = \sqrt{\sup_{\tilde{\omega}} \tilde{\omega}(A^* A)}, \qquad (1.177)$$

where the sup is taken over all states on $\mathcal{A}(\mathcal{G}_L)$;

(alternative: use GNS rep., π_ω, of $\mathcal{A}(\mathcal{G}_L)$!). Taking the closure of $\mathcal{A}(\mathcal{G}_L)$ in the norm $\|(\cdot)\|$ defined in (1.177) yields a C^*-algebra, denoted by $\mathcal{A}_L$. It is called the <u>dynamical C^*-algebra</u> associated with L.

Next, we consider the special case

$$L = \mathring{L} = \frac{m}{2}\dot{q}_s^{\,2} . \qquad (1.178)$$

Then

$$\delta \mathring{\mathcal{L}}^{q^{(0)}}(q_s) = \frac{m}{2}\dot{q}_s^{(0)\,2} - m\ddot{q}_s^{(0)} \cdot q_s , \qquad (1.179)$$

as shown in (1.164). In order to derive consequences of identity (I) in (1.174) we must consider the Green functions of the 1D Laplacian $\Delta := \frac{d^2}{ds^2}$.

<u>Retarded Green function:</u>

$$G_R(t,t') := -\Theta(t-t')(t-t'), \quad (t > t')$$

<u>Advanced Green function:</u>

$$G_A(t,t') := \Theta(t'-t)(t-t') \quad (t' > t)$$

"Feynman Green function":

$$G_F(t,t') := \frac{1}{2}\left(G_R + G_A\right)(t,t') = -\frac{1}{2}|t-t'|$$

"Commutator function":

$$G(t,t') = \left(G_R - G_A\right)(t,t') = t' - t. \qquad (1.180)$$

Then

$$\left.\begin{aligned} -\Delta G_\alpha &= -G_\alpha \Delta = 1\!\!1, \quad \alpha = R, A, F, \\ \text{and} \quad -\Delta G &= -G\Delta = 0. \end{aligned}\right\} \qquad (1.181)$$

We use these simple calculations to introduce Weyl operators and derive the Weyl relations and the CCR; (see (1.129)), We set $L = \overset{\circ}{L}$, $m = \hbar = 1$. We define

$$W_{f_0}(q_s) := f_0(s) \cdot q_s + \frac{1}{2} f_0(s) \cdot (G_F f_0)(s) \qquad (1.182)$$

where $f_0(s) \in C_0$ is an arbitrary loop. We set

$$\mathcal{W}(f_0) := S(W_{f_0}), \text{ for } L = \overset{\circ}{L}. \qquad (1.183)$$

Note that

$$S(W_{-\Delta f_0}) = S(\delta \overset{\circ}{\mathcal{L}}{}^{f_0}),$$

as follows from (1.182), (1.181) and (1.179), (with $q^{(0)} := f_0$!).

Using now (I) of (1.174) and setting $W=0$, we find that

$$S(W_{-\Delta f_0}) = S(\delta \mathring{\mathcal{L}}^{f_0}) = 1. \qquad (1.184)$$

If $f_0 = \tilde{f}_0 - \Delta g_0$, for arbitrary $\tilde{f}_0$ and g_0, a simple calculation, using (1.182), shows that

$$S(W_{f_0}) = S(W_{\tilde{f}_0}^{g_0} - \delta \mathring{\mathcal{L}}^{g_0})$$

which, after using (1.169), turns out to be equal to

$$S(W_{f_0}) = S(W_{\tilde{f}_0} + \delta W_{\tilde{f}_0}^{g_0} - \delta \mathring{\mathcal{L}}^{g_0})$$
$$\overset{(I)}{=} S(W_{\tilde{f}_0}) \qquad (1.185)$$

Using definition (1.183) of the Weyl operators $\mathcal{W}$, we get

$$\mathcal{W}(f_0) = \mathcal{W}(\tilde{f}_0), \qquad (1.186)$$

whenever $f_0 = \tilde{f}_0 - \Delta g_0$, for some g_0.

Let f_0 and h_0 belong to $\mathcal{C}_0$, and choose some $g_0 \in \mathcal{C}_0$ such that, with $f_0 = \tilde{f}_0 - \Delta g_0$, $\tilde{f}_0 \succ h_0$. That this is possible can be seen as follows: Let χ be a smoothed step function with $\chi \succ h_0$, $\chi(s) = 1$, at large times s. We set

$$\tilde{f}_0 := -\Delta \chi G_R f_0\,, \quad g_0 = (1-\chi) G_R f_0 .$$

Then

$$\tilde{f}_0 - \Delta g_0 = -\Delta G_R f_0 = f_0$$

and $\tilde{f}_0 \succ h_0$. Then (II) implies

$$\begin{aligned} S(W_{f_0})\, S(W_{h_0}) &\overset{(1.185)}{=} S(W_{\tilde{f}_0})\, S(W_{h_0}) \\ &\overset{(II)}{=} S(W_{\tilde{f}_0} + W_{h_0}) \\ &\overset{(1.182)}{=} S(W_{\tilde{f}_0 + h_0} - \langle \tilde{f}_0, G_F h_0 \rangle) \end{aligned} \qquad (1.187)$$

Next

$$\begin{aligned} \langle \tilde{f}_0, G_F h_0 \rangle &= \frac{1}{2} \langle \tilde{f}_0, G_R h_0 \rangle + \frac{1}{2} \underbrace{\langle \tilde{f}_0, G_A h_0 \rangle}_{= 0, \text{ because } \tilde{f}_0 \succ h_0!} \\ &= \frac{1}{2} \langle \tilde{f}_0, G_R h_0 \rangle - \frac{1}{2} \langle \tilde{f}_0, G_A h_0 \rangle \\ &= \frac{1}{2} \langle \tilde{f}_0, G h_0 \rangle \\ &= \frac{1}{2} \langle \tilde{f}_0 - \Delta g_0, G h_0 \rangle, \\ &\qquad \text{because } \Delta G = 0! \\ &= \frac{1}{2} \langle f_0, G h_0 \rangle \end{aligned} \qquad (1.188)$$

Putting these calculations together, we find

$$S(W_{f_0})\,S(W_{h_0}) \overset{(1.185)}{=} S(W_{\tilde{f}_0})\,S(W_{h_0})$$

$$\underset{\tilde{f}_0 \succ h_0}{\overset{(\mathrm{II})}{=}} S(W_{\tilde{f}_0} + W_{h_0})$$

$$\overset{(1.187)}{=} e^{-i\langle \tilde{f}_0, G_F h_0\rangle}\, S(W_{\tilde{f}_0 + h_0})$$

$$\overset{(1.188)}{=} e^{-\frac{i}{2}\langle f_0, G h_0\rangle}\, S(W_{\tilde{f}_0 + h_0})$$

$$\overset{(1.185)}{=} e^{-\frac{i}{2}\langle f_0, G h_0\rangle}\, S(W_{\underbrace{\tilde{f}_0 + h_0 - \Delta g_0}_{=f_0 + h_0}})$$

$$= e^{-\frac{i}{2}\langle f_0, G h_0\rangle}\, S(W_{f_0 + h_0}) \qquad (1.189)$$

In terms of <u>Weyl operators</u>:

Let f_0, h_0 be arbitrary loops in $\mathcal{C}_0$. Then the operators

$$\mathcal{W}(f_0) := S(W_{f_0}), \quad (L = \hat{L}) \qquad (1.190)$$

are unitary elements of $\mathcal{A}_L^0$ (by construction) and

$$\mathcal{W}(f_0)\cdot\mathcal{W}(h_0) \overset{(1.189)}{=} e^{-\frac{i}{2}\langle f_0, G h_0\rangle}\, \mathcal{W}(f_0 + h_0), \qquad (1.191)$$

$$\mathcal{W}(-\Delta f_0) \overset{(1.184)}{=} \mathbb{1}, \quad \forall f_0 \in \mathcal{C}_0. \qquad (1.192)$$

Let $\langle f_0, Q \rangle$ be the generator of $W(f_0)$.

Then (1.192) implies that

$$\langle -\Delta f_0, Q \rangle = 0, \quad \forall f_0 \in \mathcal{C}_0 .$$

Hence

$$Q(s) = Q + s\frac{P}{m}, \quad s \in \mathbb{R}, \quad (m=1).$$

From (1.191) we then get

$$[\langle f_0, Q\rangle, \langle h_0, Q\rangle] \underset{\substack{\uparrow \\ (1.191),(1.180)}}{=} i\langle f_0, G h_0\rangle \mathbb{1}$$

For the components Q_k, P_k, $k = 1, \cdots, f$, we then find

$$[Q_k, Q_\ell] = [P_k, P_\ell] = 0, \quad [Q_k, P_\ell] = i\delta_{k\ell}\mathbb{1}, \qquad (1.193)$$

$k, \ell = 1, \cdots, f$ (exercise!), i.e., the CCR or Heisenberg CR.

To pass from the free Lagrangian, $\overset{\circ}{L}$, to an interacting Lagrangian, L, given by

$$L(q_s, \dot{q}_s) = \overset{\circ}{L}(q_s, \dot{q}_s) - \chi(s) V(q_s),$$

supp $\chi(s)$ compact, $\chi(s) = 1$, $s \in I$.

We then set

$$S_L(W) := S_L^{\circ}(\chi V)^{-1}\, S_L^{\circ}(W + \chi V)\,, \qquad (1.194)$$

$\text{supp}\, W \subseteq I$, where $S_L^{\circ}(W) = S(W)$ is defined for the free Lagrangian. The operators $S_L(W)$ generate $\mathcal{G}_L$, and, by (1.194)

$$\mathcal{A}_L = \mathcal{A}_{L_0}\,,$$

as long as $\text{supp}\,\chi$ is compact. As $\chi \nearrow 1$, $\exists$ a homomorphism $\gamma : \mathcal{A}_L \to \mathcal{A}_{L_0}$, ...

It is straightforward to show that the standard Schrödinger picture, with $S(W)$ as in (1.167), (1.159), yields a representation of $\mathcal{G}_L$ and $\mathcal{A}_L$; (see Reed & Simon, vol. III, Thm. X.70)

1.6. Some elements of functional analysis

So far (i.e., in previous sections), we have given a picture of Quantum mechanics as a purely algebraic object — there hasn't been any talk of Hilbert spaces and the like. In passing we have mentioned C^*-algebras and the Gel'fand–Naimark–Sigal construction. Thus, let us introduce these objects in a precise way. It will then turn out that we will automatically encounter Hilbert spaces, "wave functions" and the like.

(i) What is a C^*-algebra?

A C^*-algebra over the field $\mathbb{C}$ is a normed algebra, $\mathcal{A}$, with an involution *,

$$\mathcal{A} \ni a \longmapsto a^* \in \mathcal{A} \qquad (1.195)$$

with the properties

(1) $(a^*)^* = a, \ \forall a \in \mathcal{A}$

(2) $(a\cdot b)^* = b^* \cdot a^*$, $\forall a, b$ in $\mathcal{A}$

(3) $(za+ub)^* = \bar{z}a^* + \bar{u}b^*$, $\forall a, b$ in $\mathcal{A}$, $\forall z, u$ in $\mathbb{C}$

(4) $\|a^* \cdot a\| = \|a\|^2$

$\mathcal{A}$ is abelian (commutative) iff $a\cdot b = b\cdot a$, $\forall a, b$ in $\mathcal{A}$.

Examples:

$\mathcal{B}(\mathcal{H})$; $\mathcal{K}(\mathcal{H})$ (compact linear ops. on $\mathcal{H}$);

$C(\mathfrak{X})$, where $\mathfrak{X}$ is a Hausdorff topological space; ($\nearrow$ Gel'fand isomorphism says that if $\mathcal{A}$ is abelian then $\mathcal{A} \simeq C(\mathfrak{X})$, and if $\mathcal{A}$ contains a unit, $\mathbb{1}$, then $\mathfrak{X}$ is compact); group algebras.

*Homomorphisms: $\mathcal{A}$, $\mathbb{B}$ two C^*-algebras

$$\varphi : \mathcal{A} \to \mathbb{B}$$

is a * homomorphism iff it is a homomorphism with $\varphi(a)^* = \varphi(a^*)$. Then

$$\|\varphi(a)\| \leq \|a\|, \quad \forall a \in \mathcal{A} \qquad (1.196)$$

and $\|\varphi(a)\| = \|a\|$, $\forall a \in \mathcal{A} \Leftrightarrow \varphi$ is injective.

Every finite-dimensional C^*-algebra is isomorphic to a finite direct sum of finite-dimensional matrix algebras.

A left-ideal $\mathcal{I} \subset \mathcal{A}$ is a subalgebra of $\mathcal{A}$ with the properties that $a^* \in \mathcal{I}$, $\forall a \in \mathcal{I}$, and

$$a \cdot b \in \mathcal{I}, \ \forall a \in \mathcal{A}, \forall b \in \mathcal{I}. \qquad (1.197)$$

Similarly, right-ideals are defined. A two-sided ideal is simultaneously a left- and right ideal. If $\mathcal{I}$ is a two-sided *ideal in $\mathcal{A}$ then $\mathcal{A}/\mathcal{I}$ is again a C^*-algebra.

Operations on C^*-algebras:

Direct sums, tensor products, inductive limits etc.

(ii) States on C^*-algebras

Let $\mathcal{A}$ be a C^*-algebra. A state, ω, on $\mathcal{A}$ is a bounded, positive linear functional on $\mathcal{A}$

of norm 1:

$$\|\omega\| = \sup_{a \in \mathcal{A}} \frac{|\omega(A)|}{\|A\|} \tag{1.198}$$

If $\mathcal{A}$ contains a unit, $\mathbb{1}$, then

$$\omega(\mathbb{1}) = 1. \tag{1.199}$$

The set, $\mathcal{S}(\mathcal{A})$, of all states on $\mathcal{A}$ is convex and weak*-closed in the dual, $\mathcal{A}'$, of $\mathcal{A}$. In the weak* topology, $\mathcal{S}(\mathcal{A})$ is a compact Hausdorff space, (↗Banach-Alaoglu thm.). Let $\mathcal{A} = C(\mathfrak{X})$; then

$$\mathcal{S}(\mathcal{A}) = \{\text{Borel probability measures on } \mathcal{X}\} \tag{1.200}$$

A state ω is called <u>normal</u> iff for an <u>arbitrary increasing net</u> $\{a_\alpha\} \subset \mathcal{A}$ with upper bound a

$$\omega(a_\alpha) \nearrow \omega(a) \tag{1.201}$$

A state ω is a <u>trace</u> on $\mathcal{A}$ iff

$$\omega(a \cdot b) = \omega(b \cdot a), \quad \forall a, b \text{ in } \mathcal{A} \tag{1.202}$$

For separable C^*-algebras, the set of tracial states is a <u>Choquet simplex</u>, (i.e., any tracial state is the barycenter of a unique probability measure on the set of extremal tracial states).

(iii) The Gel'fand-Naimark-Segal (GNS) construction

Let $\mathcal{A}$ be a C^*-algebra and $\omega \in \mathcal{S}(\mathcal{A})$ a state on $\mathcal{A}$. Then there exists a representation, π_ω, of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$ and a cyclic vector $\Omega \in \mathcal{H}_\omega$ such that

$$\omega(a) = \langle \Omega, \pi_\omega(a)\Omega \rangle_{\mathcal{H}_\omega}, \quad \forall a \in \mathcal{A}. \quad (1.203)$$

Proof.

$\mathcal{A}$ is a linear space equipped with a positive semi-definite inner product

$$\langle a, b \rangle := \omega(a^* \cdot b), \quad \forall a, b \text{ in } \mathcal{A}.$$

Then one has the Schwarz inequality

$$|\langle a, b \rangle|^2 \leq \langle a, a \rangle \cdot \langle b, b \rangle \quad (1.204)$$

Let $\mathcal{J}_\omega \subset \mathcal{A}$ consist of all elements a with the property that $\langle a, a \rangle = 0$. Then $\mathcal{J}_\omega$ is a left-ideal. For, if $a \in \mathcal{J}_\omega$ and $b \in \mathcal{A}$ then

$$\langle ba, ba \rangle = \omega((ba)^* ba)$$

$$= \omega(a^* b^* b a)$$

$$= \langle b^* b a, a \rangle.$$

Hence

$$0 \leq |\langle ba, ba \rangle|^2 \underset{\substack{\uparrow \\ \text{Schwarz}}}{\leq} \langle b^* ba, b^* ba \rangle \cdot \langle a, a \rangle = 0$$

Thus $ba \in \mathcal{J}_\omega$. Then

$$\mathcal{H}_\omega := \overline{\mathcal{A}/\mathcal{J}_\omega}^{\|\cdot\|}, \qquad (1.205)$$

where $\|\cdot\|$ is the norm on the linear space determined by $\langle \cdot, \cdot \rangle$. Obviously, the linear space $\mathcal{V}_\omega := \mathcal{A}/\mathcal{J}_\omega$ is norm-dense in $\mathcal{H}_\omega$. If $\psi = b + \mathcal{J}_\omega \in \mathcal{V}_\omega$, $b \in \mathcal{A}$, we set

$$\pi_\omega(a)\psi := a \cdot (b + \mathcal{J}_\omega) = a \cdot b + \mathcal{J}_\omega$$

Then π_ω defines a *homomorphism from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_\omega)$ (exercise!), and $\|\pi_\omega(a)\| \leq \|a\|$.

If $\mathcal{A} \ni \mathbb{1}$ then

$$\Omega = \mathbb{1} + \mathcal{J}_\omega \in \mathcal{H}_\omega; \qquad (1.206)$$

if $\mathcal{A} \not\ni \mathbb{1}$, may pass to

$$\tilde{\mathcal{A}} = \{ [a, z] \mid a \in \mathcal{A},\ z \in \mathbb{C} \},$$

with $\quad [a,z] \cdot [b,w] := [a \cdot b + wa + zb,\ zw]$

$$[a,z]^* := [a^*, \bar{z}].$$

This product is associative and distributive, and $[0,1] \in \tilde{\mathcal{A}}$ is the identity:

$$[a,z] \cdot [0,1] = [a,z] = [0,1][a,z]$$

□

A state ω on $\mathcal{A}$ is an <u>extreme point</u> of $\mathcal{S}(\mathcal{A})$ iff π_ω is <u>irreducible</u>, i.e. iff the commutant, $\pi_\omega(\mathcal{A})'$, in $\mathcal{B}(\mathcal{H}_\omega)$ of $\pi_\omega(\mathcal{A})$ consists of multiples of $\mathbb{1}_{\mathcal{H}_\omega}$.

(Note: A rep. π_ω is irreducible iff $\mathcal{H}_\omega$ does <u>not</u> contain any proper <u>subspace</u> $\mathcal{K} \neq \{\vec{0}\}$, $\mathcal{K} \subsetneqq \mathcal{H}_\omega$, <u>invariant</u> under $\pi_\omega(\mathcal{A})$.)

(iv) <u>The von Neumann uniqueness theorem.</u>

In (1.190)–(1.192) we have introduced the <u>Weyl</u>

operators. They have generators Q and P satisfying the canonical commutation relations (1.193). We suppose that $Q = (Q_1, \cdots, Q_f)$ and $P = (P_1, \cdots, P_f)$. Let $a \in \mathbb{R}^f$ and $b \in \mathbb{R}^f$ and set

$$W(a,b) = e^{i(a \cdot P + b \cdot Q)} \qquad (1.206)$$

Then the relations (1.191) are equivalent to

$$W(a,b)\,W(a',b') = e^{\frac{i}{2}(a \cdot b' - a' \cdot b)}\,W(a+a', b+b') \qquad (1.207)$$

In accordance with (1.206), we define

$$W(a,b)^* = e^{-i(a \cdot P + b \cdot Q)}$$

$$= W(-a,-b).$$

Then

$$W(a,b)^* W(a,b) = W(-a,-b)\,W(a,b)$$

$$= W(0,0) = 1, \cdots \qquad (1.208)$$

i.e., the operators $W(a,b)$ are <u>unitary</u>.

Let $\mathcal{A}_W := \mathcal{A}_{L_0}$ be as in Sect. 1.5. It is called the <u>Weyl algebra</u>. We propose to characterize the <u>strongly continuous representations</u> of $\mathcal{A}_W$.

<u>Notations</u> $U(a) := W(a,0)$, $V(b) := W(0,b)$. (1.209)

By (1.207)

$$\left.\begin{aligned} U(a)V(b) &= e^{\frac{i}{2}a\cdot b}\, W(a,b), \\ V(b)U(a) &= e^{-\frac{i}{2}a\cdot b}\, W(a,b), \\ \text{hence} \qquad U(a)V(b) &= e^{i a\cdot b}\, V(b)U(a) \end{aligned}\right\} \quad (1.210)$$

<u>Definition W.</u> A representation of the <u>Weyl relations</u> (1.210) on a Hilbert space $\mathcal{H}$ is a *homomorphism, γ, from the Weyl algebra $\mathcal{A}_W$ to $B(\mathcal{H})$

$$\gamma : \begin{cases} U(a) \longmapsto \gamma(U(a)) =: U_\gamma(a) \in B(\mathcal{H}), \\ V(b) \longmapsto \gamma(V(b)) =: V_\gamma(b) \in B(\mathcal{H}), \end{cases}$$

$W_\gamma(a,b) := e^{\frac{i}{2}a\cdot b}\, U_\gamma(a)V_\gamma(b)$ satisfies (1.207) and (1.208) and is <u>strongly continuous</u> in $a \in \mathbb{R}^f$ and $b \in \mathbb{R}^f$.

<u>Schrödinger representation,</u> U_S, V_S:

$\mathcal{H} = L^2(\mathbb{R}^f, d^f q)$; for $\psi = \psi(q) \in \mathcal{H}$, we set

$$(U_S(a)\psi)(q) := \psi(q+a), \qquad (1.211)$$

and

$$\left(V_S(b)\psi\right)(q) := e^{ib\cdot q}\,\psi(q) \qquad (1.3.12)$$

Theorem (von Neumann)

Every representation, γ, of the Weyl relations, in the sense of Definition W, above, is unitarily equivalent to a direct sum of Schrödinger representations.

Sketch of proof.

We set $\quad W_\gamma(f) := \int d^f a\, d^f b\; f(a,b)\, W(a,b),$

where $f \in \mathcal{S}(\mathbb{R}^{2f})$. Then

(1) $\quad \| W_\gamma(f)\| \leq \|f\|_1$

(2) $\quad W_\gamma(zf + wg) = z\,W_\gamma(f) + w\,W(g),$

for arbitrary f, g in $\mathcal{S}(\mathbb{R}^{2f})$, z, w in $\mathbb{C}$,

(3) $\quad W_\gamma(f)^* = W_\gamma(\underline{f^*}),$

where $\quad f^*(a,b) = \bar{f}(-a,-b)$

(4) $\quad W_\gamma(f) = 0 \iff f = 0$

$\Leftarrow$ is obvious.

$\Rightarrow$: $W_\gamma(f) = 0 \Rightarrow \langle \varphi, W_\gamma(a,b)\, W_\gamma(f)\, W_\gamma(-a,-b)\psi\rangle = 0,$

for all $\varphi, \psi \in \mathcal{H}$. Hence

$$\int d^f a' \int d^f b' \, e^{i(a\cdot b' - b\cdot a')} f(a', b') \langle \varphi, W_\gamma(a'; b')\psi\rangle = 0,$$

$\forall\, a, b$ in $\mathbb{R}^f$. Thus

$$f(a', b') \langle \varphi, W_\gamma(a', b')\psi\rangle \equiv 0.$$

Setting $\varphi = W_\gamma(a', b')\psi$, with $\|\psi\| = 1$, it follows that $f \equiv 0$.

(4) implies

(5) $\quad W_\gamma(f) = W_\gamma(g) \Leftrightarrow f = g.$

Let $f_0 := (2\pi)^{-f} e^{-\frac{1}{4}(a^2+b^2)}$, and $\Pi := W_\gamma(f_0)$.

Then
$$\Pi W_\gamma(a, b)\Pi = e^{-\frac{1}{4}(a^2+b^2)} \Pi$$

Hence, for $a = b = 0$,

(6) $\qquad \Pi^2 = \Pi$; and $\Pi^* = \Pi$,

by (3). Thus, Π is an orthogonal projection, with $\Pi \neq 0$, by (4).

We define
$$\mathcal{K} := \Pi \mathcal{H}. \tag{1.213}$$

For φ, ψ in $\mathcal{K}$, we have that

$$\langle W_\gamma(a',b')\varphi, W_\gamma(a,b)\psi\rangle$$

$$= \langle W_\gamma(a',b')\Pi\varphi, W_\gamma(a,b)\Pi\psi\rangle$$

$$= \langle \Pi\varphi, W_\gamma(-a',-b')W_\gamma(a,b)\Pi\psi\rangle$$

$$\overset{(1.207)}{=} e^{\frac{i}{2}(ab'-a'b)}\langle \Pi\varphi, W_\gamma(a-a', b-b')\Pi\psi\rangle$$

$$\overset{(5)}{=} e^{-\frac{1}{4}\{(a-a')^2+(b-b')^2\}} e^{\frac{i}{2}(a\cdot b'-a'\cdot b)} \langle\varphi,\psi\rangle \tag{1.214}$$

We pick a CONS, $\{\varphi_n\}$, in $\mathcal{K}$ and define subspaces

$$\mathcal{H}_n := \overline{\{W_\gamma(a,b)\varphi_n \mid (a,b)\in\mathbb{R}^{2f}\}} \subseteq \mathcal{H},$$

$n = 1,2,3,\dots$. Claim:

$$\mathcal{H}_n \perp \mathcal{H}_m, \text{ for } n\neq m \tag{1.215}$$

By (1.214),

$$\langle W_\gamma(a',b')\varphi_n, W_\gamma(a,b)\varphi_m\rangle \propto \langle\varphi_n,\varphi_m\rangle$$

$$= \delta_{nm},$$

from which (1.215) follows.

By construction, $\mathcal{H}_n$ is <u>invariant</u> under $\pi(\mathcal{A}_W)$. Thus

$$\gamma(\mathcal{A}_W) = \bigoplus_{n=1,2,3,\ldots} \gamma(\mathcal{A}_W)\Big|_{\mathcal{H}_n} \oplus \gamma(\mathcal{A}_W)\Big|_{\mathcal{H}_0}, \qquad (1.216)$$

where

$$\mathcal{H}_0 = \mathcal{H} \ominus \Big(\bigoplus_{n=1,2,3,\ldots} \mathcal{H}_n \Big).$$

We define Π_n to be the orthogonal projection onto $\mathcal{H}_n$, $n = 0, 1, 2, \ldots$.

Since $\mathcal{K} \subseteq \bigoplus_{n \geq 1} \mathcal{H}_n$, we have that

$$\mathcal{H}_0 \subseteq \mathcal{K}^\perp \qquad (1.217)$$

Thus $\qquad \Pi \mathcal{H}_0 = \{0\},$

with $\Pi = W_\gamma(f_0)$ as above. By (4)

$$W_\gamma(f_0)\Big|_{\mathcal{H}_0} = \Pi\Big|_{\mathcal{H}_0} \neq 0,$$

unless $\mathcal{H}_0 = \{0\}$. Thus

$$\gamma(\mathcal{A}_W)\Big|_{\mathcal{H}} = \bigoplus_{n \geq 1} \gamma(\mathcal{A}_W)\Big|_{\mathcal{H}_n} \qquad (1.218)$$

We define <u>intertwiners</u> $T_{nm} : \mathcal{H}_m \to \mathcal{H}_n$ by setting

$$T_{nm} W_\gamma(a,b)\varphi_n := W_\gamma(a,b)\varphi_m,$$

$n, m = 1, 2, 3, \ldots$; $\forall a, b$ in $\mathbb{R}^f$. Then

Then (1.214) implies that

$$\langle T_{nm} W_\gamma(a',b')\varphi_m, T_{nm} W_\gamma(a,b)\varphi_m\rangle$$

$$= \langle W_\gamma(a',b')\varphi_n, W_\gamma(a,b)\varphi_n\rangle$$

$$= e^{-\frac{1}{4}\{(a-a')^2+(b-b')^2\}} e^{i(a\cdot b'-a'\cdot b)} \underbrace{\langle\varphi_n,\varphi_n\rangle}_{=\langle\varphi_m,\varphi_m\rangle=1}$$

$$= \langle W_\gamma(a',b')\varphi_m, W_\gamma(a,b)\varphi_m\rangle,$$

for arbitrary n and m; i.e., the maps T_{nm} are <u>isometries.</u> It follows that the representations $\{\gamma(\mathcal{A}_W)|_{\mathcal{H}_n}\}_{n\geq 1}$ are pairwise <u>unitarily equivalent.</u> (1.219)

Now

$$\Pi W_\gamma(a,b)\varphi_n = \Pi W_\gamma(a,b)\Pi\varphi_n$$

$$= e^{-\frac{1}{4}(a^2+b^2)}\varphi_n, \quad \forall a,b \text{ in } \mathbb{R}^f,$$

by (5)!

Thus, $$\Pi\mathcal{H}_n = \{\mathbb{C}\varphi_n\}, \tag{1.220}$$

and $$\Pi\mathcal{H}_n = \underbrace{W_\gamma(f_0)\mathcal{H}_n}_{\subseteq\mathcal{H}_n} = \underbrace{\Pi_n W_\gamma(f_0)\Pi_n}_{\equiv W_\gamma^{(n)}(f_0)}\mathcal{H}_n$$

Hence, by (1.219), $W_\gamma^{(n)}(f_0)$ projects onto $\{\mathbb{C}\varphi_n\}$, i.e. the range of $W_\gamma^{(n)}(f_0)$ is <u>one-dimensional.</u>

<u>Claim.</u> $\gamma(\mathcal{A}_W)\big|_{\mathcal{H}_n}$ is <u>irreducible</u>. (1.221)

<u>Proof:</u> If $\mathcal{K}_n \subsetneqq \mathcal{H}_n$ is an <u>invariant subspace</u>

then $\mathcal{K}_n^\perp := \mathcal{H}_n \ominus \mathcal{K}_n$ is invariant, too.

Thus, by definition of $W_\gamma(f_0)$,

$$W_\gamma^{(n)}(f_0)\,\mathcal{K}_n \subseteq \mathcal{K}_n\,, \quad W_\gamma^{(n)}(f_0)\,\mathcal{K}_n^\perp \subseteq \mathcal{K}_n^\perp$$

For $\varphi \in \mathcal{K}_n$, $\psi \in \mathcal{K}_n^\perp$, (1.219) implies that

$$W_\gamma^{(n)}(f_0)\varphi = z_\varphi \varphi_n\,, \quad W_\gamma^{(n)}(f_0)\psi = w_\psi \varphi_n\,,$$

for some z_φ, w_ψ in $\mathbb{C}$. But

$$\langle \underbrace{W_\gamma^{(n)}(f_0)\varphi}_{\in \mathcal{K}_n}, \underbrace{W_\gamma^{(n)}(f_0)\psi}_{\in \mathcal{K}_n^\perp} \rangle = 0\,,$$

hence $z_\varphi \neq 0$, or $w_\varphi \neq 0$. However, if $\mathcal{K}_n \neq \{0\}$

then $\langle \varphi, W_\gamma^{(n)}(f_0)\varphi \rangle \neq 0$, for some φ,

by (4)(!). Hence $z_\varphi \neq 0$. Thus $w_\psi = 0$, $\forall \psi \in \mathcal{K}_n^\perp$

$\Rightarrow \langle \psi, W_\gamma^{(n)}(f_0)\psi \rangle = 0$, $\forall \psi \in \mathcal{K}_n^\perp$. By (4),

$\mathcal{K}_n^\perp = \{0\}$. $\square$

<u>Corollary</u>: Up to unitary equivalence, $\exists$ only one! irreducible representation of $\mathcal{A}_W$.

<u>Proof</u>: Follows from (1.219) and (1.221)! $\square$

We know one representation of $\mathcal{A}_W$ explicitly, namely the Schrödinger representation defined in (1.211) and (1.212). For the Schrödinger representation,

$$(W_S(a,b)\psi)(q) = e^{ib\cdot(q+\frac{a}{2})}\psi(q+a)$$

For $\varphi_0(q) := \pi^{-\frac{f}{4}} e^{-\frac{1}{2}|q|^2}$, we find that

$$W_S(f_0)\psi = \langle \varphi_0, \psi \rangle \psi,$$

i.e., $W_S(f_0)$ is a one-dimensional projection; hence W_S is <u>irreducible</u>.

Let P be the generator of $U_S(a)$ and Q the generator of $V_S(b)$. Then

$$P = \frac{1}{i}\left(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_f}\right), \quad Q = (q_1, \dots, q_f) \qquad (1.221)$$

defines $2f$ self-adjoint operators with a common core $\mathcal{S}(\mathbb{R}^f) \subset L^2(\mathbb{R}^f, d^f q)$.

One may ask under what conditions a representation of the canonical commutation relations

$$[P_i, P_j] = [Q_i, Q_j] = 0, \quad [P_i, Q_j] = -i\,\delta_{ij}\,\mathbb{1} \quad (1.222)$$

on a Hilbert space $\mathcal{H}$ gives rise to a unitary representation of $\mathcal{A}_W$ on $\mathcal{H}$. The answer is essentially contained in a paper of mine that appeared in Commun. math. Phys. 54, 135–150 (1977), Sect. 7; (one chooses $N := \frac{1}{2}\sum_{j=1}^{f}\{P_j^2 + Q_j^2\}, \ldots$). This is a nice story; but it is not essential for the following chapters.

(v) States and physical quantities.

We have introduced a specific notion of states on *algebras in § (ii), Sect. 1.6. According to the GNS construction, a state ω on a C^*-algebra $\mathcal{A}$ gives rise to a Hilbert space $\mathcal{H}_\omega$, a *representation π_ω of $\mathcal{A}$ on $\mathcal{H}_\omega$ and a cyclic vector, $\Omega \in \mathcal{H}_\omega$, with

$$\omega(a) = \langle \Omega, \pi_\omega(a)\Omega \rangle_{\mathcal{H}_\omega}, \quad \forall a \in \mathcal{A}.$$

Obviously, $\pi_\omega(\mathcal{A}) \subseteq B(\mathcal{H}_\omega)$. We may thus take the weak closure

$$\overline{\mathcal{A}}^w := \overline{\pi_\omega(\mathcal{A})}^w, \qquad (1.223)$$

(a is the weak limit of a net $\{a_\alpha\}_{\alpha \in A}$ iff

$$\langle \varphi, a\psi \rangle = \lim_\alpha \langle \varphi, a_\alpha \psi \rangle, \quad \forall \varphi, \psi \text{ in } \mathcal{H}_\omega;$$

assuming that $\{\|a_\alpha\|\}_{\alpha \in A}$ is bounded.)

$\overline{\mathcal{A}}^w$ contains all weak limits of nets in $\mathcal{A}$.

Given a subalgebra $\mathcal{B} \subset B(\mathcal{H}_\omega)$, we define the commutant, $\mathcal{B}'$, of $\mathcal{B}$ by

$$\mathcal{B}' := \{ a \in B(\mathcal{H}_\omega) \mid [a, b] = 0, \ \forall b \in \mathcal{B} \}$$

Then

$$\overline{\mathcal{A}}^w = \left(\pi_\omega(\mathcal{A})'\right)' \equiv \pi_\omega(\mathcal{A})'' \qquad (1.224)$$

A vector $\psi \in \mathcal{H}_\omega$ is separating for $\mathcal{B}$ iff

$$a\psi = 0 \Rightarrow a = 0, \ \forall a \in \mathcal{B},$$

which is equivalent to saying that ψ is <u>cyclic</u> for $\mathcal{B}'$; (a vector $\varphi \in \mathcal{H}_\omega$ is <u>cyclic</u> ~~for~~ $\mathcal{B}$ iff $\{b\varphi \mid b \in \mathcal{B}\}$ is <u>dense</u> in $\mathcal{H}_\omega$).

A representation π_ω of $\mathcal{A}$ is <u>irreducible</u> iff

$$\pi_\omega(\mathcal{A})' = \{\mathbb{C}\mathbb{1}\} \qquad (1.225)$$

Then ω is <u>extremal</u>.

<u>Physical quantities ("observables"</u>

In <u>classical</u> and <u>quantum physics</u>, <u>physical quantities</u> characteristic of a system, S, are represented by a family, $\mathcal{O}_S$, of <u>abstract</u> (<u>bounded</u>) <u>self-adjoint linear operators</u>:

$$\mathcal{O}_S := \left\langle \hat{a}_i \,\middle|\, \hat{a}_i^* = \hat{a}_i,\ i \in \mathcal{I}_S \right\rangle \qquad (1.226)$$

with the property that if $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ is an arbitrary finite subset of operators in $\mathcal{O}_S$ with the property that if

$$[\hat{x}_\alpha, \hat{x}_\beta] = 0 \ , \quad \forall \alpha, \beta = 1, \ldots, n$$

which is equivalent to saying that ψ is cyclic for $\mathcal{B}'$; (a vector $\varphi \in \mathcal{H}_\omega$ is cyclic for $\mathcal{B}$ iff $\{b\varphi \mid b \in \mathcal{B}\}$ is dense in $\mathcal{H}_\omega$).

A representation π_ω of $\mathcal{A}$ is irreducible iff

$$\pi_\omega(\mathcal{A})' = \{\mathbb{C}\mathbb{1}\}. \qquad (1.225)$$

Then ω is extremal.

Physical quantities/"observables"

In classical and quantum physics, physical quantities characteristic of a system, S, are represented by a family, $\mathcal{O}_S$, of abstract (bounded) self-adjoint linear operators:

$$\mathcal{O}_S := \left\langle \hat{a}_i \,\middle|\, \hat{a}_i^* = \hat{a}_i,\ i \in \mathcal{I}_S \right\rangle \qquad (1.226)$$

with the property that if $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n$ is an arbitrary finite subset of operators in $\mathcal{O}_S$ with the property that if

$$[\hat{x}_\alpha, \hat{x}_\beta] = 0, \quad \forall \alpha, \beta = 1, \dots, n$$

and if f is an arbitrary bounded, continuous, real-valued function on $\mathbb{R}^n$ then

$$f(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) \in \mathcal{O}_S \qquad (1.227)$$

It is assumed that the *algebra, $\mathring{\mathcal{A}}_S$, generated by $\mathcal{O}_S$ is equipped with a C^*-norm $\|(\cdot)\|$.

Then

$$\mathcal{A}_S := \overline{\mathring{\mathcal{A}}_S}^{\|(\cdot)\|} \qquad (1.228)$$

is a C^*-algebra. It is easy to show that

$$\|f(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)\| \leq \|f\|_\infty .$$

If

$$[\hat{a}_\alpha, \hat{a}_\beta] = 0, \quad \forall \alpha, \beta \text{ in } \mathcal{I}_S, \qquad (1.229)$$

then $\mathcal{A}_S$ is an abelian C^*-algebra.

Gel'fand's isomorphism then tells us that

$$\mathcal{A}_S = C(\mathfrak{X}_S), \qquad (1.230)$$

where $\mathfrak{X}_S$ is a Hausdorff topological space; ($\mathfrak{X}_S$ is the space of characters of $\mathcal{A}_S$). If $\mathcal{A}_S$ contains and identity $\mathbb{1}$, then

$\mathcal{X}_S$ is a <u>compact Hausdorff space</u>. (1.231)

Then <u>all</u> real-valued elements of $C(\mathcal{X}_S)$ are <u>observables.</u>

If $\{\hat{a}_\iota\}_{\iota \in \mathcal{I}_S}$ is non-commutative then $\mathcal{O}_S$ is usually <u>not</u> a linear space, hence <u>not</u> an algebra, so that

$$\mathcal{A}_S \supseteq \mathring{\mathcal{A}}_S \underset{\neq}{\supseteq} \mathcal{O}_S . \qquad (1.232)$$

<u>Examples:</u>

(1) $\hat{a}_\iota := f_\iota(q_1, \dots, q_f)$,

where f_ι is an arb. real-valued continuous function on $\mathbb{R}^f$; (it suffices to consider for example Hermite functions on $\mathbb{R}^f$)

$$\Rightarrow \mathcal{A}_S = C_0(\mathbb{R}^f)$$

$$\left.\begin{aligned} &(2)\ \hat{a}_\iota := f_\iota(q_1, \dots, q_f), \\ &\text{and} \\ &\hat{b}_\iota := g_\iota\left(\frac{\hbar}{i}\frac{\partial}{\partial q_1}, \dots, \frac{\hbar}{i}\frac{\partial}{\partial q_f}\right), \end{aligned}\right\} \qquad (1.233)$$

f_ι, g_ι as above. $\Rightarrow$ <u>"Resolvent algebra"</u>

(Buchholz - Grundling) contained in Weyl algebra;
(↗ von Neumann uniqueness theorem).

(3) "Q-bits", spins

Let $(\mathcal{A}_S, \omega)$ be given, and assume that π_ω is irreducible. Then

$$\overline{\mathcal{A}}_S := \pi_\omega(\mathcal{A}_S)'' = \mathcal{B}(\mathcal{H}_\omega).$$

It is quite common (although unreasonable) to imagine that all self-adjoint operators in $\mathcal{B}(\mathcal{H}_\omega)$, in particular all orthogonal projections on $\mathcal{H}_\omega$ are "observables". In this context an interesting theorem of Andrew Gleason is relevant.

(vi) Gleason's theorem.

Let μ be a functional defined on the space of orthogonal projections on a complex Hilbert space $\mathcal{H}$ of dimension $\dim \mathcal{H} \geq 3$ with the following properties:

$$(1) \quad \Pi = \Pi^2 = \Pi^* \longmapsto \mu(\Pi) \geq 0 \qquad (1.234)$$

If $\{\Pi_n\}_{n=1}^{\infty}$ is any countable family of mutually orthogonal projections, i.e.,

$$\Pi_n = \Pi_n^*, \quad \Pi_n \Pi_m = \delta_{nm} \Pi_n, \quad \forall n, m, \qquad (1.235)$$

with $\Pi = \sum_{n=1}^{\infty} \Pi_n$, then

$$(2) \quad \mu(\Pi) = \sum_{n=1}^{\infty} \mu(\Pi_n). \qquad (1.236)$$

$$(3) \quad \mu(\mathbb{1}) = 1.$$

μ is called a (normalized) <u>positive measure</u> on $\mathcal{H}$.

<u>Theorem</u> (Gleason)

Let μ be a normalized positive measure on $\mathcal{H}$, with $\dim \mathcal{H} \geq 3$. Then there exists a positive trace-class operator P on $\mathcal{H}$, $(P = P^* \geq 0)$ with $\operatorname{tr} P = 1$, such that

$$\mu(\Pi) = tr(P\Pi) \qquad (1.237)$$

for all orthogonal projections on $\mathcal{H}$. □

P is called a <u>density matrix</u> (density operator). If $P = \Pi_\psi$ is the orthogonal projection onto a vector $\psi \in \mathcal{H}$ then

$$tr(Pa) = tr(\Pi_\psi a) = \langle\psi, a\psi\rangle / \|\psi\|^2 . \qquad (1.238)$$

Note that $\Pi_\psi = \Pi_{z\psi}$, $z \in \mathbb{C}$; i.e. Π_ψ only depends on the <u>ray</u> $[\psi] = \{z\psi \mid z \in \mathbb{C}\}$.

Obviously, a density matrix, P, defines a state on $\mathcal{B}(\mathcal{H})$, and this state is <u>extremal</u> (<u>pure</u>) iff

$$P^2 = P \Rightarrow P = \Pi_\psi, \text{ for some } \psi \in \mathcal{H}. \quad (1.239)$$

(vii) <u>Symmetries.</u>

A <u>symmetry</u> of a system S with $\mathcal{A}_S = \mathcal{B}(\mathcal{H})$ is a map

$$\sigma : \begin{cases} \psi \in \mathcal{H} \longmapsto \sigma\psi \in \mathcal{H} \\ a \in \mathcal{B}(\mathcal{H}) \longmapsto \sigma a \in \mathcal{B}(\mathcal{H}) \end{cases} \qquad (1.240)$$

with the properties

(a) $\sigma : \mathcal{H} \to \mathcal{H}$ is surjective ;

(b) $\langle \sigma\psi, (\sigma a)\psi\sigma \rangle = \langle \psi, a\psi \rangle$,

$\forall \psi \in \mathcal{H}$, $\forall a = a^* \in B(\mathcal{H})$;

(c) $\sigma f(a) = f(\sigma a)$, $a = a^* \in B(\mathcal{H})$,

for all real-valued continuous functions f on $\mathbb{R}$.

Lemma (Wigner)

Let σ be a symmetry of S, with $\mathcal{A}_S = B(\mathcal{H})$.

Then

$$|\langle \sigma\varphi, \sigma\psi \rangle| = |\langle \varphi, \psi \rangle|, \qquad (1.241)$$

$\forall \varphi, \psi \in \mathcal{H}$.

A map σ satisfying (1.241) is called a (ray) correspondence.

Theorem (Wigner)

Let σ be a correspondence on $\mathcal{H}$. Then there exists a unitary or anti-unitary operator, U, on $\mathcal{H}$, unique up to a phase,

such that

$$\sigma \psi = U\psi, \ \forall \psi \in \mathcal{H}, \quad \sigma a = U a U^{-1}, \ \forall a \in B(\mathcal{H}). \tag{1.242}$$

See, e.g., V. Bargmann, J. Math. Phys. 5, 862 (1964).

We denote the symmetry appearing in (1.242) by σ_U. Then

$$\sigma_{U_1} \circ \sigma_{U_2} = \sigma_{U_1 \cdot U_2}, \quad \sigma_{\mathbb{1}} = id.$$

$\Rightarrow$ Symmetries form a group.

<u>Digression</u>. An operator a on $\mathcal{H}$ is <u>anti-linear</u> iff $a(z\varphi + w\psi) = \bar{z}\, a\varphi + \bar{w}\, a\psi$, $\forall \varphi, \psi$ in $\mathcal{H}$, $\forall z, w$ in $\mathbb{C}$. The adjoint, a^*, of a is defined by

$$\langle a\varphi, \psi \rangle =: \langle a^*\psi, \varphi \rangle, \tag{1.243}$$

$\forall \varphi, \psi$ in $\mathcal{H}$. An anti-linear operator U on $\mathcal{H}$ is <u>anti-unitary</u> iff

$$\left.\begin{aligned} &\langle U\varphi, U\psi \rangle = \langle \psi, \varphi \rangle, \quad \forall \varphi, \psi \text{ in } \mathcal{H} \\ \Leftrightarrow \quad &U^* U = U U^* = \mathbb{1} \end{aligned}\right\} \tag{1.244}$$

such that

$$\sigma\psi = U\psi, \ \forall \psi \in \mathcal{H}, \quad \sigma a = U a U^{-1}, \ \forall a \in B(\mathcal{H}). \tag{1.242}$$

See, e.g., V. Bargmann, J. Math. Phys. 5, 862 (1964).

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$$\sigma_{U_1} \circ \sigma_{U_2} = \sigma_{U_1 \cdot U_2}, \quad \sigma_{\mathbb{1}} = id.$$

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Digression. An operator a on $\mathcal{H}$ is anti-linear iff $a(z\varphi + w\psi) = \bar{z}\, a\varphi + \bar{w}\, a\psi$, $\forall \varphi, \psi$ in $\mathcal{H}$, $\forall z, w$ in $\mathbb{C}$. The adjoint, a^*, of a is defined by

$$\langle a\varphi, \psi \rangle =: \langle a^*\psi, \varphi \rangle, \tag{1.243}$$

$\forall \varphi, \psi$ in $\mathcal{H}$. An anti-linear operator U on $\mathcal{H}$ is anti-unitary iff

$$\left.\begin{aligned} &\langle U\varphi, U\psi \rangle = \langle \psi, \varphi \rangle, \quad \forall \varphi, \psi \text{ in } \mathcal{H} \\ \Leftrightarrow \quad &U^*U = UU^* = \mathbb{1} \end{aligned}\right\} \tag{1.244}$$

If G is a group a projective representation, U, of G on $\mathcal{H}$ by unitary and anti-unitary operators on $\mathcal{H}$ is a map

$$G \ni g \longmapsto U_g \text{ (anti-)unitary on } \mathcal{H}$$

such that

$$\left.\begin{aligned} &U_{g_1} U_{g_2} = \gamma(g_1, g_2)\, U_{g_1 \cdot g_2}, \\ &\text{with} \\ &|\gamma(g_1,g_2)| = 1, \quad \forall g_1, g_2 \text{ in } G. \end{aligned}\right\} \qquad (1.245)$$

Associativity of operator multiplication implies

$$\begin{aligned} (U_{g_1} \cdot U_{g_2}) \cdot U_{g_3} &= \gamma(g_1, g_2)\, U_{g_1 \cdot g_2} U_{g_3} \\ &= \gamma(g_1,g_2)\, \gamma(g_1 \cdot g_2, g_3)\, U_{g_1 \cdot g_2 \cdot g_3} \\ &= U_{g_1} \cdot (U_{g_2} \cdot U_{g_3}) \\ &= \gamma(g_1, g_2 \cdot g_3)\, \gamma(g_2, g_3)\, U_{g_1 \cdot g_2 \cdot g_3}, \end{aligned}$$

$\forall g_1, g_2, g_3$ in G. Hence

$$\boxed{\gamma(g_1,g_2)\,\gamma(g_1 \cdot g_2, g_3) = \gamma(g_1, g_2 \cdot g_3)\,\gamma(g_2, g_3),} \qquad (1.246)$$

i.e., γ is a G-cocycle.

If we set $\hat{U}_g = \varphi(g)\, U_g$, $|\varphi(g)| = 1, \forall g \in G$

then

$$\hat{U}_{g_1} \cdot \hat{U}_{g_2} = \frac{\varphi(g_1)\varphi(g_2)\gamma(g_1, g_2)}{\varphi(g_1 \cdot g_2)}\, \hat{U}_{g_1 \cdot g_2}$$

One says that γ is trivial (γ a co-boundary) iff $\exists$ a $U(1)$-valued function φ on G such that

$$\gamma(g_1, g_2) = \frac{\varphi(g_1 \cdot g_2)}{\varphi(g_1) \cdot \varphi(g_2)} \qquad (1.247)$$

Then $\hat{U}$, with $\hat{U}_g = \varphi(g)\, U_g$, is a <u>unitary</u> representation of G.

There are plenty of groups that have projective representations that are <u>not</u> unitary.

<u>Example:</u> G = additive group of $\mathbb{R}^{2f}$.

For $x \in \mathbb{R}^{2f}$, let $x^- := (x_1, \dots, x_f)$ and $x^+ := (x_{f+1}, \dots, x_{2f})$. We set

$$U_x := W(x^-, x^+), \qquad (1.248)$$

where W is a Weyl operator.

Then, according to (1.207)

$$U_{x_1} \cdot U_{x_2} = W(x_1^-, x_1^+)\, W(x_2^-, x_2^+)$$
$$= e^{\frac{i}{2}(x_1^- \cdot x_2^+ - x_2^- \cdot x_1^+)}\, W(x_1^- + x_2^-, x_1^+ + x_2^+)$$
$$= e^{\frac{i}{2}\langle x_1, \Omega x_2\rangle}\, U_{x_1 + x_2}, \qquad (1.249)$$

where

$$\Omega = \left(\begin{array}{c|c} 0 & \mathbb{1}_f \\ \hline -\mathbb{1}_f & 0 \end{array}\right),$$

i.e.,

$$\gamma(x_1, x_2) = e^{\frac{i}{2}\langle x_1, \Omega x_2\rangle}. \qquad (1.250)$$

Since $\mathbb{R}^{2f}$ is <u>abelian</u>, but the Weyl algebra is <u>non-commutative</u>, the cocycle γ in (1.250) is <u>non-trivial</u>.

<u>Physical example</u>: "Magnetic translations" in $\mathbb{R}^2$.

If G is a compact connected group then every continuous projective representation of G comes from a continuous unitary representation of the universal covering group, $\tilde{G}$. (See V. Bargmann, Ann. Math. <u>59</u>, 1 (1954).)

Example. $G = SO(3)$, $\tilde{G} = SU(2)$

(viii) Time evolution

We consider time t to be a real number. The time axis consists of the real numbers $\mathbb{R}$, which is an abelian group, with

$$\mathbb{R} \times \mathbb{R} \ni (t_1, t_2) \longmapsto t_1 + t_2 = t_2 + t_1 \in \mathbb{R}$$

If S is an autonomous physical system then time translations

$$\tau_t : s \longmapsto s + t$$

$$\tau_{t_1} \circ \tau_{t_2} = \tau_{t_1 + t_2}$$

are a symmetry of S represented projectively on the Hilbert space, $\mathcal{H}$, of pure state vectors of S: If $[\psi]$ denotes the state of S at
ray rep. by ψ!
some time t_0, then

$$[\psi_{t_1 + t_0}] = [U_{t_1} \psi]$$

denotes the state of S at time $t_1 + t_0$, where

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$$[\psi_{t_1 + t_0}] = [U_{t_1} \psi]$$

denotes the state of S at time $t_1 + t_0$, where

$\{U_t\}_{t\in\mathbb{R}}$ is a projective representation of $\mathbb{R}$ on $\mathcal{H}$; i.e.,

$$U_{t_1} \cdot U_{t_2} = \gamma(t_1,t_2)\, U_{t_1+t_2} \qquad (1.251)$$

The cocycle property (1.246) implies that

$$\gamma(r,s)\,\gamma(r+s,t) = \gamma(r,s+t)\,\gamma(s,t), \qquad (1.252)$$

$\forall\ r,s,t$ in $\mathbb{R}$.

For $s=t=0$, we have that

$$\begin{aligned}(U_r \cdot U_0)\cdot U_0 &= \gamma(r,0)\, U_r \cdot U_0 \\ &= \gamma(r,0)^2\, U_r \\ &= U_r \cdot (U_0 \cdot U_0) \\ &= U_r\, \gamma(0,0)\, U_0 \\ &= \gamma(r,0)\,\gamma(0,0)\, U_r\end{aligned}$$

Hence $\gamma(r,0) = \gamma(0,0)$ (1.253)

Exchanging the roles of r and t, one finds that

$$\gamma(0,r) = \gamma(0,0) \qquad (1.254)$$

Defining

$$U'_t = \gamma(0,0)^{-1} U_t$$

we find that

$$U'_t U'_0 = \gamma(0,0)^{-2} \gamma(t,0) U_t$$

$$\overset{(1.253)}{=} \gamma(0,0)^{-1} U_t$$

$$= U'_t .$$

Denoting U'_t henceforth by U_t, $t \in \mathbb{R}$, we can assume w.l.o.g. that

$$\gamma(t,0) = \gamma(0,t) = \gamma(0,0) = 1. \qquad (1.255)$$

<u>Lemma.</u> $\exists$ a function φ on $\mathbb{R}$ such that

$$\gamma(r,s) = \frac{\varphi(r+s)}{\varphi(r)\cdot\varphi(s)} \qquad (1.256)$$

i.e., γ is a trivial cocycle; (see (1.246)).

<u>Proof.</u> If (1.256) holds then

$$\varphi(0) = 1, \qquad (1.257)$$

by (1.255). With $\varphi(t)$

$$\tilde{\varphi}(t) := \varphi(t) e^{i\alpha t}, \quad \alpha \in \mathbb{R},$$

also satisfies (1.256). Thus, we may impose the condition

$$\frac{d\varphi}{dt}(0) = 0 \tag{1.258}$$

Eq. (1.256) would then imply that

$$\left.\frac{\partial \gamma}{\partial s}(r,s)\right|_{s=0} = \frac{\varphi'(r)}{\varphi(r)} - \frac{\varphi'(0)}{\varphi(0)^2}$$

$$= \frac{d}{dr} \ln \varphi(r), \tag{1.259}$$

by (1.257), (1.258). (Here we assume that $\gamma(r,s)$ is differentiable in s.) Eq. (1.259) suggests to <u>define</u> $\varphi(r)$ as the solution of the ODE (1.259) with initial condition $\varphi(0) = 1$. Differentiating (1.252) in s and setting $t = 0$ yields, after using (1.255) and (1.259),

$$\gamma(r,s) \frac{\partial}{\partial s} \ln \varphi(r+s) = \frac{\partial}{\partial s} \gamma(r,s)$$
$$+ \gamma(r,s) \frac{d}{ds} \ln \varphi(s),$$

hence

$$\frac{\partial}{\partial s} \ln \gamma(r,s) = \frac{\partial}{\partial s}\left\{ \ln \varphi(r+s) - \ln \varphi(s) \right\}$$

$$= \frac{\partial}{\partial s} \ln \frac{\varphi(r+s)}{\varphi(r)\,\varphi(s)}\,, \qquad (1.260)$$

because $\frac{\partial}{\partial s} \ln \varphi(r) = 0$. At $s = 0$

$$\ln \gamma(r,0) = 0 = \ln \frac{\varphi(r)}{\varphi(r)\varphi(0)}\,,$$

by (1.255) and (1.257). Together with (1.260) this implies that

$$\ln \gamma(r,s) = \ln \frac{\varphi(r+s)}{\varphi(r)\varphi(s)}\,,$$

whence (1.256); (integrate (1.260) in s!). $\square$

<u>Time evolution for non-autonomous systems</u>

We suppose here (and this will be criticised shortly) that time evolution of states of a physical system S maps a vector ψ_{t_1} representing a pure state of S at time t_1 to a vector ψ_{t_2} at any other time t_2, and that this map is given by a symmetry; assuming that S is isolated for all times between $\min(t_1,t_2)$ and $\max(t_1,t_2)$, in particular no "measurements" are carried out in this interval of times. Then there exists a unitary or anti-unitary operator $U(t_2,t_1)$ on $\mathcal{H}$ such that

$$\psi_{t_2} = U(t_2,t_1)\,\psi_{t_1}, \quad \forall \psi_{t_1} \in \mathcal{H}. \qquad (1.261)$$

Let t_3 be a third time and assume that S is isolated for all times between $\min(t_1,t_2,t_3)$ and $\max(t_1,t_2,t_3)$. Then

$$\psi_{t_3} = U(t_3,t_2)\,\psi_{t_2} = U(t_3,t_2)\,U(t_2,t_1)\,\psi_{t_1},$$
$$= U(t_3,t_1)\,\psi_{t_1},$$

for an arbitrary initial condition $\psi_{t_1} \in \mathcal{H}$, i.e.,

$$U(t_3,t_1) = U(t_3,t_2)\,U(t_2,t_1) \qquad (1.262)$$

We will always assume that

$$\lim_{t' \to t} U(t',t) = \mathbb{1}, \quad \forall t \in \mathbb{R}. \qquad (1.263)$$

By (1.263), $U(t_2, t_1)$ is <u>unitary</u> if $|t_2 - t_1|$ is small enough (dep. on t_1). Then (1.262) implies that $U(t_2, t_1)$ is unitary for arbitrary t_1, t_2, assuming that S remains isolated for all times between t_1 and t_2.

We now assume that there exists a domain $\mathcal{D}$ dense in $\mathcal{H}$ with the property that $U(t, t_0)\psi$ is strongly differentiable in t, $\forall \psi \in \mathcal{D}$. We define

$$H(t) := i\hbar\left(\frac{\partial}{\partial t} U(t, t_0)\right)\cdot U(t_0, t) \qquad (1.264)$$

$$= i\hbar\left(\frac{\partial}{\partial t} U(t, t_1)\cdot U(t_1, t_0)\right) U(t_0, t_1)\, U(t_1, t)$$

$$= i\hbar\left(\frac{\partial}{\partial t} U(t, t_1)\right) U(t_1, t),$$

for arbitrary t_0, t_1; i.e., $H(t)$ is <u>indep.</u> of t_0.

Conversely, suppose that a family of time-dependent Hamiltonians $\{H(t) = H(t)^*\}_{t\in\mathbb{R}}$ is given.

Under what conditions on $\{H(t)\}_{t\in\mathbb{R}}$ does the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi_t = H(t)\psi_t, \quad \psi_{t=t_0} = \psi \in \mathcal{D},$$

for some dense domain $\mathcal{D}$, have a unique solution? A sufficient condition for an affirmative answer is given in the following theorem.

<u>Theorem</u> (Reed & Simon, vol. III, Theorem X.70)
<u>Existence of propagators</u>

$\mathcal{H}$ separable, $I \subseteq \mathbb{R}$ an interval of times, $\{H(t)\}_{t\in\mathbb{R}}$ a family of time-dependent Hamiltonians with the properties

(a) all operators $H(t)$ are defined and essentially self-adjoint on a common domain $\mathcal{D}$ dense in $\mathcal{H}$, $\forall t \in I$;

(b) for all t, s in I, $\dot{H}(t)(H(s) \pm i)^{-1}$ is a bounded operator on $\mathcal{H}$ that is strongly continuous in t and s.

Under what conditions on $\{H(t)\}_{t\in\mathbb{R}}$ does the Schrödinger equation

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Then $\exists$ unitary operators $U(t,s)$ such that

$$i\hbar \frac{\partial}{\partial t} U(t,s)\psi = H(t)U(t,s)\psi,$$

$\forall \psi \in \mathcal{D}$,

$$U(t,s)U(s,r) = U(t,r),$$

and

$$\underset{t \to s}{s\text{-}\lim}\, U(t,s) = \mathbb{1}$$

For autonomous systems, the analogue of this result is Stone's Theorem:

If $\{U(t)\}_{t\in\mathbb{R}}$ is a continuous, unitary one-parameter group on $\mathcal{H}$ with $\underset{t\to 0}{s\text{-}\lim}\, U(t) = \mathbb{1}$ then there exists a densely defined self-adjoint operator H such that

$$U(t) = e^{-iHt/\hbar}, \qquad (1.265)$$

and conversely.

If P_{t_1} is the density matrix corresponding to a mixed state of S at time t_1 and $\{U(t_2,t_1)\}$ is the propagator of S then

$$P_{t_2} := U(t_2, t_1)\, P_{t_1}\, U(t_1, t_2) \qquad (1.266)$$

is the state of S at time t_2, assuming S is isolated in the interval $[\min(t_1, t_2), \max(t_1, t_2)]$.

Not treated: Projective reps. of the groupoid $\mathbb{R} \times \mathbb{R}$, (with $(t_3, t_2) \circ (t_2, t_1) = (t_3, t_1)$).

In the following we will never use the Schrodinger picture, but consistently make use of the Heisenberg picture. We will characterize systems S in terms of physical quantities, $\mathcal{O}_S$, (see (1.226)), and the algebra, $\mathcal{A}_S$, generated by $\mathcal{O}_S$, (see (1.228)).

If S is isolated time evolution of operators in $\mathcal{A}_S$ in the Heisenberg picture is defined with the help of *automorphisms

$$\alpha_{t_2, t_1} : a(t_1) \longrightarrow a(t_2) := \alpha_{t_2, t_1}(a(t_1))$$

with the interpretation that if $a(t_1)$ and $a(t_2)$ represent the same abstract physical quantity $\hat{a} \in \mathcal{O}_S$ at two different times t_1, t_2, resp., then $\exists$ a *automorphism α_{t_2,t_1} of $\mathcal{A}_S$ such that

$$a(t_2) = \alpha_{t_2,t_1}(a(t_1)). \qquad (1.267)$$

A map $\alpha : \mathcal{A}_S \to \mathcal{A}_S$ is a *automorphism iff α is a homomorphism mapping $\mathcal{A}_S$ one-to-one onto $\mathcal{A}_S$, and

$$\alpha(a)^* = \alpha(a^*). \qquad (1.268)$$

The *automorphisms $\{\alpha_{t_2,t_1}\}_{(t_1,t_2) \in \mathbb{R} \times \mathbb{R}}$ have the property that

$$\alpha_{t_3,t_2} \circ \alpha_{t_2,t_1} = \alpha_{t_3,t_1}, \qquad (1.269)$$

$\forall\, t_1, t_2, t_3$.

The system S is <u>autonomous</u> iff

$$\alpha_{t_2,t_1} =: \alpha_{t_2 - t_1}$$

only depends on the difference, $t_2 - t_1$, of the two times t_1, t_2; then $\{\alpha_t\}_{t\in\mathbb{R}}$ is a one-parameter group of *automorphisms.

The generator, δ, of a one-parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ of *automorphisms of a C^*-algebra $\mathcal{A}$ is a <u>derivation</u>:

$$i\hbar \frac{d}{dt} \alpha_t(a) =: \delta(\alpha_t(a)),$$

$\forall\, a \in \mathcal{A}_\delta$, where $\mathcal{A}_\delta$ is a norm-dense *subalgebra of $\mathcal{A}$, and

$$\delta : \mathcal{A}_\delta \to \mathcal{A}, \text{ with}$$

$$\underbrace{\delta(a\cdot b) = \delta(a)\cdot b + a\,\delta(b)}_{\text{Leibniz Rule}}, \qquad (1.270)$$

and

$$\delta(a^*) = \delta(a)^*. \qquad (1.271)$$

If

$$\mathcal{A} = C(\mathcal{X}), \quad (\mathcal{X} \text{ as above})$$

then every *automorphism $\alpha : \mathcal{A} \xrightarrow{1-1} \mathcal{A}$ corresponds to a homeomorphism

$$\varphi : \mathfrak{X} \xrightarrow{1-1} \mathfrak{X}$$

and <u>conversely</u>. A one-parameter group of *automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$ of $\mathfrak{A}$ corresponds to a flow $\{\varphi_t\}_{t\in\mathbb{R}}$ on $\mathfrak{X}$, with

$$\varphi_t : \mathfrak{X} \xrightarrow{1-1} \mathfrak{X}, \quad \forall t,$$

$$\varphi_0 = id., \; \varphi_t \circ \varphi_s = \varphi_{t+s}, \quad \forall t, s \text{ in } \mathbb{R}.$$

If $\mathfrak{X}$ has a differentiable structure and $\{\alpha_t\}_{t\in\mathbb{R}}$ is generated by a derivation δ densely defined on $\mathfrak{A}$ then δ corresponds to a globally defined vector field, X, on $\mathfrak{X}$. Non-autonomous systems give rise to analogous theorems.

Mathematically precise formulations of these statements and proofs (which are not very difficult) are omitted.

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2. New Theories as "Deformations" of Ancestor Theories

A physical system S is described – classically or quantum-mechanically – by

(1) a C^*-algebra, $\mathcal{A}_S$, generated by a family, $\mathcal{O}_S$, of abstract selfadjoint linear operators representing physical quantities characteristic of S. $\mathcal{S}(\mathcal{A}_S)$ denotes the convex space of "states of physical interest" on $\mathcal{A}_S$.

(2) A group, G_S, of symmetries of S acting as *automorphisms on $\mathcal{A}_S$:

$$G_S \ni g \longmapsto \alpha_g \in \mathrm{aut}(\mathcal{A}_S). \tag{2.1}$$

A theory of physical systems specifies the class of systems it pretends to describe and it specifies:

($\otimes$) a prescription for how to compose different systems:

$$(S, S') \longrightarrow S \vee S' \tag{2.2}$$

New theories may arise by <u>deformations</u> of $\mathcal{A}_S$, G_S and the composition law $\vee$.

<u>Deformations</u> of a mathematical object, O, are supposed to preserve the nature of O and are labelled by a <u>deformation parameter</u>, which, in physics, tends to be a dimensionful real number; (value $= 0 \leftrightarrow$ trivial deformation).

<u>Examples.</u>

- <u>Deformation of symmetries</u>, G_S:

c^{-1}: Galilei symmetry $\rightarrow$ Poincaré symmetry

Newtonian space-time $\rightarrow$ Minkowski space

External magnetic field,

B: Translations of $\mathbb{R}^2$ $\rightarrow$ magnetic translations, Heisenberg group, $\mathbb{H}^2$, with $\mathcal{G}(\mathbb{H}^2) = \mathcal{A}_W$

(Euclidian motions in $\mathbb{R}^2 \rightarrow \ldots$)

This example has been treated in Chapter

1.6, (iv).

- <u>Deformation of "observable algebra"</u>, $\mathcal{A}_S$:

$\hbar$: $C^\infty(\mathbb{R}^{2f}) \longrightarrow$ Weyl algebra $\mathcal{A}_W$ (or "resolvent algebra")

(formal "deformation quantization, à la Kontsevich – Cattaneo – Felder, sketched later)

k_B: continuum (field) theory of matter $\longrightarrow$ atomistic theory of matter;

e.g., Vlasov theory $\longrightarrow$ Hamiltonian mechanics of point particles,

or Hartree theory $\longrightarrow$ quantum many-body theory

Also: "Deformation of Geometry" (G_N, …, NC geom.)

- <u>Deformation of composition law</u>, V:

ϑ: theory of bosons/fermions $\longrightarrow$ theory of "dyons" or "anyons"

ϑ can be interpreted as coefficient of a topological term (abelian Chern–Simons term) in an action functional.

Summary: Planck–Bronstein (hyper-) cube;

see pages 1.17 and 1.18.

Next, we discuss some examples.

2.1 From Galilei symmetry to Poincaré symmetry

This is an example concerning deformations of Lie algebras and Lie groups. The deformation parameter is the inverse of the speed of light, c^{-1}.

Here are the structure relations of the Lie algebra of the Poincaré group, $\mathcal{P}_+^\uparrow = i\,SO(1,3)$:

$x^0 = ct$, t = time, $\vec{x} = (x^1, x^2, x^3)$.

$E^{\mu\nu}$: coordinate plane corresp. to

$$x^\lambda = 0, \text{ for } \lambda \neq \mu, \nu.$$

J^{ij}: generator of a rotation in E^{ij},

$$i, j = 1, 2, 3, \ (i \neq j).$$

J^{0i}: generator of a boost in E^{0i}

P^μ: generator of translation in x^μ-direction.

Then we have that

$$[J^{0i}, J^{0j}] = -\frac{1}{c^2} J^{ij} = \frac{1}{c^2} J^{ji}$$

$$[J^{0i}, J^{ij}] = -J^{0j}, \qquad \forall\, i,j = 1,2,3,\ i \neq j.$$

$$[J^{ij}, J^{kl}] = \delta^{jk} J^{il} - \delta^{ik} J^{jl} + \delta^{il} J^{jk} - \delta^{jl} J^{ik}$$

$$[P^\mu, P^\nu] = 0, \quad \forall\, \mu, \nu$$

$$[J^{ij}, P^j] = -P^i, \quad [J^{0i}, P^i] = -\frac{1}{c^2} P^0$$

$$[J^{0i}, P^0] = P^i; \tag{2.3}$$

further commutators vanish. The parameter c is the speed of light. In the limit $c \to \infty$, the Lie algebra of the Poincaré group contracts to the Lie algebra of the Galilei group, $G_+^\uparrow$, which is the symmetry group of celestial mechanics. Put differently, $\mathcal{P}_+^\uparrow$ can be viewed as a deformation of $G_+^\uparrow$. We will have to ask whether relativistic mechanics can be obtained by "deforming" Newtonian mechanics. Answer: No!

Digression

Lorentz trsf. in E^{01}

$$\left(\begin{array}{cc|cc} \cosh\chi & \sinh\chi & \multicolumn{2}{c}{0} \\ \sinh\chi & \cosh\chi & & \\ \hline \multicolumn{2}{c|}{0} & 1 & 0 \\ & & 0 & 1 \end{array}\right)\left(\begin{array}{c} ct \\ x \\ \hline y \\ z \end{array}\right)$$

$$ct' = \cosh\chi\, ct + \sinh\chi\, x$$

$$x' = \sinh\chi\, ct + \cosh\chi\, x$$

$$\boxed{\cosh\chi = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \sinh\chi = \frac{\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}}$$

$$\frac{d}{dv}\begin{pmatrix} \cos\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix} = \begin{pmatrix} \frac{2v}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}} & \frac{\frac{1}{c}\sqrt{1-\frac{v^2}{c^2}}+\frac{v^2}{c^3}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \\ * & * \end{pmatrix}$$

$$\xrightarrow[v\to 0]{} \underline{\frac{1}{c}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$\overset{01}{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \cdot \overset{02}{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} - \longleftrightarrow$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

②

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = J_{12}$$

$$(p^0, p^1) = \left(\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \right) \rightarrow 4 \text{ vector}$$

<u>Deformation to de Sitter group.</u>

Let M^{n+1} denote $(n+1)$-dim. Minkowski space.

n dim. de Sitter space, dS_n, is the hyperboloid

$$(x^0)^2 - \vec{x}^{\,2} = -r^2, \qquad (2.4)$$

$\vec{x} = (x^1, \cdots, x^n)$, $r > 0$, embedded in M^{n+1}.

Clearly, the Lorentz group, $SO(1,n)$, acts as isometries on dS_n. Its generators are denoted by $J^{\mu\nu} = -J^{\nu\mu}$, $\mu, \nu = 0, 1, \cdots, n$.

Topologically,

$$dS_n \simeq \mathbb{R} \times S^{n-1},$$

so that, for $n \geq 3$, dS_n is simply connected;

dS_n can also be defined as

$$dS_n \simeq SO(1,n)/SO(1,n-1),$$

i.e., dS_n is a Lorentzian maximally symmetric space and hence has "<u>constant curvature</u>", with

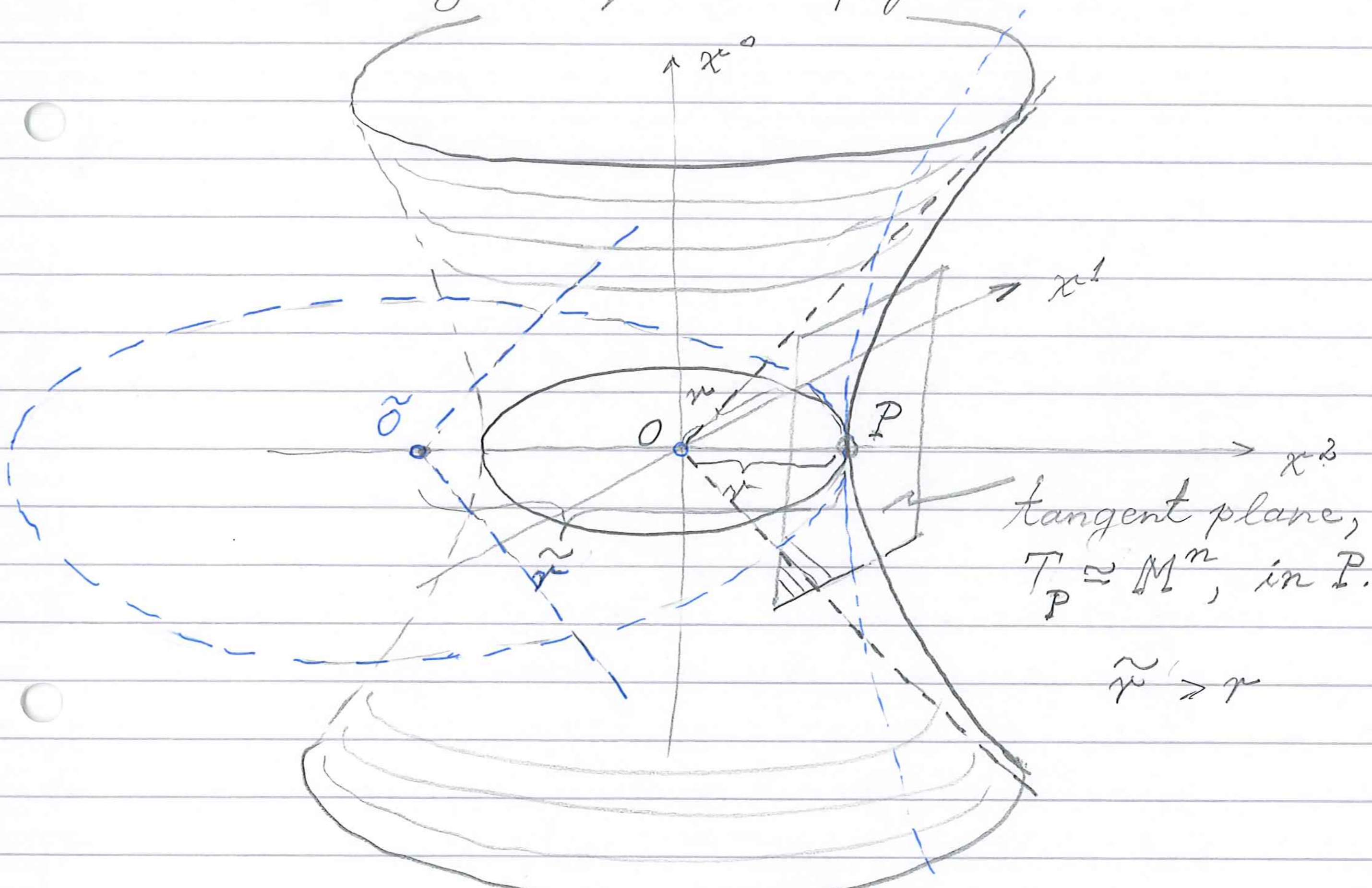
$$R_{\rho\sigma\mu\nu} = \frac{1}{r^2}\left(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu}\right),$$

hence

$$R_{\mu\nu} = \frac{n-1}{r^2} g_{\mu\nu} ,$$

$$R = \frac{n(n-1)}{r^2} = \frac{2n}{n-2} \Lambda ,$$

where Λ is the cosmological constant in Einstein's field equation of gravitation.



Note that, as $r \to \infty$, keeping P fixed, dS_n approaches M^n, $R_{\rho\sigma\mu\nu} \to 0$; the isometry group of dS_n contracts to the Poincaré group acting on $T_P \simeq M^n$.

Rescaling the generators, $M^{\mu\nu}$, $\mu,\nu = 0,1,\cdots,n$, of $SO(1,n)$

$$M_r^{n\mu} = \frac{1}{r} M^{n\mu}, \quad \mu = 0,1,\cdots,n-1,$$

we find that

$$\left.\begin{aligned} &M_r^{0n} \longrightarrow P^0, \quad M_r^{jn} \longrightarrow P^j, \quad j = 1,\cdots,n-1, \\ &M_r^{0i} \longrightarrow J^{0i}, \quad M_r^{ij} \longrightarrow J^{ij}, \quad i,j = 1,\cdots,n-1 \end{aligned}\right. \tag{2.5}$$

where the J's and the P's become the generators of the Poincaré group on T_P.

To clarify this, introduce the coordinates

$$\left.\begin{aligned} x^0 &:= r\sinh\left(\frac{\tau}{r}\right) + \frac{R^2}{2r} e^{\frac{\tau}{r}} \\ x^n &:= r\cosh\left(\frac{\tau}{r}\right) - \frac{R^2}{2r} e^{\frac{\tau}{r}} \\ x^i &:= e^{\frac{\tau}{r}} y^i, \quad i = 1,\cdots,n-1, \end{aligned}\right\} \tag{2.6}$$

with $R^2 := \sum_{i=1}^{n-1} (y^i)^2$. Then, in the coordinates

$$y^0 := \tau, \quad \vec{y} = (y^1,\cdots,y^{n-1})$$

the metric on dS_n in a neighborhood of P becomes

$$ds^2 = (dy^0)^2 - e^{2y^0/r} d\vec{y}^2,$$

In the limit $r \to \infty$, it approaches the Minkowski space metric on T_P, after a translation by r in the $-x^n$ direction.

The general theory of deformations of Lie algebras leads one to study the cohomology of Lie algebras: If $\mathcal{L}$ is a finite-dimensional Lie algebra over $\mathbb{K} := \mathbb{R}$, or $\mathbb{C}$ then its deformations are labelled by the elements of $H^2(\mathcal{L}; \mathcal{L})$, which is the cohomology group of the space of $\mathcal{L}$-valued 2-cochains, i.e., of elements of $\mathrm{Hom}_{\mathbb{K}}(\Lambda^2 \mathcal{L}, \mathcal{L})$. Elements of $\mathrm{Hom}_{\mathbb{K}}(\Lambda^2\mathcal{L}, \mathcal{L})$ describe deformations of the Lie bracket of $\mathcal{L}$. Two such deformations, φ and φ', in $\mathrm{Hom}_{\mathbb{K}}(\Lambda^2\mathcal{L}, \mathcal{L})$ are equivalent iff there exists an element $\psi \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{L}, \mathcal{L})$ such that

$$\varphi' - \varphi = d\psi,$$

where d is the differential in the complex of cochains, $\bigoplus_n \mathrm{Hom}_{\mathbb{K}}(\Lambda^n \mathcal{L}, \mathcal{L})$.

<u>Definition of d:</u>

For $T_1 \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{L}, \mathcal{L})$, i.e.,

$$T_1 : \mathcal{L} \ni X \longmapsto T_1(X) \in \mathcal{L} \quad \text{(linear)}$$

we define

$$(dT)_2(X, Y) := T_1([X, Y]), \quad \forall X, Y \text{ in } \mathcal{L},$$

and, for $T_2 \in \mathrm{Hom}(\Lambda^2 \mathcal{L}, \mathcal{L})$, we define

$$(dT)_3(X, Y, Z) := T_2([X, Y], Z) + T_2([Z, X], Y) + T_2([Y, Z], X),$$

$\forall X, Y, Z$ in $\mathcal{L}$. Then

$$\begin{aligned}(d(dT))_3(X, Y, Z) &= T_1([[X, Y], Z]) + T_1([[Z, X], Y]) \\ &\quad + T_1([[Y, Z], X]) \\ &= 0, \qquad \text{(calculation!)}\end{aligned}$$

for arbitrary X, Y, Z in $\mathcal{L}$; etc.

I will not go more deeply into this theory.

If, instead of deforming $\mathcal{L}$, one studies the

Déf. d'algèbres de Hopf et de catégories ⊗ (c.-à-d. de la notion de comp., ∨, de systèmes phys. élémentaires)

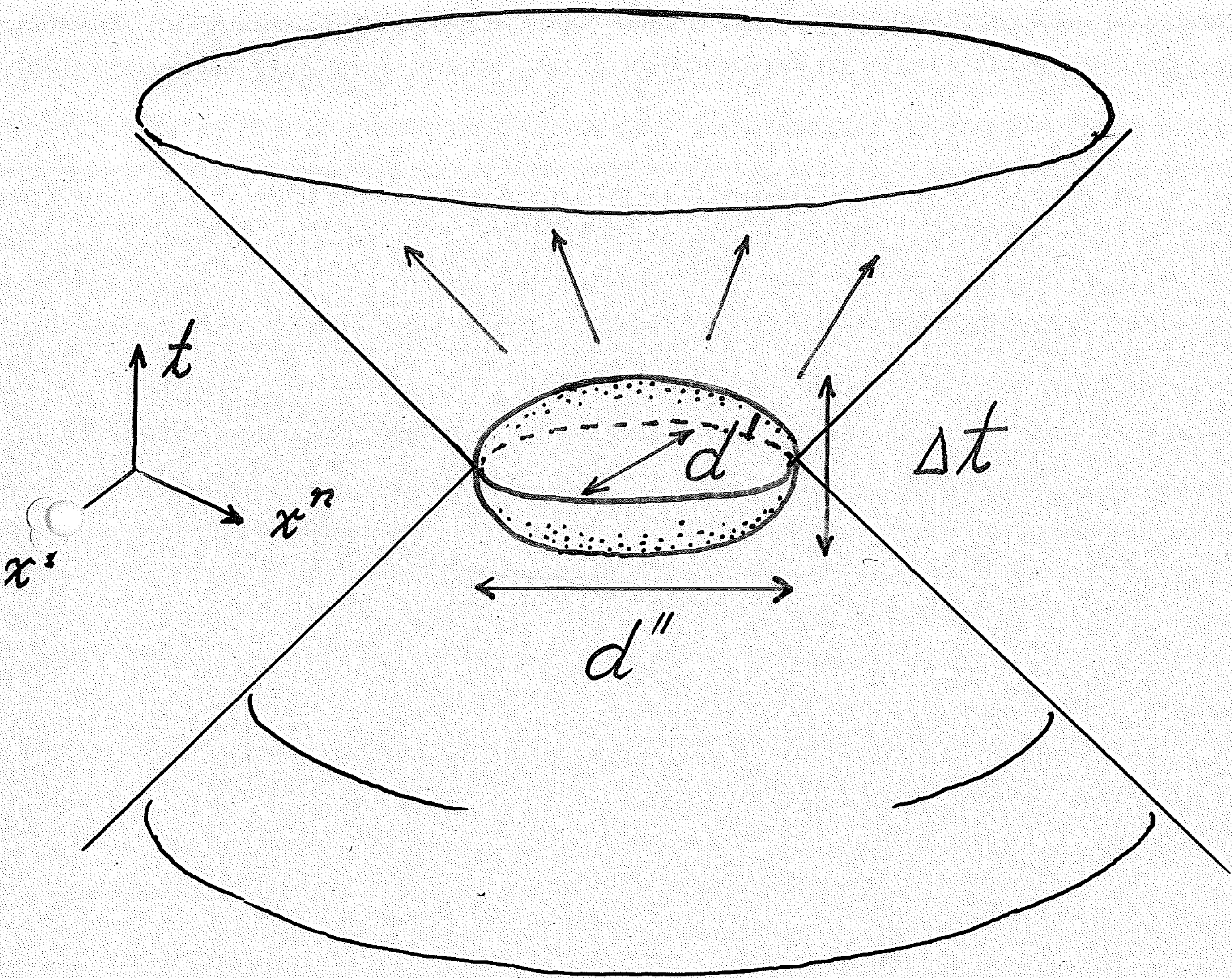
sym. de groupes → groupes quantiques

bosons
fermions } → "anyons" 2D

c^{-1}, l_P, h: Déformations de la géométrie et de la structure d'espace-temps;

$\boxed{\text{à développer !}}$

Localisation d'évènements



Relation d'incertitudes :

$$\Delta t \cdot d'^{(1)} \gtrsim l_P^2$$

$$d' \cdot d'' \gtrsim l_P^2$$

Théorie mathématique : ?

deformations of its universal enveloping algebra, $U(\mathcal{L})$, one lands on the theory of <u>quantum groups</u>. This theory plays an interesting role in the theory of deformations of V.

→ p. 2.11', 2.11''

2.2 <u>Deformations of "algebras of observables"</u>

This topic belongs to the general theory of deformations of associative algebras developed by M. Gerstenhaber and by A. Nijenhuis and R. W. Richardson in the early 60's. An important contribution relevant for the problem of quantizing general phase spaces was made by M. Kontsevich and refined by A. Cattaneo and G. Felder.

Ideally, we would like to be able to deform the algebra of smooth functions, $C^\infty(\Gamma_S)$, on a phase space Γ_S (= smooth symplectic manifold) of a physical system S to an "algebra of observables", $\mathcal{A}_S$, well suited for a quantum-mechanical description of S. To accomplish this, we must identify a family, $\mathcal{O}_S^{class.}$, of real-valued functions on Γ_S generating $C^\infty(\Gamma_S)$ (under complex linear combinations and multiplication). We should then associate an abstract self-adjoint linear operator, $\hat{a}$, with every element $a \in \mathcal{O}_S^{class.}$. We will then set

$$\mathcal{O}_S := \left\{ \hat{a} \,\middle|\, a \in \mathcal{O}_S^{class.} \right\} \qquad (2.7)$$

and define $\mathcal{A}_S$ to be the *algebra generated by $\mathcal{O}_S$. We would like $\mathcal{O}_S$ to have the property that, for a, b in $\mathcal{O}_S$, the Poisson bracket, $\{a, b\}$, belongs to $\mathcal{O}_S$, too. We might then require that

$$\widehat{\{a,b\}} = \frac{i}{\hbar}[\hat{a},\hat{b}], \tag{2.8}$$

for an appropriate choice of $\hbar$.

We have already implemented this program for

$$\Gamma_S = \mathbb{R}^{2f}, \tag{2.9}$$

(see Sect. 1.6 – von Neumann uniqueness theorem), and for

$$\Gamma_S = S^2$$

$\rightarrow$ representation theory of $SU(2)$; ($\hbar = s \in \frac{1}{2}\mathbb{Z}_+$ is quantized).

In Sect 1.5 (iv) (Stückelberg, Buchholz – Fredenhagen) we have learned (implicitly) how to quantize phase spaces

$$\Gamma_S = T^*M, \tag{2.10}$$

where M is an arbitrary (smooth Riemannian) manifold; i.e., we know how to quantize phase spaces that are general cotangent bundles; (something Schrödinger had already discovered).

We recall that, for Hamiltonians, $H(p,q)$, (p,q): Darboux coordinates on $\mathbb{R}^{2f}$, that are quadratic in p, the Dirac-Feynman path integral can be derived from a phase-space path integral: We define an action functional

$$S(p,q) := \int \left\{ \sum_{j=1}^{f} p_j(t)\, \dot{q}^j(t) - H\big(p(t), q(t)\big) \right\} dt, \tag{2.11}$$

where

$$H(p,q) = \sum_{j=1}^{f} \frac{p_j^2}{2m_j} + V(q). \tag{2.12}$$

Let

$$\text{"} \mathcal{D}p \wedge \mathcal{D}q = \text{const.} \prod_{s=T}^{t} d^f p(s)\, d^f q(s) \text{"}$$

be the formal Lebesgue measure on path space over $\mathbb{R}^{2f}$. Then

$$U(t,T)_{qQ} := \int_{\substack{q(T)=Q \\ q(t)=q}} \mathcal{D}p \wedge \mathcal{D}q \; e^{iS(p,q)} \tag{2.13}$$

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$$U(t,T)_{qQ} := \int_{\substack{q(T)=Q \\ q(t)=q}} \mathcal{D}p \wedge \mathcal{D}q \; e^{iS(p,q)} \tag{2.13}$$

has the property that the integrations over the variables $\{p(s) \mid T \leq s \leq t\}$ are independent Gaussian integrals that can be carried out explicitly, with the result

$$\int_{\substack{q(T)=Q \\ q(t)=q}} \mathcal{D}p \wedge \mathcal{D}q \; e^{iS(p,q)} =$$

$$= \int_{\substack{q(T)=Q \\ q(t)=q}} e^{i \int_T^t ds\, L(\dot{\underset{\sim}{q}}(s), \underset{\sim}{q}(s))}, \qquad (2.14)$$

where

$$L(\dot{q}, q) = \sum_{j=1}^{f} \frac{m_j}{2} (\dot{q}^j)^2 - V(q). \qquad (2.15)$$

This connection between phase-space and configuration space path integrals works for phase spaces $\Gamma_S = T^*M$, M a Riemannian manifold
← cotangent bundle of M!
with (kinetic-energy) metric, $g_{ij}(q)$, and Lagrangians

$$L(\dot{\underset{\sim}{q}}, \underset{\sim}{q}) = \frac{1}{2} \sum_{i,j} \dot{q}^i g_{ij}(q) \dot{q}^j - V(q) \qquad (2.16)$$

corresponding to a Hamiltonian

$$H(p,q) = \frac{1}{2}\sum_{i,j} p_i g^{ij}(q) p_j + V(q), \quad (2.17)$$

where $g^{ij}(q)$ is the inverse metric.

Consider a general phase space Γ_S with local Darboux coordinates (p,q), a Hamiltonian $H(p,q)$ and an action functional

$$S_\gamma := \int_\gamma \left\{ \sum_{j=1}^{f} p_j \, dq^j - H(p,q)\, dt \right\},$$

where γ is a loop in Γ_S and t a real variable parametrizing γ. We will now set $H(p,q) \equiv 0$, so that S_γ becomes a "topological action".

Let D be a 2D disk in Γ_S with

$$\partial D = \gamma.$$

We parametrize D as follows:

$$D = \left\{ X(\tau) \,\middle|\, \tau = (\tau^1, \tau^2) \in D_0 \right\}, \quad (2.18)$$

where $D_0 = \left\{ (\tau_1, \tau_2) \,\middle|\, \tau_1^2 + \tau_2^2 \leq 1 \right\}$,

$X = (p,q) \equiv (X^A)_{A=1,\dots,2f}$. Then

$$S_\gamma = \int_{\partial D} \left\{ \sum_{j=1}^{f} p_j \, dq^j \right\} \underset{\substack{\uparrow \\ \text{Stokes'}}}{=} \int_D \omega$$

$$= \frac{1}{2} \int_D \omega_{AB}(X) \, dX^A \wedge dX^B \qquad (2.19)$$

where ω is the symplectic 2-form on Γ_S.

Let $(\Omega^{AB}(X))$ be the inverse of $(\omega_{AB}(X))$; i.e., Ω defines a Poisson structure on Γ_S.

We introduce the action functional (↗ Strobl)

$$S_D := \int_{D_0} d\tau^\alpha \wedge d\tau^\beta \left\{ \frac{\partial X^A(\tau)}{\partial \tau^\alpha} \eta_{A,\beta}(\tau) - \right.$$

$$\left. - \frac{1}{2} \Omega^{AB}(X(\tau)) \, \eta_{A,\alpha}(\tau) \eta_{B,\beta}(\tau) \right\}$$

We note that S_D is invariant under reparametrizations of D_0 (integral of differential forms!) and symplectomorphisms of Γ.

The action S_D is quadratic in $\eta_{A,\alpha}$, so that

$$\int \mathcal{D}\eta \; e^{\frac{i}{\hbar} S_D(X,\eta)}$$

is a Gaussian integral. Its value is

$$\int \mathcal{D}\eta \, e^{\frac{i}{\hbar} S_D(X,\eta)} = \text{const.}\, e^{\frac{i}{2\hbar}\int_D \omega_{AB}(X)\, dX^A \wedge dX^B} \tag{2.20}$$

We would like to carry out the functional integral over η <u>and</u> over X. Since the action $S_\gamma = \int_D \omega$ is <u>"topological"</u> (because we have set $H \equiv 0$), S_D actually has an ∞-dimensional group of symmetries. Thus, to carry out the <u>formal</u> functional integral

$$[f] := \int \mathcal{D}X \mathcal{D}\eta \; e^{\frac{i}{\hbar} S_D(X,\eta)} f(X),$$

we must do some <u>gauge fixing</u>. Actually, one employs the <u>Batalin-Vilkovisky formalism</u> to solve this problem: Introduction of ghosts and anti-ghosts; (Strobl, Cattaneo-Felder, Cattaneo et al.).

We now deform the product of functions a, b in $C^\infty(T_S)$ (with support in a Darboux chart) as follows:

$$(a *_\hbar b)(X) := {}''\!\int \mathcal{D}X\,\mathcal{D}\eta\, e^{\frac{i}{\hbar}S_D(X,\eta)} \times$$

$$\times\, a(X(0))\, b(X(1))''$$

$$= \left\{ a\cdot b + \frac{i\hbar}{2}\{a,b\} + \cdots \right\}(X), \quad (2.21)$$

where $X = X(\infty)$, with $t \in \mathbb{R}$ a parametrization of ∂D_0; $X(0), X(1)$ and $X(\infty) = X$ in ∂D.

When evaluated properly (see Kontsevich; Cattaneo & Felder) (2.21) determines a deformation of $C^\infty(T_S)$ to a non-commutative algebra, in the sense of <u>formal</u> <u>power series</u> in $\hbar$.

Note that we would like $e^{\frac{i}{\hbar}\int_D \omega}$ to depend *only* on $\gamma = \partial D$. If D and D' are two disks in Γ_S with $\partial D = \partial D' = \gamma$ then $D \amalg (D')^{op}$ is a 2-sphere in Γ_S. If this 2-sphere wraps around a non-contractible cycle in Γ_S then $\int_{D \amalg (D')^{op}} \omega \neq 0$, in general. This yields a *quantization condition* on $\hbar$:

$$\frac{1}{\hbar} \int_{C_2} \omega \in 2\pi \mathbb{Z}, \qquad (2.22)$$

for any non-contractible 2-cycle C_2 in Γ_S. If $H^2(\Gamma_S)$ is trivial then (2.22) is empty, and $\hbar$ can take arbitrary values. For $\Gamma_S = S^2$, condition (2.22) is clearly non-trivial and implies the half-integer quantization of spin.

Examples of "quantized phase spaces"

1) $\Gamma_S = \mathbb{R}^{2f}$, $\Gamma_S = T^* M$ ✓

2) $\Gamma_S = S^2 \longrightarrow$ rep. theory of $SU(2)$

3) Γ_S = compact Kähler manifold; e.g.

$\Gamma_S = \mathbb{C}P^n \longrightarrow$ Bordemann, Waldmann, ...

4) Γ_S = a coadjoint orbit of a compact Lie group, G

$\rightarrow$ Bott-Borel-Weil construction of reps. of G.

$\hookrightarrow$ some extra pages!

2.3 Atomism and Quantization – deforming continuum theories of matter to atomistic theories of matter

In this section, we show (on examples) that atomism can be understood as the result of a "quantization" of a continuum theory. We have already seen that the atomistic structure of electromagnetic radiation – photons – results from quantizing the electromagnetic field.

<u>Miracle</u>: Ritz-Rydberg, B.-S.,
Bohr-Sommerfeld

Bohr's corresp. pr. <u>only approx. valid</u>

$(\alpha = e^2/\hbar c \simeq 1/137 \ll 1)$; yet,

Heisenberg's matrix mech.

is <u>consistent new mech.!</u>

$$(\mathcal{A}_0, \cdot) \longrightarrow (\mathcal{A}_\hbar, *)$$

$\hbar$: deformation parameter

$$MM \overset{?}{\sim} WM \quad \text{(Schrödinger '26)}$$

$$\sim \boxed{\text{Dirac-Feynman path int.}}$$

Action in Darboux coos.

$$S = \frac{1}{2}\int_\gamma \Big\{ \sum_j p_j\, dq^j - H(p,q)\, dt \Big\}$$

To deform $\mathcal{A}_0$ to $\mathcal{A}_\hbar$, set

$H=0 \Rightarrow q^j(t) = p_j(t) = \text{const.}$

D: surface in Γ, $\partial D =: \gamma$ (loop in Γ), $D_0 := \{(\tau^1,\tau^2) \mid |\tau| \leq 1\}$.

$$S = \frac{1}{2}\int_\gamma \Big\{\sum_j p_j\, dq^j\Big\} \underset{\substack{\uparrow \\ \text{Stokes}}}{=} \frac{1}{2}\int_D \omega$$

$$\sim \int_{D_0} d\tau^\alpha \wedge d\tau^\beta \Big\{ \frac{\partial X^A(\tau)}{\partial \tau^\alpha}\, \eta_{A,\beta}(\tau) - \frac{1}{2}\,\Omega^{AB}(X(\tau))\, \eta_{A,\alpha}(\tau)\,\eta_{B,\beta}(\tau) \Big\}$$

$S(X,\eta)$ inv. under repar. of D_0 and symplectomor. of Γ. $*$ product of two fus. f, g on Γ given by

TFT: $X(\infty) \equiv X \in \Gamma$.

$$(f \underset{\hbar}{*} g)(X) = \text{"}\int \mathcal{D}x \int \mathcal{D}\eta \; e^{\frac{i}{\hbar} S(x,\eta)}$$
$$\times f(X(0))\, g(X(1))\text{"}$$
$$= \left(f \cdot g + \frac{i\hbar}{2}\{f,g\} + \dots\right)(X),$$

$0, 1, \infty \in \partial D_0$.

∞ - dim. symmetries of S require gauge fixing (B-V formalism): Problem solved in sense of formal power series in $\hbar$ by Kontsevich; Cattaneo-Felder.

$H_2(\Gamma) \leftrightarrow$ "quant." of $\hbar$.

Shall apply this to C-S!

Concrete examples:

1) $\Gamma = T^*M$ ess. Schrödinger '26

$\Gamma = S^2$ B-H-J '25

$\Gamma = T^*\mathbb{R}^3 \times S^2$ Pauli '25, '27

+ Pauli's exclusion pr.
(Pauli '25, Heisenberg '26)

$\rightarrow$ th. of NR spinning e^-.

2) $\Gamma =$ (compact) Kähler mf.

$\Gamma =$ coadj. orbit

Borel-Weil-Bott, geom. qu.

3) $\Gamma =$ phase space over G^{S^1}
from 3D TFT (C-S & BF theories)

2.3. Atomism as Quantization – *k_B as a deformation parameter*

I show how the *atomistic structure of matter* arises as a consequence of *quantization*. – For example, the Hamiltonian mechanics of point particles can be understood as the "quantization" of Vlasov theory, and bosonic many-body theory turns out to be the quantization of Hartree theory; etc.

—

I also show how continuum theories of matter can be understood as "classical" or "mean-field" limits of atomistic theories of matter. This will be the consequence of a *Egorov-type theorem*.

4 deformation parameters

(Planck, 1900)

$\hbar$: as described.

k_B : cont. th. → atomism
thermodyn. → stat. mech.

c^{-1} : Galilei → Lorentz-Poincaré ...

l_P : th. without gravity → th. incl. gravity

In nature,

$\hbar, k_B, c^{-1}, l_P \neq 0, =1$

"quantum gravity"

Need a new "Heisenberg"!

Here we will show that the Hamiltonian mechanics of an arbitrary number of identical point particles can be understood as the canonical quantization of Vlasov theory; and that quantum many-body theory of bosons is the quantization of Hartree theory. The deformation parameter can be taken to be $N_{Avogadro}^{-1} \propto k_B$.

The Hamiltonian nature of Vlasov- and Hartree theory will be exhibited.

We will also show the converse:
Vlasov theory is the "classical (or mean-field) limit" of the Hamiltonian mechanics of point particles, as $N_A^{-1} \to 0$, and, similarly, Hartree theory is the classical (or mean-field) limit of the quantum theory of Bose gases; (Braun-Hepp, Hepp).

A Egorov-type theorem will be proven, (following Fr-Knowles-Schwarz). It says that, for these systems, <u>time evolution</u> and <u>quantization commute</u>, <u>up to terms of</u> $O(N_A^{-1})$.

<u>Quotes:</u>

"The crucial step was to write down elements in terms of their atoms... I don't know how they could do chemistry beforehand, it didn't make any sense." (Sir Harry Kroto)

"Hier (namely in Quantum Theory) liegt der Schlüssel der Situation, der Schlüssel nicht nur zur Strahlungstheorie, sondern auch zur molekularen Konstitution der Materie" (Arnold Sommerfeld, in: "Das Plancksche Wirkungsquantum und seine allgemeine Bedeutung für die Molekularphysik".)

<u>"Proof" of atomism</u>: Chemistry; spectroscopy; Brownian motion, (A. Einstein, M. Smoluchowski, J. Perrin),

Contents of Sect. 2.3

1. Atomism — by quantization of continuum field theories of matter.

2. Hamiltonian continuum theories of matter as mean-field limits of atomistic theories — a Egorov theorem.

3. Newtonian "(point) particles" as solitary waves of continuum theories.

Applications in Physics and Mathematics.

Collaborators:

W. K. Abou Salem,
S. Graffi, S. Gustafson, B.L.G. Jonsson, A. Knowles, E. Lenzmann, A. Pizzo, S. Schwarz, I.M. Sigal, T.-P Tsai, H.-T. Yau.

Similar work by:

K. Hepp, (H. Spohn), L. Erdös, B. Schlein, H.-T. Yau; P. Pickl, and others.

Mathematical themes underlying this section

(1) Yet one further example of "quantization" and its consequences. It will illustrate Dirac's recipe:

$$\{\cdot,\cdot\} \longrightarrow \frac{i}{\alpha}[\cdot,\cdot], \quad \alpha = \text{def. parameter},$$

and give us a chance to introduce Fock space; (here $\alpha = \frac{1}{N_A}$).

(2) Semi-classical analysis for systems with ∞ many degrees of freedom: Mean-field limit, Egorov-type theorem

(3) Non-linear Hamiltonian evolution equations; solitary-wave dynamics; applications (e.g. to problems in astrophysics).

Emphasize that atomism arises as result of quantization! First example: Photons, ($\alpha = \hbar$).

Introduction

Here is what this lecture will be about:

1. Atomism by deformation-quantization-of cont. field theories of matter
2. Hamiltonian continuum theories of matter as mean-field limits of atomistic theories - a Egorov theorem
3. Newtonian "point particles as "solitons" of continuum theories

Many applications to Physics & Math!

Mathematical themes underlying this lecture

(1) Quantization $\longleftrightarrow$ deformations of assoc. algebras (Poisson br. $\rightarrow$ commutators; Fock space)

(2) Semi-class. analysis for systs. w. ∞ many degs. of freedom (Egorov thm. ...)

(3) NL Hamiltonian evolution equations, soliton dyn.

Atomism

Über das Boltzmann'sche Prinzip und einige unmittelbar aus demselben fliessende Folgerungen.

Die Thermodynamik beruht ~~bekanntlich~~ auf zwei Prinzipen, dem Energieprinzip (auch 1. Hauptsatz genannt) und dem Prinzip von der Nichtumkehrbarkeit des Naturgeschehens (auch 2. Hauptsatz genannt). ~~Letzteres~~ ~~ist Prinzip [illegible] sagt aus,~~ Der Inhalt des letzteren Prinzip lässt sich ~~[illegible] Planck'schen~~ nach Planck so aussprechen

Alle Naturwissenschaft ist auf die Voraussetzung der vollständigen ~~lückenlosen~~ kausalen Verknüpfung jeglichen Geschehens begründet. Nehmen wir an ~~Wenn~~ Galilei hätte bei seinen ~~Fall und~~ Pendelversuchen gefunden ~~hätte~~, dass dasselbe Pendel so schwingt, dass die Dauer einer Schwingung in unregelmässiger Weise wechselt, Nehmen wir ferner an, ~~ohne~~ dass dieser Wechsel mit dem Wechsel ~~anderer~~ irgendwelcher anderer beobachtbarer Verhältnisse nicht hätte in Verbindung gebracht werden können. Dann wäre es Galilei unmöglich gewesen, ~~wohl kaum eingefallen~~ seine Beobachtungen zu einem Gesetze zu vereinigen. Hätten alle uns zugänglichen Erscheinungen einen derart unregelmässigen Charakter, wie wir dies in dem soeben fingierten Falle uns vorgestellt haben, so wären die Menschen gewiss nie auf naturwissenschaftliche Bestrebungen verfallen.

Welchen Charakter mussten die Erscheinungen haben, damit Wissenschaft möglich sei? Darauf möchte man zuerst etwa folgendermassen antworten: Bringen wir ein System in einen bestimmten Zustand, so ist, falls dies System von anderen Systemen – etwa durch grosse räumliche Entfernung getrennt ist, so ist der zeitliche Ablauf der Zustände dieses Systems vollkommen bestimmt; d. h. bringen wir ~~zwei~~ beliebig viele gleichbeschaffene isolierte Systeme in genau denselben Zustand und überlassen wir diese Systeme sich selbst, so ist für alle diese Systeme der zeitliche Ablauf der Erscheinungen genau derselbe.

A. Einstein

1911

"Hier (in quantum th.) liegt der Schlüssel der Situation der Schlüssel nicht nur zur Strahlungstheorie, sondern auch zur molekularen (atomistic) Konstitution der Materie ..."

A. Sommerfeld, "Das Plancksche Wirkungsquantum & seine allg. Bedeutung für die Mole<u>kular</u>physik".

Spectroscopy; Brownian motion (Einstein, Perrin, ...)

1.1. Newtonian Mechanics as "quantization" of Vlasov Mech

"Stellar dust" descr. as class continuous medium; states given by mass density

$$M f(x,p)\, dx\, dp$$

on $\mathbb{R}^3_{position} \times \mathbb{R}^3_{velocity}$, with

$$\int f(x,p)\, dx\, dp = \nu$$

(ν : # moles of dust)

Time-dependence of $f(x,p)$ given by Vlasov Equation.

This is a model of matter as a continuous medium!

Vlasov Eq.

$$\partial_t f_t(x,p) = -\frac{1}{M}(p\cdot\nabla_x f_t)(x,p)$$
$$+ (\nabla V_{eff}[f_t]\cdot\nabla_p f_t)(x,p),$$

$$V_{eff}(x) := V(x) + \int dy\, \phi(x-y) \times$$
$$\times \int dp\, f_t(y,p)$$

ϕ : reg. Newtonian pot.

& Neunzert

Braun-Hepp: Vlasov is mean-field limit of $n = \nu N_A$ point particles w. mass $m = \frac{M}{N_A}$, 2-body pot. $\frac{1}{N_A}\phi$, as $N_A \to \infty$.

Vlasov dynamics is Hamiltonian dynamics:

$$f(x,p) = \overline{\alpha(x,p)} \cdot \alpha(x,p),$$

$$\alpha \in \Gamma = H^1(\mathbb{R}^6), \; \{\alpha^\#, \alpha^\#\} = 0,$$

$$\{\alpha(x,p), \overline{\alpha(x',p')}\} = -i\,\delta(x-x')\,\delta(p-p'),$$

$$\mathcal{H}_V(\bar{\alpha}, \alpha) = i \iint dx\,dp\; \bar{\alpha}\left[\frac{1}{M} p \cdot \nabla_x - \nabla V \cdot \nabla_p\right]\alpha$$

$$- i \iint dx\,dp\, \bar{\alpha}\left[\iint dy\,dr\, \nabla\phi(x-y) |\alpha(y,r)|^2\right] \cdot \nabla_p \alpha$$

Hamiltonian Eqs. of motion,

$$(1)\; \dot{\alpha}_t(x,p) = \{\mathcal{H}_V, \alpha_t(x,p)\}, \; \dot{\bar{\alpha}}_t(x,p) = \cdots$$

$\Rightarrow$ Vlasov Eqs. for $f_t(x,p)$!

Wick quantization

$$\alpha(x,p) \to a_N(x,p)\ ,\ \overline{\alpha(x,p)} \to a_N^*(x,p)$$

$$[a_N^\#, a_N^\#] = 0\ ,\quad (N \equiv N_A)$$

$$[a_N(x,p), a_N^*(x',p')] = \frac{1}{N}\,\delta(x-x')\,\delta(p-p')$$

$$\sim \frac{i}{N}\{\alpha(x,p), \overline{\alpha(x',p')}\}\ (\text{Dirac})$$

a_N, a_N^* act on <u>Fock space</u>

$$\mathcal{F}_V := \bigoplus_{n=0}^{\infty} \mathcal{F}_V^{(n)},$$

$$\mathcal{F}_V^{(0)} = \mathbb{C}\,|0\rangle\ ,\ |0\rangle \text{ vacuum}$$

$$a_N(x,p)|0\rangle = 0,\ \forall x,p.$$

$$\mathcal{F}_V^{(n)} := \left\langle \int\cdots\int \varphi_n(x_1,p_1,\dots,x_n,p_n) \prod_{i=1}^{n} a_N^*(x_i,p_i)|0\rangle \right\rangle$$

$f_n := |\varphi_n|^2 =$ symm. density on

n-point-part. phase space $\Gamma^{(n)}$

Hamilton op. : $\hat{\mathcal{H}}_V := :\mathcal{H}_V(a_N^*, a_N):$

Schrödinger Eq. :

$$iN^{-1}\partial_t \Psi_t = \hat{\mathcal{H}}_V \Psi_t, \quad \Psi_t \in \mathcal{F}_V$$

$\Leftrightarrow$ Liouville's Eq. of motion for symm. n-particle densities, $f_n = \overline{\varphi_n} \cdot \varphi_n$, on $\Gamma^{(n)}$, 2-body potential $\frac{1}{N}\phi$, $n = 0, 1, 2, \cdots$.

Apparently, atomistic Newtonian mech. of point-part.

= quantization of continuum theory given by Vlasov eq.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \|$

(B-H) "classical limit" of Newtonian mech.

Rephrasing Braun-Hepp

$$\Psi \equiv \Psi^{(n)}(\alpha) := cst. \int \cdots \int \prod_{i=1}^{n} \alpha(x_i, p_i) a_N^*(x_i, p_i) |0\rangle$$

For $n \simeq \nu N$,

$$e^{-itN\hat{\mathcal{H}}_\nu} \Psi^{(n)}(\alpha) \simeq \Psi^{(n)}(\alpha_t) + O\left(\frac{1}{N}\right),$$

where α_t solves (1), i.e., $f_t = \nu \bar{\alpha}_t \cdot \alpha_t$ solves Vlasov, with $\int f_t = \nu$.

More precise statement in form of a <u>Egorov Theorem</u>.

Alas, description of stars in terms of Vlasov Eq. leads to instabilities

$\longrightarrow$ Quantize ($\hbar$)!

1.2 Quant. gases as "2^{nd} quantization" of Hartree mechanics

Replace $f(x,p) = \overline{\alpha(x,p)} \cdot \alpha(x,p)$

by

(2) $f_\hbar(x,p) := \frac{1}{(2\pi)^3} \int dy\, e^{-iy \cdot p}\, \overline{\psi(x - \frac{\hbar y}{2})} \cdot \psi(x + \frac{\hbar y}{2})$

$f_\hbar$ is Wigner trsf. of ψ

Dynamics of ψ:

(3) $i\hbar \partial_t \psi_t = \left[-\frac{\hbar^2}{2m} \Delta + V \right] \psi_t + \left[|\psi_t|^2 * \phi \right] \psi_t$

Hartree Equation

If solution ψ_t of (3) is plugged into (2) then

$$\lim_{\hbar \searrow 0} f_{\hbar,t}(x,p)$$

solves Vlasov Eq. (1);

($\nearrow$ Narnhofer – Sewell)

Hartree is Hamiltonian Eq. of motion on phase space $\Gamma = H^1(\mathbb{R}^3)$; Poisson brackets

$$\{\psi^\#, \psi^\#\} = 0, \quad \{\psi(x), \overline{\psi(y)}\} = i\delta(x-y)$$

Hamilton functional

$$\mathcal{H}_H(\bar{\psi}, \psi) := \hbar^{-1} \int dx\, \overline{\psi(x)} \left[-\frac{\hbar^2}{2m} \Delta_x + V \right] \psi(x)$$

$$+ \frac{\hbar^{-1}}{2} \int dx \int dy\, |\psi(x)|^2 \phi(x-y) |\psi(y)|^2$$

$$(3') \quad \dot{\psi}_t(x) = \{\mathcal{H}_H, \psi_t(x)\}, \quad \dot{\bar{\psi}}_t(x) = \cdots$$

<u>Continuum</u> (field) theory of a <u>quantum</u> gas

"Second" quantize:

$\psi(x) \to \hat{\psi}_N(x)$, $\overline{\psi(x)} \to \hat{\psi}^*_N(x)$, w.

$[\hat{\psi}^\#_N, \hat{\psi}^\#_N] = 0$, $[\hat{\psi}_N(x), \hat{\psi}^*_N(y)] = \frac{1}{N}\delta(x-y)$

Fock space

$$\mathcal{F}_H = \bigoplus_{n=0}^{\infty} \mathcal{F}_H^{(n)}, \quad \mathcal{F}_H^{(0)} = \mathbb{C}|0\rangle,$$

$$\hat{\psi}_N(x)|0\rangle = 0, \ \forall x$$

$$\mathcal{F}_H^{(n)} := \left\langle \int \cdots \int \varphi_n(x_1, \cdots, x_n) \prod \hat{\psi}^*_N(x_i)|0\rangle \right\rangle$$

Many-body Hamiltonian

$$\hat{\mathcal{H}}_N := :\mathcal{H}_H(\hat{\psi}^*_N, \hat{\psi}_N): \tag{4}$$

$$iN^{-1}\partial_t \Psi_t = \hat{\mathcal{H}}_N \Psi_t$$

is Schrödinger Eq. for Bose gas of $n = 0, 1, 2, 3, \cdots$ quantum point particles, <u>"atoms"</u>, with 2-body potential $\frac{1}{N}\phi$.

On n-particle subspace $\mathcal{F}_H^{(n)}$, (4) is equivalent to

$$
\begin{aligned}
H^{(n)} &= \sum_{j=1}^{n}\left[-\frac{\hbar^2}{2m}\Delta_j + V(x_j)\right] + \frac{1}{N}\sum_{1\le i<j\le n}\phi(x_i - x_j) \\
i\hbar\,\partial_t \varphi_t^{(n)}(x_1,\dots,x_n) &= \left[H^{(n)}\varphi_t^{(n)}\right](x_1,\dots,x_n)
\end{aligned}
\tag{4'}
$$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\text{QM many-body theory}} \arrow[r, shift left, "\hbar"'] \arrow[d, shift right, "\frac{1}{N}"'] & \boxed{\text{Class. mech of pt. part.}} \arrow[l, shift left] \arrow[d, shift right] \\
\boxed{\text{Hartree dynamics}} \arrow[u, shift right] \arrow[r, shift right] & \boxed{\text{Vlasov dynamics}} \arrow[l, shift right, "\hbar"'] \arrow[u, shift right, "\frac{1}{N}"']
\end{tikzcd}
$$

Atomism $\longleftrightarrow$ "2^{nd}" Quantization

2. A Egorov Theorem

Continuum theory of quantum gases: "Observables" ~ gauge-inv. functions on Γ

$$A(a^{(p)}) = \int \cdots \int \prod_{1}^{p} \overline{\psi(x_i)}\, dx_i \, a^{(p)}(\underline{x};\underline{y}) \prod_{1}^{p} \psi(y_j)\, dy_j$$

Time evolution given by flow, $\Phi_t : \overset{(-)}{\psi} \to \overset{(-)}{\psi}_t$, where $\overset{(-)}{\psi}_t$ solves Hartree Eq. (3),(3')

$$A(a^{(p)}) \to A_t(a^{(p)}) := A(a^{(p)}) \circ \Phi_t$$

Many-body theory of quantum gases: $\psi \mapsto \hat{\psi}_N$, $\overline{\psi} \mapsto \hat{\psi}_N^*$, with CCR; $A(a^{(p)}) \mapsto \hat{A}_N(a^{(p)})$,

$$\hat{A}_N(a^{(p)}) = \int\cdots\int \prod_1^p \hat{\psi}_N^*(x_i)\,dx_i\; a^{(p)}(\underline{x},\underline{y}) \prod_1^p \hat{\psi}_N(y_j)\,dy_j$$

gauge-inv., $\hat{\psi}_N^{\#} \longmapsto e^{\pm i\theta}\hat{\psi}_N^{\#}$,

i.e., preserves particle #.

Time evolution (Heisenberg)

$$\hat{A}_N(a^{(p)}) \longrightarrow$$

$$\hat{A}_{N,t}(a^{(p)}) = e^{itN\hat{\mathcal{H}}_N}\hat{A}_N(a^{(p)})e^{-itN\hat{\mathcal{H}}_N}$$

Conservation laws:

Gauge-inv. $\longleftrightarrow$ $\mathcal{N}(\bar{\psi},\psi) = \int |\psi|^2\,dx$

$$\downarrow$$

$$\hat{\mathcal{N}} = \text{part. \#}$$

Time-transl.-inv. $\longleftrightarrow$ $\mathcal{H}_H(\bar{\psi},\psi)$

$$\downarrow$$

$$\hat{\mathcal{H}}_N$$

Egorov Theorem (new!)

For $n \lesssim \nu N$, $(\nu < \infty)$,

$$\hat{A}_{N,t}(a^{(p)})\Big|_{\mathcal{J}^{(n)}} \underset{N\to\infty}{=} \left(\widehat{A(a^{(p)})\circ\Phi_t}\right)_N\Big|_{\mathcal{J}^{(n)}} + o(1)$$

Idea of proof:

$\hat{A}_{N,t}(a^{(p)})$ in "interaction pict.";

expand in Lie-Schwinger series; $<$ tree terms / loop terms

Using conservation laws,

$$|\text{loop terms}| \sim O\left(\frac{1}{N}\right);$$

$$\underbrace{\sum \text{tree terms}}_{\text{abs. conv., } |t| \text{ small, unif. in } N.} = \left(\widehat{A(a^{(p)})\circ\Phi_t}\right)_N$$

Related story for Fermions

$\psi \to \hat{\psi}_N$, $\bar{\psi} \to \hat{\psi}_N^*$, with

$$[\hat{\psi}_N^{\#}, \hat{\psi}_N^{\#}]_+ = 0, \; [\hat{\psi}_N(x), \hat{\psi}_N^*(y)]_+ = \frac{1}{N}\delta(x-y),$$

$$[A,B]_+ := AB + BA.$$

$$\mathcal{H}_H \to \hat{\mathcal{H}}_N^f, \quad A(a^{(p)}) \to \hat{A}_N^f(a^{(p)}),$$

$a^{(p)}(x_1, \dots, x_p; y_1, \dots, y_p)$ tot. anti-symm in x's & in y's.

There's again a Egorov-type theorem. But "continuum theory" given by

?

Hartree-Fock Eq.

for $n \sim \nu N$ orbitals.

3. Some details on Egorov Thm. for Bosons

$$\mathcal{H}_H = \mathcal{H}_0 + \mathcal{V},$$

$$\mathcal{H}_0(\bar{\psi},\psi) = \hbar^{-1}\int dx\ \bar{\psi}(x)\left[-\frac{\hbar^2}{2m}\Delta + V(x)\right]\psi(x)$$

$$\mathcal{V}(\bar{\psi},\psi) = \frac{\hbar^{-1}}{2}\iint dx\,dy\ |\psi(x)|^2\phi(x-y)|\psi(y)|^2$$

$$= \frac{\hbar^{-1}}{2} A(\phi^{(2)}),$$

$$\phi^{(2)}(x_1,x_2;y_1,y_2) = \phi(x_1-y_1)\delta(x_2-y_1)\delta(y_2-x_1),$$

$$\|\phi^{(2)}\|_{op.} = \|\phi\|_\infty .$$

$$\hat{\mathcal{H}}_N = \hat{\mathcal{H}}_{0,N} + \hat{\mathcal{V}}_N \quad (\text{quant. of } \mathcal{H}_H)$$

Lemma.

$$e^{itN\hat{\mathcal{H}}_{0,N}}\hat{A}_N(a^{(p)})e^{-itN\hat{\mathcal{H}}_{0,N}}$$

$$= \hat{A}_N\Big(\underbrace{e^{itH_0^{(p)}/\hbar}a^{(p)}e^{-itH_0^{(p)}/\hbar}}_{=:\,a_t^{(p)}}\Big)$$

$$= \left(A(a^{(p)})\circ\Phi_0^t\right)^{\wedge}_N$$

$$\|a_t^{(p)}\| = \|a^{(p)}\|$$

Time evol. in interaction pict

$$e^{itN\hat{\mathcal{H}}_N}\hat{A}_N(a^{(p)})e^{-itN\hat{\mathcal{H}}_N}$$

$$= \hat{A}_N(a_t^{(p)}) + \int_0^t ds\, e^{isN\hat{\mathcal{H}}_N}e^{-isN\hat{\mathcal{H}}_{0,N}}$$

$$\times \frac{iN}{2\hbar}\left[\hat{A}_N(\phi_s^{(2)}),\hat{A}_N(a_t^{(p)})\right]$$

$$\times\; e^{isN\hat{\mathcal{H}}_{0,N}}e^{-isN\hat{\mathcal{H}}_N}$$

#

Iterate $\to$ Schwinger-Dyson ser

conv. on $\mathcal{F}^{\leq n}$, $\forall n < \infty$, $\forall t$.

Note:

$$\frac{iN}{2\hbar}\left[\hat{A}_N(\phi_s^{(2)}), \hat{A}_N(a_t^{(p)})\right]$$

$$= \sum_{\ell=1}^{2} \binom{p}{\ell}\binom{2}{\ell} \frac{i\ell}{2\hbar N^{\ell-1}} \hat{A}_N\left(\left[\phi_s^{(2)} \lrcorner_\ell\, a_t^{(p)}\right]\right),$$

$\ell = 1$: tree terms

$\ell = 2$: loop terms, $\propto \frac{1}{N}$!

When a loop term is gen. in iteration of #: stop expanding & estimate it, using unitarity of $e^{\pm isN\mathcal{H}_{(0),N}}$!

This yields:

$$e^{itN\hat{\mathcal{H}}_N}\hat{A}_N(a^{(p)})e^{-itN\hat{\mathcal{H}}_N}$$

$$= \hat{T}_N(a^{(p)},t) + \hat{L}_N(a^{(p)},t),$$

with

$$\hat{T}_N(a^{(p)},t) = \sum_{k=0}^{\infty}\left(\frac{i}{\hbar}\right)^k \binom{p+k-1}{p-1} k! \times$$

$$\times \int_0^t dt_1 \cdots \int_0^{t_{k-1}} dt_k \; \hat{A}_N\Big([\phi^{(2)}_{t_k} \underset{1}{\sqcup}$$

$$[\phi^{(2)}_{t_{k-1}} \underset{1}{\sqcup} \cdots [\phi^{(2)}_{t_1} \underset{1}{\sqcup} a^{(p)}_t] \cdots]\Big) \; *$$

$\hat{L}_N(a^{(p)},t)$: similar sum of terms, but with 1 loop $\propto \frac{1}{N}$

Easy to show: On $\mathcal{F}^{\leq \nu N}$, series for $\hat{T}_N, \hat{L}_N$ converge in norm if

$$|t| \leq \frac{1}{4\nu \|\phi\|_\infty},$$

independently of p!

Now, compare $\hat{T}_N(a^{(p)}, t)$ with classical time evolution, in interaction picture:

From $*$ $\Rightarrow$

$$\hat{T}_N(a^{(p)}, t) = \left(A(a^{(p)}) \circ \Phi^t\right)^\wedge_N$$

on $\mathcal{F}^{\leq \nu N}$, $|t| \leq \frac{1}{4\nu \|\phi\|_\infty}$.

Arbitary t: Iterate exp., using indep. of p & unit.!

3. Newtonian point particles as "solitons" of cont. theories

Consider, e.g., Hartree Eq. as q.t. model of cont. medium:

$$i\hbar \partial_t \psi_t(x) = \left(T + V(\varepsilon x)\right)\psi_t(x) - g\left(|\psi_t|^2 * \phi\right)(x)\, \psi_t(x) \tag{5}$$

$$T = -\frac{\hbar^2 \Delta}{2m}, \sqrt{-\hbar^2\Delta + m^2}, \dots$$

V: e.g., grav. pot. of central "star", $\|V\|_\infty < \infty$.

$$\phi(x) = \frac{1}{|x|}, \frac{e^{-\mu|x|}}{|x|}, \dots$$

$$\|\psi_t\|_2^2 =: \nu = O(1), \; g > 0.$$

(Model of a "Boson star.")

Hamilton functional:

$$\mathcal{H}(\bar{\psi},\psi;\varepsilon)=\int dx\left\{\overline{\psi(x)}(T\psi)(x)+\left[V(\varepsilon x)-g\int dy\,|\psi(y)|^2\phi(y-x)\right]|\psi(x)|^2\right\}$$

Conservation laws:

- $\mathcal{N}(\bar{\psi},\psi):=\int|\psi(x)|^2dx \leftrightarrow$ gauge inv.

For $\varepsilon=0$,

- $\mathcal{P}(\bar{\psi},\psi):=-i\hbar\int(\bar{\psi}\nabla\psi)(x)\,dx$

  $\leftrightarrow$ translation inv.

Consider "energy funct."

$$\mathcal{E}_v(\psi):=\mathcal{H}(\bar{\psi},\psi;\varepsilon=0)+v\cdot\mathcal{P}(\bar{\psi},\psi)$$

$v\in\mathbb{R}^3$: C-of-M velocity.

(For pseudo-relat. T, $|v|<1$)

<u>Var. problem</u>: Construct minimizer, $\varphi_{v,\mu}$, for $\mathcal{E}_v$, with $\|\varphi_{v,\mu}\|_2^2 = \nu(\mu)$.

Solves eq.

$$T\varphi_{v,\mu} + iv\cdot\nabla\varphi_{v,\mu} - g(|\varphi_{v,\mu}|^2 * \phi)\varphi_{v,\mu} + \mu\varphi_{v,\mu} = 0$$

(Subaddit. + concentr.-comp.)

Then

$$\psi_t(x) := e^{i\theta(t)}\varphi_{v,\mu}(x-q-vt) \tag{6}$$

solves Hartree eq. (5):

solitary wave sol.

describes giant "molecule",

e.g., a "Boson star", of bound matter travelling inertially w. velocity v.

(For $T = -\frac{\hbar^2 \Delta}{2m}$, φ_v obtained from φ_0 by Galilei boost.)

Solu. (6) of (5) depends on 8 parameters:

$$\xi := (q, v, \mu, \theta) \in S \subset \mathbb{R}^8.$$

ξ: coos. of point in 8-dim. surface in $\Gamma = H^1(\mathbb{R}^3)$:

$$\mathcal{M}_S := \left\{ \varphi_{v,\mu}(\cdot - q) \,\middle|\, \varphi_{v,\mu}(0) = e^{i\theta}, \ \|\varphi_{v,\mu}\|_2^2 = \nu(\mu) \right\}$$

"soliton manifold"

$\omega|_{M_s}$: non-degenerate

Let ψ_t be solu. of (5) for $\varepsilon > 0$, w. $\mathrm{dist}(\psi_0, M_s) < O(\varepsilon)$

Let φ_{ξ_t} be "skew-orth." proj. of ψ_t onto M_s

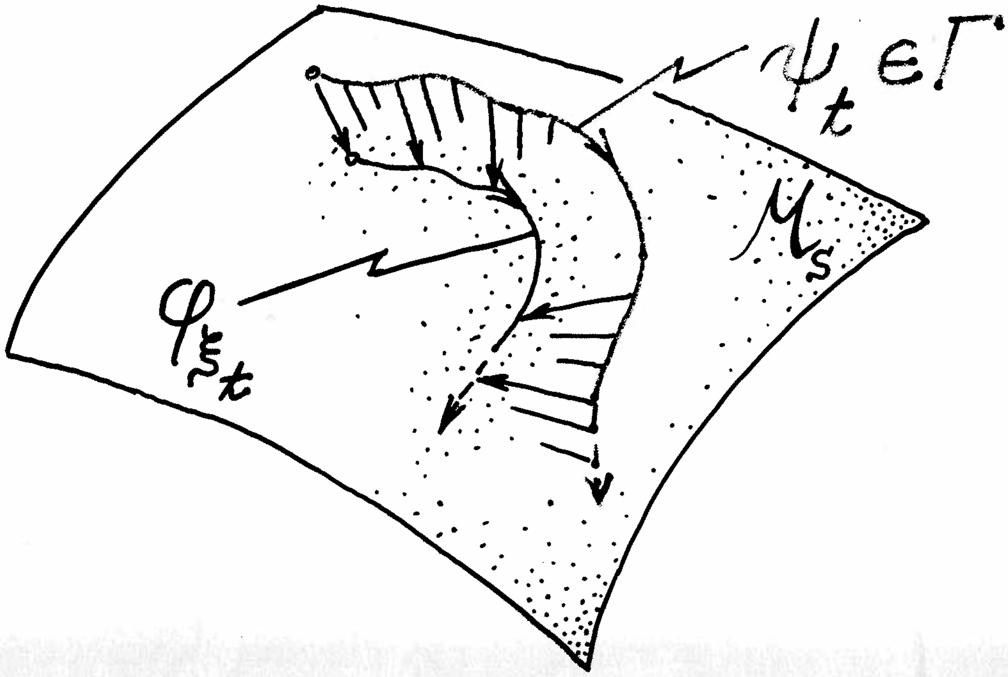
Theorem. For $|t| < O(\varepsilon^{-1\cdots})$,

$\mathrm{dist}(\psi_t, M_s) \sim O(\varepsilon) \Rightarrow$

φ_{ξ_t} well def. & unique;

(N)
$$\boxed{\begin{aligned} \dot{q}_t &= v_t + O(\varepsilon^2) \\ \gamma\, \dot{v}_t &= -\varepsilon (\nabla V)(\varepsilon q_t) + O(\varepsilon^2) \end{aligned}}$$

$\dot{\mu} = O(\varepsilon^2)$, $\dot{\Theta} = \mu - V(\varepsilon q) + O(\varepsilon^2)$



<u>Interpretation</u>: In an ext. pot. $V(\varepsilon\,\cdot)$, center of mass, q_t, of "molecule" in state ψ_t is solution of Newton's eqs. of motion for point particle in pot. $V(\varepsilon\,\cdot)$ with friction force $\sim O(\varepsilon^2)$

$\downarrow$

structure formation

($\nearrow$ e.g. T. Tao)

Further results.

(1) Motion of several inter-
acting solitons (J.F.; A-S, F, S)

(2) Asy. stability & scattering
(R, S, S; G.P.; E.L.)

(3) Ext. to Hartree-Fock &
BHF $\longrightarrow$
dynamical approach to
Chandrasekhar limit
for white dwarfs;

? BCS pairing of neutrons
in groundstate of
neutron star; ...

2-body problem: Krieger, Martel,
Raphaël

Quantum spins

$\mathbb{Z}^d \ni x \mapsto \vec{S}_x$, quantum spin of spin s

$$\hat{\vec{S}}_x := \frac{1}{s}\vec{S}_x$$

Hamiltonian (Heisenberg)

$$H = \vec{h}\cdot\sum_x \hat{\vec{S}}_x - \sum_{x,y} J(x-y)\hat{\vec{S}}_x\cdot\hat{\vec{S}}_y + \cdots$$

$$\hat{\vec{S}}_x(t) := e^{istH}\hat{\vec{S}}_x e^{-istH}$$

Class. limit ~ large-s limit

$$\text{“}\hat{\vec{S}}_x \xrightarrow[s\to\infty]{} \vec{M}_x\text{”}, \quad \hbar \leftrightarrow \frac{1}{s}$$

$\vec{M}_x \in S^2$ classical spin

$\vec{M}_x \xmapsto[s]{} \hat{\vec{S}}_x$: geom. quant.

ordering prescription!

$$\{M_x^i, M_y^j\} = \delta_{xy}\, \varepsilon^{ijk} M_x^k$$

$$\longleftrightarrow \quad [\hat{S}_x^i, \hat{S}_y^j] = \frac{i}{S}\, \delta_{xy}\, \varepsilon^{ijk} \hat{S}_x^k$$

Result: If ^ denotes quant.

$$\hat{\vec{S}}_x(t) = \left(\vec{M}_x(t)\right)^{\wedge} + \underbrace{\varepsilon_S}_{\to 0,\text{ as } S \to \infty},$$

where

$$\dot{\vec{M}}_x(t) = \vec{h} \wedge \vec{M}_x(t) - \left(\sum_y J(x-y) \vec{M}_y(t)\right) \wedge \vec{M}_x(t)$$

Landau-Lifschitz Eq.

Is Hamiltonian Eq. of motion,

$$\mathcal{H} = \vec{h} \cdot \sum_x \vec{M}_x - \sum_{x,y} J(x-y) \vec{M}_x \cdot \vec{M}_y$$

Technicalities

1. Intuition conc. M-F-L

$$H^{(n)} = \sum_{j=1}^{n} \left[-\Delta_j + V(x_j) \right] + \frac{1}{2N} \sum_{i \neq j} \phi(x_i - x_j)$$

on $\mathcal{H}^{(n)} = P_+ L^2(\mathbb{R}^3, d^3x)^{\otimes n}$;

P_+ : symmetrization (bosons!)

Schrödinger Eq.:

$$i \dot{\Psi}_t^{(n)} = H^{(n)} \Psi_t^{(n)}$$

solved by

$$\Psi_t^{(n)} = e^{-itH^{(n)}} \Psi^{(n)},$$

$\Psi^{(n)} \in \mathcal{H}^{(n)}$. If

$$\Psi^{(n)}(x_1, \dots, x_n) = \prod_{j=1}^{n} \psi(x_j) \qquad (1)$$

expect that

$$\Psi_t^{(n)}(x_1, \dots, x_n) = \prod_{j=1}^{n} \psi_t(x_j),$$

where ψ_t solves HE

$$i\dot{\psi}_t(x) = \left(-\Delta + V(x)\right)\psi_t(x)$$
$$+ \nu \int d^3y \, |\psi_t(y)|^2 \phi(y-x)\psi_t(x)$$

with $\nu = \dfrac{n}{N}$. (2)

"Pot. felt by one part. ≃ average pot. generated by particle density".

Goal: Get rid of special initial condition (1).

c

2. History

1969 – Egorov (can. trsfs. of pseudodiff. ops., ...)

1974 – Hepp (coherent states, following Schrödinger)

1977 – Braun & Hepp, Neunzert (Vlasov)

1979 – Ginibre & Velo (Hepp for $t \to \infty$)

1981 – Narnhofer & Sewell (Vlasov "for fermions")

1998 – 2009 – J.F. et al. ($\longrightarrow$)
Yau et al.

2009/10 – Pickl

D

Egorov: "Time evolution & quantization commute in semi-classical limit."

$$\Gamma = T^*M,\ \mathcal{A} = C^\infty(\Gamma),\ H,$$

Φ_t: flow on Γ gen. by H.

$$\widehat{(\cdot)}_\hbar : A \in \mathcal{A} \mapsto \hat{A}_\hbar \text{ on } L^2(M,$$

(Weyl quantization)

$$[\hat{A}_\hbar, \hat{B}_\hbar] = \frac{\hbar}{i} \widehat{\{A,B\}}_\hbar + O(\hbar^2)$$

$$\widehat{(A\circ\Phi_t)}_\hbar = e^{it\hat{H}_\hbar} \hat{A}_\hbar e^{-it\hat{H}_\hbar}$$

$$+ o_\hbar(1).$$

Hepp can be cast in this language!

In our model: $\hbar = \frac{1}{N}$,

$\Gamma = \mathcal{H}^1(\mathbb{R}^3)$ (Sobolev).

many-body (bosons) $\xrightarrow[N\to\infty]{}$ Hartree Eq.

3. Notation & framework

$\Gamma = \mathcal{H}^1(\mathbb{R}^3)$ w. complex coos

$\psi, \bar{\psi}$; $\{\psi^\#, \psi^\#\} = 0$,

$\{\psi(x), \bar{\psi}(y)\} = i\delta(x-y)$

$$\mathcal{H}(\bar{\psi},\psi) = \int d^3x\ \bar{\psi}(x)\underbrace{[-\Delta + V]}_{\equiv h}\psi(x)$$

$$+ \frac{1}{2}\int d^3x \int d^3y\ |\psi(x)|^2 \phi(x-y) |\psi(y)|$$

HE becomes

$$\dot{\psi}_t(x) = \{\mathcal{H}, \psi_t(x)\}$$

(e.g., Lenzmann) $\longrightarrow$ can. flow Φ_t.

Conservation laws

- $\mathcal{H}(\bar{\psi}_t, \psi_t) = \mathcal{H}(\bar{\psi}, \psi)$;
- $\mathcal{N}(\bar{\psi}_t, \psi_t) = \mathcal{N}(\bar{\psi}, \psi)$,

where $\mathcal{N}(\bar{\psi}, \psi) = \|\psi\|_2^2$

$\longrightarrow$ Flow Φ_t leaves

$$\Gamma_\nu := \{(\bar{\psi}, \psi) \in \Gamma \mid \mathcal{N}(\bar{\psi}, \psi) = \nu\}$$

invariant.

"Observables"

$$A(a^{(p)}) := \int \prod_1^p \bar{\psi}(x_j) d^3x_j \int \prod_1^p \psi(y_j) d^3 y_j$$
$$\times\, a^{(p)}(x_1, \dots, x_p ; y_1, \dots, y_p),$$

$a^{(p)} \in B(\mathcal{H}^{(p)})$.

Time evolution:

$A(a^{(p)}) \circ \Phi_t$ obtained by

$\psi \mapsto \psi_t, \bar{\psi} \mapsto \bar{\psi}_t$ in $A(a^{(p)})$.

Quantization

$$\psi(x) \longmapsto \hat{\psi}_N(x),\ \bar{\psi}(x) \longmapsto \hat{\psi}_N^*(x),$$

with CCR (Heisenberg, Dirac)

$$[\hat{\psi}_N^\#(x), \hat{\psi}_N^\#(y)] = 0,$$

$$[\hat{\psi}_N(x), \hat{\psi}_N^*(y)] = \frac{1}{N}\delta(x-y)$$

<u>Fock space</u> $\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$,

$\mathcal{H}^{(0)} = \mathbb{C}\Omega$, Ω : vacuum,

$$\psi_N(x)\Omega \equiv 0 .$$

$$\mathcal{H}^{(n)} \ni \Phi^{(n)} = \frac{N^{n/2}}{\sqrt{n!}} \int \varphi^{(n)}(x_1, \dots, x_n) \times \prod_1^n \hat{\psi}_N^*(x_j) d^3x_j \, \Omega$$

$$\hat{A}_N(a^{(p)}) := \int \prod_1^p \hat{\psi}_N^*(x_j) d^3x_j \times \int \prod_1^p \psi_N(y_j) d^3y_j \, a^{(p)}(x_1, \dots, x_p; y_1, \dots, y_p)$$

<u>Hamilton operator</u>

$$H_N = N \hat{\mathcal{H}}_N = \underbrace{N(\hat{A}_N(h)}_{H_N^0} + \underbrace{\tfrac{1}{2}\hat{A}_N(\phi))}_{W_N}$$

$$= \bigoplus_{n=0}^{\infty} H^{(n)} .$$

Quantum time evolution

$$\hat{A}_N(a^{(p)}) \longmapsto e^{itH_N}\hat{A}_N(a^{(p)})e^{-itH_N}$$

4. Goal

If $V, \phi \in L^\infty + L^3_{weak}$,

(e.g. Coulomb pot.), and

for arb. ν, $0<\nu<\infty$,

arb. $a^{(p)} \in B(\mathcal{H}^{(p)})$, $p = 1,2,3,\cdots$,

$$e^{itH_N}\hat{A}_N(a^{(p)})e^{-itH_N}\Big|_{\mathcal{F}^{\le \nu N}}$$

$$= \left(\widehat{A(a^{(p)})\circ\Phi_t}\right)_N\Big|_{\mathcal{F}^{\le \nu N}} + o_N(1), \qquad (3)$$

where $\mathcal{F}^{\le \nu N} = \bigoplus_{n=0}^{[\nu N]} \mathcal{H}^{(n)}$.

Moreover, for

$$\Phi^{(n)}(\varphi) := \frac{N^{n/2}}{\sqrt{n!}} \left(\hat{\psi}_N^*(\varphi)\right)^n \Omega,$$

$$\left\langle \Phi^{([\nu N])}(\varphi), e^{itH_N} \hat{A}_N(a^{(p)}) e^{-itH_N} \Phi^{([\nu N])}(\varphi) \right\rangle$$

$$= \left\langle \Phi^{([\nu N])}(\varphi_t), \hat{A}_N(a^{(p)}) \Phi^{([\nu N])}(\varphi_t) \right\rangle + o_N(1)$$

$$= \int \prod_1^p \bar{\psi}_t(x_j) dx_j \int \prod_1^p \psi_t(y_j) dy_j \; a^{(p)}(x_1, \cdots, x_p; y_1, \cdots, y_p) + o_N(1) \quad (4$$

where $\varphi_t = \frac{\psi_t}{\sqrt{\nu}}$ and $\psi_t \in \Gamma_\nu$ is a solution of HE.

Note that $(3) \Rightarrow (4)$.

5. Key ideas of proof

(i) Schwinger–Dyson exp. of $e^{itH_N}\hat{A}_N(a^{(p)})e^{-itH_N}$.

(ii) "Kato smoothing" plus combinatorial estimates

$\Rightarrow$ S-D exp. conv. on $\mathcal{F}^{\leq \nu N}$ for $|t|$ small enough, indep. of p, unif. in N

ℓ-loop terms $\propto N^{-\ell}$;

($\ell = 0$: "tree terms")

(iii) "Kato smoothing" plus comb. estimates $\Rightarrow$

iterative solu. of HE converges, for small $|t|$

$\rightarrow$ ex. of flow Φ_t on Γ_ν, for small $|t|$, dep. on ν,

$\rightarrow$ for all $|t|$, using cons laws.

(iv) $\left(\widehat{A(a^{(p)}) \circ \Phi_t}\right)_N$

= Sum tree $(\ell = 0)$ terms in (ii), for $|t|$ small

(v) Extend (ii), (iii) to arb. $|t|$, using unitarity, cons. laws, indep. of conv. in (ii), (iii) of p.

6. Algebra of observables

$$\mathcal{A} := \{ A(a^{(p)}) \mid a^{(p)} \in \mathcal{B}(\mathcal{H}^{(p)}), p \in \mathbb{N} \}$$

pointwise mult.

$$\hat{\mathcal{A}} := \{ \hat{A}_N(a^{(p)}) \mid a^{(p)} \in \mathcal{B}(\mathcal{H}^{(p)}), p \in \mathbb{N} \},$$

ops. def. on $\mathcal{F}^{fin.}$, operator mult.

(i) $$\hat{A}_N(a^{(p)})^* = \hat{A}_N(a^{(p)*}) = \left(\widehat{A(a^{(p)})}\right)_N$$

(ii) $$\hat{A}_N(a^{(p)})\hat{A}_N(b^{(q)}) = \sum_{r=0}^{p\wedge q} \binom{p}{r}\binom{q}{r} \frac{r!}{N^r} \hat{A}_N(a^{(p)} \underset{r}{-} b^{(q)}) = \underbrace{\left(\widehat{A(a^{(p)})\, A(b^{(q)})}\right)_N}_{\hat{A}_N(a^{(p)} \otimes b^{(q)})} + O\left(\frac{1}{N}\right),$$

where

$$a^{(p)} \underset{r}{\rightharpoondown} b^{(q)} = P_+ \left(a^{(p)} \otimes 1^{(q-r)}\right) \times$$

$$\left(1^{(p-r)} \otimes b^{(q)}\right) P_+ \in B(\mathcal{H}^{(p+q-r)}$$

(iii) $\Gamma(u^{-1}) \hat{A}_N (a^{(p)}) \Gamma(u)$

$$= \hat{A}_N \left((u^{-1})^{\otimes p} a^{(p)} u^{\otimes p}\right),$$

in Segal's notations.

(iv) $\| \hat{A}_N (a^{(p)}) \restriction_{\mathcal{J} \le n} \| \le \left(\frac{n}{N}\right)^p \| a^{(p)} \|$

<u>Notation</u>

$$[a^{(p)}, b^{(q)}]_r := a^{(p)} \underset{r}{\rightharpoondown} b^{(q)} - b^{(q)} \underset{r}{\rightharpoondown} a^{(p)}$$

(v) $[\hat{A}_N (a^{(p)}), \hat{A}_N (b^{(q)})]$

$$= \sum_{r=1}^{p \wedge q} \binom{p}{r} \binom{q}{r} \frac{r!}{N^r} \hat{A}_N \left([a^{(p)}, b^{(q)}]_r\right)$$

Note that (iii) $\Rightarrow$

<u>Lemma.</u>

$$e^{itH_N^0}\hat{A}_N(a^{(p)})e^{-itH_N^0}$$

$$= \hat{A}_N(a_t^{(p)}), \quad \text{where}$$

$$a_t^{(p)} = (e^{ith})^{\otimes p} a^{(p)} (e^{-ith})^{\otimes p}$$

In the following,

$$W_{N,t} := e^{itH_N^0} W_N e^{-itH_N^0}$$

$$\overset{\text{Lemma}}{=} \frac{1}{2}\hat{A}_N\left((e^{ith})^{\otimes 2}\phi(e^{-ith})^{\otimes 2}\right)$$

center-of-mass motion drops out!

"Kato smoothing":

$$\int \|\phi_t \psi\|^2 dt \leq \pi \|\psi\|^2$$

7. Schwinger–Dyson exp.

First assume, V, ϕ bounded

$$e^{itH_N}\hat{A}_N(a^{(p)})e^{-itH_N}$$

$$= e^{isH_N} e^{-isH_N^0}\hat{A}_N(a_t^{(p)})e^{isH_N^0}e^{-isH_N}\Big/_{s=t}$$

$$= \hat{A}_N(a_t^{(p)}) + \int_0^t dt_1\, e^{it_1 H_N} e^{-it_1 H_N^0} \times$$

$$iN[W_{N,t_1}, \hat{A}_N(a^{(p)})]e^{it_1 H_N^0}e^{-it_1 H_N} \quad (5)$$

on $\mathcal{F}^{fin.}$. Iterate (5) $\rightarrow$

$$e^{itH_N}\hat{A}_N(a^{(p)})e^{-itH_N}$$

$$= \sum_{k=0}^{\infty} \int_{\Delta^k(t)} d\underline{t}\, (iN)^k [W_{N,t_k}, \dots,$$

$$[W_{N,t_1}, \hat{A}_N(a_t^{(p)})] \dots]$$

$\uparrow$ simplex

conv. like

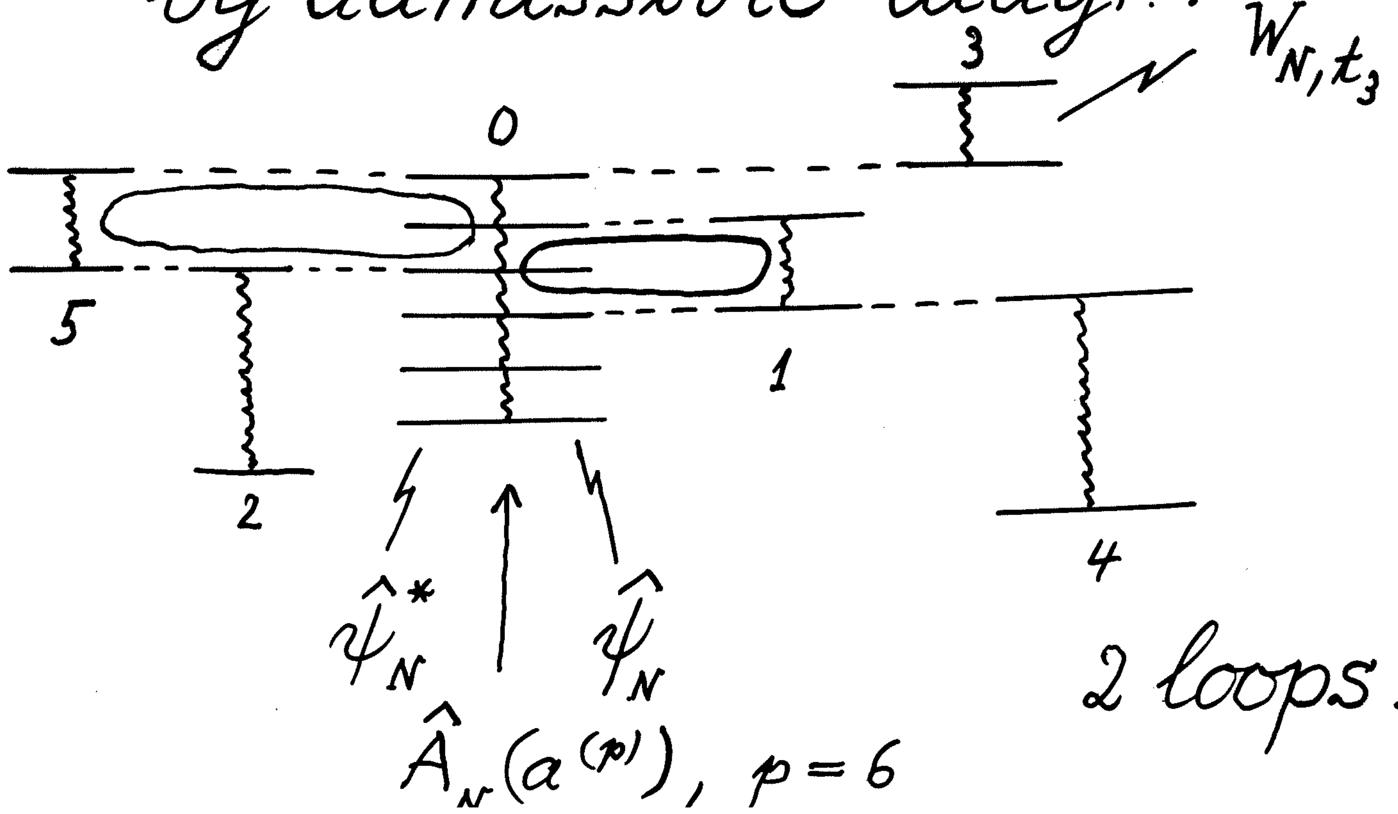
Q

$$\sum_k \frac{(|t| n^2 \|\phi\|_\infty N^{-1})^k}{k!} \left(\frac{n}{N}\right)^p \|a^{(p)}\|$$

on $\mathcal{H}^{(n)}$ $\longrightarrow$ <u>inadequate</u> to study M-F-L, $n = \nu N$, $N \to \infty$.

Use that multiple commutators only yield "connected terms" (because $[a^{(p)}, b^{(q)}]_{r=0} = 0$).

These terms can be labelled by "admissible" diagr.:



2 loops!

R

Next

$$(iN)^k \left[W_{N,t_k}, \ldots, \left[W_{N,t_1}, \hat{A}_N(a_t^{(p)}) \right] \ldots \right]$$

$$= \sum_{\ell=0}^{k} \frac{1}{N^\ell} \hat{A}_N \left(G_{t,\underline{t}}^{(k,\ell)} (a^{(p)}) \right),$$

where

$$G_{t,\underline{t}}^{(k,\ell)}(a^{(p)}) = i(p+k-\ell-1)\left[\phi_{t_k}, G_{t,\underline{t}}^{(k-1,\ell)}(a^{(p)})\right]_1$$

$$+ i\binom{p+k-\ell}{2}\left[\phi_{t_k}, G_{t,\underline{t}}^{(k-1,\ell-1)}(a^{(p)})\right]_2, \quad (6$$

$\underline{t} := \{t_1, \ldots, t_{k(-1)}\}$; $G_t^{(0,0)}(a^{(p)}) := a_t^{(p)}$.

"Adm.", connected diagr., $\mathcal{G}$, $\longleftrightarrow$ "elementary" contributions to $G_{t,\underline{t}}^{(k,\ell)}$; $k, \ell = 0, 1, 2, \ldots$, with $\ell \leq k$, $p+k-\ell \leq$ particle number.

8. Convergence of S-D exp. on $\mathcal{F}^{\leq \nu N}$, unif. in N

$G^{(k,\ell)}_{t,\underline{t}}(a^{(p)})$ is a $(p+k-\ell)$-particle kernel $\leftrightarrow$ op. on $\mathcal{H}^{(p+k-\ell)}$, with (see (6)!)

$$\|G^{(k,\ell)}_{t,\underline{t}}(a^{(p)})\| \leq 2^k \binom{k}{\ell} (p+k-\ell)^\ell \times$$
$$\times (p+k-1)\cdots p \, \|\phi\|^k \, \|a^{(p)}\|$$

(crude combinatorial estimate,

Lemma. $0 < \nu < \infty$, $|t| < (8\nu\|\phi\|)^{-1}$

Then S-D exp. on $\mathcal{F}^{\leq \nu N}$ converges in norm, uniformly in N.

Proof. Recall that $p+k-\ell \overset{!}{\leq} [\nu N]^{T}$

$$\Rightarrow \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{1}{N^\ell} \int_{\Delta^k(t)} d\underline{t} \, \| \hat{A}_N (G^{(k,\ell)}_{t,\underline{t}} (a^{(p)})) \restriction_{\mathcal{F}^{\leq \nu N}} \| \qquad (7.$$

$$\leq \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \underbrace{\frac{(p+k-\ell)^\ell}{N^\ell} \chi_{\{p+k-\ell \leq [\nu N]\}}}_{\leq \nu^\ell \chi_{\{\cdots\}}} \frac{1}{k!} \times$$

$$\times \left(2\|\phi\| \, |t|\right)^k \binom{k}{\ell} \underbrace{(p+k-1)\cdots p}_{= \binom{p+k-1}{k} k!} \, \nu^{p+k-\ell} \|a^{(p)}\|$$

$$\leq \sum_{k=0}^{\infty} \left(8\nu \|\phi\| \, |t|\right)^k (2\nu)^p \|a^{(p)}\|,$$

for all $N \geq 1$. ■

Corollary.

$$e^{itH_N} \hat{A}_N (a^{(p)}) e^{-itH_N} \restriction_{\mathcal{F}^{\leq \nu N}}$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{N^{\ell}} \sum_{k=\ell}^{\infty} \int_{\Delta^k(t)} d\underline{t}\, \hat{A}_N \left(G_{t,\underline{t}}^{(k,\ell)}(a^{(p)})\right)\Big|_{\mathcal{F}^{\leq}}^{U}$$

converges in norm, for $|t| < (8\nu\|\phi\|)^{-1}$ and all suff. large N (dep. on $|t|, p$).

<u>Extension to arbitrary t</u>

Given t, choose $m = 1, 2, 3, \dots$ s.t. $\left|\frac{t}{m}\right| \leq (16\nu\|\phi\|)^{-1}$.

$$e^{itH_N} \hat{A}_N(a^{(p)}) e^{-itH_N}$$

$$= e^{i(t-\frac{t}{m})H_N} \left(e^{i\frac{t}{m}H_N} \hat{A}_N(a^{(p)}) e^{-i\frac{t}{m}H_N}\right) e^{-i(t-\frac{t}{m})H_N}$$

$$= e^{i(t-\frac{t}{m})H_N} \left(\sum_{k=0}^{K_1} \sum_{\ell=0}^{k} \int_{\Delta_k(\frac{t}{m})} d\underline{t}\, \frac{1}{N^{\ell}} \times \right. \qquad (8)$$

$$\hat{A}_N\left(G_{\frac{t}{m},\underline{t}}^{(k,\ell)}(a^{(p)})\right)\Big) e^{-i(t-\frac{t}{m})H_N} + \mathcal{E}_N(K)$$

Using unitarity,

$$\| \mathcal{E}_N(K_1)\restriction_{\mathcal{J}\leq \nu N} \| \leq C_p \delta^{K_1}, \ \delta < 1.$$

Since radius of conv. of S-D exp. is indep. of p, may iterate (8); truncate again, etc.

Corollary

$$e^{itH_N} \hat{A}_N(a^{(p)}) e^{-itH_N} \Big|_{\mathcal{J}\leq \nu N}$$

$$= \text{(truncated) sum over ``tree terms''} + o_N(1)$$

$$= \big(\widehat{A(a^{(p)}) \circ \Phi_t} \big)_N \Big|_{\mathcal{J}\leq \nu N} + o_N(1)$$

iterative solu. of Hartree Eq.

W

9. Convergence for Coulomb potentials.

Idea: Use "Kato smoothing",

$$\int \| |x|_\varepsilon^{-1} e^{it\Delta}\psi\|^2 dt \leq \pi \|\psi\|^2,$$

$\forall \varepsilon$, where $|x|_\varepsilon^{-1} = \begin{cases} |x|^{-1}, & |x| \geq \varepsilon \\ \varepsilon^{-1}, & \text{elsew.} \end{cases}$

We set $\phi_\varepsilon(x) := \kappa |x|_\varepsilon^{-1}$ (2-body pot.) Then

$$\int \|\phi_\varepsilon(x_k - x_\ell) e^{it(\sum_j \Delta_j)} \Phi^{(n)}\|^2 dt$$

$$\leq \frac{\pi\kappa^2}{2} \|\Phi^{(n)}\|^2, \quad \forall \varepsilon.$$

Pf. W.l.o.g., set $k=1, \ell=2$, & use c-o-m coos., $X = \frac{x_1 + x_2}{2}$, $\xi = x_1 - x_2$. Then

$$\Delta_1 + \Delta_2 = \frac{1}{2}\Delta_X + 2\Delta_\xi\,,\quad [\Delta_X, \phi_\varepsilon(\xi)] = 0.$$

Using unitarity, we find that

$$\int dt \,\|\phi_\varepsilon(\xi) e^{it(\sum_j \Delta_j)} \Phi^{(n)}\|^2$$

$$\overset{\text{Fub.}}{=} \int d^3X \prod_3^n d^3x_j \int dt\, d^3\xi \,\Big|\phi_\varepsilon(\xi) e^{2it\Delta_{\xi_X}} \times \Phi^{(n)}\big(X+\tfrac{\xi}{2}, X-\tfrac{\xi}{2}, \cdots\big)\Big|^2$$

$$\overset{\text{Ksm.}}{\leqslant} \frac{\pi\varkappa^2}{2} \|\Phi^{(n)}\|^2.$$

Cauchy-Schwarz $\Rightarrow$

$$\int_0^t ds \,\|\phi_\varepsilon(\xi) e^{-is(\sum_j \Delta_j)} \Phi^{(n)}\|$$

$$\leqslant \left(\frac{\pi\varkappa^2 t}{2}\right)^{1/2} \|\Phi^{(n)}\| \qquad (9)$$

Apply this iteratively to

bound terms in (7), using unitarity, and:

(i) $G^{(k,l)}_{t,\underline{t}}(a^{(p)}) = \sum_{\text{adm.}\,\mathcal{G}_{k,l}} G^{\mathcal{G}_{k,l}}_{t,\underline{t}}(a^{(p)})$

(ii) For $\pi \in \mathcal{S}_k$, $\mathcal{G}_{k,l}$ a diagr. (as above) of order k w. l loops, $\mathcal{G}^{\pi}_{k,l}$ obtained from $\mathcal{G}_{k,l}$ by applying π to time-ordering of vertices.

(iii) $\mathcal{G}_{k,l}$ admissible $\not\Rightarrow$ $\mathcal{G}^{\pi}_{k,l}$ admissible; but true, e.g., for $l = 0$

(and, dep. on l, for π in "large" subgroups of γ_k).

(iv) $\mathcal{G}_1 \sim \mathcal{G}_2$ iff $\mathcal{G}_1 = \mathcal{G}_2^{\pi}$, for some permutation π; (equiv classes denoted by $[\mathcal{G}]$).

Then we find that

$$\int_{\Delta_k(t)} d\underline{t} \, \left\| G^{(k,l)}_{t,\underline{t}}(a^{(p)}) \Phi^{(n)} \right\|$$

$$\leq \sum_{\text{adm. } \mathcal{G}_{k,l}} \int_{\Delta_k(t)} d\underline{t} \, \left\| G^{\mathcal{G}_{k,l}}_{t,\underline{t}}(a^{(p)}) \Phi^{(n)} \right\|$$

$$\leq \sum_{\pi \in \gamma_k} \sum_{\substack{[\mathcal{G}_{k,l}] \\ \mathcal{G}_{k,l} \text{ adm.}}} \int_{\Delta_k(t)} d\underline{t} \, \left\| G^{\mathcal{G}^{\pi}_{k,l}}_{t,\underline{t}}(a^{(p)}) \Phi^{(n)} \right\|$$

$$= \sum_{\substack{[\mathcal{G}_{k,\ell}] \\ \mathcal{G}_{k,\ell}\ \text{adm.}}} \int_0^t dt_1 \cdots \int_0^t dt_k \left\| G_{t,\underline{t}}^{\mathcal{G}_{k,\ell}} (a^{(p)}) \Phi^{(n)} \right\|$$

$$\overset{(9)}{\leq} \left| \{ [\mathcal{G}_{k,\ell}] \mid \mathcal{G}_{k,\ell}\ \text{adm.} \} \right| \left(\frac{\pi \kappa^2 |t|}{2} \right)^{k/2}$$

$$\times \|a^{(p)}\| \, \|\Phi^{(n)}\| \quad (10)$$

A fairly well known combi.

estimate is

$$\left| \{ [\mathcal{G}_{k,\ell}] \mid \mathcal{G}_{k,\ell}\ \text{adm.} \} \right| \leq 2^k \binom{k}{\ell} \binom{2p+3k}{k}$$

$$\times \underbrace{(p+k-\ell)}_{\leq n_{max.} = [\nu N]}{}^{\ell}$$

(11)

(10) & (11) yield

Theorem. For $\phi_\varepsilon(x) = x|x|_\varepsilon^{-1}$, $\varepsilon \geq 0$, and $|t|$ small enough, [indep. of ε & p!],

$$e^{itH_N}\hat{A}_N(a^{(p)})e^{-itH_N}\restriction_{\mathcal{F}^{\leq \nu N}}$$

$$= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} \frac{1}{N^\ell}\int_{\Delta_k(t)} d\underline{t}\, \hat{A}_N\left(G_{t,\underline{t}}^{(k,\ell)}(a^{(p)})\right)\restriction_{\mathcal{F}^{\leq \nu N}}$$

converges in norm, unif. in $N \geq 1$.

Extension to arb. $|t|$, & convergence of pert. solu. of HE on Γ_ν ($|t|$ small enough) as before, using K sm. (9).

10. Sketch of Pickl's method

Instead of the Heisenberg-
employ the Schrödinger
picture.

$$i\dot{\Psi}_t^{(N)} = H_N \Psi_t^{(N)}$$

$$H_N = \sum_1^N \left(-\Delta_j + V_t(x_j)\right) + \sum_{i<j} \underbrace{\phi_N(x_i - x_j)}_{O(N^{-1})}$$

$$\Psi_0^{(N)} \simeq \prod_{j=1}^N \varphi(x_j)$$

Consider evolution of φ
according to HE:

$$i\dot{\varphi}_t = \left(-\Delta + V_t + \phi_{\varphi_t}\right)\varphi_t ,$$

$$\phi_{\varphi_t}(x) := \int d^3y \, |\varphi_t(y)|^2 \phi(y-x)$$

$$\overline{H}_N^{\varphi_t} := \sum_j \left(-\Delta_j + V_t(x_j) + \phi_{\varphi_t}(x_j) \right)$$

One-particle reduced density matrix:

$$\mu^{\Psi^{(N)}}(x,y) := \int \prod_{j=2}^{N} d^3x_j \times$$
$$\times \Psi^{(N)}(x, x_2, \dots, x_N) \overline{\Psi^{(N)}(y, x_2, \dots, x_N)}$$

For $\varphi \in L^2(\mathbb{R}^3)$, define $p^\varphi := |\varphi\rangle\langle\varphi|$

$$q^\varphi := 1 - p^\varphi,$$

$$p_j^\varphi := \mathbb{1} \otimes \cdots \otimes \underset{\underset{j}{\uparrow}}{p^\varphi} \otimes \cdots \otimes \mathbb{1},$$

$$q_j^\varphi := \mathbb{1}_N - p_j^\varphi$$

$$P_{N,k}^\varphi := \sum_{\substack{\underline{a}, \\ \sum_j a_j = k}} \prod_{\ell=1}^{N} (p_\ell^\varphi)^{1-a_\ell} (q_\ell^\varphi)^{a_\ell}$$

Then $\sum_k P^{\varphi}_{N,k} = \mathbb{1}_N$, and

$P^{\varphi}_{N,0}$ projects onto $\prod_{j=1}^{N} \varphi(x_j)$;

$P^{\varphi}_{N,k}$ —"— —"— states,

where k particles are in orbitals $\perp \varphi$.

Pick fu. $n(\cdot) > 0$ on $\{1, \cdots, N\}$, $\left(0 < n(k) < 1, \text{ e.g., } n(k) = \frac{k}{N}\right)$,

and define

$$\alpha^n_N(\Psi^{(N)}, \varphi) := \langle \Psi^{(N)}, P^{\varphi}_N(n) \Psi^{(N)} \rangle,$$

where

$$P^{\varphi}_N(n) := \sum_{k=0}^{N} n(k) P^{\varphi}_{N,k}.$$

If $n_0(k) = \frac{k}{N}$ then

$$P_N^{\varphi}(n_0) = N^{-1} \sum_{j=1}^{N} q_j^{\varphi}.$$

Next

(i) $\lim_{N\to\infty} \langle \Psi^{(N)}, P_N^{\varphi}(n_0) \Psi^{(N)} \rangle = 0$

$$\Leftrightarrow \ n\text{-}\lim_{N\to\infty} \mu^{\Psi^{(N)}} = p^{\varphi}$$

(ii) $\lim_{N\to\infty} \langle \Psi^{(N)}, P_N^{\varphi}(n_0) \Psi^{(N)} \rangle = 0$

$$\Leftrightarrow \lim_{N\to\infty} \langle \Psi^{(N)}, P_N^{\varphi}(n_0)^j \Psi^{(N)} \rangle = 0,$$

$j = 1, 2, 3, \cdots.$

<u>Key idea</u>

Control $\dot{\alpha}_N(t) \equiv \dot{\alpha}_N(\Psi_t^{(N)}, \varphi_t) =$

$$= i \langle \Psi_t^{(N)}, [H_N - \bar{H}_N^{\varphi_t}, P_N^{\varphi_t}(n_0)] \Psi_t^{(N)} \rangle$$

by estimating R.S., with $\Psi_0^{(N)} = \prod_{j=1}^{N} \varphi(x_j)$, hence

$$\alpha_N(0) \equiv \alpha_N(\Psi_0^{(N)}, \varphi) = 0.$$

<u>Lemma.</u>

$$|\dot{\alpha}_N(t)| \leq C_t \alpha_N(t) + O(\tfrac{1}{N}),$$

$$C_t \sim \text{const.} \|\phi\|_{2r} \cdot \|\varphi_t\|_{2s},$$

$$r \geq 1, \quad s = \frac{r}{r-1}.$$

<u>Corollary.</u>

$$\alpha_N(t) \leq e^{\int_0^t d\tau C_\tau} \alpha_N(0) + O\left(\frac{1}{N}\right)$$

$$\Rightarrow \quad n\text{-}\lim_{N\to\infty} \mu^{\Psi_t^{(N)}} = p^{\varphi_t}.$$

Note: C_t controlled by

properties of Hartree flow.

Can be used to discuss Gross-Pitaevskii limit!

Appendix to Chapters 1 and 2:

Theory of non-relativistic electrons (and positrons) with spin – coupled to external electromagnetic fields

1. FROM THE CLASSICAL HARMONICS OF NEWTON, LEIBNIZ, BACH, JACOBI ... TO HEISENBERG's QU. HARMONICS

Classical mechanics =
Hamiltonian dynamics on
class. phase space, Γ, w.
Γ = sympl. mf. equipped
w. ω, $\{\cdot,\cdot\}_P$.

Pure states = points in Γ;
mixed states = prob. meas. on Γ

"Observables": $\mathcal{A}_0 = C^\infty(\Gamma)$.

<u>Ex.</u> $\Gamma = T^*M$, $M = \mathbb{R}^f$, pt. part.

If point particles el. charged, int. "obs." is a component, x, of el. dipole moment.

System w. $\Gamma = R^{2f}$ integrable iff $\exists$ f integrals of motion

$$\underline{A} = (A_1, \cdots, A_f)$$

in involution. $\Rightarrow$ Traj. of system lie on f-D inv. tori, $T^f = \{\underline{A} = \text{const.} = \underline{A}_*\}$

Observable, x, rest. to T^f is periodic fu. of angles

$$\underline{\varphi} = \{\varphi_1, \cdots, \varphi_f\}.$$

$$\Longrightarrow \quad x = \sum_{\underline{n} \in \mathbb{Z}^f} \hat{x}_{\underline{n}} \, e^{i\underline{n}\cdot\underline{\varphi}} \, .$$

Eqs. of motion $\Longrightarrow$ $\underline{\varphi}(t) = \underline{\omega}_* \cdot t$,

where $\underline{\omega}_* = \left(\partial H / \partial \underline{A}\right)(\underline{A}_*)$,

H: Hamilton fu., $x(0) \in C(T^f(\underline{A}_*))$

$\Rightarrow$ $x(t)$ quasi-per. fu. of t

with frequencies $\omega_{\underline{n}} = \underline{n} \cdot \underline{\omega}_*$

For system of charged particles, e.m. waves of frequencies $\omega_{\underline{n}}$, $\underline{n} \in \mathbb{Z}^f$, emitted. They form abelian group: $\omega_{\underline{n}} + \omega_{\underline{m}} = \omega_{\underline{n}+\underline{m}}$.

Experimentally, this is wrong!

Ritz-Rydberg ... principle:

Frequencies of light em. by system form groupoid, $\mathcal{G}$:

$\omega = \omega_{\underline{\nu}\underline{\mu}}$, w. $\omega_{\underline{\nu}\underline{\mu}} = \omega_{\underline{\nu}\underline{\lambda}} + \omega_{\underline{\lambda}\underline{\mu}}$,

for $(\underline{\nu}\underline{\mu}), (\underline{\nu}\underline{\lambda}), (\underline{\lambda}\underline{\mu})$ in $\mathcal{G}$.

Bohr-Sommerfeld:

$$\omega_{\underline{\nu}\underline{\mu}} = \omega_{\underline{\nu}} - \omega_{\underline{\mu}},$$

$\underline{\nu}, \underline{\mu}$: allowed quantum #

$$\hbar\omega_{\underline{\nu}} = H(\underline{A}_{\underline{\nu}}), \quad \underline{A}_{\underline{\nu}} = h\underline{\nu},$$

$\underline{\nu} \in \mathbb{Z}^f$. $\longrightarrow$ Hydrogen atom, harmonic oscillator.

Heisenberg (1925): In Qu. Th., "observables", x, only dep. on data of allowed proc., $\to$ pairs of allowed quantum states $\to$ pairs $(\underline{\nu}\,\underline{\mu}) \in \mathcal{G}$, i.e.,

$x \sim \hat{x}_{\underline{n}} \mapsto$ scheme $(x_{\underline{\nu}\underline{\mu}})$, with

$\overline{x}_{\underline{\mu}\underline{\nu}} = x_{\underline{\nu}\underline{\mu}}$ ("reality cond.", emission $\to$ absorption).

$$\hat{x}_{\underline{n}} \longrightarrow x_{\underline{\nu}\underline{\mu}}$$

$$\widehat{(x^1 \cdot x^2)}_{\underline{n}} = \sum_{\underline{m} \in \mathbb{Z}^f} \hat{x}^1_{\underline{n}-\underline{m}} \cdot \hat{x}^2_{\underline{m}}$$

$$\downarrow$$

$$(x^1 * x^2)_{\underline{\nu}\underline{\mu}} = \sum_{\underline{\lambda}} x^1_{\underline{\nu}\underline{\lambda}} \cdot x^2_{\underline{\lambda}\underline{\mu}}$$

convol. prod. $\to$ matrix prod.

By R.-R., B.-S. & Maxwell th.

$$x_{\underline{\nu}\underline{\mu}}(t) = x_{\underline{\nu}\underline{\mu}}\, e^{i\omega_{\underline{\nu}\underline{\mu}} \cdot t}$$

(1)

$$= \left(e^{i(Ht)/\hbar} * x * e^{-i(Ht)/\hbar}\right)_{\underline{\nu}\underline{\mu}}$$

w: $H = \mathrm{diag}\,(\hbar\omega_{\underline{\nu}})$

In order to be able to "forget" B.-S. (e.g., to quantize non-integr. systems), find alg. generated by $x, \dot{x}, \ldots$

Thomas-Kuhn-Heisenberg sum rules: $p := m\dot{x}$, where x is a component of dipole moment, m = mass of particle (electron). Then

$$[x,p]_{\underline{\nu}\,\underline{\nu}} = [x, m\dot{x}]_{\underline{\nu}\,\underline{\nu}} \quad (2)$$

$$\stackrel{(1)}{=} 2im \sum_{\underline{\mu}} |x_{\underline{\nu}\,\underline{\mu}}|^2 \, \omega_{\underline{\nu}\,\underline{\mu}}$$

Bohr's correspondence pr.:

For $|\underline{\nu}-\underline{\mu}| \ll |\underline{\nu}|, |\underline{\mu}|, (|\underline{\nu}|, |\underline{\mu}| \gg 1)$

$x_{\underline{\nu}\,\underline{\mu}} \approx \hat{x}_{\underline{\nu}-\underline{\mu}}$, w. $\hat{x}_{\underline{\nu}-\underline{\mu}}$ calculated for traj. in $T^f(\underline{A}_{\frac{\underline{\nu}+\underline{\mu}}{2}})$

For hydrogen atom or harmonic osc., $|\hat{x}_{\underline{\nu}-\underline{\mu}}|^2$ calc. by using virial thm. and

$\omega_{\underline{\nu}\,\underline{\mu}}$ from Bohr-Sommerfeld.

$$\Longrightarrow \; [x,p]_{\underline{\nu}\,\underline{\nu}} \overset{(2)}{\approx} i\hbar$$
$$= i\hbar \; (\text{Heis.}, \text{'25})$$

Born-Jordan '25:

Matrix interpretation;

for $H = p^2/2m + \cdots + v(x, x^2, \cdots)$,

$$[H, [x,p]] = 0$$

$$\Longrightarrow \quad [x,p]_{\underline{\nu}\,\underline{\mu}} = 0, \quad \omega_{\underline{\nu}\,\underline{\mu}} \neq 0$$

$$[x,p] = i\hbar\, \mathbb{1}$$

Dirac, '25: trsf. theory,

$$\{f,g\}_P \longrightarrow \frac{i}{\hbar}[f,g]$$

B-H-J, '25: "quantum geometry"

Miracle: Ritz-Rydberg, B.-S.,

corresp. pr. only approx. valid

($\alpha = e^2/\hbar c \simeq 1/137 \ll 1$); yet,

Heisenberg's matrix mech.

consistent new mech.!

$$(\mathcal{A}_0, \cdot) \longrightarrow (\mathcal{A}_\hbar, *)$$

$\hbar$: deformation parameter

MM $\sim$ WM (Schrödinger '26)

$\sim$ Feynman path int.,
$\sim$ '43

Action in Darboux coos.

$$S = \frac{1}{2}\int_\gamma \left\{ \sum_j p_j\, dq^j - H(p,q)\, dt \right\}$$

To deform $\mathcal{A}_0$ to $\mathcal{A}_\hbar$, set

$H = 0 \Longrightarrow q^j(t) = p_j(t) = \text{const.}$

D: surface in Γ, $\partial D =: \gamma$ (loop in Γ), $D_0 := \{(\tau^1, \tau^2) \,|\, |\tau| \leq 1\}$.

$$S = \frac{1}{2} \int_\gamma \Big\{ \sum_j p_j \, dq^j \Big\} \underset{\substack{\uparrow \\ \text{Stokes}}}{=} \frac{1}{2} \int_D \omega$$

$$\sim \int_{D_0} d\tau^\alpha \wedge d\tau^\beta \Big\{ \frac{\partial X^A(\tau)}{\partial \tau^\alpha} \eta_{A,\beta}(\tau) - \frac{1}{2} \Omega^{AB}(X(\tau)) \, \eta_{A,\alpha}(\tau) \, \eta_{B,\beta}(\tau) \Big\}$$

$S(X, \eta)$ inv. under repar. of D_0 and symplectomor. of Γ. $*$ product of two fns. f, g on Γ given by

TFT: $X(\infty) \equiv X \in \Gamma$.

$$(f \star_{\hbar} g)(X) = \text{"}\int \mathcal{D}X \int \mathcal{D}\eta \; e^{\frac{i}{\hbar} S(X,\eta)}$$

$$\times f(X(0))\, g(X(1))\text{"}$$

$$= \left(f \cdot g + \frac{i\hbar}{2}\{f,g\} + \cdots\right)(X),$$

$0, 1, \infty \in \partial D_0$.

∞-dim. symmetries of S require gauge fixing (B-V formalism): Problem solved in sense of formal power series in $\hbar$ by Kontsevich; Cattaneo-Felder.

$H_2(\Gamma) \leftrightarrow$ "quant." of $\hbar$.

Shall apply this to C-S!

Concrete examples:

1) $\Gamma = T^*M$ ess. Schrödinger '26

$\Gamma = S^2$ B-H-J '25

$\Gamma = T^*\mathbb{R}^3 \times S^2$ Pauli '25, '27

+ Pauli's exclusion pr.
(Pauli '25, Heisenberg '26)

$\rightarrow$ th. of NR spinning e^-.

2) Γ = (compact) Kähler mf.

Γ = coadj. orbit

Borel-Weil-Bott, geom. qu.

3) Γ = phase space over G^{S^1}
from 3D TFT (C-S & BF theories)

4 deformation parameters

(Planck, 1900)

$\hbar$: as described.

k_B : cont. th. $\rightarrow$ atomism

thermodyn. $\rightarrow$ stat. mech.

c^{-1}: Galilei $\rightarrow$ Lorentz-Poincaré ...

l_P : th. without gravity $\rightarrow$ th. incl. gravity

In nature,

$\hbar, k_B, c^{-1}, l_P \neq 0, = 1$

"quantum gravity"

Need a new "Heisenberg"!

1. Introduction to Spin & the Pauli Principle

Point of departure: $e^{\pm}, p, n,$ particles w. spin $\frac{1}{2}$, mag. moment, obeying "Pauli".

Fund. forces \ grav. med. by quanta w. spin 1, bosons.

$\Downarrow$

Vast number of phenomena:

(1) Chemistry, "nuclear chemistry", "chemistry of hadrons". $\rightarrow \infty$ applic.

(2) Exp. detection of spin: Stern-Gerlach (at); Spin Res. (e^-, nucl.) → Imaging, MRI, …

(3) Spin-pol. e^--emission → e.g., particle physics; e^--spin precession in Weiss exchange field → magn. switching

(4) Magnetism, magnetic order – dia, para, ferro, ferri, anti-ferro, … – conspiracy of e^- magn. moment,

Pauli, Coulomb repulsion;
math. understanding: "?"
→ magn. data storage
(5) Giant & colossal
magneto-resistance: 2007
NP in physics → other lect.
(6) "Spintronics" → ...
quantum info. & computing
?
(7) Pairing mech. in high-T_c
(8) Physics of "white dwarfs"
& of neutron stars – Chandra

2. History of discovery of spin & Pauli principle

1894 – 1925

Discovery of e^-: Cathode rays, **Zeeman spectroscopy** – J.J. Thomson, Wiechert, Kaufmann; Zeeman / Larmor, Lorentz, … Einstein (theory)

e/m, e, $|\vec{v}| < c$, line splitting (in ext. mag. field)

Model of atom

Balmer, Hertz, Rydberg, Rutherford; Bohr-($-S^o, E^p, E^i, \dots$), Ritz, Hund, Landé, Stoner, ...

(1) Atoms have discr., station. states; labelled by quantum #: $n, (L, M_L), (S, M_S), (J, M)$,

$J = L+S, \cdots, |L-S|,\ M = -J, \cdots, J,$

$L = 0, 1, 2, \cdots;\ S = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$

Selection rules:

$\Delta L = \pm 1,\ \Delta J = 0, \pm 1$ in trans.

Z even $\longleftrightarrow$ S, J integer

Z odd $\longleftrightarrow$ S, J half-integer

(2) Bohr's frequency cond.

$$h\nu_{mn} = E_m - E_n$$

mag. field

(3) Line splittings in ext. $\vec{H}$

Paschen-Back: If $\mu_0 |\vec{H}| \gg |\vec{S}| \omega$

$\mu_0 = \frac{e\hbar}{2mc}$, then

[energy]

$$\Delta E \simeq (M_L + 2 M_S)\,\mu_0 |\vec{H}| \qquad (1)$$

Anomalous Zeeman: $\mu_0 |\vec{H}| \lesssim |\vec{S}|\omega$

$$\Delta E \simeq M g\, \mu_0 |\vec{H}|, \qquad (2)$$

$$g = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)} \qquad (3)$$

(Landé). Pauli derives (2), (3) from (1) & Sum Rule;

$\rightarrow$ $\underline{g_S = 2}$, contrad. w. Larmor:

$$|\vec{M}| / |\vec{L}| = \frac{\text{charge}}{2 \times \text{mass} \times c} \; ; \; g = 1$$

mag. moment

M, J half-integer contrad. Bohr-Sommerfeld.

Pauli's discovery of el. spin:

- S cannot be angular mom. of core of atom ($g_{core} \lesssim 1 < 2!$)
- Filled shells have $\vec{M}_{tot.} = 0!$
- S must be property of <u>valence el.</u> (in alkalis) $\rightarrow$

"Class. not descr. <u>2-valuedness</u>";
(Kronig, Urey, Uhl. & Goudsmit)

$g_e = 2$ + fine str. $\rightarrow$ <u>Thomas precession</u> (1926)

<u>Pauli Principle</u> (1925):

Stoner: # states in an el. config. w. given L: $\underbrace{2(2L+1)}_{\text{\# Zeeman spl.}}$;
rôle of J.

Pauli: States of an el. in atom classified by quantum #

$$n,\ (L, M_L),\ m_s = \pm \frac{1}{2}$$

Each state occupied by at most one el.! → struct. of atoms

Fermi, Dirac, Wigner

QM of angular mom. & spin ⟷ unitary reps. of qm rotation group, SU(2):

"Dreimännerarbeit" (B, H, J)

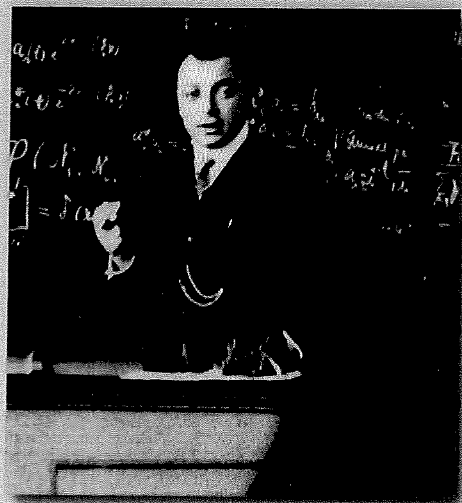
NR electron eq. : Pauli, 1927

⟶ condensed-matter phys., mathematics.

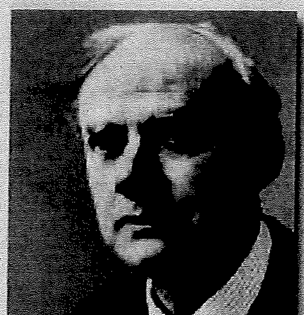
Rel. e^- eq. : Dirac (Kramers), 1927

SPIN & PAULI

EXCLUSION PRINCIPLE



W. Pauli



P.A.M. Dirac

E. Fermi

F. Bloch

3. Mathematics of Spin, and Pauli Equation

Phys. space (homogeneous &) isotropic $\longrightarrow$ space rotations are dyn. symm. of autonomous phys. systems. In QM descr. by unitary reps. of

$$Spin(n) := \widetilde{SO(n)},$$

$n = 2, 3, \cdots$ dim. of space;

($\nearrow$ Wigner, Bargmann). "Spin"s characterizes irreps of Spin(n)

$$n = 2: \ Spin(2) = \mathbb{R} \rightarrow \boxed{s \in \mathbb{R}}$$

$n=3$: $Spin(3) = SU(2) \rightarrow \boxed{s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots}$

For $n \geq 3$, $Spin(n)$ is 2-fold cover of $SO(n)$; if π is irrep. of $Spin(n)$ w. "spin" s then

$$e^{i2\pi s} = 1 \leftrightarrow \pi \text{ rep. of } SO(n)$$

$$\underset{\substack{\uparrow \\ (2\pi\text{-rotation})}}{e^{i2\pi s}} = -1 \leftrightarrow \pi \text{ "double-valued" rep. of } SO(n)$$

<u>Pauli equation</u>: (M = phys. space)

$\{\gamma^j\}_{j=1}^n$: Pauli-Dirac matrices,

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}$$

size $2^{[n/2]} \times 2^{[n/2]}$; for $n=3$,

$$\vec{\gamma} = \vec{\sigma} \quad \text{(Pauli matrices)}$$

A: em vector pot. w. $dA = B$

2. THE "STANDARD MODEL" OF NR QUANTUM THEORY

Consider 1 NR e^-, spin $\frac{1}{2}$.

$$\Gamma = T^*\mathbb{R}^3 \times S^2 \xrightarrow[\hbar]{} \mathcal{H}_e, \vec{p} = \frac{\hbar}{i}\vec{\nabla}, \ldots$$

Hilbert space of state vectors

$$\mathcal{H}_e = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$$

Spin (from S^2): $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$,

$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$: Pauli matrices

Schrödinger-Pauli eq.:

$$i\hbar \frac{\partial}{\partial t}\psi_t = H_t \psi_t,$$

$\psi_t \in \mathcal{H}_e$, $H_t = H_t^*$ given by

$$H_t := \frac{1}{2m}\vec{\pi}^{\,2} -$$

kE

$$\frac{1}{2mc}\left\{\vec{\pi};\left(\vec{\mu}-\frac{e}{2mc}\vec{S}\right)\wedge\vec{E}_t\right\} -$$

S-O

$$e\phi_t - \vec{\mu}\cdot\vec{B}_t \,,$$ where

pE Z

$$\vec{\pi} \equiv m\vec{V} := \frac{\hbar}{i}\vec{\nabla}_x - \frac{e}{c}\vec{A}(\underline{x},t)$$

$$\vec{\mu} := \frac{g\,\mu_{Bohr}}{\hbar}\vec{S}\,, \quad \vec{S} = \frac{\hbar}{2}\vec{\sigma}\,,$$

$$g = 2,\ \mu_{Bohr} = -\frac{e\hbar}{2mc} \simeq 5.79\times10^{-9}\,\frac{eV}{Gauss}$$

$\vec{E}_t$: external electric field

$\vec{B}_t$: ext. magnetic field

$\vec{A}_t$: em vector potential

$U(1)_{em} \times SU(2)_{spin}$ gauge inv.

$$D_\mu := \frac{\partial}{\partial x^\mu} + i a_\mu + w_\mu \,, \quad \mu = 0,1,2,3$$

$$a_0 = \frac{e}{\hbar c}\phi \,, \quad \vec{a} = -\frac{e}{\hbar c}\vec{A} \,,$$

$$w_\mu = \sum_{A=1}^{3} w_{\mu A}\, i\sigma_A \,, \quad \text{with}$$

$$w_{0A} = -\frac{g\mu_{Bohr}}{2\hbar c} B_A + \cdots$$

$$w_{kA} = \left(-\frac{g\mu_{Bohr}}{2\hbar c} + \frac{e}{4mc^2}\right)\sum_{M=1}^{3} \varepsilon_{kAM} E_M + \cdots$$

Adding a term $\propto O(\vec{w}^{\,2})$,

Pauli eq. becomes :

$$i\hbar c D_0 \psi_t = -\frac{\hbar^2}{2m} \vec{D}^2 \psi_t \quad \text{(PE)}$$

Form of this eq. <u>invariant</u> under $U(1) \times SU(2)$ gauge transformations:

$$a \longmapsto a + d\chi$$

$$w \longmapsto g w g^{-1} + g d g^{-1}$$

$$\psi \longmapsto e^{-i\chi} g \psi$$

$$\chi = \chi(\underline{x}, t) \in \mathbb{R}, \quad g = g(\underline{x}, t) \in SU(2)$$

ψ section of vector bundle assoc. to principal $U(1) \times SU(2)$-

bundle; base space $\mathbb{R}^3$.

$a \leftrightarrow U(1)$ connection A-B

$w \leftrightarrow SU(2)$ — " — A-C

effects

<u>General form of a, w</u>

Equip $\mathbb{R}^3$ with <u>curved metric</u>; study Pauli eq. in <u>moving frame</u> det. by $\vec{V}(\underline{x},t)$, $\vec{\nabla}\cdot\vec{V} \equiv 0$.

$$a_0 = \frac{e}{\hbar c}\phi - \frac{m}{2}\vec{V}^2 - \frac{e}{\hbar c}\vec{V}\cdot\vec{A}$$

$$a_k = -\frac{e}{\hbar c}A_k - \frac{m}{\hbar}V_k$$

$$w_{0A} = -\left\{ \frac{g\mu}{2\hbar c} B_A - W_A + \left(-\frac{g\mu}{2\hbar c} + \frac{e}{4mc^2} \right) \frac{1}{c} (\vec{V} \wedge \vec{E})_A + \frac{1}{c} \Omega_A \right\} + (\propto \omega)$$

where

$\mu = \mu_{Bohr}$, $\vec{W}$: Weiss exchange field

$\vec{\Omega} = \frac{1}{2} \vec{\nabla} \wedge \vec{V}$: vorticity

$$w_{kA} = \cdots + \omega_{kA}$$

ω : spin conn. assoc. to ∇_{LC}

$w \longleftrightarrow$ Magnetism, high-T_c

Pauli eq. is Euler-Lagr. eq. derived from $U(1)\times SU(2)$ gauge-inv. action funct.

$$S(\psi^*, \psi; a, w) =$$

$$\int dt \int_{\mathbb{R}^3} dvol. \left\{ i\hbar c\, \psi^* D_0 \psi - \frac{\hbar^2}{2m} (\vec{D}\psi)^* \cdot \vec{D}\psi - U(\psi^*, \psi) \right\}$$

ext. pot, 2-body int.

Starting point for funct. integral approach to many-body theory, incorp. Pauli Excl. Principle.

U(1)×SU(2) gauge invariance common to all systems of non-relat. qm particles w. spin and, w. PP, underlies following phys. phenomena:

U(1) gauge invariance

$\longleftrightarrow$ A-B, spectroscopy – R & M
metals, el. conductivity
quantum Hall effect
London eq., supercon-ductivity, super-fluidity, …

SU(2) gauge invariance

$\leftrightarrow$

- Aharonov-Casher $\leftrightarrow \vec{E}$
- Zeeman effect $\leftrightarrow \vec{B}$
- Bloch spin res. / Einstein-deHaas -Barnett effect $\leftrightarrow \vec{B}, \vec{\Omega}$
- magnetism $\leftrightarrow \vec{B}, \vec{W}$
- high-T_c superconductivity $\leftrightarrow \vec{W}$

QM Larmor theorem

System of electrons in ext. $\vec{B}$-field, vector pot. $\vec{A}$, $\vec{\nabla}\cdot\vec{A}=0$.

To 1^{st} order in $\vec{B}$, can eliminate $\vec{B}$ by passing to moving frame w. $\vec{V} = -\frac{e}{mc}\vec{A}$

e.g. $\Longrightarrow$ • Einstein-deHaas-Barnett,

• London equation:

$$\vec{J}_s \equiv q n_s \vec{V} = -\frac{q^2 n_s}{Mc}\vec{A}$$

$$q = 2e, \quad M \simeq 2m.$$

• Quant. flux of vortices: supercond. - superfluids

$$\Phi_{\vec{B}} = \frac{hc}{q} n \longleftrightarrow \Phi_{\vec{V}} = \frac{h}{M} n,$$

$$n \in \mathbb{Z}$$

$$q = 2e \qquad M \overset{e.g.}{=} m_{^4He}, 2m_{^3He}$$

3. PAULI's ELECTRON $\leftrightarrow$ SUSY QM $\leftrightarrow$ DIFF. GEOMETRY & -TOPOLOGY

Consider an e^- in $(\mathbb{R}^3, g)$.

$$\mathcal{H}_e = L^2(\mathbb{R}^3, dvol_g) \otimes \mathbb{C}^2.$$

Introduce Pauli–Dirac op.:

$$\mathcal{D} := \sum_{k=1}^{3} \sigma^k (-i \nabla_k)$$

∇: spin conn. on spinor bundle over $(\mathbb{R}^3, g)$ assoc. w LC

Neglect electrostat. potential, spin-orbit interactions & exchange field. Then

$$\nabla_k = \partial_k - i \frac{e}{\hbar c} A_k + \frac{1}{2} \omega_{k,ab} [\sigma^a, \sigma^b]$$

spin connection

Susy form of Pauli eq.:

$$i\hbar \partial_t \Psi_t = \frac{\hbar^2}{2m} \mathcal{D}^2 \Psi_t .$$

$\leftrightarrow$ $g=2!$ inv. under time-indep. $U(1)\times SU(2)$ gauge trsfs.

Set $$Q = \frac{\hbar}{\sqrt{2m}} \mathcal{D}$$

Q is "supercharge" of $N=1$ susy QM, with $H = Q^2$!

- Analogous theory for NR positrons: Pass to conjug. spinor bundle, $e \to -e$.
- Positronium = electron-positron groundstate

State space:

$$L^2(\mathbb{R}^3, dvol_g) \otimes (\Lambda_+ \oplus \Lambda_-),$$

$$\Lambda_\pm := (\Lambda_0 \oplus \Lambda_1)_{\pi=\pm}$$

square-int. diff. forms on $\mathbb{R}^3$

$\longrightarrow$ $N=2$ susy QM, super-charges d, d^*,

$$H = (2M)^{-1}(dd^* + d^*d).$$

Generalize this theory:

$$(\mathbb{R}^3, g) \longrightarrow (M, g)$$

M: gen. Riem. (spin^c) mf.

Can recover all of diff. top. & diff. geom. from susy QM of "electron" & "positronium"; special geom. $\longleftrightarrow$ higher susy.

(M,g) : smooth, orientable Riemannian spin^c mf.

$$\begin{tikzcd} \text{cotangent bundle} & T^*M \arrow[dr] & & \Lambda^\bullet M \arrow[dl] & \text{forms} \\ & & M & & \end{tikzcd}$$

$\Omega^\bullet(M) = \{\text{smooth sects. of } \Lambda^\bullet M\}$

differential forms

Module for $\mathcal{A} := C^\infty(M)$

$\mathcal{H} = L^2(\Omega^\bullet(M) \otimes \mathbb{C}, dvol_g)$

For $\xi \in \Omega^1(M) \otimes \mathbb{C}$, set

$X := g^{-1}\xi \in \Gamma(TM)$

vector fields

Creation- & annihilation ops

$\xi, \eta, \cdots \in \Omega^1(M) \otimes \mathbb{C}$, $\Phi \in \mathcal{H}$.

$$a^*(\xi)\Phi := \xi \wedge \Phi,$$

$$a(\xi)\Phi := \bar{X} \lrcorner \Phi.$$

Then

$$\{a^\#(\xi), a^\#(\eta)\} = 0,$$

CAR

$$\{a(\xi), a^*(\eta)\} = g^{-1}(\bar{\xi}, \eta),$$

$$a(\xi)\Omega^0 = 0, \ a^*(\xi)\Omega^n = 0,$$

where $n = \dim M$.

$$\Gamma(\xi) := a^*(\xi) - a(\xi)$$

$$\bar{\Gamma}(\xi) := i(a^*(\xi) + a(\xi))$$

Then

$$\{\Gamma(\xi), \bar{\Gamma}(\eta)\} = 0$$

$$\{\overset{(-)}{\Gamma}(\xi), \overset{(-)}{\Gamma}(\eta)\} = -2\,\mathrm{Re}\, g^{-1}(\bar{\xi}, \eta)$$

two anticomm. sects. of Clifford bundle.

(M, g) spin^c $\Leftrightarrow \exists$ hermitian vector bundles $W, \bar{W}$, $\overset{(-)}{S} := \Gamma(\overset{(-)}{W})$, spinors, s.t.

dim M even:

$$\Omega^{\cdot}(M) \otimes \mathbb{C} \simeq \bar{S} \otimes_{A} S,$$

$\Gamma(\xi) = 1 \otimes c(\xi)$, $\bar{\Gamma}(\xi) = \bar{c}(\xi) \otimes \gamma$,

Clifford generators

$\xi \in \Omega^1(M)$.

dim M odd:

$$\Omega^{\cdot}(M)\otimes\mathbb{C} \simeq \bar{S}\otimes_{A} S \otimes \mathbb{C}^2$$

$$\Gamma(\xi) = 1\otimes c(\xi)\otimes\sigma^3,\ \bar{\Gamma}(\xi) = \bar{c}(\xi)\otimes 1\otimes\sigma^1$$

QM of Pauli electron and positron on (M,g)

$$\mathcal{H}_e := L^2(S, dvol_g),\ \mathcal{H}_p := L^2(\bar{S}, dvol_g)$$

∇^S_A : spin^c connection on S compatible with ∇_{LC}, i.e.,

em

$$\nabla^S_A(c(\xi)\psi) = c(\nabla_{LC}\xi)\psi + c(\xi)\nabla^S_A\psi$$

$\nabla^{\bar{S}}_A$: spin^c connection on $\bar{S}$ conjugate to ∇^S_A

Ambiguity of ∇^S_A:

$$(\nabla^S_{A_1} - \nabla^S_{A_2})\psi = i\alpha \otimes \psi, \quad \alpha \in \Omega^1(M).$$

$$R_{\nabla^S_A}(X,Y) = \nabla^S_{A,X}\nabla^S_{A,Y} - \nabla^S_{A,Y}\nabla^S_{A,X} + \nabla^S_{A,[X,Y]}$$

$$F_A(X,Y) := 2^{-[\frac{n}{2}]} tr\left(R_{\nabla^S_A}(X,Y)\right)$$

em field strength

Pauli electron:

$D_A := c \circ \nabla^S_A$ on $\mathcal{H}_e$

Hamiltonian

$$H := \frac{1}{2m} D_A^2 + \phi$$

$$= \frac{1}{2m}\left(-\Delta + \frac{r}{4} + c(F_A)\right) + \phi$$

sa on $(\mathcal{D} \subseteq)\, \mathcal{H}_e$, $g = 2$

"Observables":

$$\mathcal{A} = C^\infty(M) \quad \left(\hat{\mathcal{A}} = \psi DO(T^*M \times G_{n,2})\right)$$

"Pauli positron"

$$\bar{D}_A := \bar{c} \circ \nabla^{\bar{S}}_A \quad \text{on } \mathcal{H}_p$$

$$H = \frac{1}{2m} \bar{D}_A^{\,2} - \phi$$

N = 1 spectral data

$$(\mathcal{H}, \mathcal{A}, D, (\gamma))$$

$$\mathcal{H} = \mathcal{H}_e, \mathcal{H}_p, \quad D = D_A, \bar{D}_A$$

"$\gamma = \gamma_5$": $\mathbb{Z}_2$ grading, n even,

$$\{\gamma, D\} = 0 = [\gamma, a], \quad a \in \mathcal{A}.$$

$e \leftrightarrow p$: "charge conjug."

Theorem (A.C.)

$(\mathcal{H}, \mathcal{A}, D, \gamma)$ encodes Riem. geometry of spin^c mf. (M, g) completely, (incl. de Rham th., ...).

Open problem: $\mathcal{A} \to \hat{\mathcal{A}}$?

QM of NR positronium

Positronium = groundstate of bound electron-positron

$$\Rightarrow \mathcal{H}_{e-p} = \mathcal{H}_p \otimes_{\mathcal{A}} \mathcal{H}_e \, (\otimes \mathbb{C}^2)$$

$$\simeq L^2(\Omega^{\cdot}(M) \otimes \mathbb{C}, dvol_g)$$

For $\Psi = \bar{\Psi}_1 \otimes_A \Psi_2$, $\Psi_1 \in \mathcal{H}_p$, $\Psi_2 \in \mathcal{H}_e$,

$$\nabla \Psi := (\nabla_A^{\bar{S}} \bar{\Psi}_1) \otimes \Psi_2 + \bar{\Psi}_1 \otimes_A \nabla_A^S \Psi_2$$

$$= \nabla_{LC} \Psi \quad \text{indep. of } A$$

$$\mathcal{D} := \Gamma \circ \nabla, \quad \bar{\mathcal{D}} := \bar{\Gamma} \circ \nabla.$$

$$\{\mathcal{D}, \bar{\mathcal{D}}\} = 0, \quad \mathcal{D}^2 = \bar{\mathcal{D}}^2. \quad \#$$

$$H = \frac{1}{2\mu} \mathcal{D}^2 = \frac{1}{2\mu} \bar{\mathcal{D}}^2 \text{ on } \mathcal{H}_{e-p}$$

$N=(1,1)$ spectral data

$$\left(\mathcal{H}_{e-p}, \mathcal{A}, \mathcal{D}, \bar{\mathcal{D}}, \underbrace{(\gamma, \bar{\gamma})}_{\text{"real structure"}}\right)$$

Conventional interpretation:

$$d := \frac{1}{2}(\mathcal{D} - i\bar{\mathcal{D}}), \quad d^* = \frac{1}{2}(\mathcal{D} + i\bar{\mathcal{D}})$$

$$\# \Longleftrightarrow d^2 = (d^*)^2 = 0.$$

d: exterior derivative

4. Stability of Matter in arb. ext. magnetic fields

A first illustration of phys. importance of el. spin & $g_e = 2$.

System of N el. & K (static) nuclei w. charges $eZ_1, \dots, eZ_K$,

$$\sum_{k=1}^{K} Z_k \sim N = 1, 2, 3, \dots$$

Units: Energy in Ry, length in $\frac{1}{2}\times$Bohr radius (a), magn. fields in $4e/\alpha a^2$, $\alpha \simeq 1/137$.

Magn. field energy

$$\frac{1}{2\alpha^2}\int \vec{B}(x)^2 d^3x \quad (Ry)$$

Hilbert space

$$\mathcal{H}^{(N)} := \underbrace{\mathcal{H}_e \wedge \cdots \wedge \mathcal{H}_e}_{N \text{ factors}}$$

Hamiltonian

$$H^{(N)}_{\vec{A}} := \sum_{j=1}^{N} \Big\{ \overbrace{\left[\vec{\sigma}_j \cdot \left(-i\vec{\nabla}_j + \vec{A}(x_j)\right)\right]^2}^{\text{"}\frac{m}{2}\vec{v}_j^{\,2} + \text{Zeeman"}} \qquad \text{KE}$$

$$- \sum_{k=1}^{K} \frac{Z_k}{4\pi |x_j - X_k|} \Big\} \qquad \text{e-n CA}$$

$$+ \sum_{1 \leq i < j \leq N} \frac{1}{4\pi |x_i - x_j|} \qquad \text{e-e CR}$$

$$+ \sum_{1 \leq k < l \leq K} \frac{Z_k Z_l}{4\pi |X_k - X_l|} \qquad \text{n-n CR}$$

Coulomb gauge:

$$\vec{\nabla} \cdot \vec{A}(x) = 0, \quad \forall x \in \mathbb{E}^3.$$

$$\leftarrow \quad \vec{\nabla} \wedge \vec{A} = \vec{B} \in L^2 \quad (\text{F. En.} < \infty)$$

Results.

(1) For $g_e < 2$, stability (D & L, L & T)

For $g_e > 2$, even H-atom unstable

$$\boxed{g_e = 2}$$

(2) Ex. of 0-modes of $D_A = \vec{\sigma}\cdot(-i\vec{\nabla}+\vec{A})$

$\Rightarrow \; D_A^2 \psi = 0, \; \psi \in \mathcal{N}, \; \dim \mathcal{N} \geq 1$

$\rightarrow$ "Magnetic bottles"

Can F. En. offset Coulomb attr. ?

Consider energy functional

$$\mathcal{E}^{(N)}(\Psi, \vec{A}, \underline{Z}, \underline{X}) :=$$

$$\langle \Psi, H_{\vec{A}}^{(N)} \Psi \rangle + \frac{1}{2\alpha^2} \int \vec{B}(x)^2 d^3x,$$

$$\Psi \in \mathcal{H}^{(N)}, \quad \vec{\nabla}\cdot\vec{A} = 0, \; \vec{B} = \vec{\nabla}\wedge\vec{A} \in L^2$$

(3) Instability of single-el ions:

Ex. $Z_c = O(\alpha^{-2}) < \infty$ such that

for $Z < Z_c$,

$E_0(\alpha, Z) := \inf_{\psi, \vec{A}} \mathcal{E}^{(1)}(\psi, \vec{A}, Z, 0) > -CZ^2 > -\infty$

for $Z > Z_c$,

$$E_0(\alpha, Z) = -\infty$$

(4) Instab. of single-el molecules.

If $\alpha > \alpha_c$ (~ 6.7) then $\exists$ K &

$Z = Z_1 = \cdots = Z_K$ such that

$$\inf_{\psi, \vec{A}} \mathcal{E}^{(1)}(\psi, \vec{A}, \underline{Z}, \underline{X}) = -\infty$$

Inconsistencies for large Z

and large α!

Punchline $\longrightarrow$

(5) Stability of Matter in arb. ext. magnetic fields (Fe & Fr)

Theorem (L-L-S)

If $Z_k \leq Z$, $k = 1, \cdots, K$, and

$$Z\alpha^2 < 0.04 \,, \quad \alpha < 0.06$$

then

$$\inf_{\Psi, \vec{A}} \mathcal{E}^{(N)}(\Psi, \vec{A}, \underline{Z}, \underline{X}) \geq -C \cdot (N+K),$$

for arb. N, K (TD stability!)

What about the behaviour of the theory after quantization of the electromagnetic field?

$\boxed{\text{NR QED}}$

5. NR QED: Lamb shift & radiative corr to m_{el}, g_e, ...

Quantize em field $\longrightarrow$

Hilbert space $\mathcal{H}^{(N)} \otimes \mathcal{F}_{photons}$

Hamiltonian $H_\Lambda = H^{(N)}_{\vec{A}_\Lambda} + H_f$,

Λ a UV cutoff ($\Lambda \sim m_{el} c^2$)

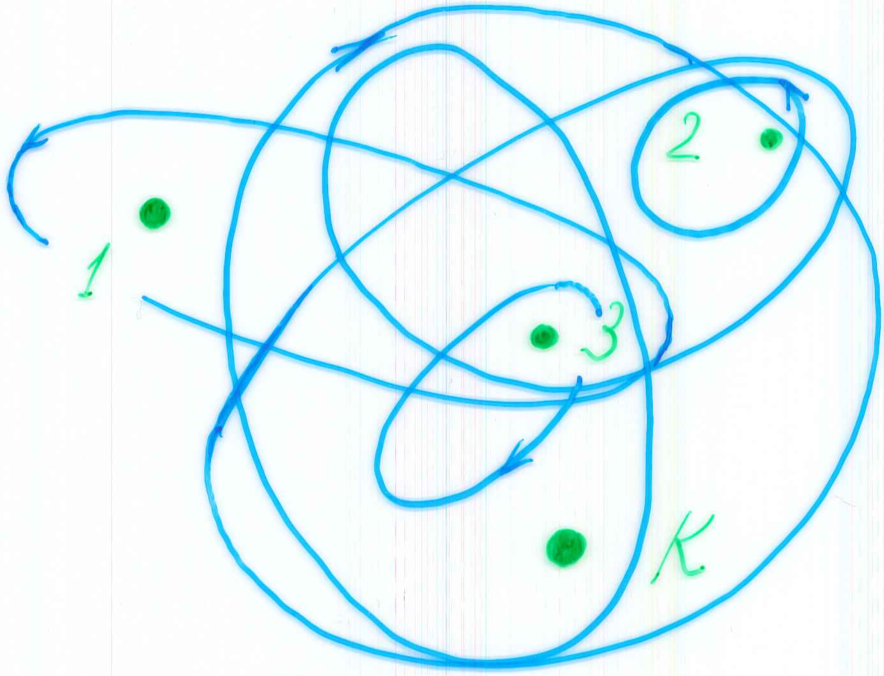
Then, for arb. $Z < \infty$, $0 < \alpha < \infty$,

$$H_\Lambda \geq - C_{\alpha, Z} \, K \Lambda$$

Conv. algorithm to calculate Lamb shift & life times, Ampl. for Rayleigh scatt. ($\rightarrow$ Bohr), Compton scattering i

Matter (Heisenberg 1925)

N electrons, K nuclei (static)



$$H_{matter} = \sum_{j=1}^{N} -\frac{\hbar^2}{2M}\Delta_j + \sum_{1\le i<j\le N} \frac{e^2}{|\vec{X}_i - \vec{X}_j|}$$

$$- \sum_{\substack{j=1 \\ \ell=1,\dots,K}}^{N} \frac{e^2 Z_\ell}{|\vec{X}_j - \vec{R}_\ell|} + \sum_{k\neq \ell} \frac{e^2 Z_k Z_\ell}{|\vec{R}_k - \vec{R}_\ell|}$$

on Hilbert space

$$\mathcal{H}_{\text{matter}} = \big(L^2(\mathbb{R}^3) \otimes \underset{\uparrow \atop \text{spin}}{\mathbb{C}^2}\big)^{\underbrace{\wedge N}_{\uparrow \atop \text{Pauli principle}}}$$

70 years of math. research

<u>Schrödinger</u> ... Rellich, Kato, ..., Hunziker, ..., Dyson-Lenard, ..., Simon, Lieb, Thirring, ..., Enss, ... Sigal, Soffer, Graf, Derezinski, ...

em field photons

$$E = \hbar\omega\,, \quad \vec{P} = \hbar\vec{K}$$

(Planck 1900) Einstein 1905

$$\vec{E} = -\vec{\nabla}\phi + \frac{1}{c}\dot{\vec{A}}$$

$$\vec{B} = \mathrm{curl}\,\vec{A}$$

Coulomb gauge: $\vec{\nabla}\cdot\vec{A} = 0$.

Fourier rep. (normal modes)

$$\vec{A}(\vec{X},t) = \frac{1}{(2\pi)^{3/2}}\sum_{\lambda=\pm}\int\frac{d^3K}{\sqrt{2|\vec{K}|}}\left\{a^*_\lambda(\vec{K})\times \vec{\varepsilon}_\lambda(\vec{K})\,e^{i(\omega t-\vec{K}\cdot\vec{X})} + h.c.\right\}$$

$$H_f = \frac{1}{2}\int\left\{\vec{E}(\vec{X},t)^2 + \vec{B}(\vec{X},t)^2\right\}$$

$$= \sum_{\lambda=\pm} \int d^3K\, a_\lambda^*(\vec{K})(c|\vec{K}|)a_\lambda(\vec{K})$$

From now on, set $c=1$, $\hbar = 1$, $\frac{e^2}{\hbar c} =: \alpha \simeq \frac{1}{137}$;

introduce dimensionless variables:

$\vec{X} = \beta\vec{x}$, $\vec{R} = \beta\vec{r}$, $\vec{K} = \mu\vec{k}$,

$[\beta] = [\mu^{-1}] = $ length;

$\beta = (2m\alpha)^{-1}$, $\mu = 2m\alpha^2$, $\beta\mu = \alpha$,

m: phys. electron mass.

v

Quantum theory of em field: (Einstein, Jordan, Dirac, ...)

Normal modes of em field, labelled by pol. λ, wave vector $\vec{k}$, are <u>harmonic oscillators</u> with frequ. $\omega = |\vec{k}|$ ($c = 1$!). $\longrightarrow$

Quantize canonically (Heisenberg 1925):

$$[a^{\#}_{\lambda}(\vec{k}), a^{\#}_{\lambda'}(\vec{k}')] = 0,$$

$$[a_\lambda(\vec{k}), a^*_{\lambda'}(\vec{k}')] = \delta_{\lambda\lambda'}\delta(\vec{k}-\vec{k}')$$

Vacuum: Ω, $\langle \Omega, \Omega \rangle = 1$,

$$a_\lambda(\vec{k})\Omega = 0, \ \forall \lambda, \vec{k}.$$

$$\overline{\langle \sum \prod a^*(h_j)\Omega \rangle} =: \mathcal{F}_\gamma$$

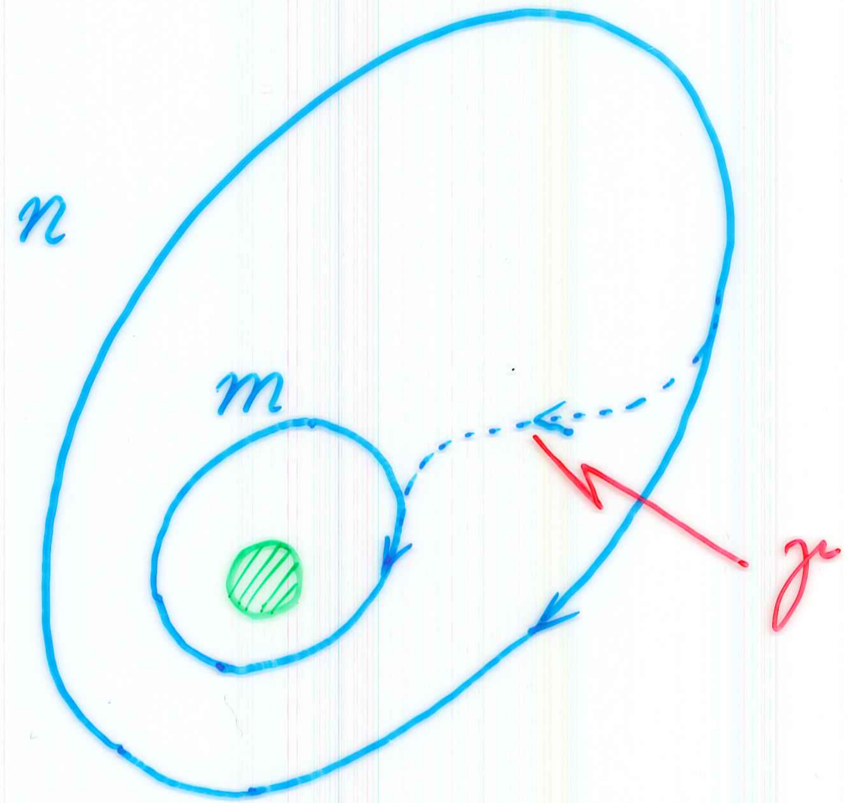
<u>Fock space</u> of photons.

$\downarrow$

Theory of quantized, free em field.

Matter interacting with (quantized) em field?

Bohr (1913), Sommerfeld, ...
Einstein (1917), ..., <u>Heisenberg</u> (1925), ...



$$\hbar\omega_{nm} = E_n - E_m \quad (1)$$

What are these "<u>quantum jumps</u>"; what is status of Bohr's eq. (1), ... ?

Prior to ~1993: No math. rigorous results on atomic spectroscopy, Rayleigh scattering, Compton effect, ...

Knew a lot about "dark" atoms, but very little about "shiny" atoms.

I will report on progress made since 1993.

(1) STANDARD MODEL OF ATOMS INTERACTING WITH EM FIELD

Hilbert space of total system is

$$\mathcal{H} := \left(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2\right)^{\wedge N} \otimes \mathcal{F}_\gamma$$

Dynamics given in terms of Hamiltonian $H_{\alpha,\Lambda}$ coupling electrons to em field;

α : strength of coupling,

Λ : UV cutoff.

$\swarrow g_{bare} = 2\,!$

$$H_{\alpha,\Lambda} = \sum_{j=1}^{N} \left\{ \frac{1}{M_\Lambda} \left[\vec{\sigma}_j \cdot \left(-i\vec{\nabla}_j + \alpha^{3/2} \vec{A}_\Lambda(\alpha \vec{x}_j) \right) \right]^2 - \mu_\Lambda \right\}$$

$$+ V_{Coulomb}(\underline{x}, \underline{r}) + H_f \,,$$

M_Λ : "bare" mass of e^-,

μ_Λ : chem. potential,

$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$: Pauli matrices,

$$\vec{A}_\Lambda(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=\pm} \int \frac{d^3k}{\sqrt{2|\vec{k}|}} \varkappa_\Lambda(\vec{k}) \times$$

$$\times \left\{ a_\lambda^*(\vec{k}) \vec{\varepsilon}_\lambda(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} + h.c. \right\},$$

$$V_{Coulomb} = \sum_{i<j} \frac{1}{|\vec{x}_i - \vec{x}_j|} - \sum_{i,\ell} \frac{Z_\ell}{|\vec{x}_i - \vec{r}_\ell|} + \cdots \,,$$

$$H_f = \sum_{\lambda=\pm} \int d^3k \, a_\lambda^*(\vec{k}) |\vec{k}| a_\lambda(\vec{k})$$

$$\varkappa_\Lambda(\vec{k}) \overset{e.g.}{=} (2\pi\Lambda^2)^{-3/2} \exp\left(-\frac{|\vec{k}|^2}{2\Lambda^2}\right)$$

Problems. $H_{\alpha,\Lambda} = H^*_{\alpha,\Lambda}$,

spec$(H_{\alpha,\Lambda})$, props. of $e^{-itH_{\alpha,\Lambda}}$, scattering th., $\lim \Lambda \to \infty$, ... ?

i) $\Lambda \to \infty$

- no positrons → no vacuum polarization $\Rightarrow$ $\alpha_\Lambda \equiv \alpha$ independent of Λ.
- $\mu_\Lambda \sim M_\Lambda^{-1} \Lambda^{2-\delta}$, $0 \leq \delta \sim$?
- $M_\Lambda \sim \Lambda^{-8\pi\alpha + O(\alpha^2)}$

  (perturbative RG)
- Lamb shift finite, as $\Lambda \to \infty$.

. . . .

No nonperturbative results on $\Lambda \to \infty$, yet!

From now on, fix $\Lambda \sim M_\Lambda \sim m < \infty$.

ii) Stability for fixed Λ.

(Lieb et al., BFG, FFG, ...)

K nuclei, $Z_\ell \leq Z$, $\forall \ell$

N electrons

Groundstate energy $\underset{K \to \infty}{\geq} -C_\Lambda K$

(class. field: $\Lambda \to \infty$ understood)

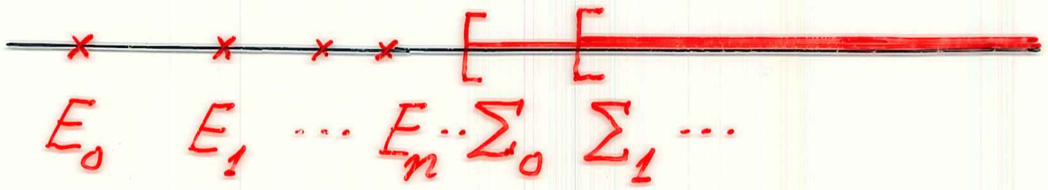
$$g_{eff} \equiv g := (\alpha \Lambda m^{-1})^{3/2} \ll 1.$$

$$H_{\alpha,\Lambda} \equiv H_g = \underbrace{H_{atom} + H_f}_{H_0} + W_g \quad (2)$$

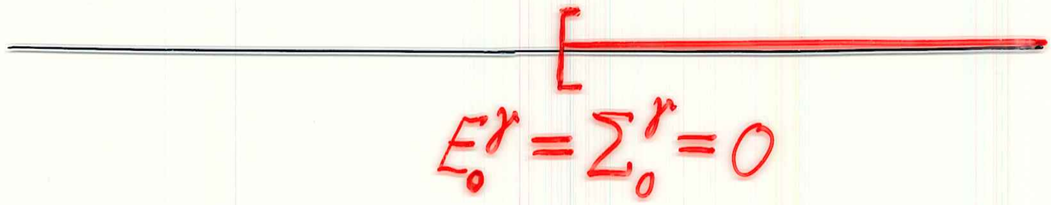
$$\mathrm{spec}(H_g) \sim \, ?$$

Consider e.g. single atom + radiation field.

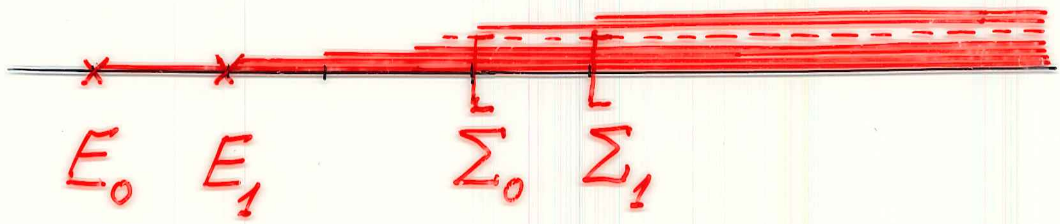
spec(H_{atom})



spec(H_f)



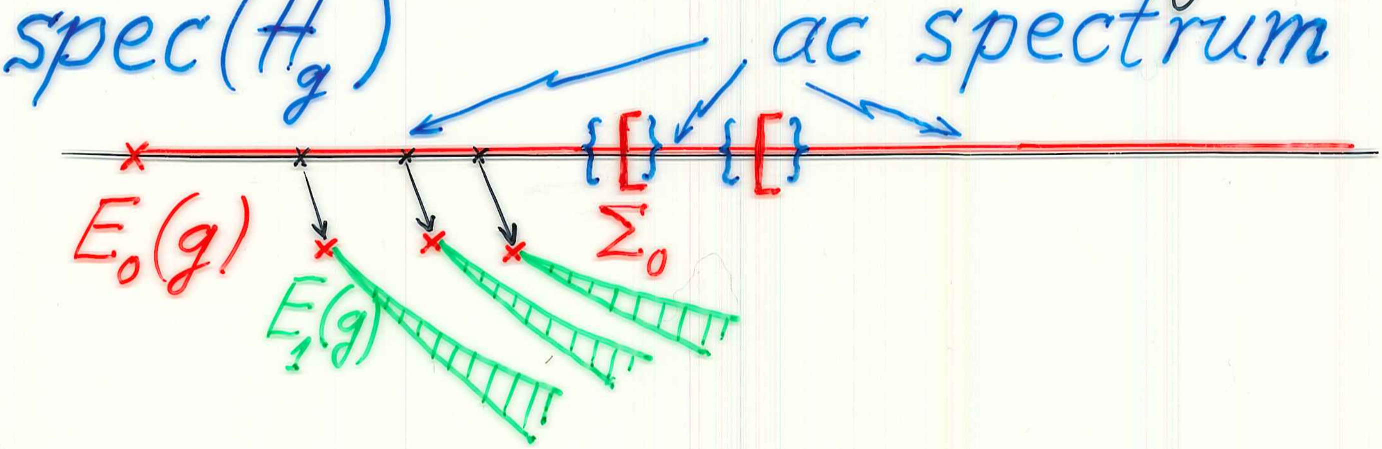
spec(H_0)



spec(H_0) contains <u>embedded e.v.'s</u> $(E_1, E_2, \cdots)$ at <u>thresholds of cont. spect.</u> (from H_f).

Standard pert. th. inapplicable!

Results when pert. W_g is turned on; (Bach-F-Sigal)



$E_i(g)$: resonance energies at tip of "branch cut".

(1) stable groundstate:

$\sigma_{pp}(H_g) = \{E_0(g)\}$, below Σ_0

(B-F-S, G-L-L; F ← G-J / B-F-P)

"bound states": For $I \subset (-\infty, \Sigma_0)$, g small enough, $\exists\, \xi < \infty$ such that

$$\| e^{|\vec{x}|/\xi} \otimes 1_{\mathcal{F}}\, \chi_I(H_g) \| < \text{const}_I$$

expo. decay in $\vec{x}$!

(2) spec$(H_g) \setminus \{E_0(g)\}$ is purely ac, except(?) near Σ's.

(3) Re $E_i(g) \sim$ Bethe

Im $E_i(g) \sim$ Fermi $\quad i = 1, 2, \ldots$

+ cuts

(4) Scattering theory

B - Heisenberg - Wheeler ...

(D - G, F - Griesemer - Schlein, [Spohn])

(i) Ex. of asymp. e.m. fields:

$$e^{iH_g t} e^{-iH_f t} a^{\#}(f) e^{iH_f t} e^{-iH_g t} \xrightarrow[t \to \pm\infty]{s} a_{\pm}^{\#}(f)$$

on bound states (or ones where prop. vel. of e^-'s $< c$)

(ii) AC for Rayleigh scattering

Let ψ be any bound state; $\psi_0(g)$ groundstate $(g \neq 0)$.
Then, (with arb. tiny IR cutoff)

$$\psi = s\text{-}\lim \sum_{\alpha} a^*_{\underset{(-)}{+}}(f_1^\alpha) \cdots a^*_{\underset{(-)}{+}}(f_{n_\alpha}^\alpha)\psi_0(g)$$

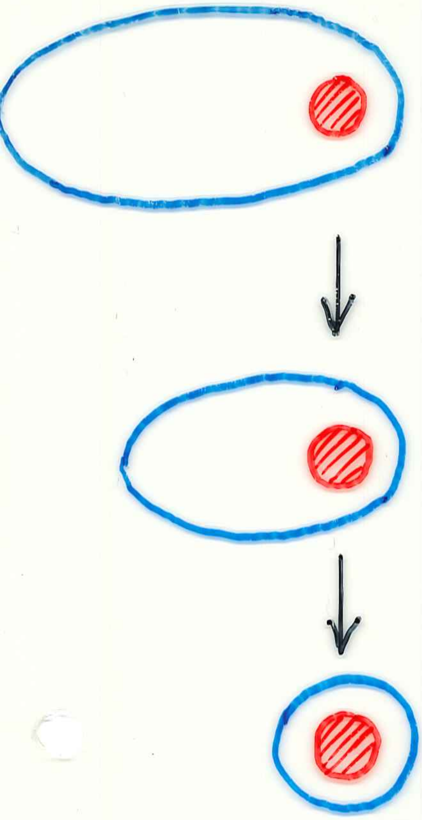
$\langle N_\gamma \rangle_{\psi_t}$?

(iii) Cor.: Relaxation to g.s.

If ψ is a bound state, and A any "local observable" then

$$\langle e^{-itH_g}\psi, A e^{-itH_g}\psi \rangle \xrightarrow[t \to \pm\infty]{} \langle \psi_0(g), A\psi_0(g) \rangle$$

Isolated atom always relaxes to groundstate!



$$|\langle\psi_\alpha, e^{-itH_g}\psi_\alpha\rangle|$$

ψ_α $\lesssim a e^{-tg^2\Gamma_\alpha} + O_N(g^4)\, t^{-N}$

$\downarrow \rightarrow \gamma$ $\quad h\nu_{\alpha\beta} \approx E_\alpha - E_\beta$

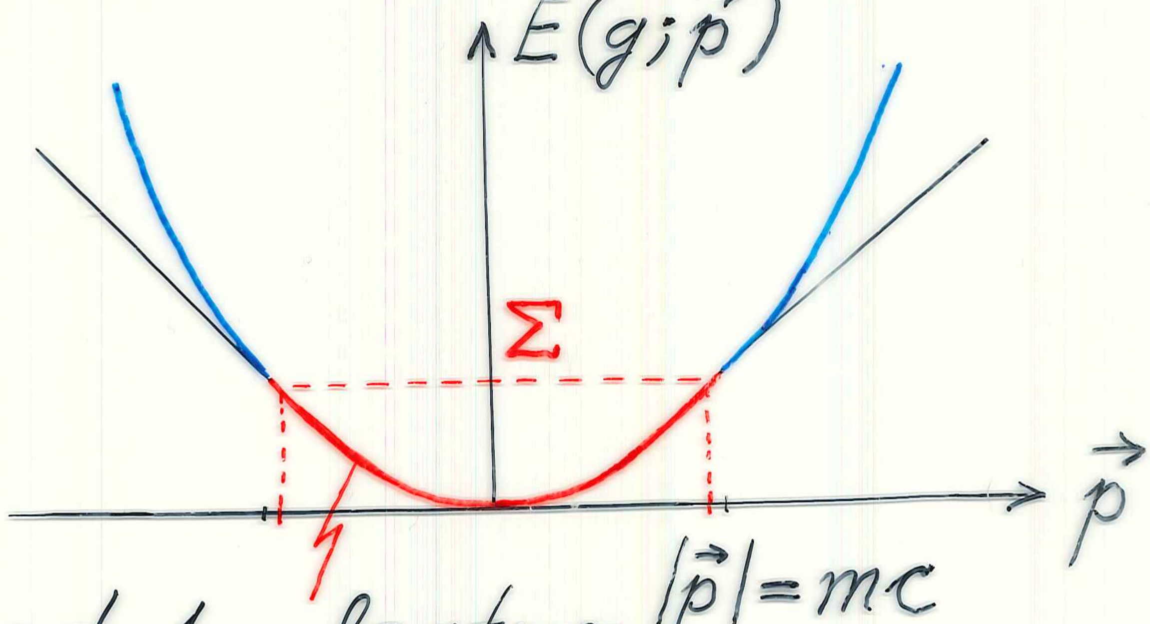
ψ_β (Ri-Ryd, Bohr)

$\downarrow \rightarrow \gamma$ $\quad$ <u>from S-matrix</u>

ψ_0 No (quasi-)per. int

(Bach-K-Z, J.F., A.P., …)

(iv) <u>AC for Compton scattering</u>



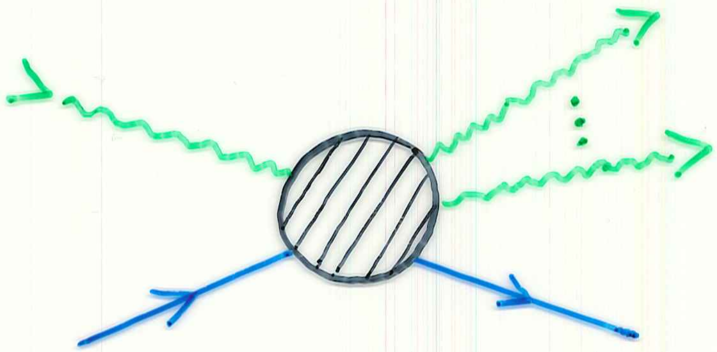
dressed 1-electron state $\psi_{DES}(\vec{p})$, energy

$$E(g,\vec{p}) = \frac{1}{2m_{REN}}\vec{p}^{\,2} + O(\vec{p}^{\,4})$$

Consider one (freely moving) e^-; no nuclei; arb. tiny IR cutoff. Let ψ be arb. state with tot. energy below Σ; g small enough (dep. on Σ). Then

$$\psi = s\text{-}\lim \sum_{\alpha} \prod_{1}^{n_\alpha} a^*_{\substack{+\\(-)}}(f_i^\alpha)\,\psi_{DES}(h),$$

$(\mathrm{suppt}\; h \subset \{\vec{p} \mid |\vec{p}| < \gamma mc!\})$



Genuine IR problem when IR cutoff $\to 0$! (F, Pizzo)

BCFS: Exp. for $m_{ren.}$

Removal of IR cutoff

$$\sigma \to 0$$

(1) Ex. of dressed one-e^- states : ✓ (J.F., 1972, ...)

IR rep. is <u>coherent state rep.</u> (... T.C., A.P.)

Regularity of $E_g(p)$ (up to 2 derivatives)

(2) Møller wave ops. for Compton scattering & "Bremsstrahlung": ✓ (J.F., ..., A.P.)

(3) Infrared-finite, "finite" algorithms to calculate $E_0(g), \cdots$, based on multi-scale analysis:

$$E_0(g) = E_0 + \sum_{j=1}^{N} \varepsilon_j(g)\, g^j + O(g^{N+\varepsilon})$$

$\varepsilon_j(g) = o(g^{-1})$ compt. in terms of finite # of conv. integrals; but infrared logs in g! (B-F-P)

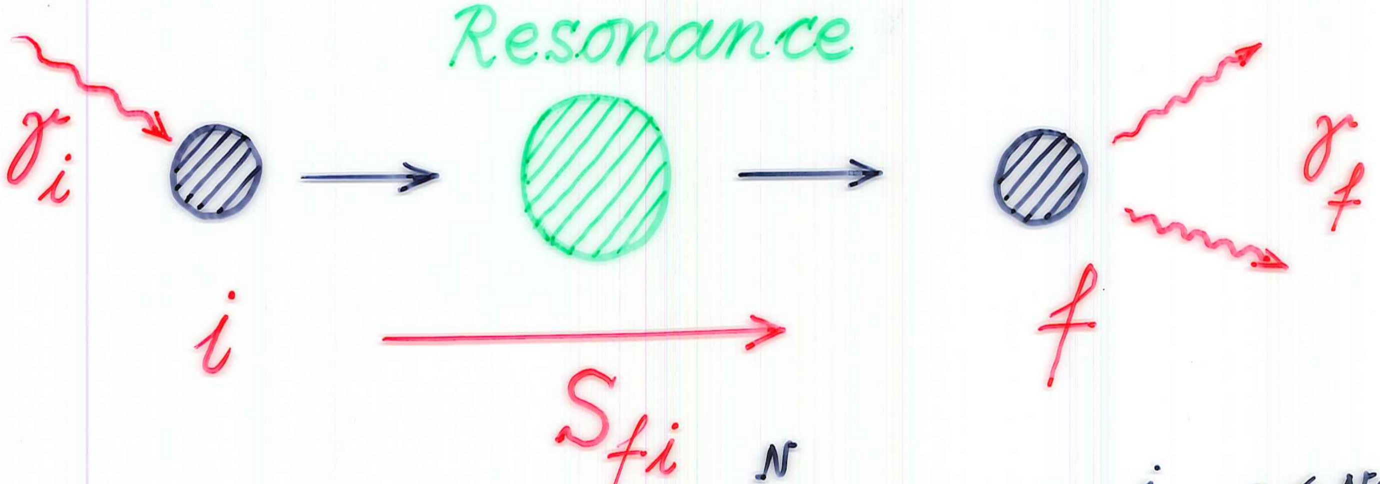
$E_0(g)$ pres. not C^∞ in g, $|g| < g_*$!

(4) Infrared-finite algor. for scattering amplitudes

of Rayleigh- (& Compton-?) scattering:

i : atom in groundst., γ in h_i

f : " " " , γ in h_f



$$S_{fi}(g) = \langle f | i \rangle + \sum_{j=1}^{N} \underbrace{\sigma_{fi}(g)}_{\text{IR-logs}} g^j + O(g^{N+\varepsilon})$$

$\rightarrow$ Derivation of (Einstein-) Bohr frequency cond., ... !

But no info. on Lamb shift & life times, yet, because spread in energy of states h_i and h_f of γ_i, γ_f, resp., is $O(1) \rightarrow$ not suff. accurate for Lamb shift, ⋯.

Fundamental problems:

- Control of $e^{-i(tH_g)/\hbar}$ up to times $t \gtrsim O(g^{-2})$, $\nearrow$ E, S, Y.
- Show that, for Ψ in range of $\chi(H_g < \Sigma_0)$, $\Psi \in D(N_\gamma^{1/2})$,

$$|\langle e^{-i(tH_g)/\hbar}\Psi, N_\gamma e^{-i(tH_g)/\hbar}\Psi\rangle| \leq \text{const.}, \ \forall t.$$

2. RETURN TO EQUILIBRIUM & THERMAL IONIZATION

(1) R to E Consider finitely many "toy atoms" with finite-dim. state space coupled to black-body radiation at temp. $T > 0$; (Einstein 1917). Suppose "dipole matrix elements" $x_{\alpha\beta} \neq 0$, for some $\beta < \alpha$, $\forall \alpha > 0$; (where α, β label atomic eigenstates).

Let ρ be an arb. initial state of coupled system which is "normal" w.r. to equilibrium state ρ_f^β of black-body radiation, (i.e., very far from atoms, ρ is $\simeq$ thermal state w. temp. $T = (k_B \beta)^{-1}$).

Then, for suff. small $|g|$, $\forall T > 0$,

$$\text{"}\lim_{t\to\infty} \rho\left(e^{itH_g} A e^{-itH_g}\right) = \rho^\beta(A),\text{"}$$

where A is arb. "local obs.", and ρ^β is unique equ. state of coupled system.

R to E, as $t \to \infty$, is expo fast, for some "realistic" couplings;

$$\rho^{\beta}\big|_{atom} = Z_{\beta}^{-1} \sum_{\alpha} e^{-\beta E_{\alpha}} P_{\alpha} + O(|g|)$$

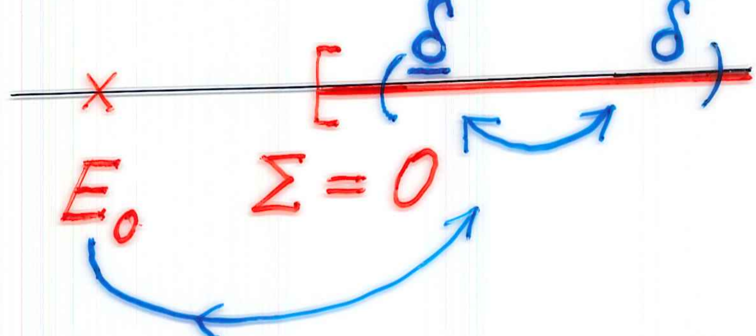
---

(2) Thermal ionization

Couple more "realistic" atom, with coexistence of point- and cont. spectrum, to black-body radiation at $T>0$.

$$H_{atom} \simeq E_0\big|_{\mathbb{C}} \oplus (-\Delta)\big|_{L^2(\mathbb{R}^3)}$$

$spec(H_{atom})$



Interaction W_g assumed

to only make transitions between bound state ψ_0 (E_0) and states ψ_E with energy $E \in [\underline{\delta}, \overline{\delta}]$, $0 < \underline{\delta} < \overline{\delta} < \infty$.

Then, for $|g|$ small, $T > 0$,

$$\lim_{t \to \infty} \rho\left(e^{itH_g} P_0 e^{-itH_g}\right) = 0,$$

for an arb. initial state ρ normal w.r.to ρ_f^β;

$\nexists$ any stationary state!

Atom always entirely ionized!

<u>OPEN PROBLEMS</u>

Fate of electrons, after ionization?

Conjecture: "Quantum Brownian motion"

$$\rho([\vec{x}(t)-\vec{x}(0)]^2) \underset{t\to\pm\infty}{\sim} D_{\beta,g}\,|t|,$$

for some $0 < D_{\beta,g} < \infty$.

(↗ L. Erdős)

Other results on ionisation:

- by laser pulses
  (F-K-S; F-Schlein: Kramer's model)

Need non-pert. methods to study large-field QED, low-T, long-t behaviour

3. CONCLUSIONS

(1) Discovery of QM of atoms, etc. by (B, E) Heisenberg, B, J, Dirac, Schrödinger is miracle: Approx. valid data & rules, phys. "wrong" idealizations, ... led to correct, consistent theory.

(2) Quantum theory of atoms, ... coupled to radiation field (spectroscopy, qu. optics, lasers) still rich in math. (analytical) challenges! IR probl. large $\vec{E}$, $\vec{B}$

(3) Results described in this lecture lead to understanding of:

dissipation & friction through disp. radiation → irreversible behaviour

Relaxation to g.s., R to E, TI, approach to NESS, Decay of resonances (metastable states), ...

Why interesting (or important)? Foundations of thermodynamics

(4) Irreversible behaviour from reversible dynamics.
Retrieval of info. about nature through experiments always acc. by dissipation & friction, irrev. beh. ($S\uparrow$), decay of metastable states.
Results may help to develop a "quantum th. of exp.", with hope that QM (of open systs. w. ∞ many degs. of freedom) will provide its own interpretation!

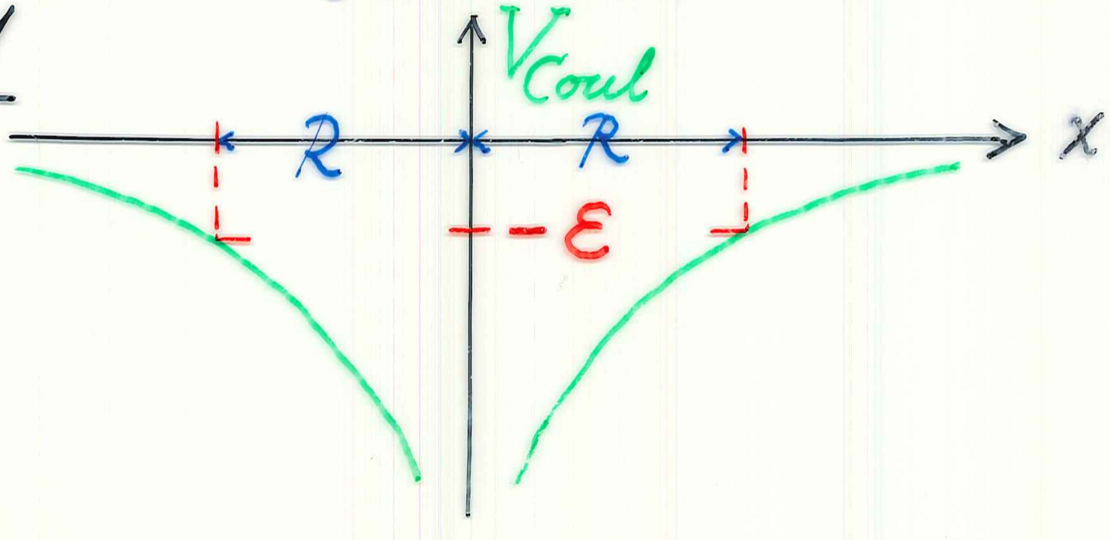
4. SOME IDEAS OF PROOFS

(1) Ex. of stable groundstates

(unique, up to ↑↓, for $N=1$)
No "IR catastrophe" for boundstates, $\psi \leftrightarrow$ "soft-γ bounds": $\langle N^\gamma \rangle_\psi \leq C_\psi g^2 \int \frac{d^3k}{|\vec{k}|}$
→ "compactness arg.", (F '72 ← G-J).

Expo. decay of boundstates

$N=1$



ψ bd. state, max. energy $-E_\psi < 0$

R so large that $V_{Coul}(|\vec{x}| \geq R) \geq -\varepsilon \gg -E_\psi$

Then

$$-E_\psi \geq \langle \psi, H_g \psi \rangle = \langle \chi_< \psi, H_g \chi_< \psi \rangle$$
$$+ \langle \chi_> \psi, H_g \chi_> \psi \rangle - \varepsilon$$
$$\geq -|E_0(g)| \, \|\chi_< \psi\|^2 - 2\varepsilon - a|g|$$

$$\Rightarrow \|\chi_< \psi\|^2 \geq \frac{E_\psi - 2\varepsilon - a|g|}{|E_0(g)|} \cdots \geq 1 - o(1)$$

[$\Rightarrow$ Expo. decay of ψ from standard resolvent estimates!]

(2), (3) Nature of spec(H_g); resonances

Dilatation analyticity ($N=1$)

$$\mathcal{H} = (L^2(\mathbb{R}^3)\otimes\mathbb{C}^2)\otimes\mathcal{F}$$

For $\psi(\vec{x}) \in L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$,

$$(U_{el}(\theta)\psi)(\vec{x}) = e^{\frac{3\theta}{2}}\psi(e^{\theta}\vec{x})$$

For $(\varphi_n)_{n=0,1,2,\dots} \in \mathcal{F}$,

$$(U_f(\theta)\varphi_n)(\vec{k}_1,\dots,\vec{k}_n)$$
$$= e^{-3n\theta/2}\varphi_n(e^{-\theta}\vec{k}_1,\dots,e^{-\theta}\vec{k}_n)$$

For $\Lambda<\infty$ (κ_Λ in C^∞),

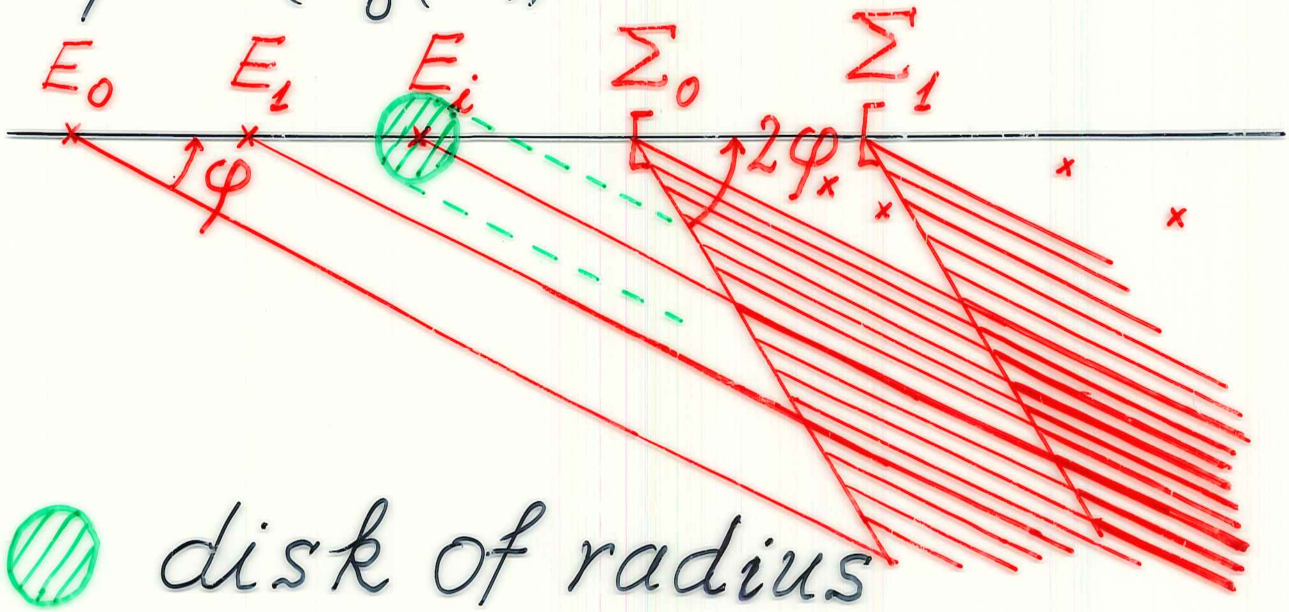
$$H_g(\theta) \equiv U(\theta)\, H_g\, U(\theta)^*,$$
$$U(\theta) = U_{el}(\theta)\otimes U_f(\theta),$$

"analytic family" in θ.

$$H_f(\theta) = e^{-\theta} H_f \; ; \quad -\Delta(\theta) = e^{-2\theta}(-\Delta)$$

Set $\theta = i\varphi$, $\varphi > 0$. Then

$\mathrm{spec}(H_0(\theta))$:



$\mathcal{D}_{\rho_0} \equiv$ disk of radius

$$\tfrac{1}{2}\rho_0 \sim g^{2-2\varepsilon}, \quad \varepsilon > 0$$

$$\rho_0 \ll \mathrm{dist}(E_i, \mathrm{spec}(H_{atom}) \setminus \{E_i\})$$

Interested in

$$\mathrm{spec}(H_g(\theta)) \cap \mathcal{D}_{\rho_0}, \quad \theta = i\varphi,$$

($|g|$ small).

Why interesting?

$$\langle \Psi, (z - H_g)^{-1} \Phi \rangle$$

$$= \langle \Psi(\bar{\theta}), (z - H_g(\theta))^{-1} \Phi(\theta) \rangle,$$

for all real θ; $(U(\theta)^* = U(\theta)^{-1}!)$

If Ψ, Φ dilatation-analyt.,

R.S. well def. on complem. of $\mathrm{spec}(H_g(\theta))$, $\theta = i\varphi \Rightarrow$

analyt. cont. of L.S. in z from $\{\mathrm{Im}\, z > 0\}$ across cut to compl. of $\mathrm{spec}(H_g(i\varphi))$.

$\Rightarrow$ Find <u>resonance energies</u>, $E_i(g)$, + nature of cuts.

Tool to analyze spec$(H_g(i\varphi))$:
Iterative perturbation th. (RG), based on "Feshbach map" (B-F-S)

$$\mathcal{H},\ P,\ H,\ \overline{P} = \mathbb{1} - P$$

$$H_P = PHP,\ H_{\overline{P}} = \overline{P}H\overline{P} \text{ on } \overline{P}\mathcal{H}.$$

Assume that

$z \in \mathrm{Res}(H_{\overline{P}})$, $\|R_{\overline{P}}(z)HP\|$ & $\|PHR_{\overline{P}}(z)\| < \infty$ *

Feshbach map:

$$(z, H) \longmapsto f_{z,P}(H) = \left[H_P - PHR_{\overline{P}}(z) \cdot HP\right]\Big|_{P\mathcal{H}}$$

Theorem. Assume $*$

(i) $z \in \mathrm{spec}(H) \Leftrightarrow z \in \mathrm{spec}(f_{z,P}(H))$

(ii) $z \in \sigma_{pp}(H) \Leftrightarrow z \in \sigma_{pp}(f_{z,P}(H))$

(iii) $[P_1, P_2] = 0 \Rightarrow f_{z,P_1} \circ f_{z,P_2} = f_{z,P_1 \cdot P_2}$

("RG property")

Application to "standard model": $\mathcal{H}$ as above,

$P = P_i^{atom}(i\varphi) \otimes \chi_{H_f \leq \rho_0}$, $\rho_0 \sim g^{2-2\varepsilon}$,

$\varepsilon > 0$, $|g|$ small; $H = H_g(i\varphi)$,

$\varphi > 0$, $\mathcal{D}_{\rho_0}$ as above;

$\mathcal{D}_{\rho_0} \subset \mathrm{Res}(H_{\bar{P}}) \Longrightarrow$

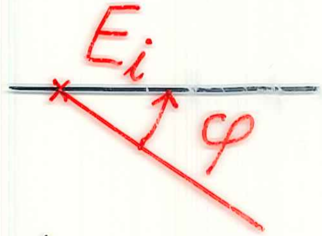
$f_{z,P}(H_g(i\varphi))$ well def.; (i), (ii), (iii)!

$$f_{z,P}(H) = P(E_i + e^{-i\varphi} H_f)P \qquad \mathrm{I}$$

$$+ P W_g P \qquad \mathrm{II}$$

$$+ P W_g \overline{P}(z - H_{\overline{P}})^{-1} \overline{P} W_g P \qquad \mathrm{III}$$

$\mathrm{I} = E_i + e^{-i\varphi} H_f$ on $P\mathcal{H}$



$$\mathrm{II} \simeq \Delta_1 E_i(g) P, \quad \Delta_1 E_i(g) \in \mathbb{R}$$

$$\mathrm{III} \simeq P W_g \overline{P}(z - H_{0,P})^{-1} \overline{P} W_g P$$

$$\simeq (-i g^2 \Gamma_i + \Delta_2 E_i(g)) P$$

$$\Gamma_i = \sum_{j<i} \int d^3k \, \delta(|\vec{k}| + E_j - E_i) \times$$

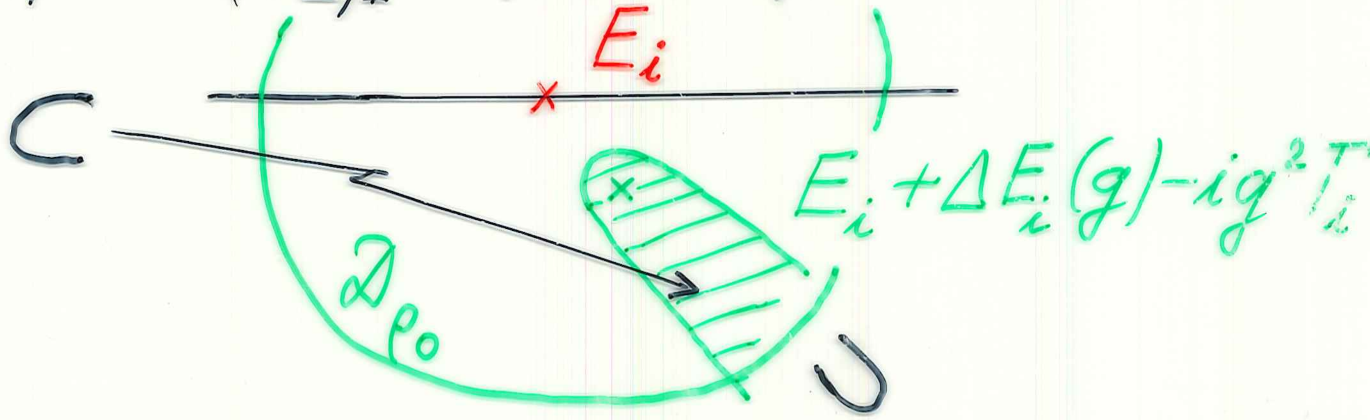
$$\times \langle G(\vec{k})\psi_i, P_j^{atom} G(\vec{k})\psi_i \rangle$$

"form factor" of "lin. term" in W_g; $\Gamma_i > 0$!

"Error terms" are $O(|g|^{3-3\varepsilon})$!

$\Rightarrow$ For $|g|$ small $(0<\varepsilon<\frac{1}{3})$,

$$\mathrm{spec}(f_{z,P}(i\varphi)) \approx \mathrm{spec}(\mathrm{I}+\mathrm{II}+\mathrm{III})$$



By (ii), $\mathrm{spec}(H_g(i\varphi)) \cap \mathcal{D}_{\rho_0}$!

$$\Rightarrow \left|\langle \Psi_i, e^{-itH_g}\Psi_i\rangle\right|$$

$$\leq a e^{-tg^2\Gamma_i} + O_N(g^4)t^{-N}$$

Iterative improvement

(RG); see B-F-S!

4. Analytical Methods, I: Feshbach-Schur Map

$$H\psi = z\psi \ , \ P: \text{orth. proj.}$$

Then

$$(PHP + PHP^{\perp})\psi = zP\psi \qquad i$$

$$(P^{\perp}HP + P^{\perp}HP^{\perp})\psi = zP^{\perp}\psi \qquad ii$$

Let $\bar{H} := P^{\perp}HP^{\perp}$, $z \in \rho(\bar{H})$.

Then $ii \Rightarrow$

$$P^{\perp}\psi = (z - \bar{H})^{-1}P^{\perp}HP\psi$$

& $i \Rightarrow$

$$F_P(H - z)\psi = 0, \qquad iii$$

$$F_P(H-z) := P(H-z)P - \qquad iv$$
$$- PHP^{\perp}(\bar{H} - z)^{-1}P^{\perp}HP$$

Properties of F-S Map:

(1) $z \in \sigma_{pp}(H) \Leftrightarrow 0 \in \sigma_{pp}(\mathcal{F}_P(H-z))$

(2) $z \in \rho(H) \Leftrightarrow 0 \in \rho(\mathcal{F}_P(H-z))$

(3) $H\psi = z\psi \Leftrightarrow \mathcal{F}_P(H-z)\varphi = 0,$

$$\psi = Q(z)\varphi \qquad \varphi = P\psi$$

$$Q(z) = P + (z-\bar{H})^{-1}P^{\perp}HP$$

(4) $P(H-z)^{-1}P = \mathcal{F}_P(H-z)^{-1}$

(5) If $P' < P$

$$\mathcal{F}_{P'}(H-z) = \mathcal{F}_{P'}(\mathcal{F}_P(H-z))$$

& Isospectrality

Application:

$$H = H(\theta),\ \mathrm{Im}\,\theta > 0,\ P_0 = P_{e_i}^{at} \otimes 1(H_f \le 1)$$
$$i \ge 1$$

$$z \in D_{\delta_0}(e_i)\,,\quad \delta_0 = 1 - O(\alpha^{\cdots})$$

$$H^{(0)} := \mathcal{F}_{P_0}\left(H(\theta) - z\right) \qquad (6)$$

acting on $1(H_f \leq 1)\mathcal{F}$

Apply resolvent exp. & re-Wick ordering (+ "pull-through formula") to R.S. of (6):

$$H^{(0)} = T^{(0)}(z) + \sum_{\substack{n,m \\ n+m \geq 1}} W^{(0)}_{n,m}(z) + \mathcal{E}^{(0)}(z)\,,$$

$$T^{(0)}(z) = T(H_f, z)\,,\quad \left| T(\varepsilon, z)/\varepsilon \right| \leq 1 + c_0\,,$$

$$W_{n,m}(z) = 1(H_f \leq 1) \int d\underline{k} \int d\underline{\tilde{k}} \prod_1^n a^*(k_i) \times$$

$$\times \underbrace{w_{n,m}(\underline{k}, \underline{\tilde{k}}; z)}_{\text{bd. by } \prod_1^n |k_i|^{-1/2+\mu} \prod_1^m |\tilde{k}_j|^{-1/2+\mu}} \prod_1^m a(\tilde{k}_j)\, 1(\cdots)$$

$\mathcal{E}^{(0)}(z)$: scalar, analyt. fu. of z,

with $\mathcal{E}^{(0)}(z^{(0)}) = 0$,

for some $z^{(0)} \in D_{\delta_0}(e_i)$, $\operatorname{Im} z^{(0)} < 0$.

Next, apply F–S map, with

$P = P_1 = 1(H_f \leq \rho)$, $0 < \rho < 1$, to $H^{(0)}$,

with $z \in D_{\delta_1}(z^{(0)}) \subset D_{\delta_0}(e_i)$:

$$H^{(1)} := \mathcal{F}_{P_1}(H^{(0)})$$

$$= T^{(1)}(z) + \sum_{\substack{n,m \\ n+m \geq 1}} W^{(1)}_{n,m}(z) + \mathcal{E}^{(1)}(z)$$

$\| W^{(1)}_{n,m}(z) 1(H_f \leq \rho) \| \leq cst.\, \xi^{n+m} \rho^{1+\mu}$,

$0 < \xi < 1$, $\ldots$, $\mathcal{E}^{(1)}(z^{(1)}) = 0$,

for some $z^{(1)} \in D_{\delta_1}(z^{(0)})$, $\operatorname{Im} z^{(1)} < 0$.

...

Inductive construction:

$P_k = 1\left(H_f \leq \rho^{(1+\frac{\mu}{2})^{k-1}}\right)$,

ρ small enough (dep. on μ),

α " " (" " μ).

$\rightarrow H^{(k)}, D_{\delta_k}(z^{(k-1)}) \subset \cdots \subset D_{\delta_1}(z^{(0)}) \subset D_{\delta_0}(e_1)$

with $H^{(k)} \simeq (1+c_k) H_f \, 1\left(H_f \leq \rho^{(1+\frac{\mu}{2})^{k-1}}\right)$,

$c_k \to c_\infty, \; |c_\infty| < 1,$

$\delta_k \to 0$ superexpon.

$z^{(k)} \to e_i(\alpha)$, $\operatorname{Im} e_i(\alpha) < 0$,

for $i \geq 1$; ($\operatorname{Im} e_0(\alpha) = 0 \Rightarrow$

$e_0(\alpha)$ eigenvalue of H_α!)

All this clearly works for

complex c.c., too ($\to$ certain analyticity props.; albeit not in $\alpha^{1/2}$, for standard model).

"Smooth F-S map": Replace $1(H_f \leq (\cdot))$ by C^∞-approx. to char. fu.. Complicates alg., but simplifies analysis. Can be combined with Mourre theory (FGS).

Obsession: Cast all this in form of RG map on Banach space of Hamiltonians: l.o.

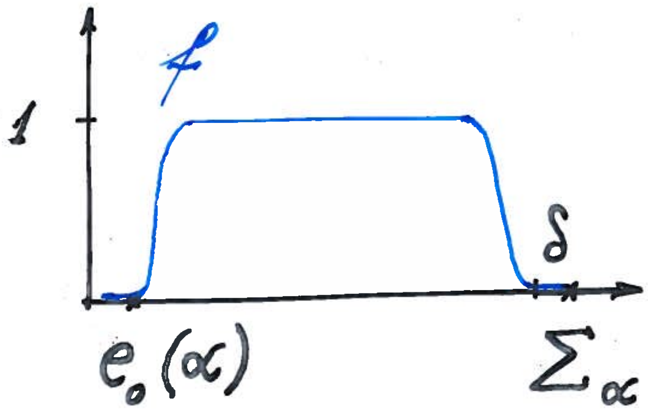
5. Properties of Time Evol.

$>\alpha$ small $<$

1. Survival prob. & life times of resonances: As above (Hunziker's arguments).

2. Escape of photons: Mourre th. $\Rightarrow$ limited absorption principle; namely, for

$$\psi = f(H_\alpha)\varphi \,, \quad \varphi \in D(a^\alpha)$$



$$\| \langle a \rangle^{-\alpha} e^{-itH_\alpha} \psi \| \xrightarrow[t\to\infty]{} 0$$

3. Relaxation to groundstat

Asy. completeness of Rayleigh scattering (FGS, Fau-Si, DeR-Kup)

$$\overset{FGS}{\Longrightarrow} \langle e^{-itH_\alpha}\psi, a e^{-itH_\alpha}\psi\rangle$$

$$\underset{t\to\infty}{\longrightarrow} \langle \psi_\alpha^{(0)}, a\,\psi_\alpha^{(0)}\rangle,$$

$\forall a \in \mathcal{A}_{loc.}$, provided

$$\psi \in \mathrm{Ran}\, P_{H_\alpha}([e_0(\alpha), \Sigma_\alpha - \delta])$$

and

$$e^{-itH_\alpha}\psi \underset{t\to\infty}{\simeq} \sum_i \prod_{l_i} F^+_{\mu\nu}(h^{\mu\nu}_{l_i,t})\,\psi_\alpha^{(0)}$$

4. Rel. to g.s. & R to E in gen. spin-boson models

$H = H_{at} + H_f + \lambda D$ on

$\mathcal{H} = \mathbb{C}^N \otimes \mathcal{F}$;

$H_{at} = M \otimes \mathbb{1}$,

M an $N \times N$ matrix w. e.v.'s

$$e_0 < e_1 < \cdots < e_{N-1}$$

D : FGR cond. $\Rightarrow$ all e_i, $i \geq 1$, move off real axis in 2^{nd} order p.t.; (see above). Then

for small enough λ

$$\langle e^{-itH}\psi, a\, e^{-itH}\psi\rangle \underset{t\to\infty}{\longrightarrow} \langle \psi^{(0)}, a\,\psi^{(0)}\rangle,$$

$\forall a \in \mathcal{A}_{loc}$;

$$\langle e^{-itH}\psi, N_f\, e^{-itH}\psi\rangle$$

unif. bounded in t;

(FGS $\longrightarrow$ De Roeck-Kup.)

+ Return to Equilibrium (Jakšić-Pillet, BFS, ...)

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5. Quantum Brownian motion (De Roeck-F)

. . . . . .

6. Conclusions

"In all my films, I have been faithful to these suspension points in the conclusions. Besides, I have never written the word 'END' on the screen."
(Federico Fellini)

"Everyone wants to understand art (physics)*. Why don't we try to understand the song of a bird? Why do we love the night, the flowers, everything around us, without trying to understand them? But in the case of a painting* (result in physics)*, people think they have to understand." (Pablo Picasso)*

Thank you for listening!

3. The Question of Existence of Hidden Variables in Quantum Mechanics.

Einstein, Schrödinger, von Laue, de Broglie ...:

"Final theory" (superseding Quantum Mechanics) should provide a realistic description of Nature:

> "Alle Naturwissenschaft ist auf die Voraussetzung der vollständigen kausalen Verknüpfung jeglichen Geschehens begründet." (A. Einstein, talk at "Physikalische Gesellschaft", Zürich, 1910.)

Probabilities should enter into Quantum Physics the way they enter into Classical Physics, namely as an expression of ignorance (of initial conditions).

Knowledge of complete initial conditions may be fundamentally impossible, as in (classical) relativistic theories with an event horizon.

(Also refer to de Broglie – Bohm theory.)

3.1 A brief recapitulation of the Copenhagen interpretation

We temporarily use the so-called Schrödinger picture: Let S be a physical system. Quantum-mechanically, states of S, i.e., density matrices on a Hilbert space $\mathcal{H}$, evolve in time according to the Schrödinger-Liouville (or Landau-von Neumann) equation

$$P_t = U(t,s)\, P_s\, U(s,t), \quad t,s \text{ in } \mathbb{R}, \qquad (3.1)$$

where P_t is the state of S at time t, and U is the propagator of S

UNLESS a physical quantity $\hat{X}$ of S is measured, at time t, say. A measurement of $\hat{X}$ interrupts the Schrödinger-Liouville evolution. The state of S right after a measurement of $\hat{X}$ is an "eigenstate" of the self-adjoint linear operator X on $\mathcal{H}$ representing $\hat{X}$. In the Schrödinger picture, X is <u>time-independent</u>.

Let us suppose that the state of S at time t just before $\hat{X}$ is measured is given by the density matrix P_t and that a measurement of $\hat{X}$ at time t has yielded a measured value in an interval $\Delta \subset \mathbb{R}$. Let $\{\Pi^X(\Delta), \Delta \subset \mathbb{R}\}$ denote the spectral projections of X.

Let $P_t^{\hat{X}}(\Delta)$ be the state of S <u>right after</u> a measurement of $\hat{X}$ that has yielded a measured value in Δ.

"Copenhagen" is based on the following <u>two</u> <u>Postulates</u>:

(PI) If $\hat{X}$ is immediately measured again, after the first measurement of $\hat{X}$ has yielded a value in Δ, then the measured value in the second measurement of $\hat{X}$ belongs to Δ with certainty. In other words,

$$tr\left(P_t^{\hat{X}}(\Delta)\,\Pi^X(\tilde{\Delta})\right) = 1, \qquad (3.2)$$

whenever $\tilde{\Delta} \supseteq \Delta$.

(PII) Let Π be an orthogonal projection with

$$\text{Range}\,\Pi \subseteq \text{Range}\,\Pi^X(\Delta) \Leftrightarrow \Pi\cdot\Pi^X(\Delta) = \Pi^X(\Delta)\Pi = \Pi. \qquad (3.3)$$

(We will then write $\Pi \prec \Pi^X(\Delta)$.) Then

$$tr\left(P_t\,\Pi\right) = tr\left(P_t^{\hat{X}}(\Delta)\,\Pi\right)\cdot tr\left(P_t\,\Pi^X(\Delta)\right) \qquad (3.4)$$

Lemma.

Suppose (PI) and (PII) hold. Then

$$P_t^{\hat{X}}(\Delta) = \frac{\Pi^X(\Delta)\,P_t\,\Pi^X(\Delta)}{tr\left(P_t\,\Pi^X(\Delta)\right)} \qquad (3.5)$$

Proof. Let $\mathcal{R}(\Delta) := \Pi^X(\Delta)\mathcal{H} \subseteq \mathcal{H}$ be the range of $\Pi^X(\Delta)$. Let $A : \mathcal{R}(\Delta) \to \mathcal{R}(\Delta)$ be an arbitrary self-adjoint operator on $\mathcal{R}(\Delta)$. We set $A = 0$ on $\mathcal{R}(\Delta)^\perp$. Let Π^A be an arbitrary spectral projection of $A/_{\mathcal{R}(\Delta)}$. Then (3.3) holds for

$\Pi = \Pi^A$. Hence

$$tr\left(P_t \Pi^A\right) \overset{(3.3)}{=} tr\left(P_t \Pi^X(\Delta) \Pi^A \Pi^X(\Delta)\right)$$
$$= tr\left(\Pi^X(\Delta) P_t \Pi^X(\Delta) \Pi^A\right)$$
$$= tr\left(\frac{\Pi^X(\Delta) P_t \Pi^X(\Delta)}{tr\left(P_t \Pi^X(\Delta)\right)} \cdot \Pi^A\right) tr\left(P_t \Pi^X(\Delta)\right) \qquad (3.6)$$

Since $A = A^* : \mathcal{R}(\Delta) \to \mathcal{R}(\Delta)$ is arbitrary, and Π^A is an arbitrary spectral projection of A, (3.6), (3.4) and the definition of $P_t^{\hat{X}}(\Delta)$ imply (3.5).

If $P_t = \Pi_\psi$ is a pure state (i.e., Π_ψ is the orthogonal projection onto a ray $[\psi]$, $\psi \in \mathcal{H}$) then

$$\left.\begin{aligned} P_t^{\hat{X}}(\Delta) &= \Pi_{\psi(\Delta)}, \quad \text{where} \\ \psi(\Delta) &:= \Pi^X(\Delta)\psi \big/ \|\Pi^X(\Delta)\psi\| \end{aligned}\right\} \qquad (3.7)$$

("collapse of the wave function")

The "Copenhagen interpretation" of QM is completed by requiring one further postulate:

(P III) Born's Rule

Given the state, P, of S right before $\hat{X}$ is measured, the probability, $prob_{\hat{X}}(\Delta / P)$, to measure a value for $\hat{X}$ in $\Delta \subset \mathbb{R}$ is given by

$$prob_{\hat{X}}(\Delta / P) := tr\left(P \, \Pi^{X}(\Delta)\right) \qquad (3.8)$$

Are these postulates reasonable? The answer will be (yes and) no!

Suppose we carry out two subsequent measurements of observables $\hat{X}$ and $\hat{Y}$ at times s and t, resp., with $s < t$. Let us assume for simplicity that the spectra of $\hat{X}$ and $\hat{Y}$ are pure-point. Then

$$X = \sum_{\xi} \xi \, \Pi^{X}_{\xi} \,, \quad Y = \sum_{\eta} \eta \, \Pi^{Y}_{\eta} \qquad (3.9)$$

are their spectral decompositions. Let P_s be the state of S right before $\hat{X}$ is measured, and let $\{U(t,s) \mid t, s \text{ in } \mathbb{R}\}$ be the propagator of S.

Let $P_s^{\hat{X}}$ be the state of S after the measurement of $\hat{X}$ at time s, and assume that the measurement of $\hat{Y}$ at time t is the first measurement made after time s. Suppose it yields the value η. According to Born's Rule (PIII),

$$\mathrm{prob}_{\hat{Y}}(\eta \mid P_s^{\hat{X}}) = \mathrm{tr}\left(U(t,s)\, P_s^{\hat{X}}\, U(s,t)\, \Pi_\eta^{Y}\right)$$

$$= \mathrm{tr}\left(P_s^{\hat{X}}\, U(s,t)\, \Pi_\eta^{Y}\, U(t,s)\right) \quad (3.10)$$

Let us suppose that S was prepared in a state P at some initial time t_{in} and that the first measurement after t_{in} was the one of $\hat{X}$. Then, by (PI) and (PII),

$$P_{s,\xi}^{\hat{X}} = Z_s^X(\xi)\, \Pi_\xi^X\, U(s,t_{in})\, P\, U(t_{in},s)\, \Pi_\xi^X, \quad (3.11)$$

for some $\xi \in \mathrm{spec}\,\hat{X}$, where

$$Z_s^X(\xi) := \mathrm{tr}\left(U(s,t_{in})\, P\, U(t_{in},s)\, \Pi_\xi^X\right)^{-1}$$

Let us denote this state by $P_s^{\hat{X}}(\xi/P)$. The probability, $\mathrm{prob}_{\hat{X}}(\xi/P)$, to measure ξ is given by

$$\text{prob}_{\hat{X}}(\xi \,|\, P) = Z_s^X(\xi)^{-1};$$

see (3.8). Thus the joint probability to measure ξ for $\hat{X}$ at time s and η for $\hat{Y}$ at time t, given the initial condition P at time $t_{in} < s$, is given by

$$\text{prob}_{\hat{X},\hat{Y}}(\xi, \eta \,|\, P) = \text{tr}\left(U(t,s)\, P^{\hat{X}}_{s,\xi}\, U(s,t)\, \Pi^Y_\eta\right)$$
$$\times\ \text{prob}_{\hat{X}}(\xi \,|\, P).$$

Setting

$$\Pi^Z(r) := U(t_{in}, r)\, \Pi^Z\, U(r, t_{in}), \tag{3.12}$$

for any operator Z and an arbitrary spectral projection, Π^Z, of Z (<u>Heisenberg picture</u>), we find

$$\text{prob}_{\hat{X},\hat{Y}}(\xi, \eta \,|\, P) = \text{tr}\left(\Pi^Y_\eta(t)\, \Pi^X_\xi(s)\, P\, \Pi^X_\xi(s)\, \Pi^Y_\eta(t)\right) \tag{3.13}$$

(<u>Lüders</u>' generalization of Born's Rule!)

The <u>Copenhagen interpretation</u> of QM would be just <u>fine</u> if

$$\sum_{\xi \in \mathrm{spec}\hat{X}} \mathrm{prob}_{\hat{X},\hat{Y}}(\xi,\eta|P) = \mathrm{prob}_{\hat{Y}}(\eta|P), \quad (3.14)$$

for any initial condition P at some time $t_{in} < t$ and any measurement of an observable $\hat{X}$ at an intermediate time $s \in (t_{in}, t)$. (Compare to class. Markov processes, e.g., Brownian motion.)

Alas, (3.14) usually fails, because of "interference terms": We first note that

$$\mathrm{prob}_{\hat{Y}}(\eta|P) = \mathrm{tr}\left(P\,\Pi^{Y}_{\eta}(t)\right)$$

$$= \mathrm{tr}\left(\Pi^{Y}_{\eta}(t)\,P\,\Pi^{Y}_{\eta}(t)\right)$$

$\sum_{\xi}\Pi^{X}_{\xi}(s) = \mathbb{1} \longrightarrow$

$$\overset{\downarrow}{=} \sum_{\xi,\xi'} \mathrm{tr}\left(\Pi^{Y}_{\eta}(t)\,\Pi^{X}_{\xi}(s)\,P\,\Pi^{X}_{\xi'}(s)\,\Pi^{Y}_{\eta}(t)\right),$$

$$\overset{?}{=} \sum_{\xi} \mathrm{tr}\left(\Pi^{Y}_{\eta}(t)\,\Pi^{X}_{\xi}(s)\,P\,\Pi^{X}_{\xi}(s)\,\Pi^{Y}_{\eta}(t)\right) \quad (3.15)$$

$$= \sum_{\xi} \mathrm{prob}_{\hat{X}\,\hat{Y}}(\xi,\eta|P),$$

(3.15) holds, <u>no matter what will be measured</u> <u>after</u> time t, iff the two states,

$$Z_t^{Y}(\eta)\, \Pi_\eta^{Y}(t)\, P\, \Pi_\eta^{Y}(t),$$

and

$$Z_t^{Y}(\eta)\, \Pi_\eta^{Y}(t) \Big\{ \sum_{\xi \in \operatorname{spec} \hat{X}} \Pi_\xi^{X}(s)\, P\, \Pi_\xi^{X}(s) \Big\}\, \Pi_\eta^{Y}(t),$$

with $Z_t^{Y}(\eta)$ as in (3.11) (with $X \to Y$, $\xi \to \eta$, $s \to t$), coincide. Assuming that this is true for all initial conditions, P, we conclude that

$$\sum_{\xi \in \operatorname{spec} \hat{X}} \Pi_\xi^{X}(s)\, f(Y(t))\, \Pi_\xi^{X}(s) = f(Y(t)),$$

where $Y(t) = \sum_\eta \eta\, \Pi_\eta^{Y}(t) = U(t_{in}, t)\, Y\, U(t, t_{in})$, for an arbitrary real-valued continuous function, f, on $\mathbb{R}$. It follows that

$$\left[\Pi_\eta^{Y}(t),\, \Pi_\xi^{X}(s) \right] = 0, \qquad (3.16)$$

$\forall\, \xi \in \operatorname{spec} \hat{X}$, $\forall\, \eta \in \operatorname{spec} \hat{Y}$.

This analysis can easily be extended to an arbitrary number, n, of consecutive measurements of observables, $\hat{X}_1, \dots, \hat{X}_n$, at times $t_1 < \dots < t_n$

$$Z_t^{Y}(\eta)\, \Pi_\eta^{Y}(t)\, P\, \Pi_\eta^{Y}(t),$$

and

$$Z_t^{Y}(\eta)\, \Pi_\eta^{Y}(t) \Big\{ \sum_{\xi \in \mathrm{spec}\,\hat{X}} \Pi_\xi^{X}(s)\, P\, \Pi_\xi^{X}(s) \Big\}\, \Pi_\eta^{Y}(t),$$

with $Z_t^{Y}(\eta)$ as in (3.11) (with $X \to Y$, $\xi \to \eta$, $s \to t$), coincide. Assuming that this is true for all initial conditions, P, we conclude that

$$\sum_{\xi \in \mathrm{spec}\,\hat{X}} \Pi_\xi^{X}(s)\, f\big(Y(t)\big)\, \Pi_\xi^{X}(s) = f\big(Y(t)\big),$$

where $Y(t) = \sum_\eta \eta\, \Pi_\eta^{Y}(t) = U(t_{in}, t)\, Y\, U(t, t_{in})$,

for an arbitrary real-valued continuous function, f, on $\mathbb{R}$. It follows that

$$\big[\Pi_\eta^{Y}(t),\, \Pi_\xi^{X}(s)\big] = 0, \qquad (3.16)$$

$\forall\, \xi \in \mathrm{spec}\,\hat{X}$, $\forall\, \eta \in \mathrm{spec}\,\hat{Y}$.

This analysis can easily be extended to an arbitrary number, n, of consecutive measurements of observables, $\hat{X}_1, \dots, \hat{X}_n$, at times $t_1 < \cdots < t_n$.

Extending formula (3.13) we define the probability of measuring the value $\xi^{(i)}$ for the observable $\hat{X}_i$, at time t_i, $i = 1, \ldots, n$ by

$$\mathrm{prob}_{\hat{X}_1, \ldots, \hat{X}_n}\left(\xi^{(1)}, \ldots, \xi^{(n)} \middle| P\right)$$

$$= tr\left(\left(\prod_{i=1}^{n} \Pi^{X_i}_{\xi^{(i)}}(t_i)\right)^{*} P \prod_{i=1}^{n} \Pi^{X_i}_{\xi^{(i)}}(t_i)\right), \quad (3.17)$$

(formula of <u>Schwinger</u> & <u>Wigner</u>)

where mention of the measurement times $t_1, \ldots, t_n$ is suppressed on the left side. If we require that

$$\sum_{\xi^{(i)}} \mathrm{prob}_{\hat{X}_1, \ldots, \hat{X}_i, \ldots, \hat{X}_n}\left(\xi^{(1)}, \ldots, \xi^{(i)}, \ldots, \xi^{(n)} \middle| P\right)$$

$$= \mathrm{prob}_{\hat{X}_1, \ldots, \check{\hat{X}}_i, \ldots, \hat{X}_n}\left(\xi^{(1)}, \ldots, \check{\xi}^{(i)}, \ldots, \xi^{(n)} \middle| P\right),$$

for arbitrary P, arbitrary $\xi^{(i+1)}, \ldots, \xi^{(n)}$, <u>and for</u> <u>any</u> i, we conclude that

$$\left[\Pi^{\hat{X}_i}_{\xi^{(i)}}(t_i),\ H(\underline{\xi}^{n}_{i+1})\, H(\underline{\xi}^{n}_{i+1})^{*}\right] = 0, \quad (3.18)$$

$\forall\, \xi^{(i)} \in \mathrm{spec}\, \hat{X}_i$, where

$$H\left(\underline{\xi}_{i+1}^{n}\right) := \prod_{k=i+1}^{n} \Pi_{\xi^{(k)}}^{\hat{X}_k}(t_k), \qquad (3.19)$$

$\forall\ \underline{\xi}_{i+1}^{n} \equiv \left(\xi^{(i+1)}, \dots, \xi^{(n)}\right)$.

Conditions (3.18), with H given by (3.19), $\forall i = 1, \cdots, n-1$, define what Griffiths calls a "consistent history" of measurements.

Let us simplify our notations:

$$P_j := \Pi_{\xi^{(j)}}^{\hat{X}_j}(t_j), \quad H_j := \prod_{k=j+1}^{n} \Pi_{\xi^{(k)}}^{\hat{X}_k}(t_k). \qquad (3.20)$$

Definition. A history of consecutive measurements of observables $\hat{X}_1, \dots, \hat{X}_n$ at times $t_1 < \cdots < t_n$ is said to be δ-consistent iff

$$\frac{1}{2} \left\| \left[P_j, H_j H_j^*\right] \right\| \leq 1 - \delta, \qquad (3.21)$$

for some $\delta \in [0, 1]$, for all choices of $P_1, \dots, P_n$ as specified in (3.20).

We define a sequence of positive numbers, $\{C_n\}_{n=0,1,2,\cdots}$, by $C_1 = 0$,

$$C_n = 6\left(4\sum_{k=1}^{n-1} C_k + 1\right), \qquad n \geq 2.$$

Lemma 3.1 Let $\{P_1, \cdots, P_n\}$ be a family of n orthogonal projections on a Hilbert space $\mathcal{H}$, and let

$$H_j := \prod_{k=j+1}^{n} P_k .$$

We assume that

$$\| [P_j, H_j H_j^*] \| < \varepsilon, \qquad (3.21)$$

for a sufficiently small ε and all $j = 1, \cdots, n-1$. Then there exists a family $\{\tilde{P}_1, \cdots, \tilde{P}_n\}$ of orthogonal projections with the properties that

$$\| \tilde{P}_j - P_j \| < C_{n+1-j} \cdot \varepsilon, \qquad (3.22)$$

and

$$[\tilde{P}_j, \tilde{H}_j \tilde{H}_j^*] = 0 \qquad (3.23)$$

where $\tilde{H}_j := \prod_{k=j+1}^{n} \tilde{P}_j$, for all $j = 1, \cdots, n$.

The operators $\tilde{H}_j$ are orthogonal projections, $\forall j$.

Corollary 3.2 If $0 \leq 1-\delta$ is small enough then there is a consistent history of modified measurements close (in norm – see (3.22)) to a δ-consistent history.

The above lemma is a straightforward consequence of the following

Lemma 3.3

Let $P = P^*$ be a bounded operator on $\mathcal{H}$, and let $0 < \varepsilon < \frac{1}{4}$. If $\|P^2 - P\| < \varepsilon$ then $\exists$ an orthogonal projection, $Q = Q^2 = Q^*$, on $\mathcal{H}$ such that

$$\|Q - P\| < 2\varepsilon .$$

The operator Q can be chosen to be a function of P.

Proof. Let $A = A^*$ be a self-adjoint operator on $\mathcal{H}$, and let $\sigma(A)$ denote its spectrum. If f is a real-valued function on $\mathbb{R}$ then $\sigma(f(A)) = f(\sigma(A))$.

We set $A := P$, $f(x) = x^2 - x$. Then

$$\|f(P)\| = \sup_{\lambda \in \sigma(P)} |\lambda^2 - \lambda| < \varepsilon ,$$

by hypothesis. This implies that

$$\sigma(P) \subset \underbrace{[-2\varepsilon, 2\varepsilon]}_{=\Delta_0} \cup \underbrace{[1-2\varepsilon, 1+2\varepsilon]}_{=\Delta_1},$$

for $\varepsilon < \frac{1}{4}$, as is easily checked. By the spectral theorem

$$P = \int_{\sigma(P)} \lambda \, d\Pi_P(\lambda),$$

where Π_P are the spectral projections of P. We set

$$Q := \Pi_P(\Delta_1) \tag{3.24}$$

Clearly $Q = Q^2 = Q^*$. Moreover

$$Q - P = \int_{\Delta_1} (\lambda - 1) \, d\Pi_P(\lambda) + \int_{\Delta_0} \lambda \, d\Pi_P(\lambda)$$

$$\left\| \int_{\Delta_1} (\lambda - 1) \, d\Pi_P(\lambda) \right\| \leq \max_{\lambda \in \Delta_1} |\lambda - 1| < 2\varepsilon$$

$$\left\| \int_{\Delta_0} \lambda \, d\Pi_P(\lambda) \right\| \leq \max_{\lambda \in \Delta_0} |\lambda| < 2\varepsilon.$$

Since $\Delta_0 \cap \Delta_1 = \emptyset$, for $\varepsilon < \frac{1}{4}$, the lemma is proven. $\square$

We now prove Lemma 3.1 for $n = 2$. Then its hypothesis tells us that

$$\| [P_1, P_2] \| < \varepsilon. \qquad (3.25)$$

We fix $H_1 := P_2 P_2^* = P_2$ and define

$$P := P_2 P_1 P_2 + P_2^\perp P_1 P_2^\perp , \quad P_2^\perp := 1 - P_2$$

Then

(i) $\|P - P_1\| \leq 2 \| [P_1, P_2] \| < 2\varepsilon$,

because $P_2 P_2^\perp = P_2^\perp P_2 = 0$ and by (3.25); and

(ii) $P^2 = P + P_2 [P_1, P_2] P_1 P_2 + P_2^\perp [P_1, P_2^\perp] P_1 P_2^\perp$,

hence $\|P^2 - P\| < 2\varepsilon$, by (3.25).

By Lemma 3.3, $\exists$ a projection $Q = Q^2 = Q^*$, with $[Q, P] = 0$, hence $[Q, P_2] = 0$, such that $\|Q - P\| < 4\varepsilon$. Thus

$$\tilde{P}_1 := Q \ , \quad \tilde{P}_2 := P_2$$

satisfies Lemma 3.1. $\square$

The general case of Lemma 3.1 is treated in [F-Schubnel, Schrödinger lecture].

The formalism of consistent histories (or δ-consistent histories) is not satisfactory, at all:
The choice of an initial state $P = P_{t_{in}}$ and of a propagator $\{U(t,s) \mid t,s \text{ in } \mathbb{R}\}$ does not enable one to make predictions about "events" or about what kind of "observables" will be measured.
In addition to (P, U), one has to specify a sequence of consistent measurements.
But the theory does not enable one to do such a thing; and there are many different sequences of consistent measurements that are <u>incompatible</u>, i.e., usually two such sequences cannot be composed to yield another sequence of consistent measurements.

Clearly, we should <u>abandon</u> this particular extension of the Copenhagen interpretation!

<u>More on these matters in Sect. 4!</u>

3.2 The Kochen - Specker Theorem

In this section, we accept the predictions of quantum mechanics made on the basis of the Copenhagen interpret.

We ask: $\exists$ hidden - variables theory?

This would mean that $\exists$ measure space $(\Omega, \mathcal{F})$ and maps

$$\begin{cases} A = A^* \in \mathcal{O}_S \longmapsto f_A : \Omega \to \mathbb{R} \\ \Psi \in \mathcal{H} \longmapsto \rho_{[\Psi]} : \text{prob. measure on } (\Omega, \mathcal{F}) \end{cases}$$

with the properties:

(P1)

$$\boxed{\begin{gathered} \langle \Psi, E^A(\Delta) \Psi \rangle / \|\Psi\|^2 = \rho_{[\Psi]}\left(f_A^{-1}(\Delta)\right) \\ \Delta \subset \mathbb{R} \text{ a Borel set} \\ E^A(\Delta) : \text{spectral projections of } A \end{gathered}}$$

$$\Rightarrow \langle \Psi, A \Psi \rangle = \int_\Omega f_A(\omega) \, d\rho_{[\Psi]}(\omega)$$

(P2)

$$\boxed{\begin{gathered} \text{If } u : \mathbb{R} \to \mathbb{R} \text{ is any bounded measurable function then} \\ f_{u(A)} = u \circ f_A \end{gathered}}$$

Note:

(P1) & (P2) are compatible:

$$\langle \Psi, E^{u(A)}(\Delta)\Psi \rangle = \langle \Psi, E^{A}(u^{-1}(\Delta))\Psi \rangle$$

$$\overset{(P1)}{=} \rho_{[\Psi]}\left(f_A^{-1}\left(u^{-1}(\Delta)\right)\right)$$

$$= \rho_{[\Psi]}\left((u \circ f_A)^{-1}(\Delta)\right)$$

$$\overset{(P2)}{=} \rho_{[\Psi]}\left(f_{u(A)}^{-1}(\Delta)\right)$$

(P3) $\boxed{\begin{array}{l}\text{Given any abelian algebra } \mathfrak{M} \text{ of} \\ \text{commuting, self-adjoint operators, then} \\ \qquad f : A \in \mathfrak{M} \longmapsto f_A \in L^\infty(\Omega) \\ \text{is an algebra homomorphism; i.e.,} \\ \qquad f_{A_1 A_2} = f_{A_1} \cdot f_{A_2}, \\ \text{for all } A_1, A_2 \text{ in } \mathfrak{M}.\end{array}}$

(Claim: (P3) follows from (P1), (P2).
If $\dim \mathcal{H} < \infty$ this is easy to prove: Ex.!)
Straumann, p. 373.

- If $\dim \mathcal{H} = 2$ a hidden-variables theory with (P1)–(P3) exists; (QM of spin $\frac{1}{2}$).

<u>Theorem.</u> (Kochen & Specker)

- If $\dim \mathcal{H} \geq 3$ a hidden-variables theory with (P1)–(P3) does <u>not</u> exist.

<u>Proof.</u> (following N. D. Mermin)

<u>3 spin-½ particles:</u>

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

<u>6 observables:</u>

$$A_1 = \sigma_x \otimes 1 \otimes 1, \; A_2 = 1 \otimes \sigma_x \otimes 1, \; A_3 = 1 \otimes 1 \otimes \sigma_x$$

$$B_1 = \sigma_y \otimes 1 \otimes 1, \; B_2 = 1 \otimes \sigma_y \otimes 1, \; B_3 = 1 \otimes 1 \otimes \sigma_y$$

Consider

$$Q_1 = A_1 B_2 B_3, \; Q_2 = B_1 A_2 B_3, \; Q_3 = B_1 B_2 A_3$$

(i) The Q_i's are <u>s.a.</u> ops.

$$\sigma_x \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_z$$

$$\sigma_y \sigma_x = \underbrace{\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}}_{\sigma_y} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_z$$

$$\Rightarrow \boxed{\sigma_x \sigma_y = -\sigma_y \sigma_x}$$

$$\Rightarrow \text{(ii)} \quad [Q_i, Q_j] = 0, \quad \forall\, i,j$$

Finally,

$$\text{(iii)} \quad Q_i^{\,2} = 1, \quad \forall i.$$

Construct a joint eigenvector with e.v. = 1:

$$\Psi_1 := \frac{1}{\sqrt{2}} \left[|\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle - |\downarrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle \right]$$

$$\Rightarrow \text{(iv)} \quad Q_j \Psi_1 = \Psi_1, \quad \forall j.$$

Assume that $\exists$ hidden-variable theory reproducing this system:

Def. $\boxed{q_j := f_{Q_j}}$

Then

$$(v) \quad \int_{\Omega} q_j(\omega)\, d\rho_{[\Psi_1]}(\omega) \overset{(P1)}{=} \langle \Psi_1, Q_j \Psi_1 \rangle \overset{(iii)}{=} 1$$

Because $Q_j^2 = 1 \overset{(P2)}{\Longrightarrow} q_j^2 = 1 \Longrightarrow$

$$q_j(\omega) = \pm 1, \text{ a.e.}$$

With $(v) \Longrightarrow \quad q_j(\omega) = 1$, a.e. on $\operatorname{supp} d\rho_{[\Psi_1]}$.

$\Longrightarrow \quad q_1 q_2 q_3 = 1$, a.e. on $\operatorname{supp} d\rho_{[\Psi_1]}$.

Because A_1, B_2, B_3 <u>commute</u> (+ cycl.!)

$$\Longrightarrow \quad (vi) \quad q_1 \overset{(P3)}{=} f_{A_1} \cdot f_{B_2} \cdot f_{B_3} = a_1 b_2 b_3, \ (\text{+ cycl.})$$

(vii) $q_1 q_2 q_3 = a_1 b_2 b_3 \, b_1 a_2 b_3 \, b_1 b_2 a_3$

$= a_1 a_2 a_3$,

because $b_i^2 = f_{B_i}^2 \overset{(P2)}{=} f_{B_i^2} = f_{\ldots \underbrace{\sigma_y^2}_{=1} \ldots} = 1$.

<u>But</u> $Q_1 \cdot Q_2 \cdot Q_3 = -A_1 A_2 A_3$ (calculation!)

$\Rightarrow$

$$q_1 q_2 q_3 = f_{Q_1 Q_2 Q_3} = f_{-A_1 A_2 A_3}$$

$$= -f_{A_1 A_2 A_3} = -f_{A_1} f_{A_2} f_{A_3}$$

$$= -a_1 a_2 a_3$$

$\longrightarrow$ $\boxed{\text{Contradiction!}}$

<u>The original Kochen-Specker argument</u>

$\mathcal{H} = \mathbb{C}^3$, spin-1 particle.

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(vii) $q_1 q_2 q_3 = a_1 b_2 b_3 \, b_1 a_2 b_3 \, b_1 b_2 a_3$

$= a_1 a_2 a_3$,

because $b_i^2 = f_{B_i}^2 \overset{(P2)}{=} f_{B_i^2} = f_{\underbrace{\cdots \sigma_y^2 \cdots}_{=1}} = 1$.

But $Q_1 \cdot Q_2 \cdot Q_3 = - A_1 A_2 A_3$ (calculation!)

$(\Rightarrow$

$$q_1 q_2 q_3 = f_{Q_1 Q_2 Q_3} = f_{-A_1 A_2 A_3}$$

$$= - f_{A_1 A_2 A_3} = - f_{A_1} f_{A_2} f_{A_3}$$

$$= - a_1 a_2 a_3$$

$(\longrightarrow$ $\boxed{\text{Contradiction!}}$

The original Kochen-Specker argument

$\mathcal{H} = \mathbb{C}^3$; spin-1 particle.

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The operators

$$1-S_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 1-S_2^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 1-S_3^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are mutually commuting orthogonal projections of rank 1 whose sum $= 1_3$. More generally,

$$P(\vec{e}) = 1_3 - (\vec{e}\cdot\vec{S})^2, \quad |\vec{e}| = 1,$$

is a rank-1 proj. projecting onto $\vec{e}$ $\Rightarrow$

$$P(\vec{e}) = |\vec{e}\rangle\langle\vec{e}|$$

$$\Rightarrow \quad P(\vec{e})_{ij} = e_i e_j \qquad \text{(in above basis)}$$

If $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is any orthonormal dreibein then

(viii) $$\sum_{j=1}^{3} P(\vec{e}_j) = 1, \quad P(\vec{e}_i)\cdot P(\vec{e}_j) = \delta_{ij}\, P(\vec{e}_i)$$

(use rotation invariance!)

The ops. $P(\vec{e}_1), P(\vec{e}_2), P(\vec{e}_3)$ are functions of

$$A = \sum_{j=1}^{3} \alpha_j\, P(\vec{e}_j), \qquad \alpha_1 < \alpha_2 < \alpha_3 .$$

→ Can use (P3)! (See page 12.)

Suppose Kochen – Specker hypotheses, (P1) – (P3), are <u>correct</u>. Then

(ix) $P(\vec{e}) \longmapsto f_{P(\vec{e})} = \chi_{\vec{e}}$,

a <u>characteristic function</u> on $(\Omega, \mathcal{F})$, (because $P(\vec{e})^2 = P(\vec{e})$!)

By (viii), page 17,

(x) $$\sum_{j=1}^{3} \chi_{\vec{e}_j} = 1, \quad \text{a.e.}$$

If $d\mu$ is any probability measure on $(\Omega, \mathcal{F})$ then

$$\mathbb{E}_\mu \chi_{\vec{e}} := \int \chi_{\vec{e}}(\omega)\, d\mu(\omega)$$

satisfies

$$\left.\begin{array}{l} 0 \leq \mathbb{E}_\mu \chi_{\vec{e}} \leq 1 \\ \text{and, by (x), above,} \\ \displaystyle\sum_{j=1}^{3} \mathbb{E}_\mu \chi_{\vec{e}_j} = 1. \end{array}\right\} \text{(xi)}$$

If $d\mu(\omega)$ is given by a Dirac δ-function on Ω then

$$\varphi(\vec{e}) := \mathbb{E}_{\mu=\delta}(\chi_{\vec{e}}) = 0 \text{ or } 1, \quad (xii)$$

for all $\vec{e}$, with $\sum_{j=1}^{3} \mathbb{E}_\mu \chi_{\vec{e}_j} = 1, \quad (xiii)$

for arbitrary orthonormal dreibeins $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Functions φ with the above properties do not exist!

Proof 1: K–S

Proof 2: Use Gleason's thm.:

$$\varphi : \vec{e} \in S^2 \longmapsto \varphi(\vec{e}) \quad (= 0 \text{ or } 1, \forall \vec{e})$$

After complexification: φ is an additive (see (xiii)!) measure, on the lattice of projections acting on $\mathbb{C}^3$.

$\xrightarrow{\text{Gleason}}$ $\exists$ a s.a. matrix $X \geq 0$, with

$tr\, X = 1$ such that

$$\varphi(\vec{e}) = tr(XP(\vec{e})) = (\vec{e}, X\vec{e})$$

$\Rightarrow$ For some $\vec{e}$, $0 < \varphi(\vec{e}) < 1$,

contradicting (xii)!

Compare to Kakutani's thm. ($\nearrow$ Dyson):

f a real-valued continuous function on n-dim. sphere, S^n, in $\mathbb{R}^{n+1}$ centered at O.

Then $\exists$ $n+1$ points, $x_1, x_2, \cdots, x_{n+1}$, on S^n such that the vectors $\overrightarrow{Ox_1}, \ldots, \overrightarrow{Ox_{n+1}}$ in $\mathbb{R}^{n+1}$ are mutually orthogonal, and

$$f(x_1) = f(x_2) = \cdots = f(x_n)$$

Pf.: Dyson, Ann. Math. 54, 534–536 (1951)

$\mathrm{tr}\, X = 1$ such that

$$\varphi(\vec{e}) = \mathrm{tr}(X P(\vec{e})) = (\vec{e}, X\vec{e})$$

$\Rightarrow$ For some $\vec{e}$, $0 < \varphi(\vec{e}) < 1$,

contradicting (xii)!

Compare to Kakutani's thm.; (→ Dyson):

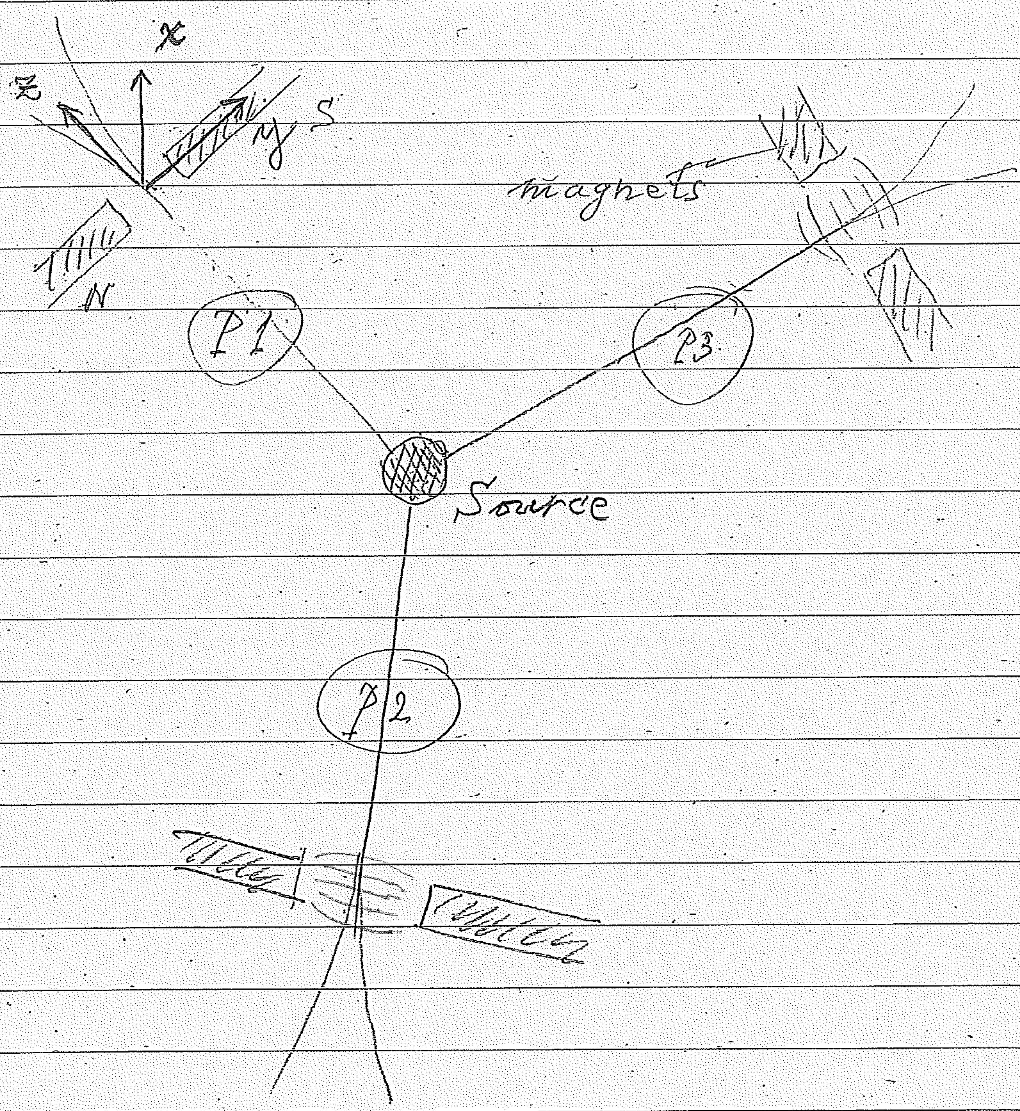
f a real-valued continuous function on n-dim. sphere, S^n, in $\mathbb{R}^{n+1}$ centered at O.

Then $\exists$ $n+1$ points, $x_1, x_2, \ldots, x_{n+1}$, on S^n such that the vectors $\overrightarrow{Ox_1}, \ldots, \overrightarrow{Ox_{n+1}}$ in $\mathbb{R}^{n+1}$ are mutually orthogonal, and

$$f(x_1) = f(x_2) = \cdots = f(x_n)$$

Pf.: Dyson, Ann. Math. 54, 534–536 (1951)

3.3 EPR & Bell



Q_1, Q_2, Q_3 as in ~~...~~ page 13; $\Big(Q_1 = A_1 B_2 B_3$

$\sigma_x \otimes 1 \otimes 1$

$1 \otimes \sigma_y \otimes 1 \ldots$

$$\Psi_1 = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle\otimes|\uparrow\rangle\otimes|\uparrow\rangle - |\downarrow\rangle\otimes|\downarrow\rangle\otimes|\downarrow\rangle\right]$$

Source produces 3 spin-½ particles in state Ψ_1 (repeatedly!).

EPR notions of "reality" and "locality"

(R) Quantities (of a system) whose values can be predicted with certainty are "elements of reality".

realism

(L) Elements of reality of a system, S_1, cannot be influenced by measurements at another system, S_2, space-like separated from S_1.

locality

For Ψ_1:

$$Q_1 \Psi_1 = Q_2 \Psi_1 = Q_3 \Psi_1 = \Psi_1$$

→ Measure σ_x for P1, σ_y for P2

⇒ σ_y for P3 determined → hence

σ_y an "element of reality."

Measure σ_y for P1, σ_y for P2

⇒ σ_x for P3 determined → hence

σ_x an "element of reality".

Since $[\sigma_x, \sigma_y] = \sigma_z \neq 0$ ⇒ σ_x and σ_y for P3 __cannot__ simultaneously be "elements of reality".

__Problem__: (L) not valid in strict form!

Bell's analysis:

Assume local hidden variables theory.

$\Rightarrow \exists$ random variables $a_i(\omega) := f_{A_i}(\omega)$,

$b_i(\omega) := f_{B_i}(\omega)$, $i = 1, 2, 3$, such that

$$1 = \langle \Psi_1, \underbrace{A_1 B_2 B_3}_{Q_1} \Psi_1 \rangle = \int_\Omega a_1(\omega)\, b_2(\omega)\, b_3(\omega)\, d\rho_{[\Psi_1]}(\omega) \qquad (xiv)$$

$$\left.\begin{array}{l} 1 = f_1 = f_{A_i^2} = f_{A_i}^2 = a_i^2 \\ 1 = f_1 = f_{B_i^2} = f_{B_i}^2 = b_i^2 \end{array}\right\} \forall i$$

$\Rightarrow a_i(\omega) = \pm 1$, $b_i(\omega) = \pm 1$, $\forall \omega, \forall i$,

$\Rightarrow$ By (xiv),

$$a_1 b_2 b_3 = 1, \text{ a.e.}, (+\text{ cycl.}) \qquad (xv)$$

on $\operatorname{supp} d\rho_{[\Psi_1]} \neq \emptyset$.

(xv) + cycl. $\Longrightarrow$

$$a_1\, b_2\, b_3\;\; b_1\, a_2\, b_3\;\; b_1\, b_2\, a_3 \;=\; a_1\, a_2\, a_3 \;=\; 1$$

$$\Longrightarrow \quad a_1\, a_2\, a_3 = 1\,, \quad \underline{\text{on } \operatorname{supp} d\rho_{[\Psi_1]}}$$

On the other hand,

$$1 = \langle \Psi_1, Q_1\, Q_2\, Q_3\, \Psi_1 \rangle = \langle \Psi_1, -A_1\, A_2\, A_3\, \Psi_1 \rangle$$

$$= -\int a_1(\omega)\, a_2(\omega)\, a_3(\omega)\, d\rho_{[\Psi_1]}(\omega)$$

$$\Longrightarrow \quad a_1\, a_2\, a_3 = -1\,, \quad \underline{\text{on } \operatorname{supp} d\rho_{[\Psi_1]}}$$

$\longrightarrow$ $\boxed{\text{Contradiction!}}$

Bell inequality. 2 spin-½ particles

$$\left.\begin{aligned} A_{\vec{e}_1} &:= \vec{\sigma}\cdot\vec{e}_1 \otimes \mathbb{1}, \\ B_{\vec{e}_2} &:= \mathbb{1} \otimes \vec{\sigma}\cdot\vec{e}_2 \end{aligned}\right\}$$

Clearly, $[A_{\vec{e}_1}, B_{\vec{e}_2}] = 0$

Initial state:

$$\Psi := \frac{1}{\sqrt{2}}\left(|\uparrow\rangle\otimes|\downarrow\rangle - |\downarrow\rangle\otimes|\uparrow\rangle\right)$$

(total spin $= 0$!)

Then

$$\boxed{\langle \Psi, A_{\vec{e}_1}\cdot B_{\vec{e}_2}\, \Psi\rangle = -\vec{e}_1\cdot\vec{e}_2} \qquad (*)$$

(Choose $\vec{e}_1 = \begin{pmatrix}0\\0\\1\end{pmatrix}$, $\vec{e}_2 = \vec{e} = \begin{pmatrix}\sin\vartheta\sin\varphi\\ \sin\vartheta\cos\varphi\\ \cos\vartheta\end{pmatrix}$,

with $\varphi = 0$. Then $\vec{e}_1\cdot\vec{e}_2 = \cos\vartheta$!)

If local hidden variables existed

$$E(\vec{e}_1, \vec{e}_2) \equiv \langle \Psi, A_{\vec{e}_1} \cdot B_{\vec{e}_2} \Psi \rangle$$

$$= \int_\Omega a_{\vec{e}_1}(\omega)\, b_{\vec{e}_2}(\omega)\, d\rho_{[\Psi]}(\omega). \quad (xvi)$$

Since $A_{\vec{e}_1}^2 = 1$, $B_{\vec{e}_2}^2 = 1$,

$$|a_{\vec{e}_1}(\omega)| = 1, \quad |b_{\vec{e}_2}(\omega)| = 1.$$

Consider

$$\boxed{F(\vec{e}_1, \vec{e}_2, \vec{e}_1{}', \vec{e}_2{}') := E(\vec{e}_1, \vec{e}_2) + E(\vec{e}_1, \vec{e}_2{}') + E(\vec{e}_1{}', \vec{e}_2) - E(\vec{e}_1{}', \vec{e}_2{}').}$$

Since

$$-2 \leq xy + xy' + x'y - x'y' \leq 2 \quad {}^{*)}$$

for arb. x, y, x' and y' in $[-1, 1]$,

it follows from (xvi) that

$$-2 \leq F(\vec{e}_1, \vec{e}_2, \vec{e}_1{}', \vec{e}_2{}') \leq 2. \qquad (BI)$$

---

*) $xy + xy' + x'y - x'y' = x(y + y') + x'(y - y')$

$|x| \leq 1$; $|y + y'| + |y - y'| \leq 2 \longrightarrow$ Done!

But choosing

$$\vec{e}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -\frac{1}{\sqrt{2}} \end{pmatrix},$$

$$\vec{e}_2{}' = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \quad \vec{e}_1{}' = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{e}_1 \cdot \vec{e}_2 = -\frac{1}{\sqrt{2}}, \quad \vec{e}_1 \cdot \vec{e}_2{}' = -\frac{1}{\sqrt{2}}$$

$$\vec{e}_1{}' \cdot \vec{e}_2 = -\frac{1}{\sqrt{2}}, \quad \vec{e}_1{}' \cdot \vec{e}_2{}' = \frac{1}{\sqrt{2}}$$

By $(*)$, page 26,

$$\left.\begin{aligned} F(\vec{e}_1, \vec{e}_2, \vec{e}_1{}', \vec{e}_2{}') &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ &= \frac{4}{\sqrt{2}} = 2\sqrt{2} \end{aligned}\right\}$$

$\longrightarrow$ This violates (BI)!

<u>Exercise:</u> Details of Bell + <u>CHSH</u>.

3.4 Tsirelson's work on Bell inequalities

Conider two independent system, S_1 and S_2, and the composed system $S = S_1 \vee S_2$. The Hilbert spaces for S_1 and S_2 are denoted by $\mathcal{H}_1$ and $\mathcal{H}_2$, resp., and $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is the Hilbert space of S; $\mathcal{H}$ denotes the category of finite-dimensional Hilbert spaces. We define families, $\mathcal{O}_i^q$, of operators in $\mathcal{O}_{S_i}$, $i = 1, 2$:

(a) $$\mathcal{O}_i^q := \{A^{(i)} \in \mathcal{O}_{S_i} \subset B(\mathcal{H}_i) \mid A^{(i)*} = A^{(i)}, \|A^{(i)}\| \leq 1\}, \quad (17)$$

$i = 1, 2$.

(b) The convex set of density matrices, P, on $\mathcal{H}$ is denoted by $\mathcal{S}_{\mathcal{H}}$.

(c) Let (Ω, Σ) be a measure space. The family of probability measures, μ, on (Ω, Σ) is denoted by Prob. The set of real-valued, bounded random variables on (Ω, Σ) *) bounded

*) If choice of σ-algebra Σ is obvious it will not be mentioned, anymore!

in absolute value by 1 is denoted by $\mathcal{O}^c$.

Let K and L be two natural numbers, $k \in \{1, \dots, K\}$, $l \in \{1, \dots, L\}$ two indices.

<u>Quantum Correlation Matrices</u>

$$\mathcal{M}^q_{K,L} := \Big\{ \Gamma \in M_{K\times L}(\mathbb{R}) \,\Big|\, \Gamma_{kl} := tr\big(P A^{(1)}_k \otimes A^{(2)}_l\big), \tag{18}$$
$$A^{(1)}_k \in \mathcal{O}^q_1,\ A^{(2)}_l \in \mathcal{O}^q_2,\ k = 1, \dots, K,\ l = 1, \dots, L \Big\},$$

for an arbitrary $P \in \mathcal{S}_{\mathcal{H}}$, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\mathcal{H}_1, \mathcal{H}_2$ arbitrary Hilbert spaces.

A subset of $\mathcal{M}^q_{K,L}$ is the set of

<u>Classical Correlation Matrices</u>

$$\mathcal{M}^c_{K,L} := \Big\{ \Gamma \in \mathcal{M}^q_{K,L} \,\Big|\, [A^{(i)}_m, A^{(i)}_n] = 0,\ \forall m, n,\ i = 1,2 \Big\} \tag{19}$$

In this case $A^{(1)}_1, \dots, A^{(1)}_K, A^{(2)}_1, \dots, A^{(2)}_L$ can be <u>simultaneously diagonalized</u> and can be viewed as r.-v. functions, $a^{(1)}_1, \dots, a^{(1)}_K, a^{(2)}_1, \dots, a^{(2)}_L$ on their <u>joint spectrum</u>, denoted Ω, with

$$\| a^{(i)}_j \|_\infty \leq 1, \quad \forall j,\ i = 1,2. \tag{20}$$

The set of N-tuples of unit vectors in $\mathbb{R}^M$ is denoted by $T_{N,M}$.

Lemma T. The spaces $\mathcal{M}^q_{K,L}$ and $\mathcal{M}^c_{K,L}$ can be characterized as follows:

(i) $$\mathcal{M}^q_{K,L} = \{ \Gamma \in \mathbb{M}_{K\times L}(\mathbb{R}) \mid \Gamma_{k\ell} = x_k \cdot y_\ell, \text{ where } (x_1, \dots, x_K, y_1, \dots, y_L) \in T_{K+L, K+L} \} \quad (21)$$

(ii) $$\mathcal{M}^c_{K,L} = \{ \Gamma \in \mathcal{M}^q_{K,L} \mid \Gamma_{k\ell} = \int_\Omega a^{(1)}_k(\omega)\, a^{(2)}_\ell(\omega)\, d\mu(\omega), \; a^{(i)}_j \in \mathcal{O}^c, \forall j, i=1,2; \; \mu \in \text{Prob} \} \quad (22)$$

(iii) $\mathcal{M}^q_{K,L}$ is a convex*) compact subset of $\mathbb{M}_{K\times L}(\mathbb{R})$, (consequence of (i)); and $\mathcal{M}^c_{K,L}$ is a convex polytope in $\mathbb{M}_{K\times L}(\mathbb{R})$, (dep. on choice of Ω, ...).

Definition. A convex polytope, $\mathcal{P}$, in $\mathbb{R}^n$ is a convex compact set with a finite number of extremal points $\Longleftrightarrow$ $\mathcal{P}$ is the convex hull of a finite set of points in $\mathbb{R}^n$.

*) Exercise! (Use direct sums, ...)

<u>Proposition.</u> A convex polytope $\mathcal{P} \subset \mathbb{R}^n$ is a finite intersection of closed half-spaces

$$H(y_i, \alpha_i) := \{x \in \mathbb{R}^n \mid x \cdot y_i \leq \alpha_i\},$$

$\alpha_i \in \mathbb{R}$, $\vec{0} \neq y_i \in \mathbb{R}^n$, $i = 1, 2, \cdots, m$, for some m.

The proof of this Proposition is ess. obvious, the Lemma is proven in Tsirelson's paper. (Use $tr(P(\cdot))$ to define a scalar product on the vector space

$$B(\mathcal{H}) \supseteq Vec(\mathcal{O}_1^q) \dotplus Vec(\mathcal{O}_2^q),$$

where $Vec(\mathcal{O}_i^q)$ is the vector space generated by $\mathcal{O}_i^q$; [recall that $\dim \mathcal{H} < \infty$!]. The dimension of $Vec(\mathcal{O}_1^q) \dotplus Vec(\mathcal{O}_2^q)$ is $\leq K + L$, and $\{A_1^{(1)}, \ldots, A_K^{(1)}, A_1^{(2)}, \ldots, A_L^{(2)}\}$ is a generating system. $\cdots \Rightarrow$ (i)! (ii) & (iii) are then quite straightforward.)

In order to state Tsirelson's main result, we have to invoke the following theorem due to <u>A.</u>

Grothendieck:

Theorem G. Let $h = (h_{ij})$ be some real $n \times n$ matrix, for an arbitrary $n \in \mathbb{N}$. If, for arbitrary

$\underline{s} = (s_1, \dots, s_n)$ and $\underline{t} = (t_1, \dots, t_n)$ in $\mathbb{R}^n$,

$$\left|\underline{s} \cdot h\,\underline{t}\right| = \left|\sum_{i,j=1}^{n} h_{ij}\, s_i\, t_j\right| \leq \max_i |s_i| \cdot \max_j |t_j| \tag{23}$$

then

$$\left|\sum_{i,j=1}^{n} h_{ij} \langle x_i, y_j \rangle_{\mathcal{H}}\right| \leq K_n \max_i \|x_i\| \cdot \max_j \|y_j\|, \tag{24}$$

for arbitrary vectors $x_1, \dots, x_n, y_1, \dots, y_n$ in a Hilbert space $\mathcal{H}$. In (24), K_n is a constant strictly > 1.

$$K_G := \sup_n K_n \leq 1.782 \quad \text{(Grothendieck's constant)} \tag{25}$$

"Normalization condition C". We choose $A_k^{(i)}$ in (17) and (19) such that (e.g.) $\left(A_k^{(i)}\right)^2 = \mathbb{1}$, $\forall k$, $i = 1, 2$.

Theorem T. Suppose the normalization condition C holds. Then

$$T \in \mathcal{M}^{q}_{K,L} \Rightarrow K_G^{-1}\, T \in \mathcal{M}^{c}_{K,L}, \text{ for arbitrary } K, L. \tag{26}$$

Proof. Since $\mathcal{M}^c_{K,L}$ is a convex polytope in $M_{K\times L}(\mathbb{R}) \simeq \mathbb{R}^{K\cdot L}$, $\exists\, d < \infty$ vectors, $y^{(1)}, \dots, y^{(d)}$,

such that

$$\Gamma \in \mathcal{M}^{c}_{K,L} \iff \Gamma \cdot y^{(j)} \leq \alpha_j, \quad \forall j = 1, \cdots, d, \quad (27)$$

for some constants $\alpha_1, \cdots, \alpha_d$; (see Proposition).

If $\Gamma \in \mathcal{M}^{c}_{K,L}$ then $-\Gamma \in \mathcal{M}^{c}_{K,L}$; (replace $A^{(1)}_k$ by $-A^{(1)}_k$, $k = 1, \cdots, K$, and $A^{(2)}_\ell$ kept fixed, $\ell = 1, \cdots, L$). Thus, for $\Gamma \in \mathcal{M}^{c}_{K,L}$, we also have that

$$\left.\begin{aligned} &\Gamma \cdot y^{(j)} \geq -\alpha_j, \\ \iff \quad &\Gamma \cdot \left(-y^{(j)}\right) \leq \alpha_j, \quad j = 1, \cdots, d \end{aligned}\right\} \quad (28)$$

Hence

$$y^{(1)}, \cdots, y^{(d)} = y^{(1)}, -y^{(1)}, \cdots, y^{(\frac{d}{2})}, -y^{(\frac{d}{2})}.$$

Compactness of $\mathcal{M}^{c}_{K,L}$ then implies that

$$\left|\Gamma \cdot y^{(j)}\right| \leq |\alpha_j|, \quad j = 1, \cdots, \frac{d}{2}. \quad (29)$$

We now set

$$\boxed{h^{(j)} := \alpha_j^{-1} y^{(j)}, \quad j = 1, \cdots, \frac{d}{2}.}$$

Since $h^{(j)} \in \mathbb{R}^{K \cdot L}$, it can be written as a real $K \times L$ matrix, $h^{(j)} = \left(h^{(j)}_{k\ell}\right)$. Then (29) implies:

$$\left|T \cdot h^{(j)}\right| = \left|\sum_{k,l} T_{kl}\, h^{(j)}_{kl}\right| \leq 1. \qquad (30)$$

Inequality (30) must hold in the special case where $T_{kl} = s_k \cdot t_l$,*) for vectors $\underline{s} = (s_1, \dots, s_K)$ and $\underline{t} = (t_1, \dots, t_L)$, with $|s_k| \leq 1$, $|t_l| \leq 1$, $\forall k, l$; (since $\|A_j^{(i)}\|, \|a_j^{(i)}\|_\infty \leq 1$, $\forall j$, $i = 1, 2$!).

Thus, for all $j = 1, \dots, \frac{d}{2}$

$$\left|\sum h^{(j)}_{kl}\, s_k \cdot t_l\right| \leq 1$$

Let $N := \max(K, L)$ and regard $h^{(j)}$ as an $N \times N$ matrix, $\tilde{h}^{(j)}$, by setting the added matrix elements in $\tilde{h}^{(j)}$ to 0. Then, for any $\underline{s}, \underline{t}$ in $\mathbb{R}^N$, with $|s_i|, |t_i| \leq 1$, $\forall i = 1, \dots, N$, we have that

$$\left|\sum_{k,l=1}^{N} \tilde{h}^{(j)}_{kl}\, s_k\, t_l\right| \leq 1. \qquad (31)$$

Now, by Lemma T, (i), an arbitrary correlation

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*) This is seen by choosing arbitrary product measures.
$\Longrightarrow \{a_k^{(1)}\}, \{a_l^{(2)}\}$ indep. from each other!

matrix $\Gamma \in \mathcal{M}^{q}_{K,L}$ can be represented as

$\Gamma = (\Gamma_{kl})$, with $\Gamma_{kl} = x_k \cdot y_l$,

for some vectors $x_1, \dots, x_K, y_1, \dots, y_L$ in $\mathbb{R}^{K+L}$ of norm (≤ 1). We enlarge this family of vectors to $2N$ vectors by adding $N - \min(K,L)$ 0-vectors. Grothendieck's theorem, Theorem G, says that

$$\Big|\sum_{k,l=1}^{N} \tilde{h}^{(j)}_{kl}\, x_k \cdot y_l\Big| \leq K_N,$$

for all $j = 1, \dots, \frac{d}{2}$; i.e.,

$$\Big|\sum_{k,l} \tilde{h}^{(j)}_{kl}\, \frac{1}{K_N}\, \Gamma_{kl}\Big| \leq 1,$$

for all $j = 1, \dots, \frac{d}{2}$. Hence $\frac{1}{K_N}\Gamma$ belongs to the convex polytope characterized by (29), i.e., it is a classical correlation matrix. $\square$

We have seen in Sect 3.3, (xvi), (BI) and p. 28, that

$$\mathcal{M}^{c}_{2,2} \subsetneqq \mathcal{M}^{q}_{2,2}$$

"Exercise" Provide a general abstract argument showing that, assuming (C),

$$\mathcal{M}^{c}_{K,L} \subsetneq \mathcal{M}^{q}_{K,L}, \qquad (32)$$

for arbitrary K, L.

A cheap way is to embed Bell's example (two sets of 2×2 matrices) into higher dim. examples, which is easy.

4. **Some properties and puzzling features of Quantum Mechanics**

Contents of Chapter 4

4.1. Inadequacy of the Schrödinger equation ...

To begin with, we return to the Copenhagen interpretation of QM. A reasonable idea of how to render it more precise is to say that an "observable" $\hat{X}$ is measured at time t iff the state of the system S, P_t, at time t commutes with the operator X on $\mathcal{H}$ repr. $\hat{X}$ in the Schrödinger picture; i.e., if the state of S at time t is described by the density matrix P_t then

$$[X, P_t] = 0. \qquad (4.1)$$

This might be weakened to

$$\|[X, P_t]\| < \varepsilon \ll 1. \qquad (4.2)$$

Condition (4.2) implies that $\exists$ a self-adjoint operator $\tilde{X}$, with

$$\|\tilde{X} - X\| < o(1), \text{ as } \varepsilon \searrow 0, \qquad (4.3)$$

and with the property that

$$[\tilde{X}, P_t] = 0. \qquad (4.4)$$

One can then say that, instead of X, an "observable" $\tilde{X}$ close to X in norm is measured at time t.

Let us suppose that $\hat{X}$ has discrete spectrum, so that

$$X = \sum_{\xi \in \sigma(\hat{X})} \xi \, \Pi_{\xi}^{X} .$$

Then (4.1) implies that

$$P_t = \sum_{\xi \in \sigma(\hat{X})} \Pi_{\xi}^{X} P_t \Pi_{\xi}^{X}, \qquad (4.5)$$

as one easily verifies. Thus P_t is a mixed state <u>unless</u>

$$P_t = \Pi_{\xi}^{X} P_t \Pi_{\xi}^{X},$$

<u>for some</u> $\xi \in \sigma(\hat{X})$, in which case P_t may be a <u>pure state</u>, (but is not necessarily pure if $\dim \Pi_{\xi}^{X} > 1$).

In the <u>Heisenberg picture</u>, (4.5) says that

$$P = \sum_{\xi \in \sigma(\hat{X})} \Pi^{X}_{\xi}(t)\, P\, \Pi^{X}_{\xi}(t) \qquad (4.5')$$

In the Schrödinger picture, physical quantities / "observables" are represented by <u>time-independent linear operators</u> acting on $\mathcal{H}$ generating a C^*-algebra $\mathcal{A}_S$, with

$$\overline{\mathcal{A}_S}^{w} = B(\mathcal{H}). \qquad (4.6)$$

If the initial state, P, of S were <u>pure</u> ($P = \Pi_{\psi}$, $\psi \in \mathcal{H}$) and time evolution of states of S were given by a <u>unitary propagator</u>, $\{U(t,s) \mid t,s \text{ in } \mathbb{R}\}$, then the state of S at time t, P_t, would be <u>pure</u>, too. Then Eq. (4.5) would imply that

$$P_t = \Pi^{X}_{\xi} P_t \Pi^{X}_{\xi}, \text{ for } \underline{\text{some}}\ \xi \in \sigma(\hat{X}),$$

i.e., P_t would be an <u>eigenstate</u> of X! Then <u>Born's Rule would become empty</u>, and QM would be a <u>deterministic theory</u>. Assuming (4.6) and unitarity of the propagator, the same conclusions are reached in the Heisenberg picture.

Now, the property of a state to be pure or mixed depends on the algebra to which it is applied. If $\overline{\mathcal{A}_S}^{w} \subsetneq B(\mathcal{H})$, and $X = X^* \in \mathcal{A}_S$ then (4.5) may hold and P_t may be mixed even if the initial state $P = \Pi_\psi$, $\psi \in \mathcal{H}$, might be pure. This is a consequence of entanglement! There are two possible incarnations of this scenario:

(I) Decoherence; (not viable!)

(II) In the Heisenberg picture, $\mathcal{A}_S$ might depend on time t and may shrink, as t increases.*)

(I) Decoherence: Purity of states "lies in the eye of the beholder". We assume that

$$\mathcal{H} = \mathcal{H}_E \otimes \mathcal{H}_O , \tag{4.7}$$

where "E" stands for "environment", or "measuring equipment", and "O" for "observables".

*) See Chapter 6.

We assume that the system S and our abilities to measure "observables" characteristic of S are such that the "observables" O_S (see (1.226)) of S generate the algebra

$$\mathcal{A}_S = \{ \mathbb{1}|_{\mathcal{H}_E} \otimes X|_{\mathcal{H}_O} \mid X \in B(\mathcal{H}_O) \} \qquad (4.8)$$

Then the restriction of a pure state $P = \Pi_\psi$, $\psi \in \mathcal{H}$, to the algebra $\mathcal{A}_S$, $P|_{\mathcal{A}_S}$, as given by

$$tr_{\mathcal{H}_O}(P|_{\mathcal{A}_S} X) = \langle \psi, \mathbb{1} \otimes X \psi \rangle / \|\psi\|^2$$

may be <u>mixed</u>; a consequence of <u>entanglement</u> between E and O, i.e., $\psi \neq \varphi_E \otimes \varphi_O$.

Let us see how this might solve the so-called "<u>measurement problem</u>"! Suppose the system is in an initial state $P_E \otimes P_O$ at time t_{in}, where P_E and P_O are density matrices on $\mathcal{H}_E$ and $\mathcal{H}_O$, resp.. Suppose the time evolution of S is given by a propagator, $U(t,s)$, of the form

$$U(t,s) = \sum_{\xi \in \sigma(\hat{X})} V_\xi(t,s) \otimes \Pi_\xi^X, \qquad (4.9)$$

for some $X = X^* \in \mathcal{A}_S$, $(\hat{X} \in \mathcal{O}_S)$, with the property that

$$V_\xi(s,t)\, V_{\xi'}(t,s) \neq \mathbb{1}_{\mathcal{H}_E}, \quad \xi \neq \xi', \quad (4.10)$$

for $s \neq t$. The state of S at time t is then given by

$$P_t = \sum_{\xi,\xi'} V_\xi(t, t_{in})\, P_E\, V_{\xi'}(t_{in}, t) \otimes \Pi_\xi^X P_0 \Pi_{\xi'}^X$$

When evaluated on operators $\mathbb{1} \otimes A \in \mathcal{A}_S$ the state P_t looks like

$$tr_{\mathcal{H}}(P_t\, \mathbb{1} \otimes A) = \sum_{\xi,\xi'} tr_{\mathcal{H}_E}\left(P_E\, V_{\xi'}(t_{in}, t)\, V_\xi(t, t_{in})\right) \times$$

$$\times\ tr_{\mathcal{H}_0}\left(\Pi_\xi^X P_0 \Pi_{\xi'}^X A\right). \qquad (4.11)$$

For $\xi' = \xi$, the first factor under the sum on the R.S. of (4.11) is equal to 1. But, for

$\xi' \neq \xi$, property (4.10) can be sharpened to

$$tr_{\mathcal{H}_E}\left(P_E\, V_{\xi'}(t_{in}, t)\, V_{\xi}(t, t_{in})\right) \approx 0, \qquad (4.12)$$

if $t - t_{in}$ is "large", and $\underline{E}$ "macroscopic"; (typically E a system with ∞ many degrees of freedom). It is easy to construct models (E: ∞ many degrees of freedom & "gapless" modes) with the property that

$$\lim_{t \to \infty} tr_{\mathcal{H}_E}\left(P_E\, V_{\xi'}(t_{in}, t)\, V_{\xi}(t, t_{in})\right) = 0, \qquad (4.13)$$

whenever $\xi' \neq \xi$. (Explicit models have been constructed by K. Hepp (1972) and by B. Schubnel & J.F. (2014).)

Property (4.13) proves that, although $P_{t_{in}}\big|_{\mathcal{A}_S}$ is pure, $P_t\big|_{\mathcal{A}_S}$ tends to be mixed for $t > t_{in}$, is mixed, for t large enough, and

$$\left.\begin{aligned} &P_t\big|_{\mathcal{A}_S} \longrightarrow P_\infty^X, \text{ as } t \to \infty, \\ \text{with } \quad &P_\infty^X = \sum_{\xi \in \sigma(\hat{X})} \Pi_\xi^X\, P_0\, \Pi_\xi^X, \end{aligned}\right\} \qquad (4.14)$$

i.e., P_∞^X is an incoherent mixture of eigenstates of $X \Leftrightarrow [P_\infty^X, X] = 0$;

and one can now invoke Born's Rule:

$$\text{prob}_{\hat{X}}(\xi / P) = tr_{\mathcal{H}_\mathcal{O}}\left(P_\mathcal{O}\, \Pi_\xi^X\right). \qquad (4.15)$$

Clearly,

$$\text{prob}_{\hat{X}}(\xi / P) \geq 0, \ \forall \xi, \text{ and } \sum_\xi \text{prob}_{\hat{X}}(\xi / P) = 1; \qquad (4.16)$$

$\text{prob}_{\hat{X}}(\xi / P)$ is the probability to find the value ξ in a (very late) measurement of $\hat{X}$, given that the initial state of S is $P = P_E \otimes P_\mathcal{O}$, where P_E satisfies the decoherence condition (4.13).

This type of reasoning goes back to Ludwig. I find it rather silly. For, the "Heisenberg cut" between E and $\mathcal{O}$, i.e., the factorization of $\mathcal{H}$ into $\mathcal{H}_E \otimes \mathcal{H}_\mathcal{O}$, and property (4.9) look rather arbitrary.

4.2. Lindblad dynamics and quantum measurement problem.

We suppose that the "observable degrees of freedom", S_σ, of a system S are coupled to an environment, or measuring equipment, E, as in (4.7), and let $\{U_\lambda(t,s) \mid t,s \text{ in } \mathbb{R}\}$ be the propagator of $E \vee S_\sigma = S$. We suppose that U_λ is generated by a Hamiltonian

$$H_\lambda = H^E \otimes \mathbb{1} + \mathbb{1} \otimes H^\sigma + \lambda V, \quad \lambda \in \mathbb{R}, \qquad (4.17)$$

so that $U_\lambda(t,s) = U_\lambda(t-s) = e^{-i(t-s)H_\lambda}$, ($\hbar = 1$!)

(S is autonomous — this just simplifies our discussion but is not a crucial assumption).

We rescale time: $\tau = \lambda^2 t$ (4.18)

E. B. Davies has shown that, under "reasonable" hypotheses on H_λ and P_E, the following holds:

Let $V_\lambda(t) := U_\lambda(\lambda^{-2} t)$, and, for $X \in B(\mathcal{H}_\sigma)$, $X_t := e^{-i\lambda^{-2} t H^\sigma} X e^{i\lambda^{-2} t H^\sigma}$. Then

$$tr\left(V_\lambda(t) P_E \otimes P_\sigma V_\lambda(-t)\, \mathbb{1} \otimes X_t\right)$$

$$\xrightarrow[\lambda \to 0]{} tr\left(\Lambda_t(P_\sigma)\, X\right) = tr\left(P_\sigma\, \Phi_t(X)\right) \qquad (4.19)$$

where Φ_t is a so-called <u>quantum dynamical semi-group</u> acting on $\mathcal{A}_S = B(\mathcal{H}_0)$, Λ_t is the dual semi-group acting on density matrices, (i.e., positive ops. in $\mathcal{J}_1(\mathcal{H}_0)$ of trace $= 1$).

<u>Definition L</u> (Lindblad)

A <u>quantum-dynamical semi-group</u> is a family of maps $\{\Phi_t\}_{t\geq 0}$, $\Phi_t : \mathcal{A} \to \mathcal{A}$, where $\mathcal{A}$ is a von Neumann algebra containing $\mathbb{1}$, with the following properties:

(i) Φ_t is "<u>completely positive</u>" (CP), $\forall t \geq 0$

(ii) $\Phi_t(\mathbb{1}) = \mathbb{1}$

(iii) $\Phi_t \circ \Phi_s = \Phi_{t+s}$

(iv) $\lim_{t\to 0} \| \Phi_t - id_{\mathcal{A}} \| = 0$ *)

(v) Φ_t is <u>ultraweakly continuous</u> (normal), $\forall t$.

We will now explain the mathematical terms used in this definition.

*) somewhat <u>unphysical</u>, unless $\dim \mathcal{H}_0 < \infty$.

1. The *ultraweak topology* on $\mathcal{A} = \mathcal{B}(\mathcal{H})$ is the weakest topology on $\mathcal{B}(\mathcal{H})$ with the property that all trace-class operators on $\mathcal{H}$ define linear functionals on $\mathcal{B}(\mathcal{H})$ that are continuous in this topology. (It can be defined by the family of semi-norms on $\mathcal{B}(\mathcal{H})$: $|tr(WX)|$, $W \in \mathcal{J}_1(\mathcal{H})$, $\forall X \in \mathcal{B}(\mathcal{H})$.)

2. *Definition CP.* A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras is *completely positive* (CP) iff

$$\Phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$$

takes *positive elements* of $M_n(\mathcal{A}) := \left\{ \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}, \; A_{ij} \in \mathcal{A} \right\}$ (4.20) to positive elements of $M_n(\mathcal{B})$.

The family of CP maps $\Phi : \mathcal{A} \to \mathcal{B}$ is denoted by $CP(\mathcal{A}, \mathcal{B})$. We also use the notations

$$\boxed{CP(\mathcal{A}) := CP(\mathcal{A}, \mathcal{A}), \quad CP(\mathcal{H}) := CP(\mathcal{B}(\mathcal{H})).}$$

Remarks. (a) If $\Phi : \mathcal{A} \to \mathcal{B}$ is *positive* then

$$\Phi(A)^* = \Phi(A)^*, \; \forall A \in \mathcal{A}.$$

(b) If U is a unitary operator on $\mathcal{H}$ then $\Phi_U : X \mapsto U^* X U$, $X \in \mathcal{B}(\mathcal{H})$, belongs to $CP(\mathcal{H})$. Such CP maps are called "pure".

(c) If $\mathcal{A} = C(\mathfrak{X})$ then a positive map $\Phi : \mathcal{A} \to \mathcal{A}$ is automatically CP.

In the following it is always assumed that $\mathcal{A}$ contains an identity, $\mathbb{1}$.

Lemma S (Stinespring)

Let $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, where $\mathcal{A}$ is a C^*-algebra with $\mathbb{1}_{\mathcal{A}}$, with the property that

(i) Φ is linear and positive

(ii) Φ is "normalized", i.e., $\Phi(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{H}}$.

Then Φ is CP iff $\exists$ a *repr. π of $\mathcal{A}$ on a Hilbert space $\mathcal{K}$ and an isometry $V : \mathcal{H} \to \mathcal{K}$ such that

$$\Phi(A) = V^* \pi(A) V, \quad \forall A \in \mathcal{A}.$$

A simple consequence of Lemma S is:

<u>Corollary</u>. Let $\tilde{\mathcal{H}} := \mathbb{C}^n \otimes \mathcal{H}$, $\mathcal{A} = B(\mathcal{H})$, and $\tau : B(\tilde{\mathcal{H}}) \to B(\mathcal{H})$ given by

$$\tau : K \otimes A \to tr_n(K) A,$$

where tr_n is the normalized trace on $M_n(\mathbb{C}) \cong B(\mathbb{C}^n)$. Let U be a unitary operator on $\tilde{\mathcal{H}}$. Then Φ, def. by

$$\Phi(A) := \tau\left(U^* (\mathbb{1}_n \otimes A) U\right), \quad A \in B(\mathcal{H}), \qquad (4.21)$$

belongs to $CP(\mathcal{H})$.

The proof is an exercise.

<u>Lemma K</u> (Kraus)

Let $\Phi \in CP(\mathcal{H})$ be <u>ultraweakly continuous</u>.

Then

$$\Phi(X) = \sum_\alpha V_\alpha^* X V_\alpha, \quad \forall X \in B(\mathcal{H}), \quad (4.22)$$

where V_α and $\sum_\alpha V_\alpha^* V_\alpha$ belong to $B(\mathcal{H})$ and the sum contains at most countably many terms. Conversely, Φ defined by (4.22) is ultraweakly continuous and CP.

$$\Phi \text{ normalized} \iff \sum_\alpha V_\alpha^* V_\alpha = \mathbb{1}.$$

Pf. of Corollary:

$$\mathcal{A} = \mathcal{B}(\mathcal{H}_2), \quad (\mathcal{H}_1 = \mathbb{C}^n)$$

$$\mathcal{H} = Mat_n(\mathbb{C}) \otimes \mathcal{H}_2,$$

with scalar product:

$$\langle K \otimes \psi, L \otimes \varphi \rangle = tr_n(K^* L) \langle \psi, \varphi \rangle$$

normalized trace

π on $\mathcal{H}$ given by

$$\pi(A) K \otimes \varphi := K \otimes A\varphi, \quad \varphi \in \mathcal{H}_2$$

$V : \mathcal{H}_2 \to \mathcal{H}$ given by

$$\psi \mapsto \mathbb{1} \otimes \psi$$

$$\langle K \otimes \varphi, V\psi \rangle$$

$$= \langle K \otimes \varphi, \mathbb{1} \otimes \psi \rangle$$

$$= tr_n(K^*) \langle \varphi, \psi \rangle$$

$\Longrightarrow$ $V^* : \mathcal{H} \to \mathcal{H}_2$ given by

$$V^* : K \otimes \varphi \mapsto tr_n(K)\, \varphi$$

$$V V^* : K \otimes \varphi \to tr_n(K)\, \mathbb{1} \otimes \varphi,$$

Etc. — — — — — — $\square$

See Proposition 2 of Lindblad, Commun. Math. Phys. 48, 119–130 (1976).

Lemma. Let $\Phi \in CP(\mathcal{A})$ be normalized. Then

$$\Phi(A^*A) \geq \Phi(A^*)\cdot\Phi(A), \quad \forall A \in \mathcal{A}. \qquad (4.23)$$

Pf. Use Lemma S, and recall that VV^* is an orthogonal projection, so that $VV^* \leq \mathbb{1}$. $\square$

It follows from (4.23) that, for $\mathcal{A} = \mathcal{B}(\mathcal{H})$,

$$\|\Phi(A)\| \leq \|A\|, \quad \forall A \in \mathcal{B}(\mathcal{H}). \qquad (4.24)$$

Pf. In the following, Ψ is always a unit vector in $\mathcal{H}$.

$$\begin{aligned}
\|\Phi(A)\|^2 &= \sup_{\Psi} \|\Phi(A)\Psi\|^2, \quad (\|\Psi\| = 1) \\
&= \sup_{\Psi} \langle \Psi, \Phi(A)^*\Phi(A)\Psi\rangle \\
&= \sup_{\Psi} \langle \Psi, \Phi(A^*)\Phi(A)\Psi\rangle \\
&\leq \sup_{\Psi} \langle \Psi, \Phi(A^*A)\Psi\rangle \\
\text{Lemma S} \rightarrow\ &= \sup_{\Psi} \langle V\Psi, \pi(A^*A)V\Psi\rangle \\
&\leq \|V\Psi\|^2 \|\pi(A^*A)\| \\
&= \|A\|^2
\end{aligned}$$

$\square$

Let Φ_t be a quantum-dynamical semi-group, in the sense of Definition L. Then

$$\Phi_t = e^{t\mathcal{L}},$$

where, by (iv), $\mathcal{L}: \mathcal{A} \to \mathcal{A}$ is a <u>bounded</u> map, with

$$\lim_{t \to 0} \| \mathcal{L} - t^{-1}\{\Phi_t - id_{\mathcal{A}}\}\| = 0. \qquad (4.24)$$

The generator $\mathcal{L}$ is <u>ultraweakly continuous</u>.

Note that if $\|H\| < \infty$ then $\mathcal{L} := -i\, ad_H$ $\left(ad_H(A) := [H, A]\right)$ generates a quantum-dynamical semi-group.

Next, we want to characterize generators of quantum-dynamical semi-groups more precisely.

<u>Completely dissipative operators.</u>

For simplicity, we only consider the special case $\mathcal{A} = \mathcal{B}(\mathcal{H})$, (although the theory is more general). Let $M_n(\mathcal{A})$ be the algebra of $n \times n$ matrices with matrix elements in $\mathcal{A}$, Let $\{\Phi_t\}_{t\geq 0}$ be a quantum-dynamical semi-group with generator $\mathcal{L}$. We set $\Phi_{t,n} := \Phi_t \otimes \mathbb{1}_n$ and $\mathcal{L}_n := \mathcal{L} \otimes \mathbb{1}_n$, with $\mathcal{L}_n = \frac{d}{dt} \Phi_{t,n}\Big|_{t=0}$.

By (4.23),

$$\frac{d}{dt} \Phi_{t,n}(A^*A)\Big|_{t=0} \geq \frac{d}{dt}\left(\Phi_{t,n}(A^*) \cdot \Phi_{t,n}(A)\right)\Big|_{t=0},$$

$\forall A \in M_n(\mathcal{A})$. Thus

$$\mathcal{L}_n(A^*A) \geq \mathcal{L}_n(A^*) \cdot A + A^* \cdot \mathcal{L}_n(A), \quad \forall A \in M_n(\mathcal{A}), \tag{4.25}$$

and conversely.*) This motivates the following

<u>Definition D</u> (dissipation function)

Let $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ be bounded. We define

$$D_n(\mathcal{L}; A, B) := \mathcal{L}_n(A^*B) - \mathcal{L}_n(A^*) \cdot B - A^* \mathcal{L}_n(B), \tag{4.26}$$

for A, B in $M_n(\mathcal{A})$, which is a <u>sesquilinear map</u> from $M_n(\mathcal{A}) \times M_n(\mathcal{A})$ to $M_n(\mathcal{A})$.

<u>Remark.</u> $\mathcal{L}_n$ is a *map, i.e., $\mathcal{L}_n(A)^* = \mathcal{L}_n(A)$.

For $\mathcal{L}_n(A)^* = \frac{d}{dt} \Phi_{t,n}(A)^*\Big|_{t=0} = \frac{d}{dt} \Phi_{t,n}(A^*)\Big|_{t=0}$

$$= \mathcal{L}_n(A^*),$$

using that $\Phi_{t,n}$ is positive on $M_n(\mathcal{A})$, hence a *map.

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* G. Lindblad, Commun. Math. Phys. <u>48</u>, 119–130 (1976).

Note that

$$D_1(\mathcal{L}; A, B) = 0,\ \forall A, B \Leftrightarrow \mathcal{L} = ad_H; \qquad (4.27)$$

see Lindblad, loc. cit.

Definition CD. A bounded *map $\mathcal{L}: \mathcal{A} \to \mathcal{A}$, with $\mathcal{L}(\mathbb{1}) = 0$ and

$$D_n(\mathcal{L}; A, A) \geq 0,\ \forall A \in M_n(\mathcal{A}),\ \forall n, \qquad (4.28)$$

is called completely dissipative (CD), and

$$CD(\mathcal{A}) := \{\mathcal{L}: \mathcal{A} \to \mathcal{A} \mid \mathcal{L} \text{ is } CD\}$$

$CD(\mathcal{A})_\sigma$ denotes the class of CD maps: $\mathcal{A} \to \mathcal{A}$ that are ultraweakly continuous.

Proposition L (Lindblad, loc. cit.)

Let $\Phi_t = e^{t\mathcal{L}}$, with $\mathcal{L}: \mathcal{A} \to \mathcal{A}$ a bounded *map.

Then

$$\Phi_t \in CP(\mathcal{A}),\ \Phi_t(\mathbb{1}) = \mathbb{1} \Leftrightarrow \mathcal{L} \in CD(\mathcal{A}).$$

This is an immediate consequence of

$$\Phi_t(A^*A) \geq \Phi_t(A^*)\Phi(A),\ \forall A \in \mathcal{A},\ \Phi_t(\mathbb{1}) = \mathbb{1}, \quad \forall t \Leftrightarrow$$

$$D_1(\mathcal{L}; A, A) \geq 0, \ \forall A \in \mathcal{A}, \ \mathcal{L}(\mathbb{1}) = 0.$$

(see Proposition 4 in Lindblad, loc. cit.)

<u>Theorem L</u> (Lindblad, loc. cit.)

(1) If $\mathcal{L} \in CD(\mathcal{A})_\sigma$ then $\exists \ \Psi \in CP(\mathcal{A})_\sigma$ and an operator $H = H^* \in \mathcal{A}$ such that, $\forall X \in \mathcal{A}$,

$$\mathcal{L}(X) = i\,ad_H(X) + \Psi(X) - \frac{1}{2}\{\Psi(\mathbb{1}), X\}, \tag{4.29}$$

where $\{A, B\} := AB + BA$

(2) If $\Psi \in CP(\mathcal{A})$, $H = H^* \in \mathcal{A}$, then the operator $\mathcal{L}$ defined in (4.29) is in $CD(\mathcal{A})$.

<u>Pf.</u> See Reuteler, or Lindblad. (In the proof one makes use of Lemma S, by Stinespring.)

<u>Corollary</u>.

If $\mathcal{L} \in CD(\mathcal{A})_\sigma$, $\mathcal{A} = B(\mathcal{H})$ then

$$\mathcal{L}(X) = i\,ad_H(X) + \sum_\alpha \left[V_\alpha^* X V_\alpha - \frac{1}{2}\{V_\alpha^* V_\alpha, X\} \right], \tag{4.30}$$

$\forall X \in \mathcal{A}$, where $H = H^* \in \mathcal{A}$, $V_\alpha \in \mathcal{A}$, $\forall \alpha$, and $\sum_\alpha V_\alpha^* V_\alpha \in \mathcal{A}$; and <u>conversely</u>.

This follows from Theorem L and Lemma K.

The generator, L, of the semi-group

$$\Delta_t : \mathcal{J}_1(\mathcal{H}) \to \mathcal{J}_1(\mathcal{H})$$

dual to the quantum-dynamical semi-group $\Phi_t = e^{t\mathcal{L}}$ on $B(\mathcal{H})$ is then given by

$$L(P) = -i\,ad_H(P) + \frac{1}{2}\sum_\alpha \left([V_\alpha P, V_\alpha^*] + [V_\alpha, P V_\alpha^*] \right),$$

for any $P \in \mathcal{J}_1(\mathcal{H})$, in particular, $\qquad (4.31)$

$P =$ arb. density matrix.

We say that $\mathcal{L} \in CD(\mathcal{A})_6$ is pure if the sum over α in (4.30) contains only <u>one</u> term; i.e.,

$$\mathcal{L}(X) = i\,ad_H(X) + V^* X V - \frac{1}{2}\{V^*V, X\} \qquad (4.32)$$

<u>Special cases.</u>

- $\mathcal{L}(X) = VXV - \frac{1}{2}\{V^2, X\}, \quad V = V^*$ (s.a.)

  ("<u>Gaussian semi-groups</u>")

- $\mathcal{L}(X) = V^* X V - X, \quad V^* = V^{-1}$ (unitary)

  ("<u>Poisson semi-groups</u>")

<u>Properties of time evolutions described by quantum-dynamical semi-groups</u> (after Baumgartner & Narnhofer, arXiv: 0806.3164, June 2008)

We consider a quantum-dynamical semi-group, $\{\Phi_t \mid \Phi_t \in CP(\mathcal{H}),\ \forall t \geq 0\}$, and its dual $\{\Lambda_t \mid \Lambda_t : \mathcal{T}_1(\mathcal{H}) \to \mathcal{T}_1(\mathcal{H})$, trace-preserving, $\forall t \geq 0\}$.

We are interested in the following questions

(1) Existence and characterization of stationary states for $\{\Lambda_t\}_{t\geq 0}$;

(2) Properties of paths $\{\Lambda_t(P)\}_{t\geq 0}$, where P is a density matrix.

<u>Remark.</u> In (4.30), we can replace

$$V_\alpha \longrightarrow V_\alpha - z_\alpha \mathbb{1},$$

for all α, by changing H:

$$H \to H + \frac{i}{2}\sum_\alpha \left[\bar{z}_\alpha V_\alpha - z_\alpha V_\alpha^*\right] =: \widetilde{H}$$

<u>Pf.</u> $$\mathcal{L}(X) = i\,ad_H(X) + \sum_\alpha \left[V_\alpha^* X V_\alpha - \frac{1}{2}\{V_\alpha^* V_\alpha, X\}\right] \qquad (4.30)$$

$$\stackrel{2}{=} i\,ad_{\tilde{H}}(X) + \sum_{\alpha} \Big[(V_\alpha^* - \bar{z}_\alpha \mathbb{1}) X (V_\alpha - z_\alpha \mathbb{1}) - \frac{1}{2} \{ (V_\alpha^* - \bar{z}_\alpha \mathbb{1})(V_\alpha - z_\alpha \mathbb{1}), X \} \Big].$$

Equality is verified by evaluating the R.S. $\square$

Thus, we can require that $tr(V_\alpha) = \overline{tr(V_\alpha^*)} = 0$ in (4.30). We should also require the operators V_α in (4.30) to be linearly independent: Let the sum in (4.30) have p terms, i.e., α ranges over $1, \cdots, p$. We then require that

$$\Big(tr\big(V_\alpha^* V_\beta\big) \Big)_{\alpha, \beta = 1, \cdots, p}$$

are the matrix elements of a positive-definite $p \times p$ matrix. The following proposition is due to Spohn.

Proposition S. Let $N < \infty$ and let $p > N^2 - 1 - \frac{N}{2}$. Then there exists a density matrix P_∞ such that

$$\lim_{t \to \infty} \Lambda_t(P) = P_\infty, \quad \forall P \text{ (DM)}; \qquad (4.33)$$

the approach to P_∞ is exponentially fast.

If we want to achieve a result such as (4.14) this result is <u>disappointing</u>: The P^X_∞ in (4.14) is <u>not</u> unique. All we would like to have is that

$$\left[\lim_{t\to\infty} \Lambda_t(P), X\right] = 0 \qquad (4.34)$$

if $\{\Lambda_t\}_{t\geq 0}$ is supposed to describe an asymptotic measurement of $X = X^* \in \mathcal{A}_S$; i.e., we would only like to have "<u>dephasing</u>".

To see whether this is possible we invoke a general result due to Baumgartner and Narnhofer.

<u>Theorem BN-1.</u>

Let us assume for simplicity that $\mathcal{H} = \mathbb{C}^N$, $N < \infty$, and that $\Lambda_t = e^{Lt}$, $t \geq 0$, with L as in (4.31). Then the following statements hold:

(1) <u>Decay</u>: There exists a minimal orthogonal projection, P_0, on $\mathcal{H}$, with $P_0^\perp := \mathbb{1} - P_0$, such that

$$\lim_{t\to\infty} P_0^\perp \Lambda_t(P) P_0^\perp = 0 \,, \quad \forall P \in \mathcal{J}_1^+, \quad (4.35)$$

($\mathcal{J}_1^+$ = pos. trace-class ops. on $\mathcal{H}$).

In the following $P_0^\perp$ is the maximal projection for which (4.35) holds; i.e. if $Q \leq P_0$, $Q > 0$, then $\exists\, P \in \mathcal{J}_1^+$ such that

$$\limsup_{t\to\infty} \operatorname{tr}\left(\Lambda_t(P)\, Q\right) \neq 0 \qquad (4.36)$$

(2) Dephasing: The subspace $P_0\mathcal{H} \subseteq \mathcal{H}$ can be decomposed in a unique way into a direct sum

$$P_0\mathcal{H} = \bigoplus_k \text{Range } Q_k \qquad (4.37)$$

of subspaces $\mathcal{H}_k := \text{Range } Q_k$, such that

$$Q_k \cdot Q_l = \delta_{kl}\, Q_k \qquad (4.38)$$

$$\lim_{t\to\infty} Q_k \Lambda_t(P)\, Q_l = 0, \quad k \neq l, \quad (4.39)$$

the projections Q_k, $k = 1, 2, \dots$, are minimal if (4.38) and (4.39) are imposed, [and the time evolutions of the blocks $Q_k\, \Lambda_t(P)\, Q_l$ are "independent" of each other.]

(3) <u>Asymptotic dynamics</u>: Each $\mathcal{H}_k$ is a tensor product

$$\mathcal{H}_k \equiv Q_k \mathcal{H} = \mathbb{C}^{n(k)} \otimes \mathcal{H}_{0k},$$

for some $n(k) = 1, 2, \cdots$, and there exist self-adjoint Hamiltonians $H_k \otimes \mathbb{1}$ and density matrices Σ_k on $\mathcal{H}_{0k}$ such that, <u>for every density matrix</u> P (<u>initial condition</u>), $\exists \{\lambda_k, P_{\infty,k}\}$, with $0 \leq \lambda_k \leq 1$, $\sum_k \lambda_k = 1$, $P_{\infty,k}$ a density matrix on $\mathbb{C}^{n(k)}$, such that

$$\lim_{t\to\infty} \Lambda_t(P) = \sum_k Q_k \left(e^{-iH_k t} P_{\infty,k} e^{iH_k t} \otimes \Sigma_k \right) Q_k \tag{4.40}$$

Stationary states have the property that

$$[H_k, P_{\infty,k}] = 0, \quad \forall k.$$

Furthermore

$$\mathrm{Rank}(\Sigma_k) = \dim \mathcal{H}_{0k}, \quad \forall k. \tag{4.41}$$

("<u>Dissipation</u>").

(4) <u>Basins</u>: The "decaying subspace" $P_0^{\perp} \mathcal{H}$ can be

split into a direct sum of "basins"

$$P_0^\perp = \sum_k P_k = \sum_{k\geq 1}\sum_l P_{kl},$$

with $P_{kl} \cdot P_{jm} = \delta_{kj}\, \delta_{lm}\, P_{kl}$ (orthogonality, or "disjointness") such that

$$P_j \Lambda_t (P_{kl} P P_{kl}) P_j \neq 0, \quad j < k$$

$$P_j \Lambda_t (P_{kl} P P_{kl}) P_j = 0, \quad j > k$$

$$P_{km} \Lambda_t (P_{kl} P P_{kl}) P_{km} = 0, \quad m \neq l$$

The time evolution of any P is out of the ranges of the projections P_{kl} into the range of a projection $P_{0l} \leq P_0$, $\forall\, k$ and l, and every Q_k is a sum of $n(l)$ "basins" P_{0l}.

Suppose that there are mutually orthogonal ("disjoint") projections, $\{\Pi_\xi\}_{\xi \in \mathcal{X}}$, such that for L as in (4.31),

$$[\Pi_\xi, H] = 0, \quad [\Pi_\xi, V_\alpha] = 0, \ \forall \alpha \qquad (4.42)$$

for all ξ. By (4.31) and (4.42), we obviously have that

$$\Lambda_t(\Pi_\xi P \Pi_{\xi'}) = \Pi_\xi \Lambda_t(P) \Pi_{\xi'}, \quad \forall t, \qquad (4.43)$$

for arbitrary ξ, ξ'. Moreover, one has the following

Proposition BN-2.

The "basins" $P_{k\ell}$, $k \geq 0$, in Theorem BN-1 can be chosen in such a way that every projection Π_ξ is a sum of some of the projections $P_{k\ell}$.

Application to "Measurement Theory":

Let X be an "observable" of S,

$$\mathcal{A}_S = B(\mathcal{H}_0), \quad X = X^* \in \mathcal{A}_S,$$

with time evolution of density matrices on $\mathcal{H}_0$ given by $\Lambda_t = e^{Lt}$, L as in (4.31).

Suppose the spectral projections $\{\Pi_\xi\}_{\xi \in \mathcal{X} \equiv \sigma(X)}$ of X satisfy (4.42); assume that L is such that $P_0^\perp = 0$, where $P_0^\perp$ is the projection appearing in statement (1) of Theorem BN-1.

By Theorem BN-1, (4), every Q_k is a sum of "basins" P_{0j}, $j \in \mathcal{J}_k$, and, by Proposition BN-2, every Π_ξ is a sum of "basins" P_{0j}, $j \in \mathcal{J}_\xi$. Since the projections P_{0j} are <u>disjoint</u> (see Theorem BN-1, (4)) they all commute with each other. Thus

$$\boxed{[\Pi_\xi, Q_k] = 0, \quad \forall \xi \in \sigma(X), \forall k.} \qquad (4.44)$$

By (4.43),

$$\Lambda_t(\Pi_\xi P \Pi_{\xi'}) = \Pi_\xi \Lambda_t(P) \Pi_{\xi'}, \qquad (4.45)$$

$\forall \xi, \xi'$. As $t \to \infty$,

$$\text{L.S. of } (4.45) \to \sum_k Q_k (\Pi_\xi P \Pi_{\xi'})_{\infty,t} Q_k$$

$$\text{R.S. of } (4.45) \to \sum_k \Pi_\xi Q_k P_{\infty,t} Q_k \Pi_{\xi'}$$

$$\overset{(4.44)}{=} \sum_k Q_k \Pi_\xi P_{\infty,t} \Pi_{\xi'} Q_k, \qquad (4.46)$$

where $(\cdot)_{\infty,t}$ is given in (4.40), for arbitrary ξ, ξ'. If $n(k) = 1$, then (4.46) vanishes unless $\xi = \xi'$ and $\Pi_\xi \geq Q_k$.

We conclude that

$$\left[\lim_{t\to\infty} \Lambda_t(P), X\right] = 0 \qquad (4.47)$$

Recalling (4.1), this implies that, asymptotically, for very large time, the observable X is measured. Notice, however, that (4.47) is only provable for very special choices of the Lindblad generator L that depend on X. One might say that X is asymptotically measurable provided E, i.e., the equipment determining L, is chosen so as to enable one to measure X.

If $n(k) > 1$, for some k, then there might be no $\xi \in \sigma(X)$ such that $Q_k \leq \Pi_\xi$. Instead

$$Q_k = \sum_{\xi \in \mathcal{X}_k \subseteq \sigma(X)} \Pi_\xi \, ,$$

and only an appropriately coarse-grained version of X will be measured, as $t \to \infty$.

4.3 State vector collapse – Bassi, Dürr & Hinrichs

It has been suggested by Ghirardi, Rimini and Weber that, in order to solve the quantum measurement problem and explain the emergence of unambiguous measurement results, one could modify the linear Schrödinger time evolution of state vectors, ψ, in a Hilbert space $\mathcal{H}$ by inserting non-linear drift terms and stochastic noise terms in the Schrödinger equation. It was shown by Gisin and elaborated upon by Polchinski that any non-linear, but deterministic modification of the Schrödinger equation entails superluminal signaling. Thus, only stochastic non-linear modifications might be viable. Collapse models are based on such modifications. In collapse models, the Schrödinger equation for the evolution of state vectors, ψ_t, in time t is replaced by a non-linear stochastic differential equation of the form*)

$$d\psi_t = \Big[-iH\,dt + \sum_{\alpha=1}^{n} \big(V_\alpha^* - v_\alpha(\psi_t)\big)\,dW_\alpha(t)$$
$$-\frac{1}{2}\sum_{\alpha=1}^{n}\Big(V_\alpha V_\alpha^* - 2\,v_\alpha(\psi_t)V_\alpha^* + v_\alpha(\psi_t)^2\Big)dt\Big]\psi_t, \qquad (4.48)$$

where

$$v_\alpha(\psi) := \frac{1}{2}\big\langle \psi, (V_\alpha^* + V_\alpha)\psi \big\rangle. \qquad (4.49)$$

Here, H is the usual Hamiltonian of the system, V_α, $\alpha = 1, \dots, n$, are linear operators determining the "preferred basis" for the collapse, and $W_\alpha(t)$, $\alpha = 1, \dots, n$ are Wiener processes on the real line. (See Bassi & Ghirardi.) Eqs. (4.48), (4.49) can be generalized by, for example, considering other kinds of noise, such as Poissonian noise. But the general structure of the equations resemble

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*) Note that $\|\psi_t\|^2 = \langle \psi_t, \psi_t \rangle$ is preserved by (4.48), (4.49).

that of (4.48).

Bassi, Dürr and Hinrichs have succeeded in classifying all Markovian evolutions of state vectors [without superluminal signaling satisfying a certain condition (4.51), below]. Here we present a sketch of their result.

Theorem BDH-1

We consider a finite-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^d$. Let $\{\psi_t\}_{t \geq 0}$ be a trajectory of normalized state vectors in $\mathcal{H}$, $(\|\psi_t\| = 1, \forall t)$, and let $\Pi_{\psi_t} = |\psi_t\rangle\langle\psi_t|$ denote the orthogonal projection onto $[\psi_t]$. We assume that the time evolution of the system is such that

(1) The time-dependence of $\{\psi_t\}_{t \geq 0}$ is time-homogeneous and Markovian.

(2) The time-dependence of Π_{ψ_t} is time-homogeneous and Markovian.

Let $\mathbb{E}$ denote an expectation with respect to the measure on path space, Ξ, determined by the Markov process that describes the stochastic evolution of the state vectors in $\mathcal{H}$; with

$$\Xi := \left\{ \{\psi_t\}_{t \geq 0} \,\middle|\, \psi_t \in \mathcal{H}, \forall t \right\} \left(= \bigtimes_{t \geq 0} \mathcal{H}_t \right) \qquad (4.50)$$

Let us consider the time evolution of a density matrix P, with

$$P = \sum_i \lambda_i \Pi_{\chi_i} = \sum_j \mu_j \Pi_{\phi_j} ,$$

$\lambda_i, \mu_j \geq 0$, $\forall i, j$, $\operatorname{tr} P \stackrel{!}{=} 1$. We assume that

$$(3) \quad P_t = \sum_i \lambda_i \mathbb{E}\left[\Pi_{\chi_{i,t}} \,\middle|\, \chi_{i,s} = \chi_i \right]$$

$$\qquad (4.51)$$

$$= \sum_j \mu_j \mathbb{E}\left[\Pi_{\phi_{j,t}} \,\middle|\, \phi_{j,s} = \phi_j \right], \quad s < t.$$

Then there exists a time-homogeneous quantum-dynamical semi-group $\{\Lambda_t = e^{Lt}\}_{t \geq 0}$ such that

$$P_t = \Lambda_{t-s}(P_s), \qquad (4.52)$$

Thus

$$\frac{dP_t}{dt} = L(P_t) = -i[H, P_t]$$
$$+ \sum_{\alpha=1}^{n} \left[V_\alpha^* P_t V_\alpha - \frac{1}{2}\{V_\alpha V_\alpha^*, P_t\}\right] \quad (4.53)$$

Remarks. (i) Eq. (4.51), condition (3), is a consequence of a result of Gisin that says that (4.51) is a necessary condition for the absence of super-luminal signaling. It implies linearity of the time evolution of the density matrices P.

(ii) The complete positivity of the linear maps Λ_t is a consequence of the Markovian evolution of the state vectors ψ_t.

(iii) Theorem BDH-1 raises the following problem: Given a diffusion process for the state vectors ψ_t, with

$$d\psi_t = \left[A(\psi_t)\,dt + \sum_{\alpha=1}^{N} B_\alpha(\psi_t)\,dW_\alpha(t)\right]\psi_t, \quad (4.54)$$

find the conditions on $A(\psi), B_1(\psi), \cdots, B_n(\psi)$, that guarantee that <u>statistical mixtures</u> (DM)

$$P_t = \sum \lambda_i \, \mathbb{E}\left[\Pi_{\psi_t}\right]$$

satisfy (4.52), (4.53).

<u>Theorem BDH-2.</u>

We assume that the operators $\mathbb{1}, V_1, \dots, V_n$ (see (4.53)) are linearly independent. Then Eqs. (4.52), (4.53) follow from Eq. (4.54) iff

- $N \geq n$, and
- $A(\psi) = -iH - \frac{1}{2}\sum_{\alpha=1}^{N}\left\{V_\alpha V_\alpha^* - 2v_\alpha(\psi)V_\alpha^*(\psi) + v_\alpha(\psi)^2\right\}$,

  with $V_{n+1} = 0, \dots, V_N = 0$, and
- $B_\alpha(\psi) = V_\alpha^*(\psi) - v_\alpha(\psi)$,

  where

$$v_\alpha(\psi) = \frac{1}{2}\langle \psi, \left(V_\alpha(\psi)^* + V_\alpha(\psi)\right)\psi\rangle,$$

  and

$$V_\alpha(\psi) = \sum_{\beta=1}^{n} u_{\alpha\beta}(\psi)\, V_\beta,$$

  where $\{u_{\alpha\beta}(\psi)\}$ are the matrix elements of a unitary $N \times N$ matrix, $U(\psi)$.

<u>Remarks.</u> (i) The ambiguity encapsulated in $\mathcal{U}(\psi)$ leaves the Lindblad equation (4.53) unchanged, but affects its <u>"unravelling"</u> in the form of Eq. (4.54). The form (4.48) of (4.54) leads to a maximal <u>collapse rate.</u>

(ii) Theorems BDH-1 and BDH-2 tell us that the "usual collapse models" are the only non-linear Markovian extensions of the Schrödinger equation compatible with the Lindblad Eq. (4.53), [~~hence with the absence of superluminal signaling, (by Gisin's result~~)].

<u>Example.</u>

$$\boxed{n = N = 1, \quad V = V_{\alpha=1} = V^{*}.}$$

Then

$$d\psi_t = -iH\psi_t\, dt + \Big\{\big(V - v(\psi_t)\big)\psi_t\, dW(t)$$

$$- \frac{1}{2}\Big[\big(V - v(\psi_t)\big)^2 \psi_t\, dt\Big\}, \qquad (4.55)$$

with $\quad v(\psi) = \langle \psi, V\psi \rangle$

The corresponding Lindblad equation for the statistical mixture, P_t, is then

$$\frac{dP_t}{dt} = -i[H, P_t] + VP_tV - \frac{1}{2}\{V^2; P_t\}$$

$$= -i\, ad_H(P_t) - \frac{1}{2}[V,[V,P_t]]. \quad (4.56)$$

$\Rightarrow$ P_t evolves towards an incoherent superposition of spectral projections of the operator V! (See, e.g., Gisin!)

Taking $\mathcal{H} = L^2(\mathbb{R}, dx)$, $H = -\frac{1}{2m}\frac{d^2}{dx^2}$, $V = x$, $\quad (4.57)$

we obtain a <u>standard collapse model.</u>

<u>Remarks.</u> (i) Consider a collapse model mapping an arbitrary initial density matrix, P, asymptotically to a state $P_\infty = \lim_{t\to\infty} \Lambda_t(P)$ given by

$$P_\infty = \sum_{\xi \in \mathcal{X}} p_\xi \frac{\Pi_\xi P \Pi_\xi}{tr(P\Pi_\xi)}, \quad (4.58)$$

where $\Pi_\xi = \Pi_\xi^2 = \Pi_\xi^*$ is an orthogonal projection, $\forall \xi \in \mathcal{X}$, with $\Pi_\xi \cdot \Pi_\eta = \delta_{\xi\eta}\Pi_\xi$, and the coefficients $p_\xi \geq 0$, with $\sum p_\xi = 1$. Since $P_\infty = \lim_{t\to\infty} \Lambda_t(P)$

(depends) linearly on the initial condition P, we conclude that

$$p_\xi = tr\left(P\Pi_\xi\right), \tag{4.59}$$

(see (4.58)), which can be interpreted as Born's Rule. Apparently, Born's Rule is the only "probability rule" in collapse models compatible with [the absence of superluminal signaling] a linear dependence of P_∞ on P.

(ii) Eq. (4.54) determines a Markovian stochastic diffusion process on the unit sphere in the Hilbert space $\mathcal{H}$. This process is uniquely characterized by its transition function

$$\Gamma_t(\psi, d\varphi) = p_t(\psi - \varphi)\, d\varphi,$$

where $d\varphi$ is the uniform measure on $S^1(\mathcal{H})$ and p_t satisfies a Fokker–Planck equation, (heat equation with drift; see Bassi–Dürr–Hinrichs).

Remark on the issue of superluminal signaling

We assume that time evolution of states is given by a flow map Φ_t on state vectors in $\mathcal{H}$:

$$\psi \longmapsto \psi_t = \Phi_t(\psi)$$

Let us ask under what conditions on Φ_t this flow can be lifted to density matrices, P_0, in a unique fashion. The problem is that P_0 can be written as a convex combination of rank-1 projections in many ways:

$$P_0 = \sum_i \lambda_i \, |\psi^i\rangle\langle\psi^i|$$
$$= \sum_j \mu_j \, |\varphi^i\rangle\langle\varphi^i| \qquad (4.60)$$

Claim.

If $$\sum_i \lambda_i \, |\psi^i_t\rangle\langle\psi^i_t| = \sum_j \mu_j \, |\varphi^i_t\rangle\langle\varphi^i_t|, \qquad (4.61)$$

whenever (4.60) holds then the map Φ_t determines a linear map, Λ_t, on density matrices: If $P = \lambda P_1 + (1-\lambda) P_2$, with

$$P_i = \sum_j \lambda_j^i \, |\psi^{j,i}\rangle\langle\psi^{j,i}| \, , \quad i = 1, 2,$$

then

$$P = \sum_{i,j} \mu_j^i \, |\psi^{j,i}\rangle\langle\psi^{j,i}| \, , \quad \text{with}$$

$$\mu_j^1 = \lambda \lambda_j^1 \, , \quad \mu_j^2 = (1-\lambda) \lambda_j^2 \, .$$

If time evolution, $\Lambda_t(P)$, of P is given by and satisfies (4.61) then it follows that

$$\Lambda_t(P) = \lambda \Lambda_t(P_1) + (1-\lambda) \Lambda_t(P_2),$$

and we may conclude linearity.

But what (4.61) has got to do with the absence of superluminal signaling remains mysterious, because Gisin's arguments are shaky.

The canonical unraveling of the Lindblad Eq.:

- Explain for discretized time, general n. → Chap. 5
- Example $n=2$; Poisson process on Bloch sphere; (with J. Faupin)

In light of Theorems S and BN-1, Prop. BN-2, NONE of this will provide a satisfactory account of interesting sequences of events in isolated systems.

Quantum Mechanics for the "kindergarden"

4.4. Wigner's Friend Paradoxon

Consider a lab containing a silver atom gun emitting a silver atom with spin in state

$$\psi_{atom} = |\rightarrow\rangle = \frac{1}{\sqrt{2}}\left\{|\uparrow\rangle + |\downarrow\rangle\right\} \qquad (4.62)$$

polarized in the $+x$-direction. Wigner's friend, F, is in the lab and measures the spin of the atom in the z-direction, using, e.g., a Stern-Gerlach experiment (equipment E). If the atom is initially in state ψ_{atom} specified in Eq. (4.62) then F predicts that, after the measurement, the spin of the silver atom will be in state

$$\left.\begin{array}{ll} & |\uparrow\rangle \quad \text{with probability } \frac{1}{2}\,, \\ \underline{\text{or}} & |\downarrow\rangle \quad \text{— " — — " —} \quad \frac{1}{2}\,. \end{array}\right\} \qquad (4.63)$$

The superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$ predicted by F for the state of the atom *after* the Stern-Gerlach experiment is *incoherent*. In fact, before F knows the outcome for the spin measurement has been, F would work with the state

$$\rho^{F}_{lab} = \frac{1}{2}|\Psi^{\uparrow}\rangle\langle\Psi^{\uparrow}| + \frac{1}{2}|\Psi^{\downarrow}\rangle\langle\Psi^{\downarrow}|, \tag{4.64}$$

where

$$\left.\begin{aligned} \Psi^{\uparrow} &:= |\uparrow\rangle \otimes |E^{\uparrow}\rangle \otimes |F^{\uparrow}\rangle \\ \text{and} \quad \Psi^{\downarrow} &:= |\downarrow\rangle \otimes |E^{\downarrow}\rangle \otimes |F^{\downarrow}\rangle, \end{aligned}\right\} \tag{4.65}$$

where $|E^{\updownarrow}\rangle$ is the state of the Stern-Gerlach equipment and $|F^{\updownarrow}\rangle$ is the state of Wigner's friend *after* completion of the spin measurement, provided its outcome is $\updownarrow \in \{\uparrow, \downarrow\}$.

Wigner, however, believes that the evolution of the state of the lab is determined by a unitary propagator. He therefore predicts that the state of the lab *after* the Stern-Gerlach experiment is a coherent superposition

$$P^W_{lab} = |\Psi_{out}\rangle\langle\Psi_{out}|, \qquad (4.66)$$

with

$$\Psi_{out} := \frac{1}{\sqrt{2}}\{\Psi^{\uparrow} + e^{i\theta}\,\Psi^{\downarrow}\}, \qquad (4.67)$$

where $e^{i\theta}$ is a relative phase determined by Wigner's propagator. The vectors $\Psi^{\uparrow}$ and $\Psi^{\downarrow}$ belong to orthogonal subspaces, $\mathcal{H}^{\uparrow}$ and $\mathcal{H}^{\downarrow}$, of $\mathcal{H}_{lab} = \mathcal{H}_{atom} \otimes \mathcal{H}_E \otimes \mathcal{H}_F$.

Now, Wigner is a very clever and powerful man; he invents an experiment measuring an observable M that has non-vanishing matrix elements between $\mathcal{H}^{\uparrow}$ and $\mathcal{H}^{\downarrow}$. If well designed, a measurement of M, assuming the lab is in state P^W_{lab} given in

(4.66), (4.67) can reveal information on the phase factor $e^{i\theta}$ through interference effects, while a measurement of M, assuming the lab is in the <u>mixed</u> state P^F_{lab} of Eq. (4.64), does <u>not</u> show any interference effects and is therefore entirely insensitive to $e^{i\theta}$!

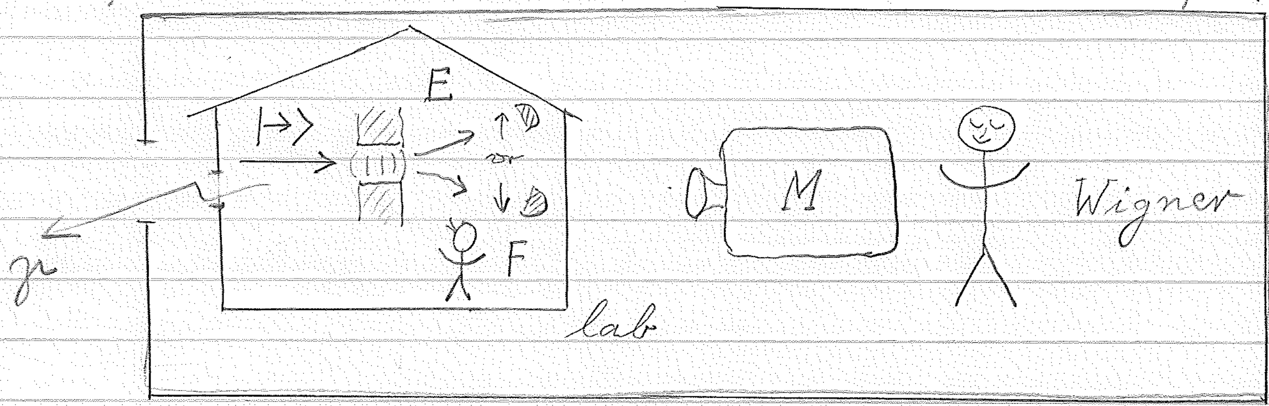
We believe that <u>Wigner will never be able to perform an experiment measuring</u> $e^{i\theta}$.

This leads to the conclusion:

(1) Time evolution of lab not given by the unitary propagator of Wigner; and/or

(2) "Observables" with the property of M, measuring the phase $e^{i\theta}$, do not exist.

If the state of the lab <u>after</u> the spin measurement is <u>either</u> $\Psi^{\uparrow}$ <u>or</u> $\Psi^{\downarrow}$ then (1) is undoubtedly correct! We will see that the reason <u>why</u> it is correct is related to the fact that observables with the property of M do not exist, i.e., to the truth of (2).



<u>Isolated</u>, but <u>open</u> system

A comment on the "many-worlds interpretation" of Quantum Mechanics

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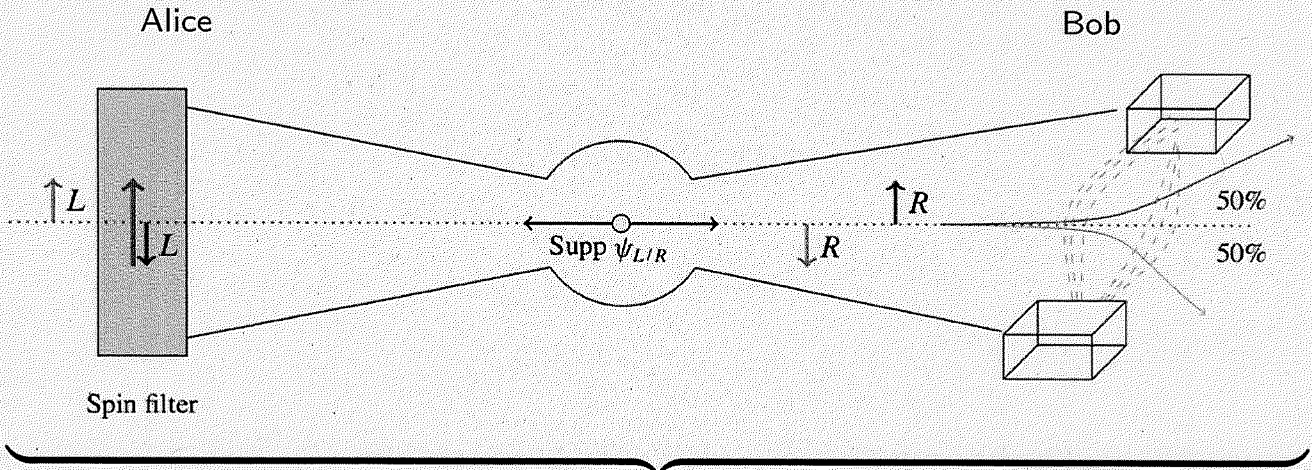
Quantum theory cannot be fully predictive, because ...

4.5 Bell-type measurements cannot be described by Schrödinger Eq.

A Gedanken-Experiment (↗ Faupin-F-Schubnel):

Two particles (electrons), P and P', prepared in a spin-singlet initial state, $\psi_{L/R}$,[2] with orbital wave functions chosen such that P propagates into the cone opening to the right, while P' propagates into the cone opening to the left and ending in the spin filter, (except for very tiny tails leaking beyond those cones).

Time evolution of P essentially *independent* of the one of Q, which includes P' and spin filter – consequence of cluster props. of propagator!



Q:={spin filter ∨ particle P'} cone opening to R:= ess. supp of orbital wave function of P

[2] nowadays called a *Bell pair of Qbits*

… Quantum theory is fundamentally probabilistic –

in spite of the *deterministic nature of the Schrödinger Eq.!* –
Temporary assumptions (leading to a contradiction):

I. P and P': Spin-$\frac{1}{2}$ particles prepared in a *spin-singlet initial state*; spin filter prepared in a poorly known initial state not (necessarily) entangled with initial state of P' and P.

II. *Dynamics of state of total system fully determined by Schrödinger equation.* In particular, initial state of spin filter determines whether P' will pass through it or not, (given that the initial state of $P' \vee P$ is a spin-singlet state, with P' and P moving into opposite cones).

III. *Correlations between outcomes of spin measurements of P' and of P are as predicted by standard quantum mechanics*, (relying on the "Copenhagen interpretation") – "non-locality" of QM.

Fact: Assuming short-range interactions, Schrödinger evolution of state of system factorizes into free evolution of P tensored with complicated evolution of $Q := \{P' \vee \text{ spin filter}\}$, up to tiny errors. This follows from our choice of initial conditions & cluster properties of time evolution! Hence *spin of P is ess. conserved before measurement!* $\Rightarrow$

Time evolution of "observables" and of states in QM

If assumptions I. and II. held then:
Expectation value of spin of $P \approx 0, \forall$ times! $\Rightarrow$ State of spin of P' after interaction of P' with spin filter does **not** bias state of spin of P when measured, (e.g., in a Stern-Gerlach exp.)!

This consequence of the first two assumptions **contradicts** *the third (last) assumption stated above!*

Thus, if the usual *correlations* between two "independent" measurements (here of z-comp. of spins of P' and of P), predicted on the basis of the projection postulate of "Copenhagen", are observed[3] then it follows that the Schrödinger equation *cannot* describe the evolution of states of systems, and hence that qm dynamics is *fundamentally stochastic*.

It turns out that one may safely assume the validity of the *Heisenberg-picture evolution* of "observables" for *isolated systems* (define!), which is perfectly *deterministic*. But, in Quantum Mechanics, the *evolution of states is stochastic*. $\Rightarrow$ Equivalence of the *Heisenberg picture* and the *Schrödinger picture* is an erroneous claim!

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[3]as suggested by the experiments of *Aspect, Gisin*, and others

4.6

1

~~4.12~~ Is Quantum Probability Th. ≡ Class. Probability Theory?

And – if not – how does it differ?

Classical

<u>Class. (topol.) dynamical syst.</u>:

M: (cp. topol.) state space;
σ-alg., Σ, of Borel sets.

$\mathcal{A} := C(M)$

$\{\tau_{t,s}\}_{t,s\in\mathbb{R}} \subset Aut(\mathcal{A})$: time evol.
$\overset{1-1}{\longleftrightarrow}$ homeos $\{\phi_{t,s}\}_{t,s\in\mathbb{R}}$ of M

$\omega, \rho, \cdots$: States = prob. meas. on (M,Σ).

2

π: meas. class; $\mathcal{A}^{\pi} := L^{\infty}(M,\pi)$

$\Pi^{(i)} := \chi_{\Omega_i}(\cdot)$, $\Omega_i \in \Sigma$ ($\neq$ null set)

$\omega \in \pi$: state

$$\mathrm{Prob}_{\omega}\{\Pi^{(0)}_{t_0}, \dots, \Pi^{(n)}_{t_n}\} := \int_M d\omega(\xi) \prod_{i=0}^{n} \chi_{\Omega_i}(\phi_{t_i,t_*}(\xi)) \quad (1)$$

ω pure $\Leftrightarrow$ $\omega = \delta_{\xi_*}$, $\xi_* \in M$
$\to$ 0-1 laws
etc.

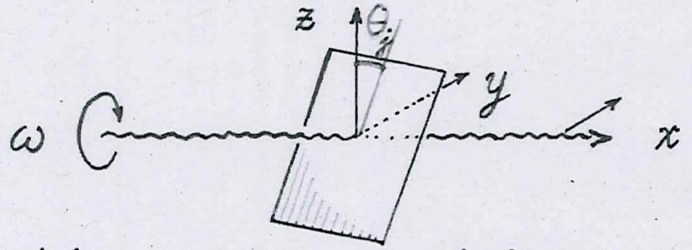
(Maassen)

<u>Example of quantum syst.</u>:

Quantum

Beam of (monochromatic) light = beam of photons, passing ~~through~~ through $n+1$ polarization filters

3



After passing j^{th} filter, photon pol. in dir. $\boxed{\theta_j := \frac{j\pi}{2n}}$

Initially, beam circ. pol.

$\Pi_+^{(j)}$: photon passes through j^{th} filter;

$\Pi_-^{(j)}$: photon absorbed in j^{th} filter.

$$\mathrm{Prob}_\omega\{\Pi_+^{(\theta)} | \Pi_+^{(\varphi)}\} = \cos^2(\theta - \varphi),$$

$$\mathrm{Prob}_\omega\{\Pi_-^{(\theta)} | \Pi_+^{(\varphi)}\} = \sin^2(\theta - \varphi),$$

$$\mathrm{Prob}_\omega\{\Pi_\pm^{(0)}\} = \frac{1}{2}.$$

4

Thus,

$$\mathrm{Prob}_\omega\{\Pi_+^{(0)}, \cdots, \Pi_+^{(n)}\} = \frac{1}{2}\left(\cos\left(\frac{\pi}{2n}\right)\right)^{2n} \xrightarrow[n\to\infty]{} \frac{1}{2} \quad (2)$$

$$\mathrm{Prob}_\omega\{\Pi_\pm^{(0)}, \Pi_+^{(n)}\} = 0$$

no intermediate filters!

If syst. were class. dyn. syst.

$$\left(\tfrac{1}{2} = \right) \mathrm{Prob}_\omega\{\Pi_+^{(0)}, \cdots, \Pi_+^{(n)}\} \leq \mathrm{Prob}_\omega\{\Pi_+^{(0)}, \Pi_+^{(n)}\} (=0),$$

because $\Pi_+^{(j)} + \Pi_-^{(j)} = 1$, $\Pi_\pm^{(j)} \geq 0$.

Interference!

$\Rightarrow$ $\boxed{\mathrm{Prob}_\omega \text{ not given by (1)!}}$

More sophisticated arguments

Kochen-Specker, Bell's <

Quantum Zeno Effect

System S, Hilbert space of state vectors $\mathcal{H}$; finite-dimensional subspace of $\mathcal{H}$ = range of orthogonal projection P, with $P^{\perp} := \mathbb{1} - P$.

Hamiltonian H with spectral proj. $dE_H(\lambda)$; $U(t) = e^{-itH/\hbar}$. We define

$$V_N(t) := \left(P\, U\!\left(\tfrac{t}{N}\right) P\right)^N, \qquad (3)$$

and

$$Z_N(t) := V_N(t)^* V_N(t) \qquad (4)$$

Facchi & Ligabò (J. Math. Phys. $\underline{51}$, 022103 (2010)) have proven the following

Theorem FL. The following statements are equivalent:

(A) $P E_H\big((-\infty, \Lambda] \cup [\Lambda, \infty)\big) P = o(\Lambda^{-1})$, as $\Lambda \to \infty$.

(B) $\frac{d}{dt} Z_1(t)\Big|_{t=0} = 0$.

(C) $\lim_{N\to\infty} Z_N(t) = P$, uniformly in t in compact subsets of $\mathbb{R}$.

<u>Interpretation.</u> We prepare S in a state represented by the vector $\Psi \in P\mathcal{H}$ at time $t=0$.

<u>Question:</u> What is the probability, $p_N(t)$, that the state of S is found (by an appropriate measurement) to belong to $P\mathcal{H}$ at <u>all</u> times $\frac{t}{N}, \frac{2t}{N}, \ldots, \frac{(N-1)t}{N}$?

According to the Lüders – Schwinger – Wigner formula, $p_N(t)$ is given by

$$p_N(t) = \| P V_N(t) \Psi \|^2 = \langle \Psi, Z_N(t) \Psi \rangle \underset{N \to \infty}{\longrightarrow} \langle \Psi, P \Psi \rangle = 1, \qquad (5)$$

assuming that (A) of Theorem FL holds.

$\Rightarrow$ Frequent measurements prevent Ψ to move out of $P\mathcal{H}$!

<u>Reasoning.</u> Let's assume that H is bounded. Then the <u>amplitude</u> for a transition

from subspace $P\mathcal{H}$ to subspace $P^{\perp}\mathcal{H}$ in a time $\frac{t}{N}$ is given by

$$\alpha_N(P \to P^{\perp}) = P^{\perp} e^{-i\frac{tH}{N\hbar}} P$$

$$\simeq \frac{1}{N} P^{\perp}\left(-\frac{itH}{\hbar}\right) P$$

$$= \frac{c(t)}{N}, \qquad (6)$$

for some t-dep. constant $c(t)$. Thus the probability for a transition from $P\mathcal{H}$ to $P^{\perp}\mathcal{H}$ is given by

$$p_N(P \to P^{\perp}) = O(1)\frac{1}{N^2}, \qquad (7)$$

which implies that

$$p_N(P \to P) = 1 - \frac{O(1)}{N^2} \qquad (8)$$

Considering the definition of $p_N(t)$ in the Question asked above, we find that

$$p_N(t) = \left(1 - \frac{\text{const.}}{N^2}\right)^N \xrightarrow[N \to \infty]{} 1 \qquad (9)$$

Time-Energy Uncertainty Relations, and Decay of Resonances

System S, Hilbert space $\mathcal{H}$, P: density matrix on $\mathcal{H}$.

(i) Standard uncertainty relations

Let $A = A^*$ and $B = B^*$ be two "observables" of S

Consider the non-negative quadratic polynomial

$$p(\lambda) := tr\left[P \underbrace{(A - i\lambda B)(A + i\lambda B)}_{\text{a positive operator!}}\right]$$

$$= \langle A^2 \rangle_P + i\lambda \langle [A,B] \rangle_P + \lambda^2 \langle B^2 \rangle_P ,$$

where $\langle (\cdot) \rangle_P := tr[P(\cdot)]$. Since $p(\lambda) \geq 0$, the discriminant

$$\langle [A,B] \rangle_P^2 - 4 \langle A^2 \rangle_P \cdot \langle B^2 \rangle_P$$

of $p(\lambda)$ must be ≤ 0. Thus

$$\langle A^2 \rangle_P \cdot \langle B^2 \rangle_P \geq \frac{1}{4} \langle [A,B] \rangle_P^2$$

In these arguments we can replace

$A \mapsto A - \langle A \rangle_P \mathbb{1}$, $B \mapsto B - \langle B \rangle_P \mathbb{1}$. This does not affect the commutator! We set

$$\Delta_P A := \sqrt{\left\langle \left(A - \langle A \rangle_P \mathbb{1}\right)^2 \right\rangle_P}\ , \quad \Delta_P B := \sqrt{\left\langle \left(B - \langle B \rangle_P \mathbb{1}\right)^2 \right\rangle_P}.$$

Then we have that

$$\Delta_P A \cdot \Delta_P B \geq \frac{1}{2} \left\langle [A, B] \right\rangle_P \qquad (10)$$

(ii) <u>Time-energy uncertainty relation</u>

We suppose S is initially in a state contained in the range of an orthogonal projection P on $\mathcal{H}$. In the absence of more precise knowledge we take the density matrix $\frac{1}{d_P} P$, $d_P = \mathrm{tr} P$, as our initial condition. Let $H = H^*$ be the Hamiltonian of S. We consider the survival probability

$$p(t) := \frac{1}{d_P} \mathrm{tr}\left(e^{-itH/\hbar} P e^{itH/\hbar} P\right)$$

$$= \frac{1}{d_P} \mathrm{tr}\left[\left(P e^{itH/\hbar} P\right)^* \left(P e^{itH/\hbar} P\right)\right]$$

$$\geq 0 \qquad (11)$$

We set

$$P(t) := e^{-itH/\hbar} P e^{itH/\hbar} = P(t)^* \qquad (12)$$

Then

$$p(t) = \frac{1}{d_P} tr\left(P\left(\tfrac{t}{2}\right) P\left(-\tfrac{t}{2}\right)\right) \qquad (13)$$

We have that

$$\dot{P}(t) = -\frac{i}{\hbar} [H, P(t)].$$

Hence

$$\dot{p}(t) = -\frac{i}{\hbar d_P} tr\left([H, P(t)] P\right) \qquad (14)$$

$$\ddot{p}(t) \overset{(11)}{=} -\frac{1}{\hbar^2 d_P} tr\left([H, [H, P(t)]] P\right)$$

$$= -\frac{1}{\hbar^2 d_P} tr\left(i[H, P(t)] \cdot i[H, P]\right). \qquad (15)$$

Using the cyclicity of the trace and $P^2 = P$, we find that

$$p(0) = 1, \quad \dot{p}(0) = 0, \qquad (16)$$

$$\ddot{p}(0) \overset{(11)}{=} -\frac{1}{\hbar^2 d_P} tr\left([H, [H, P]] P\right)$$

$$= -\frac{1}{\hbar^2 d_P} tr\left(\left(i[H, P]\right)^2\right) \leq 0$$

$$= -\frac{2}{\hbar^2 d_P} \, tr\left(P\left(H - PHP\right)^2\right)$$

$$=: -\frac{2}{\hbar^2} (\Delta E)^2 \tag{17}$$

In (15), (17) we have used that

$$tr([A,B]C) = tr([C,A]B) = -tr(B[A,C]).$$

Using the Schwarz inequality for traces and the unitarity of time evolution on the R.S. of (15) we find that

$$\ddot{p}(t) \geq \ddot{p}(0), \text{ for } t \neq 0. \tag{18}$$

By (16) this implies that

$$\dot{p}(t) = \dot{p}(0) + \int_0^t \ddot{p}(s)\,ds \geq t\ddot{p}(0) = -\frac{2t}{\hbar^2} (\Delta E)^2$$

Hence

$$p(t) = p(0) + \int_0^t \dot{p}(s)\,ds$$

$$\geq 1 + \frac{t^2}{2}\ddot{p}(0) = 1 - \frac{t^2}{\hbar^2}(\Delta E)^2 \tag{19}$$

Let $\Delta t > 0$ be the earliest time later than 0 such that $p(\Delta t) = \frac{1}{2}$. Then (19) implies

$$-\frac{1}{2} \geq -\frac{(\Delta t)^2}{\hbar^2}(\Delta E)^2,$$

i.e.,

$$\Delta t \cdot \Delta E \geq \frac{\hbar}{\sqrt{2}} \tag{20}$$

Further variants of time-energy uncertainty relations with improved constants can be found in [P. Pfeifer and J.F., Rev. Mod. Phys. $\underline{67}$, 759–779 (1995)].

Suppose the initial state of S is given by a density matrix P. We assume that

$$H = H_0 + V, \text{ with } [H_0, P] = 0 \tag{21}$$

Let E be an orthogonal projection, (E stands for "escape (states)"). We consider the escape probability

$$p_{esc}(t) := tr(P(t)E). \tag{22}$$

Then we have that

$$p_{esc}(t) = tr(PE) - \frac{i}{\hbar}\int_0^t tr\big([V,P](s)E\big) \quad (23)$$

This very simple identity yields non-trivial lower bounds on life times of "shape resonances", provided

- $p_{esc}(0) = tr(PE)$ is tiny;
- $tr\big(|[V,P]|\big)$ is tiny.

The second condition permits one to analyze situations where E has an ∞-dim range.

For applications to concrete situations see [Pfeifer & J.F.], loc. cit.

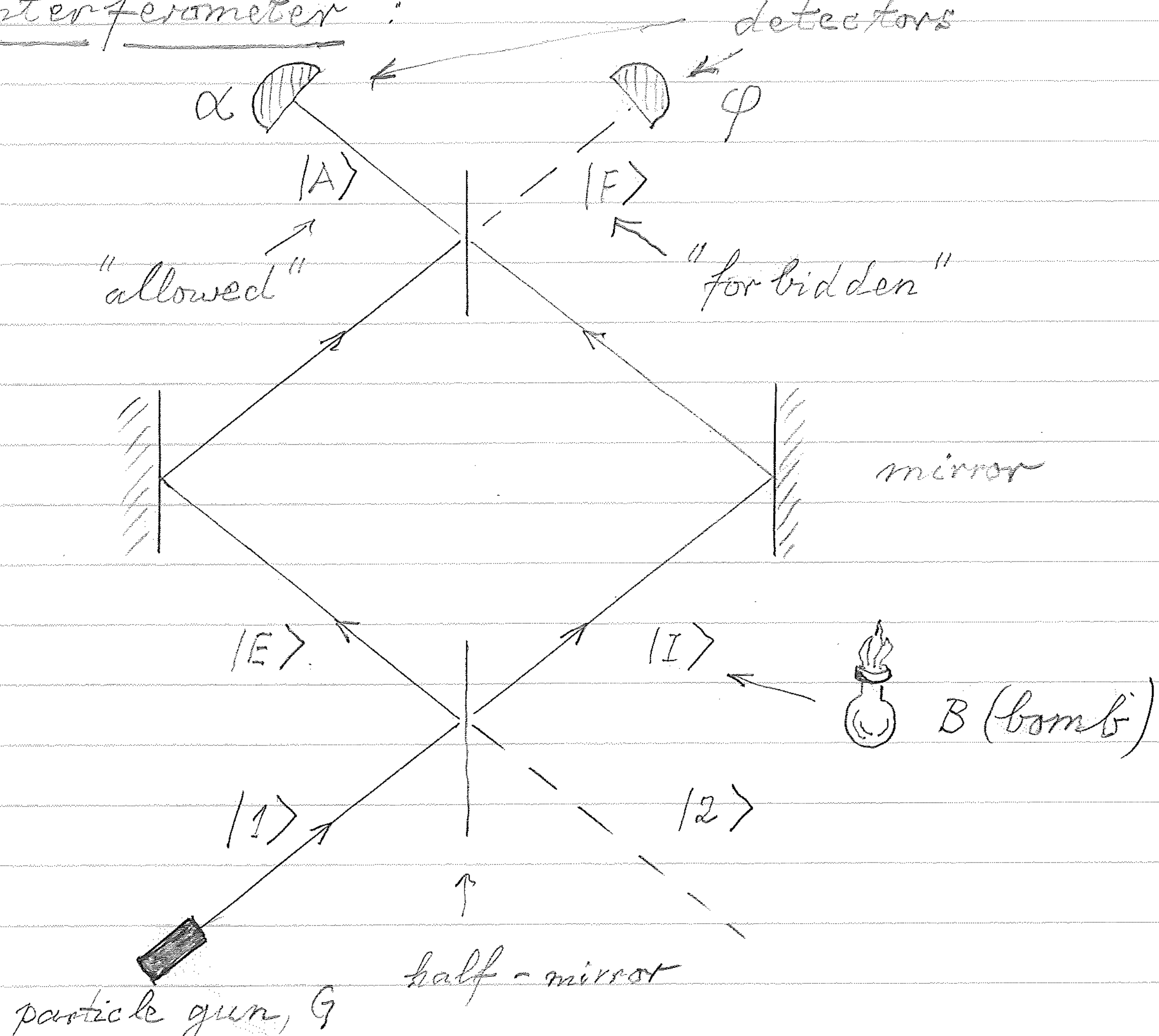
Somewhat more sophisticated than (23) are:

- The RAGE Theorem (Ruelle and followers)
- Analysis of resonances using dilatation analyticity (and renormalization group estimates):

– Aguilar – Balslev – Combes, Simon, Hunziker, ...
– Bach – JF – Sigal (Separate lectures!)

4.7 Lucien Hardy's Gedankenexperiment

We consider quantum particles – electrons, photons – moving through a Mach-Zehnder interferometer:



QM evolution of particle, P, through interferometer

$$\left.\begin{aligned} |1\rangle &\rightarrow \frac{1}{\sqrt{2}}\{|E\rangle + |I\rangle\} \\ |2\rangle &\rightarrow \frac{1}{\sqrt{2}}\{|E\rangle - |I\rangle\} \\ |E\rangle &\rightarrow \frac{1}{\sqrt{2}}\{|A\rangle + |F\rangle\} \end{aligned}\right\} \quad (1)$$

$$|I\rangle \longrightarrow \frac{1}{\sqrt{2}}\{|A\rangle - |F\rangle\} \tag{1'}$$

Thus,

$$|1\rangle \longrightarrow \frac{|E\rangle + |I\rangle}{\sqrt{2}} \longrightarrow \frac{\frac{|A\rangle+|F\rangle}{\sqrt{2}} + \frac{|A\rangle-|F\rangle}{\sqrt{2}}}{\sqrt{2}} = |A\rangle \tag{2}$$

<u>Avi Elitzur</u> & <u>Lev Vaidman</u> have proposed the following Gedankenexperiment:

There is a factory producing bombs.

- A bomb is "<u>good</u>" iff it explodes when hit by a particle, P.
- A bomb is "<u>bad</u>" iff it does <u>not</u> explode & is completely transparent for particles, P.

A bomb B is now placed on arm I of the interferometer. There are two possibilities:

(a) B is "bad" $\Rightarrow$ B has no effect on P whatsoever. $\Rightarrow$ P emitted by gun G ends up in state $|A\rangle$ with certainty, see (2), thus hits α.

(b) B is "good" $\Rightarrow$ there are 3 possibilities

B explodes

B does <u>not</u> explode

P hits α

P — " — φ

If B explodes $\Rightarrow$ P must have traveled through $|I\rangle$

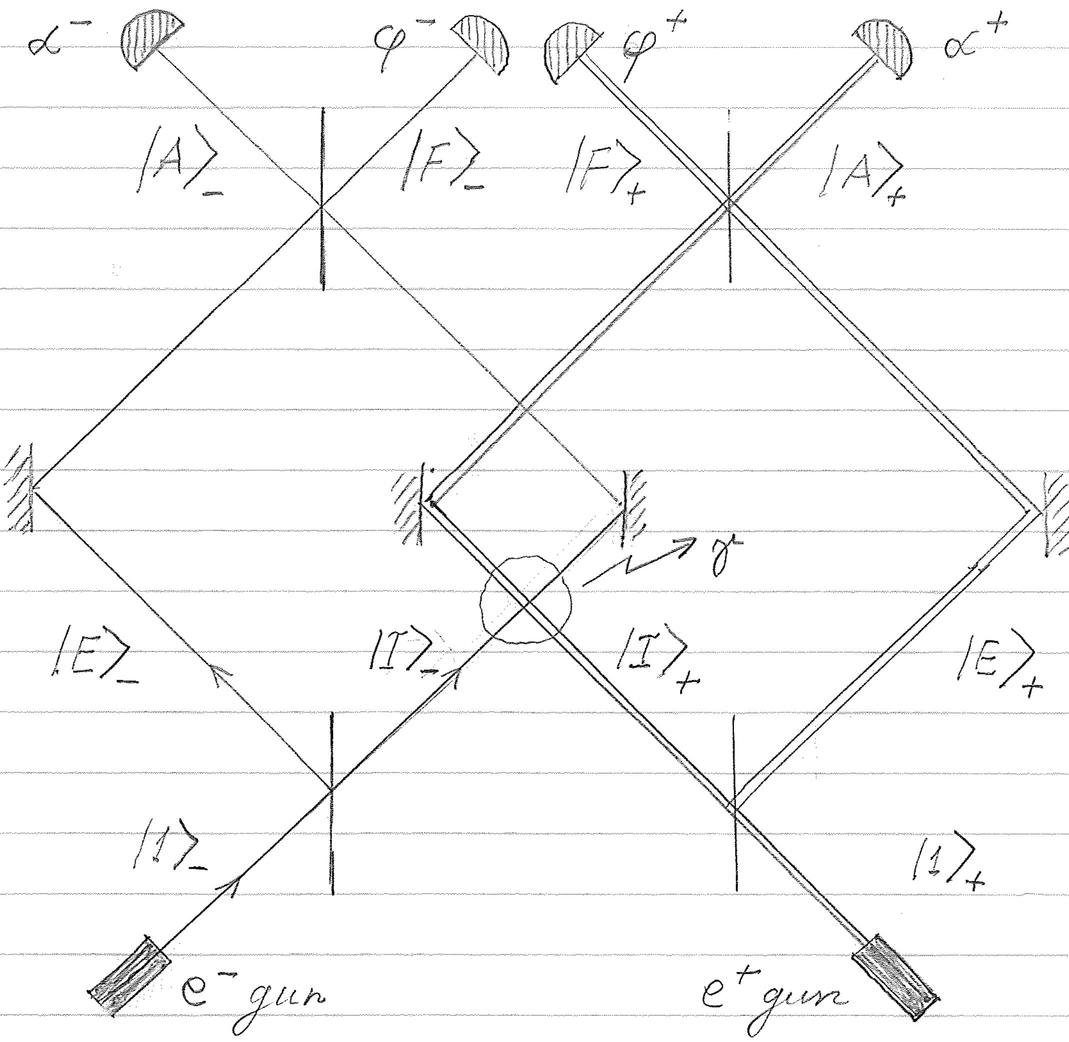
If B does <u>not</u> explode $\Rightarrow$ P must have traveled through $|E\rangle$ $\Rightarrow$ ends up in state $\frac{1}{\sqrt{2}}\{|A\rangle + |F\rangle\}$ $\Rightarrow$ hits detector α with prob $\frac{1}{2}$ and detector φ with prob $\frac{1}{2}$.

Hence if P hits φ then there must be a "good" bomb B on I, and P must have gone through E.

This is an "<u>interaction-free measurement</u>" of the property of B.

Next, we discuss a more sophisticated

Extension, due to Lucien Hardy



In the absence of positrons e^+, particle $P = e^-$ moves through the interferometer as described above. If e^- is absent, $P = e^+$ moves through the mirror-image interferometer as described above.

If e^- travels in state $|I\rangle_-$ and e^+ travels in state $|I\rangle_+$ then the process

$$e^- + e^+ \longrightarrow 2\gamma \quad (2\text{ photons})$$

takes place.

The initial state of the system is

$$|in\rangle = |1\rangle_- \otimes |1\rangle_+ \otimes |0\rangle_\gamma \tag{3}$$

$|in\rangle$ evolves to

$$\frac{1}{\sqrt{2}}\left\{|E\rangle_- + |I\rangle_-\right\} \otimes \frac{1}{\sqrt{2}}\left\{|E\rangle_+ + |I\rangle_+\right\} \otimes |0\rangle_\gamma$$

$$= \frac{1}{2}\left\{|E\rangle_- \otimes |E\rangle_+ + |E\rangle_- \otimes |I\rangle_+ + |I\rangle_- \otimes |E\rangle_+\right\} \otimes |0\rangle_\gamma$$

$$+ \frac{1}{2}\underbrace{|I\rangle_- \otimes |I\rangle_+ \otimes |0\rangle_\gamma}_{\rightarrow\ |\emptyset\rangle_- \otimes |\emptyset\rangle_+ \otimes |2\gamma\rangle} \tag{4}$$

If $e^- - e^+$ pair annihilation is <u>not</u> observed then the state of the system must be

$$|int\rangle := \frac{1}{\sqrt{3}}\left\{|E\rangle_- \otimes |E\rangle_+ + |E\rangle_- \otimes |I\rangle_+ + |I\rangle_- \otimes |E\rangle_+\right\} \otimes |0\rangle_\gamma \tag{5}$$

<u>before</u> e^- and e^+ reach the mirrors in the center of the interferometer.

The probability for e^+ and e^- to settle in the intermediate state $|int\rangle$ is $\frac{3}{4}$.

By (1), (1'), the state $|int\rangle$ evolves into

$$\frac{1}{2\sqrt{3}}\Big\{\big(|A\rangle_- + |F\rangle_-\big)\otimes\big(|A\rangle_+ + |F\rangle_+\big) + \big(|A\rangle_- + |F\rangle_-\big)\otimes \big(|A\rangle_+ - |F\rangle_+\big) + \big(|A\rangle_- - |F\rangle_-\big)\otimes\big(|A\rangle_+ + |F_+\rangle\big)\Big\}\otimes|0\rangle_\gamma$$

$$= \frac{1}{2\sqrt{3}}\Big\{3\,|A\rangle_-\otimes|A\rangle_+ + |A\rangle_-\otimes|F\rangle_+ + |F\rangle_-\otimes|A\rangle_+ - |F\rangle_-\otimes|F\rangle_+\Big\} \quad (6)$$

The probability of ending up in $|F\rangle_-\otimes|F\rangle_+$ is $\frac{1}{12}\times\frac{3}{4} = \frac{1}{16} > 0.$ (7)

One may now be tempted to argue as follows:

If e^+ hits the detector φ^+ it must have occupied $|F\rangle_+$ before reaching the detector. But then e^- must have occupied $|I\rangle_-$; for, otherwise, e^+ would <u>not</u> have felt

the presence of e^- and hence would have settled in $|A\rangle_+$ and hit detector α^+.
But since their has not been any pair annihilation (preventing e^+ from reaching φ^+), and since e^- must seemingly have occupied $|I\rangle_-$, we conclude that e^+ must have occupied $|E\rangle_+$!

If we assume that both, e^+ and e^- have hit φ^+ and φ^-, resp., i.e., have occupied $|F\rangle_- \otimes |F\rangle_+$ — an event that has probability $\frac{1}{16} > 0$ — and then exchange the roles of e^+ and e^-, we seemingly conclude, that

e^+ must have occupied $|I\rangle_+$,

and

e^- " " " $|E\rangle_-$,

which contradicts the claim made above!

The two contradicting conclusions described above are counter-factual. There has not been any measurement carried out on e^- and e^+ before they reach, or do not reach, two of the detectors $\alpha^-, \varphi^-, \alpha^+, \varphi^+$.

Suppose we place a detector, δ^-, on the arm of the interferometer carrying the state $|I\rangle_-$ of e^-. Then we have the following situation:

a1) If e^+ arrives at φ^+ then there is a chance that e^- occupies $|I\rangle_-$ in order to disturb e^+, which will then occupy $|E\rangle_+$. But if e^- occupies $|I\rangle_-$ the detector δ^- clicks, and e^+ must then occupy $|E\rangle_+$ to avoid pair annihilation into 2 photons.

a2) If e^- arrives at φ^-, but δ^- has not clicked then e^- must have occupied $|E\rangle_-$ and e^+ can occupy $|I\rangle_+$ or $|E\rangle_+$.

In either case, there is no paradox!

In the original setting, <u>without</u> δ^-, there is no paradox either: To observe e^- and e^+ arrive at the final detectors means that there has <u>not</u> been pair annihilation (an event that would happen with probability $\frac{1}{4} < 1$). This implies that e^- and e^+ must have occupied the entangled state $|int\rangle$ of Eq.(5). The transition probability for the process

$$|int\rangle \rightarrow |F\rangle_- \otimes |F\rangle_+$$

is $\frac{1}{12} > 0$.

Classical alternatives are not applicable to entangled quantum states!

4.8. Quantum teleportation (Bennett, Brassard, ...)

This discussion is based on the Copenhagen interpretation of QM: If an observable $X = \sum \xi \, \Pi_\xi$ is measured then an initial state ψ becomes the state $\Pi_\xi \psi / \| \Pi_\xi \psi \|$ after measurement of X, with probability $\| \Pi_\xi \psi \|^2 = \langle \psi, \Pi_\xi \psi \rangle$ (Born's Rule).

There are two experimentalists, Alice and Bob. Alice and Bob capture one particle of a Bell pair, each, particle A and particle B, resp.. The initial state of the Bell pair is given by

$$\psi = \frac{1}{\sqrt{2}} \left\{ |\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B \right\}. \qquad (1)$$

where $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2$, $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$.

Alice captures a second particle, C, with spin $\frac{1}{2}$ whose spin is in a state $\varphi = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{C}^2$, $|u|^2 + |v|^2 = 1$. The state φ of C is unknown to Alice. She is now asked to transmit the information on the state, φ, of C to Bob, using only transmission of classical information, such as the outcome of an experiment she is able to carry out on particles A and C, (not affecting particle B in Bob's lab). She is not supposed to send particle C to Bob, and a direct measurement carried out on particle C by Alice does not reveal what the state, φ, of C has been before the measurement.

The claim is that if A and B form a Bell pair in state ψ and if Alice measures a suitable "observable" of $A \vee C$ and transmits the outcome of the measurement to Bob by telephone (i.e., "classically"), then Bob can carry out an operation on particle B (spin precession) that maps the state of B to φ.

The initial state of $(C \vee A) \vee B$ is given by

$$\varphi \otimes \psi = \frac{1}{\sqrt{2}} \left\{ \varphi \otimes |\uparrow\rangle_A \otimes |\downarrow\rangle_B - \varphi \otimes |\downarrow\rangle_A \otimes |\uparrow\rangle_B \right\}, \quad (2)$$

see (1).

Alice measures the "observable"

$$X = \Pi_1 + 2\Pi_2 + 3\Pi_3 + 4\Pi_4, \quad \underline{\Pi_i = \Pi_{\chi_i}}, \quad (3)$$

where $\chi_1, \cdots, \chi_4$ form an orthonormal basis in $\mathbb{C}^2_C \otimes \mathbb{C}^2_A \simeq \mathbb{C}^4$ given by:

$$\underline{\chi_1 = \frac{1}{\sqrt{2}} \left\{ |\uparrow\rangle_C \otimes |\downarrow\rangle_A - |\downarrow\rangle_C |\uparrow\rangle_A \right\} \qquad s = 0}$$

$$\chi_2 = \frac{1}{\sqrt{2}} \left\{ |\uparrow\rangle_C \otimes |\downarrow\rangle_A + |\downarrow\rangle_C \otimes |\uparrow\rangle_A \right\} \qquad s = 1,\ m = 0$$

$$\chi_3 = \frac{1}{\sqrt{2}} \left\{ |\uparrow\rangle_C \otimes |\uparrow\rangle_A - |\downarrow\rangle_C \otimes |\downarrow\rangle_A \right\} \qquad s = 1$$

$$\chi_4 = \frac{1}{\sqrt{2}} \left\{ |\uparrow\rangle_C \otimes |\uparrow\rangle_A + |\downarrow\rangle_C \otimes |\downarrow\rangle_A \right\} \qquad s = 1$$

$$\left[\frac{1}{\sqrt{2}} (\chi_3 + \chi_4) = |\uparrow\rangle_C |\uparrow\rangle_A \ : \ s = 1,\ m = 1, \text{ etc.} \right]$$

The state of $(C \vee A) \vee B$ <u>after</u> measuring X

is, (ignoring normalization), one of the following states; (note: $[\psi] = \{e^{i\theta}\psi/_{\|\psi\|}\}$):

$$[\Pi_1 \varphi\otimes\psi] = \chi_1 \otimes\{-\langle\uparrow|\varphi\rangle_C |\uparrow\rangle_B - \langle\downarrow|\varphi\rangle_C |\downarrow\rangle_B\}$$

$$[\Pi_2 \varphi\otimes\psi] = \chi_2 \otimes\{-\langle\uparrow|\varphi\rangle_C |\uparrow\rangle_B + \langle\downarrow|\varphi\rangle_C |\downarrow\rangle_B\}$$

$$[\Pi_3 \varphi\otimes\psi] = \chi_3 \otimes\{\langle\uparrow|\varphi\rangle_C |\downarrow\rangle_B + \langle\downarrow|\varphi\rangle_C |\uparrow\rangle_B\} \tag{4}$$

$$[\Pi_4 \varphi\otimes\psi] = \chi_4 \otimes\{\langle\uparrow|\varphi\rangle_C |\downarrow\rangle_B - \langle\downarrow|\varphi\rangle_C |\uparrow\rangle_B\},$$

each with probability $\|\Pi_i \varphi\otimes\psi\|^2 = \frac{1}{4}$, $i=1,\dots,4$. If Alice measures the value 1 for X corresponding to Π_1 then the state of the system is given by $[\Pi_1\varphi\otimes\psi]$. Thus, particle B in Bob's lab is in state $-\varphi (\simeq \varphi)$. If Alice measures 2 the state is given by $[\Pi_2\varphi\otimes\psi]$. When Bob is told this outcome he applies a spin precession on particle B around the 3-axis by $180°$. This maps $[\Pi_2 \varphi\otimes\psi]$ to

$$\left(\mathbb{1}_{C\vee A} \otimes \sigma^3_B\right) \cdot \frac{\Pi_2\, \varphi \otimes \psi}{\|\Pi_2\, \varphi \otimes \psi\|} = \chi_2 \otimes \left\{ -\langle \uparrow | \varphi \rangle_C |\uparrow\rangle_B - \langle \downarrow | \varphi \rangle_C |\downarrow\rangle_B \right\}. \tag{5}$$

Then particle B in Bob's lab is in state $-\varphi \, (\simeq \varphi)$.

If Alice measures 3, Bob applies a spin precession around the 1-axis by an angle 180° to particle B, which is then in state φ. Finally, if Bob learns that Alice has measured 4 he lets the spin of B precess around the 2-axis by 180°, and B is then in state $-i\varphi \, (\simeq \varphi)$.

In all four cases, particle B ends up in state $[\varphi]$! Note, however, that Alice has to communicate to Bob the value of the observable X she has measured for teleportation to work. The argument can easily be extended to a situation where the particles live in higher-dimensional Hilbert spaces.

Quantum teleportation has been realized with photons by Zeilinger et al. in 1997.

Let's return to more serious matters.

4.9. Entanglement - Klyachko's Theorem

Suppose we consider n systems, $S_1, \cdots, S_n$, with Hilbert spaces $\mathcal{H}_1, \cdots, \mathcal{H}_n$, all prepared in a single lab. Then it makes sense to consider their composition to a single system

$$S = S_1 \vee \cdots \vee S_n.$$

Conventionally, the Hilbert space of S is taken to be

$$\mathcal{H}_S = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n. \tag{6}$$

Rather than attempting to present a very general discussion of composition of systems, I consider the special situation where $\mathcal{H}_S$ is given by (6), $\mathcal{A}_{S_i} = B(\mathcal{H}_i)$, $\forall i = 1, \cdots, n$, and

$$\mathcal{A}_S = \mathcal{B}(\mathcal{H}_S) = \mathcal{B}(\mathcal{H}_1) \otimes \cdots \otimes (\mathcal{H}_n). \qquad (7)$$

Let P be a density matrix on $\mathcal{H}_S$, and let P_i denote its i^{th} marginal, defined as follows: For $X \in \mathcal{B}(\mathcal{H}_i)$, we define

$$X^{(i)} := \mathbb{1} \otimes \cdots \otimes \underset{\substack{\uparrow \\ i^{th}\text{ slot}}}{X} \otimes \cdots \otimes \mathbb{1}. \qquad (8)$$

Then

$$tr(P_i X) := tr(P X^{(i)}), \quad \forall X \in \mathcal{B}(\mathcal{H}_i). \quad (9)$$

Problem. Given a density matrix P on $\mathcal{H}_S$ and arbitrary density matrices $\tilde{P}_1, \dots, \tilde{P}_n$, under what conditions on the spectra of $P, \tilde{P}_1, \dots, \tilde{P}_n$ is $\tilde{P}_i = P_i$ the i^{th} marginal of P, for $i = 1, \dots, n$?

A solution to this difficult problem has been given by Klyachko in 2004. Chistandl, Walter and co-workers have found more concrete solutions, later on.

<u>Exercise.</u> Suppose $n=2$ and $P=|\Psi\rangle\langle\Psi|$, for a unit vector $\Psi\in\mathcal{H}_1\otimes\mathcal{H}_2$. Then

$$\sigma(P_1)=\sigma(P_2)$$

$\operatorname{rank}(P_1)=\operatorname{rank}(P_2)\leq\min(\dim\mathcal{H}_1,\dim\mathcal{H}_2)$, and $\operatorname{rank}(P_1)=\operatorname{rank}(P_2)=1\iff\Psi=\varphi_1\otimes\varphi_2$, for unit vectors $\varphi_1\in\mathcal{H}_1$, $\varphi_2\in\mathcal{H}_2$.

<u>Pf.</u> Let $\dim\mathcal{H}_1\leq\dim\mathcal{H}_2=N$. We set

$$\overline{\mathcal{H}}_1=\mathcal{H}_1\oplus\mathfrak{h},$$

where $\mathfrak{h}$ is a Hilbert space of $\dim\mathfrak{h}=\dim\mathcal{H}_2-\dim\mathcal{H}_1$.

Then

$$\mathcal{H}_1\oplus\{\vec{0}\}\otimes\mathcal{H}_2$$

is a subspace in $\overline{\mathcal{H}}_1\otimes\mathcal{H}_2$, and Ψ belongs to this subspace. Let $\{\varphi_1^{(1)},\dots,\varphi_N^{(1)}\}$ be a CONS in $\overline{\mathcal{H}}_1$, (with $\{\varphi_1^{(1)},\dots,\varphi_M^{(1)}\}$ a CONS in $\mathcal{H}_1$, $M=\dim\mathcal{H}_1$). For an arbitrary Ψ in $\overline{\mathcal{H}}_1\otimes\mathcal{H}_2$, we have that

$$\Psi=\sum_{i,j}c_{ij}\,\varphi_i^{(1)}\otimes\varphi_j^{(2)}\qquad(10)$$

(where $c_{ij} = 0$, for $i > M$ if $\Psi \in \mathcal{H}_1 + \{\vec{0}\} \otimes \mathcal{H}_2$),

The matrix $C = (c_{ij})$ has a polar decomposition

$$C = DU \tag{11}$$

where $D = (d_{ij}) \geq 0$, $U = (u_{ij})$ <u>unitary</u>.

Since C has rank M, U unitary, D has rank $M \leq N$. If $\Delta_1 \geq \Delta_2 \geq \cdots \geq \Delta_N \geq 0$ are the eigenvalues of D then $\Delta_{M+1} = \cdots = \Delta_N = 0$. (12)

We define

$$\tilde{\varphi}_m^{(1)} := \sum_i u_{mi} \varphi_i^{(1)} \tag{13}$$

Then

$$\langle \tilde{\varphi}_i^{(1)}, \tilde{\varphi}_j^{(1)} \rangle_{\mathcal{H}_1} = \sum_{l,k} \overline{u}_{il} \underbrace{\langle \varphi_l^{(1)}, \varphi_k^{(1)} \rangle_{\mathcal{H}_1}}_{\delta_{lk}} u_{jk}$$

$$= \sum_k u_{jk} \overline{u}_{ik} = \delta_{ji} = \delta_{ij},$$

by unitarity of U. Now (10), (11) and (13) imply

$$\Psi = \sum_{j,m} d_{jm} \tilde{\varphi}_m^{(1)} \otimes \varphi_j^{(2)}. \tag{14}$$

Since $D \geq 0$, $\exists$ <u>unitary</u> $N \times N$ matrix $V = (v_{ij})$

diagonalizing D, i.e.,

$$D = V^*(\mathrm{diag}\, \Delta_k) V \Leftrightarrow d_{jm} = \sum_k \Delta_k \bar{v}_{kj} v_{km} \qquad (15)$$

We now set

$$\hat{\varphi}_k^{(1)} := \sum_m v_{km} \tilde{\varphi}_m^{(1)}, \quad \hat{\varphi}_k^{(2)} := \sum_j \bar{v}_{kj} \varphi_j^{(2)} \qquad (16)$$

Then, using (12),

$$\sum_{k=1}^{M} \Delta_k \hat{\varphi}_k^{(1)} \otimes \hat{\varphi}_k^{(2)} = \sum_{k=1}^{N} \Delta_k \hat{\varphi}_k^{(1)} \otimes \hat{\varphi}_k^{(2)}$$

$$= \sum_{k,m,l} \Delta_k \bar{v}_{kl} v_{km} \tilde{\varphi}_m^{(1)} \otimes \varphi_l^{(2)}$$

$$= \sum_{m,l} d_{lm} \tilde{\varphi}_m^{(1)} \otimes \varphi_l^{(2)} = \Phi .$$

Since V is unitary, the vectors $\hat{\varphi}_1^{(1)}, \dots, \hat{\varphi}_N^{(1)}$ and $\hat{\varphi}_1^{(2)}, \dots, \hat{\varphi}_N^{(2)}$ are CONS in $\tilde{\mathcal{H}}_1, \mathcal{H}_2$, resp.

This completes the solution of the exercise.

Next, I quote Klyachko's theorem without proof.

Theorem K (Klyachko)

We set $n=2$, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\dim \mathcal{H}_1 \equiv M \leq N \equiv \dim \mathcal{H}_2$.

Let P be a density matrix on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with eigenvalues $\underline{p} := \{p_1, \dots, p_{M\cdot N}\}$, $p_1 \geq p_2 \geq \dots \geq p_{M\cdot N} \geq 0$, $\sum_i p_i = 1$, and let P_1 and P_2 be the marginals of P with spectra $\underline{p}^{(1)}$ and $\underline{p}^{(2)}$, resp., where $\underline{p}^{(i)} = \{p_1^{(i)}, \dots, p_{\dim \mathcal{H}_i}^{(i)}\}$, in decreasing order, with $p_j^{(i)} \geq 0$, $\forall j$, $i = 1, 2$.

Then

$$\sum_{i=1}^{M} a_i \, p^{(1)}_{\pi(i)} + \sum_{j=1}^{N} b_j \, p^{(2)}_{\sigma(j)} \leq \sum_{k=1}^{M\cdot N} (a+b)^{\downarrow}_k \, p_{\varkappa(k)}, \tag{17}$$

for arbitrary seqs. $a_1 \geq \dots \geq a_M$, $b_1 \geq \dots \geq b_N$, with $\sum_{i=1}^{M} a_i = \sum_{j=1}^{N} b_j = 0$, arbitrary permutations π of $\{1, \dots, M\}$, σ of $\{1, \dots, N\}$ and $\varkappa$ of $\{1, \dots, M\cdot N\}$ with non-vanishing "Schubert coefficients", $c^{\varkappa}_{\pi\sigma}$, and where $\{(a+b)^{\downarrow}_k\}_{k=1}^{M\cdot N}$ is the sequence $\{(a_i + b_j) \mid i = 1, \dots, M, \ j = 1, \dots, N\}$ arranged in decreasing order.

The converse, i.e. (17) $\Rightarrow P_1, P_2$ are the marginals of P, is true, too.

Remark. To every permutation σ of $\{1,\cdots,n\}$ one can associate a so-called Schubert polynomial, $\mathfrak{S}_\sigma(a_1,\cdots,a_n)$, in n variables. It is a homogeneous polynomial in $a_1,\cdots,a_n$ of a degree $d(\sigma)$ representing the so-called Schubert cycle of σ in the cohomology of the flag manifold, $Fl(n)$, whenever n is large enough. Schubert polynomials form a basis in the polynomial ring $\mathbb{Z}[a_1,a_2,\cdots]$ in ∞ many variables. Thus, there are unique coefficients $c^{\varkappa}_{\pi\sigma}$ such that

$$\mathfrak{S}_\pi\cdot\mathfrak{S}_\sigma=\sum_{\varkappa} c^{\varkappa}_{\pi\sigma}\,\mathfrak{S}_\varkappa .$$

The coefficients $c^{\varkappa}_{\pi\sigma}$ are non-negative integers generalizing the Littlewood-Richardson coefficients (for Schur functions). There is no closed formula known for these coefficients.

The number of inequalities in (17) grows very rapidly in M and N: For $M=2$ and $N=4$, one has already 234 inequalities.

A remark on quantum statistics.

We consider a system S given by

$$S = \bigcup_{n=1}^{\infty} S_1 \vee \cdots \vee S_n ,$$

$$S_j \simeq S^{(0)}, \text{ for all } j, \forall n. \qquad (18)$$

The question is: what is the Hilbert space of $S_1 \vee \cdots \vee S_n$? It should be a subspace, $\mathcal{H}^{(n)}$, with

$$\mathcal{H}^{(n)} \subseteq \underbrace{\mathcal{H}_{S^{(0)}} \otimes \cdots \otimes \mathcal{H}_{S^{(0)}}}_{n \text{ factors}} \qquad (19)$$

Since $S_1 \simeq S_2 \simeq \cdots \simeq S_n \simeq S^{(0)}$, the permutation group $\mathcal{S}_n$ of n letters acts on $\mathcal{H}^{(n)}$. In $d \geq 3$ space dimensions, $\mathcal{S}_n$ is represented on $\mathcal{H}^{(n)}$ according to a representation corresponding to a Young diagram with

$$\left.\begin{array}{l} \leq k \text{ columns} \iff \text{para-Bose statistics,} \\ \text{or} \\ \leq k \text{ rows} \iff \text{para-Fermi statistics,} \end{array}\right\} \quad (20)$$

for some k <u>indep.</u> of n; ($\nearrow$ Doplicher–Haag–Roberts). Doplicher and Roberts have shown that there always exists a compact topological group, G, such that if one augments $\mathcal{H}_{\Sigma(o)}$ to

$$\mathcal{H}_{\Sigma(o)} \otimes \mathcal{V},$$

where $\mathcal{V}$ carries a representation of G, then

$$\mathcal{H}^{(n)} = \left(\mathcal{H}_{\Sigma(o)} \otimes \mathcal{V}\right) \otimes_{\varepsilon} \cdots \otimes_{\varepsilon} \left(\mathcal{H}_{\Sigma(o)} \otimes \mathcal{V}\right), \quad (21)$$

where

$\otimes_{\varepsilon} = \otimes_{s} \iff$ symm. tensor product $\iff$ Bose statistics

or

$\otimes_{\varepsilon} = \otimes_{a} \iff$ anti-symm. tensor product $\iff$ Fermi statistics (Pauli exclusion principle).

$$\mathcal{H}_{\Sigma} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$$

The truth is a little more subtle, and the results alluded to, above, are only theorems in the framework of <u>relativistic local quantum theory</u>

If the dimension of space is $d=2$ then the role of the permutation groups, S_n, is played by the <u>braid groups</u>, B_n, $n=2,3,\dots$. This insight has its roots in papers of Streater and Wilde, JF, Leinaas and Myrheim, and was fully formulated in work of JF et al. Its mathematical home is the

Theory of <u>monoidal</u> (C^*) <u>tensor categories</u>, and, for $d=2$, <u>braided</u> tensor categories.

<u>Remark.</u> Multi-<u>fermion</u> states are always <u>entangled</u>.

Entanglement entropy.

von Neumann entropy of a state of a physical system given by a density matrix, P, acting on a Hilbert space $\mathcal{H}$:

$$S(P) := -k\, tr(P \ln P) \tag{22}$$

The Rényi entropy of P is defined by

$$S_\alpha(P) := \frac{1}{1-\alpha} \ln tr(P^\alpha), \quad \alpha < 1. \tag{23}$$

Note: $\lim_{\alpha \nearrow 1} S_\alpha(P) = S(P)$.

Suppose now that $S = S_1 \vee S_2$, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and let P_1 and P_2 be the marginals of P.

The entanglement entropy is then defined by

$$\begin{aligned} S_\alpha(P; P_1, P_2) &:= S_\alpha(P_1 \otimes P_2) - S_\alpha(P) \\ &= S_\alpha(P_1) + S_\alpha(P_2) - S_\alpha(P) \end{aligned} \tag{24}$$

If $P = P_1 \otimes P_2$ then

$$S_\alpha(P; P_1, P_2) = 0 \tag{25}$$

Properties of entropies: ....

ETH with discrete time:

$$\Psi = \sum_i z_i \underset{\in \mathcal{H}_1}{\varphi_i} \otimes \underset{\in \mathcal{H}_2}{\psi_i} \quad , \quad \sum z_i \varphi_i = \phi$$

unit vectors

$$|\Psi\rangle\langle\Psi| = \sum_{i,j} z_i \bar{z}_j |\varphi_i \otimes \psi_i\rangle\langle\varphi_j \otimes \psi_j|$$

$$= \sum_{i,j} z_i \bar{z}_j \underbrace{|\varphi_i\rangle\langle\varphi_j|}_{P_{ij}} \otimes |\psi_i\rangle\langle\psi_j|$$

Partial trace over 2nd tensor factor:

$$\sum_n \langle \chi_n|\psi_i\rangle\langle\psi_j|\chi_n\rangle = \underbrace{\langle\psi_j|\psi_i\rangle}_{G_{ij}}$$

$$\Rightarrow \mathrm{tr}_{\mathcal{H}_2} |\Psi\rangle\langle\Psi|$$

$$= \sum_{i,j} G_{ij} z_i \underbrace{|\varphi_i\rangle\langle\varphi_j|}_{P_{ij}} \bar{z}_j$$

diagonalize with unitary map

$$G_{ij} = \overline{u_{ki}}\, \gamma_k\, u_{kj}$$

$$\tilde{\varphi}_k = \sum_j u_{kj} \varphi_j$$

$$\Rightarrow \mathrm{tr}_{\mathcal{H}_2} |\Psi\rangle\langle\Psi| = \sum \gamma_k |\tilde{\varphi}_k\rangle\langle\tilde{\varphi}_k|$$

2

$$\psi_1 = \psi$$

$$\psi_i = \psi + \varepsilon \underbrace{\delta\psi_i}_{\perp \psi\text{, depends on } \varphi_i !} , \quad i = 2, \dots, m$$

If $\dim \mathcal{H}_2 = M < N = \dim \mathcal{H}_1$,

then $\operatorname{rank}(G_{ij}) \leq M$

$$\Longrightarrow \boxed{N - M \quad \gamma_k\text{'s vanish}}$$

~~One eigenvector of G is~~ $\approx \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \frac{1}{\sqrt{N}}$

$$\underbrace{P_{ij} \longrightarrow \sum_k \underbrace{\gamma_k}_{\geq 0} \underbrace{\overline{v}_{ki}}_{\overline{u}_k} P_{ij} \underbrace{u_{kj}}_{\overline{v}_k^*}}_{\text{completely positive map}}$$

5. Theory of Indirect Measurements in Quantum Mechanics

Outline

After a short introduction to some of the conundrums afflicting Quantum Mechanics we study the effective quantum dynamics of systems interacting with a long chain of independent probes, which, afterwards, are subject to direct (projective) measurements and are then lost.
This first leads us to develop a theory of indirect measurements of *time-independent* quantities (non-demolition measurements).
Subsequently, the theory of indirect weak measurements of *time-dependent* quantities is outlined, with emphasis on the description of Markov processes whose state spaces are spectra of quantities to be measured.

The Founding Fathers of Quantum Mechanics (QM):

5. Indirect Measurements in Quantum Mechanics

The theory presented in this chapter was pioneered by E.B. Davies, K. Kraus, H. Maassen & B. Kümmerer; with more recent followers in France, at ETH Zurich and elsewhere.

In this chapter, relativity theory is not essential. We study small systems with components whose "free energies" are tiny as compared to their rest energies ($\sim mc^2$). We can then assume that one can introduce an absolute time to describe dynamical processes exhibited by such systems. Absolute time is modeled by $\mathbb{R}$. An interval, $I \subseteq \mathbb{R}$, of times is called an "era".

Definition of "isolated systems."

A system S is isolated during an era I iff interactions between S and its complement, S^c,

$(S^c = \text{Universe} \setminus S)$ are so weak that they can be ignored for all practical purposes (FAPP), throughout the era I.

The notion of isolated systems is central in Quantum Mechanics, because only for isolated systems we have a conceptually clear idea about <u>time evolution of operators</u> representing physical quantities in the <u>Heisenberg picture</u>.

We provisionally adopt the following description of isolated systems, S:

- States of S are described by density matrices, P, acting on a Hilbert space $\mathcal{H}$. A state, P, of S is <u>pure</u> iff P is a rank-1 orthogonal projection.
- Physical quantities pertaining to S are described by self-adjoint linear operators

belonging to a family $\mathcal{O}_S$; $\mathcal{O}_S$ generates a (C)* algebra $\mathcal{A}_S$. At every time $t \in \mathbb{R}$, there is a <u>faithful *representation</u> $\gamma_t : \mathcal{A}_S \to B(\mathcal{H})$. If $\hat{X} \in \mathcal{A}_S$

$$\gamma_t(\hat{X}) =: X(t) \in B(\mathcal{H}), \quad \gamma_t(\hat{X}^*) = X(t)^*, \qquad (5.1)$$

for all $\hat{X} \in \mathcal{A}_S$. Any two representations, γ_t and γ_s, are <u>unitarily equivalent</u>, with

$$X(t) = U(s,t)\, X(s)\, U(t,s), \qquad (5.2)$$

for t, s in $\mathcal{I}$, where $\{U(t,s)\}_{t,s \in \mathcal{I} \subseteq \mathbb{R}}$ is the <u>propagator</u> of S during era $\mathcal{I}$; i.e.,

- $U(t,s)$ is a unitary operator on $\mathcal{H}$, for arbitrary times t, s in $\mathcal{I}$;
- $U(t,s)\, U(s,r) = U(t,r)$, $\forall\, t,s,r$ in $\mathcal{I}$;
- $U(t,s)$ is strongly continuous in t and s, and $U(t,t) = \mathbb{1}$, $\forall\, t,s$ in $\mathcal{I}$. $\qquad (5.3)$

- S is <u>autonomous</u> during an era $\mathcal{I}$ iff there exists a self-adjoint, time-independent

Hamiltonian, $H = H_{\mathcal{Y}}$, such that

$$U(t,s) = e^{-i(t-s)H_{\mathcal{Y}}/\hbar}. \tag{5.4}$$

In the following, we choose units such that $\hbar = 1$, and we suppress reference to a specific era $\mathcal{Y}$.

We will also consider descriptions involving a <u>discrete time</u> $t = \tau n$, $n \in \mathbb{Z}$, for some "firing time" $\tau > 0$. Then we will not speak of a Hamiltonian H, but say that, for an autonomous system,

$$U(n\tau, m\tau) = U^{n-m}, \quad n, m \in \mathbb{Z}, \tag{5.5}$$

for some unitary operator U on $\mathcal{H}$.

In this section, we adopt a very naive view of <u>measurements/observations</u> of physical quantities: Let $\hat{X} \in \mathcal{O}_S$, and let $X(t) = X(t)^*$ be the operator representing $\hat{X}$ at time t. We assume, for simplicity, that $\hat{X}$ has pure-point spectrum, so that

$$X(t) = \sum_{\xi \in \sigma(\hat{X})} \xi \, \Pi_\xi(t), \quad \forall t, \qquad (5.6)$$

where $\Pi_\xi(t)$ is the spectral projection of $X(t)$ corresponding to the eigenvalue $\xi \in \sigma(X(t)) = \sigma(\hat{X})$.

If P is the state of S right before a projective (direct) measurement of $\hat{X}$ at some time t with outcome ξ then the state of S right after the measurement of $\hat{X}$ is given by

$$P_\xi := \frac{\Pi_\xi(t) \, P \, \Pi_\xi(t)}{tr\left(P \Pi_\xi(t)\right)} \qquad (5.7)$$

If the measurement of $\hat{X}$ is repeated on n independent identical systems $\simeq S$, all prepared in the same state P, then the frequency of measuring the value $\xi \in \sigma(\hat{X})$ is given by

$$\text{prob}_P\{\xi\} = \text{tr}\left(P\,\Pi_\xi(t)\right), \qquad (5.8)$$

(t = time of measurement).

Eq. (5.7) is the "collapse postulate", eq. (5.8) is Born's Rule.

Obviously, this view of "direct measurements" is totally inadequate! The main goal of these lectures is to replace it with a more mature, logically coherent quantum-mechanical theory of measurements (a "quantum theory of experiments"). However, to develop a precise "theory of indirect (weak) measurements", the framework described in (5.1) through (5.8), above, is viable.

We will not develop a completely general theory, but discuss a certain series of examples.

5.1. The Haroche - Raimond experiment

(Maassen - Kümmerer / Guerlin et al., Bauer - Bernard, Ballesteros - Fraas - JF - Schubnel, ...)

This is an experiment on an isolated system S, with

$$S = P \vee E, \tag{5.9}$$

where

P is a cavity in which a standing wave (coherent state) of electromagnetic radiation is excited.

E consists of experimental equipment used to indirectly measure an observable of P; (actually the number of photons stored in P).

E consists of:

(1) Probes $A_1, A_2, A_3, \cdots$ = identical Rydberg atoms, all prepared in the same superposition of two highly excited internal states, $|\uparrow\rangle$ and $|\downarrow\rangle$; (e.g., initial state of

every atom $= |\uparrow\rangle$.)

Atoms A_i, $i = 1, 2, 3, \ldots$, are <u>independent</u> of each other; do <u>not</u> interact with each other.

During time interval $[(m-1)\tau, m\tau)$, m^{th} atom streams through cavity P and is subsequently subjected to a projective measurement in:

(2) a <u>detector</u>, D, (τ: duration of a measurement cycle)

Measurement in D corresponds, for example, to measuring value, $\xi = \pm 1$, of observable X,

$$X := \sum_{\xi} \xi \, \pi_{\xi} ,$$

with

$$\left.\begin{aligned} \xi = 1 &\Leftrightarrow \pi_1 = |\uparrow\rangle\langle\uparrow| \\ \xi = -1 &\Leftrightarrow \pi_{-1} = |\downarrow\rangle\langle\downarrow| \end{aligned}\right\}$$

$S = P \vee E$

$\mathcal{O}_P = \{ \text{functions of } N \}$,

where N is the photon number operator

assoc. with cavity P.

$\mathcal{O}_E = \left\langle \mathbb{1}_P \otimes \mathbb{1}_{A_1} \otimes \cdots \otimes X_{A_m} \otimes \mathbb{1}_{A_{m+1}} \otimes \cdots \,\middle|\, m = 1,2,3,\cdots \right\rangle$

The Rydberg atoms A_i are out of resonance

w.r. to cavity P $\Rightarrow$ they do not emit or

absorb a photon from P during their

passage through P.

$\Rightarrow$ Time evolution commutes with N!

However, evolution of internal state of

every atom, A_m, $m = 1, 2, 3, \cdots$, during its passage

through P depends on the number of

photons in P.

§5.1.1

Assume, for simplicity, that only

$0, 1, 2, \cdots, N < \infty$ photons, (all of the same

frequency ω_0), can be excited in P.

Then

$$\mathcal{H}_P = \mathbb{C}^{N+1}$$

$$\mathbb{1}\Big/_{\mathcal{H}_P} = \sum_{\nu=0}^{N} \Pi_\nu ,$$

Π_ν : orth. projection on state with ν photons

$$\mathcal{N}\Big/_{\mathcal{H}_P} = \sum_{\nu=0}^{N} \nu \, \Pi_\nu$$

$$\mathcal{H}_{A_m} = \mathbb{C}^2 , \quad \forall \, m$$

$$\mathcal{H}_S = \mathcal{H}_P \otimes \mathcal{H}_D \otimes \mathbb{C}^2_{A_1} \otimes \mathbb{C}^2_{A_2} \otimes \mathbb{C}^2_{A_3} \otimes \cdots ,$$

with reference state

$$\omega = \omega_P \otimes \varphi_D \otimes |\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle \otimes \cdots , \quad (1)$$

where ω_P is an arbitrary density matrix on $\mathbb{C}^{N+1}$, φ_D is a state of D which it relaxes to exponentially fast after each proj. measurement of observable $X_m = X|_{\mathbb{C}^2_{A_m}}$, $m = 1, 2, 3, \cdots$, with a rate $\gg 1/\tau$;

$|\uparrow\rangle \in \mathbb{C}^2_{A_m}$: initial state of A_m.

Time evolution of S in time-interval $[(m-1)\tau, m\tau)$:

$$U^{(m)} := \sum_{\nu=0}^{N} \Pi_\nu|_{\mathcal{H}_P} \otimes R|_{\mathcal{H}_D} \otimes 1|_{\mathbb{C}^2_{A_1}} \otimes \cdots \otimes 1|_{\mathbb{C}^2_{A_{m-1}}}$$

$$\otimes u_\nu|_{\mathbb{C}^2_{A_m}} \otimes 1_{\mathbb{C}^2_{A_{m+1}}} \otimes \cdots ,$$

where u_ν is a unitary operator on $\mathbb{C}^2$, R is the time evolution of D driving its state back to φ_D after proj. measurement of $X_m = X|_{\mathbb{C}^2_{A_m}}$.

<u>Born's Rule for probes:</u>

$p(\xi|\nu)$ = probability for X ($= X_m$, for some $m = 1, 2, 3, \dots$) to have value ξ ($= \pm 1$), given that cavity, P, contains ν photons.

$$\boxed{p(\xi|\nu) = \langle \uparrow | u_\nu^* \pi_\xi u_\nu | \uparrow \rangle} \qquad (2)$$

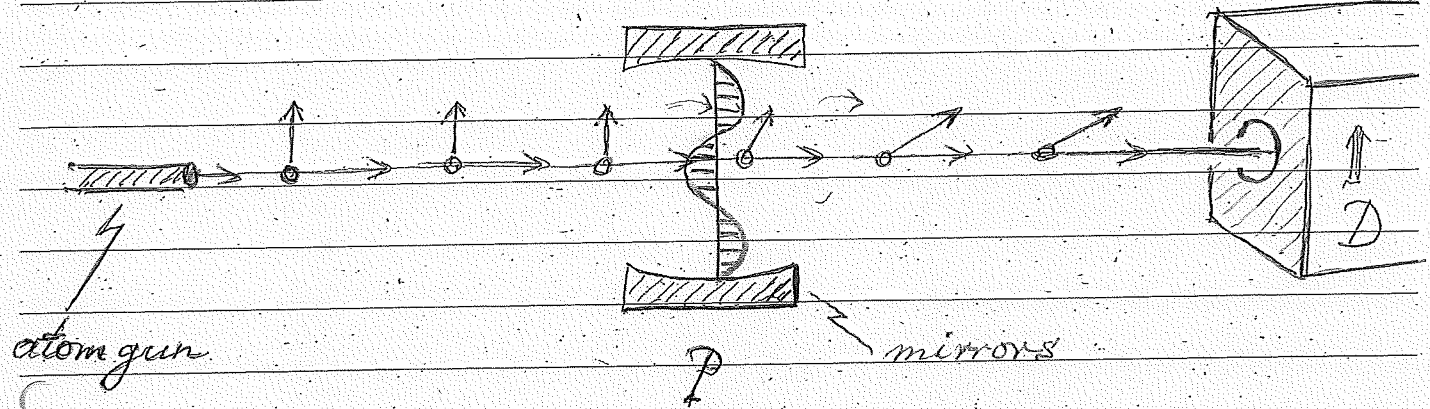
$$\Rightarrow \boxed{\sum_\xi p(\xi, \nu) = 1, \forall \nu.}$$

<u>Born's Rule for cavity:</u>

Probability to find ν photons in cavity P, given that P has been prepared in state ω_P:

$$p_\omega(\nu) := \omega_P(\Pi_\nu) \qquad (3)$$

Sketch of experiment:



Atom A_m precesses in space spanned by $|\uparrow\rangle, |\downarrow\rangle$ during passage through P, $\forall m$. Precession only depends on ν! (=photon number).

Example of a detector, D:
States $|\uparrow\rangle$ of D all occupied $\Rightarrow$ A_m in state $|\uparrow\rangle$ transmitted; A_m in state $|\downarrow\rangle$ absorbed (assuming A_m is a fermion $\to$ Pauli principle).

"Histories"

History of length r consists of a measurement protocol

$$\underline{\xi}_r = (\underline{\xi}_{r-1}, \xi_r)$$
$$= (\xi_1, \xi_2, \dots, \xi_r)$$

of values ξ_m of observable X_m measured by D on probe A_m after its passage through P.

$\mu(\underline{\xi}_r \mid \nu)$: probability of a history $\underline{\xi}_r$ of length r, given that there are precisely ν photons inside P.

Then

$$\mu(\underline{\xi}_r \mid \nu) = \prod_{m=1}^{r} p(\xi_m \mid \nu) \qquad (4)$$

(see (2), page ~~5~~.6)

$\mu_\omega(\underline{\xi}_r)$: probability of history $\underline{\xi}_r$, given that S has been prepared in state ω; (see (1), page 5.4).

Then, by (3), page 5.6, and (4):

$$\mu_\omega(\underline{\xi}_r) = \sum_{\nu=0}^{N} p_\omega(\nu)\, \mu(\underline{\xi}_r \mid \nu) \tag{5}$$

---

§ 5.1.3

Conditional probability for N to have the value ν, given a history $\underline{\xi}_r$ of length r, and given that S has been prepared in state ω:

$$p_\omega^{(r)}(\nu \mid \underline{\xi}) := \frac{p_\omega(\nu)\, \mu(\underline{\xi}_r \mid \nu)}{\mu_\omega(\underline{\xi}_r)} \tag{6}$$

where $\underline{\xi} := \underline{\xi}_{r=\infty} = (\xi_1, \xi_2, \xi_3, \cdots)$

Ξ : space of all histories, $\underline{\xi}$,
equipped with σ-algebra of
cylinder sets

Properties of functions $p_\omega^{(r)}(\nu \,|\, \cdot)$ on Ξ:

(i) $0 \leq p_\omega^{(r)}(\nu \,|\, \underline{\xi}) \leq 1$,

$$\sum_{\nu=0}^{N} p_\omega^{(r)}(\nu \,|\, \underline{\xi}) = 1 .$$

(ii)
$$p_\omega^{(r)}(\nu \,|\, \underline{\xi}) = p_\omega(\nu) \frac{\prod_{m=1}^{r} p(\xi_m \,|\, \nu)}{\sum_{\nu'} p_\omega(\nu') \prod_{m=1}^{r} p(\xi_m \,|\, \nu')}$$

$$= \underbrace{\frac{p_\omega(\nu) \prod_{m=1}^{r-1} p(\xi_m \,|\, \nu)}{\sum_{\nu'} p_\omega(\nu') \prod_{m=1}^{r-1} p(\xi_m \,|\, \nu')}}_{p_\omega^{(r-1)}(\nu \,|\, \underline{\xi})} \cdot \frac{p(\xi_r \,|\, \nu)}{\sum_{\nu''} \underbrace{p_\omega(\nu'') \frac{\prod_{m=1}^{r-1} p(\xi_m \,|\, \nu'')}{\sum_{\nu'} p_\omega(\nu') \prod_{m=1}^{r-1} p(\xi_m \,|\, \nu')}}_{p_\omega^{(r-1)}(\nu'' \,|\, \underline{\xi})} p(\xi_r \,|\, \nu'')}$$

Thus

$$\boxed{p_\omega^{(r)}(\nu|\underline{\xi}) = p_\omega^{(r-1)}(\nu|\underline{\xi}) \cdot \frac{p(\xi_r/\nu)}{\sum_{\nu''} p^{(r-1)}(\nu''|\underline{\xi})\, p(\xi_r/\nu'')}} \qquad (7)$$

(iii) <u>Conditional expectation</u>

$$\mathbb{E}_\omega\left[p_\omega^{(r)}(\nu|\cdot)\,\middle|\,\underline{\xi}_{r-1}\right]$$

$$= \left(\sum_{\xi_r} p_\omega^{(r)}(\nu|\underline{\xi})\,\mu_\omega(\underline{\xi}_{r-1}, \xi_r)\right) \Big/ \mu_\omega(\underline{\xi}_{r-1})$$

$$\overset{(4),(5),(6)}{=} \sum_{\xi_r} \frac{p_\omega(\nu)\prod_{m=1}^{r} p(\xi_m/\nu)}{\mu_\omega(\underline{\xi}_r)} \cdot \frac{\mu_\omega(\underline{\xi}_{r-1}, \xi_r)}{\mu_\omega(\underline{\xi}_{r-1})}$$

$$= p_\omega^{(r-1)}(\nu|\underline{\xi}),$$

<u>because</u> $\boxed{\sum_{\xi_r} p(\xi_r/\nu) = 1, \ \forall \nu, \ \text{by } (2).}$

Hence

$$\mathbb{E}_\omega\left[p_\omega^{(r)}(\nu\,|\,\cdot\,)\,\middle|\,\underline{\xi}_{r-1}\right] = p_\omega^{(r-1)}(\nu\,|\,\underline{\xi}_{r-1}), \quad \forall\,\nu \tag{8}$$

where $\quad \mathbb{E}_\omega[F] := \sum_{\underline{\xi}} F(\underline{\xi})\,\mu_\omega(\underline{\xi}). \qquad (8')$

(i) & (iii) $\Longrightarrow$ $\left\{p_\omega^{(r)}(\nu\,|\,\cdot\,)\right\}_{\nu=0,\dots,N}$ are

bounded martingales on Ξ.

By the martingale convergence theorem,

$$p_\omega^{(r)}(\nu\,|\,\cdot\,) \underset{r\to\infty}{\longrightarrow} p_\omega^{(\infty)}(\nu\,|\,\cdot\,), \tag{9}$$

μ_ω - almost everywhere on Ξ, $\forall\,\nu$ -

Next, using (ii) (see (7), page 5.11), we find ~~the~~ identity:

$$p_\omega^{(\infty)}(\nu\,|\,\underline{\xi}) = p_\omega^{(\infty)}(\nu\,|\,\underline{\xi})\,\frac{p(\xi\,|\,\nu)}{\sum_{\nu''} p_\omega^{(\infty)}(\nu''\,|\,\underbrace{\underline{\xi},\xi}_{(\xi_1,\dots,\xi_\infty,\xi)})\,p(\xi\,|\,\nu'')} \tag{10}$$

Suppose $\underset{\sim}{\xi} \in \Xi$ is such that

$$p_{\omega}^{(\infty)}(\nu \mid \underset{\sim}{\xi}) \neq 0, \quad \forall \nu \in I_{\underset{\sim}{\xi}} \subseteq \{0, 1, \dots, N\}$$

(but possibly $\subsetneq$!)

Then (i), page 5.11, implies that

$p_{\omega}^{(\infty)}(\cdot \mid \underset{\sim}{\xi})$ is a probability distribution on $I_{\underset{\sim}{\xi}}$. Moreover, (10) then implies that

$$p(\xi \mid \nu) = \sum_{\nu' \in I_{\underset{\sim}{\xi}}} p_{\omega}^{(\infty)}(\nu' \mid \underset{\sim}{\xi}, \xi)\, p(\xi \mid \nu')$$

$$= \overline{p(\xi \mid \cdot)}^{\,p_{\omega}^{(\infty)}}, \tag{11}$$

average w.r. to prob. measure $p_{\omega}^{(\infty)}$!

$\forall \nu \in I_{\underset{\sim}{\xi}}$.

$*$ Suppose that $p(\xi \mid \nu_1) = p(\xi \mid \nu_2)$ implies that $\nu_1 = \nu_2$. (Remember that ξ only takes 2 values, $\xi = \pm 1$, and $\sum_{\xi = \pm 1} p(\xi \mid \nu) = 1$, $\forall \nu$. Thus, if $*$ holds for $\xi = +1$ it also holds for $\xi = -1$!)

Then (11) implies that

$$I_{\underset{\sim}{\xi}} = \{ \nu(\underset{\sim}{\xi}) \}, \quad \text{and}$$

$$\boxed{p_\omega^{(\infty)}(\nu \,|\, \underset{\sim}{\xi}) = \delta_{\nu\, \nu(\underset{\sim}{\xi})}\,,} \tag{12}$$

for some $\nu(\underset{\sim}{\xi}) \in \{0, 1, \dots, N\}$.

$\boxed{\text{Purification!}}$

Exercise: Show that, under assumption $*$, (11) implies (12).

[Hint: If $p(\underset{\sim}{\xi} | \nu_*) = \max\limits_{\nu \in I_{\underset{\sim}{\xi}}} p(\underset{\sim}{\xi} | \nu)$ then

$p(\underset{\sim}{\xi} | \nu_*) = p(\underset{\sim}{\xi} | \cdot) \longrightarrow p_\omega^{(\infty)}$ is compatible

with $*$ iff $I_{\underset{\sim}{\xi}} = \{\nu_*\}$.]

Thm: Born's Rule

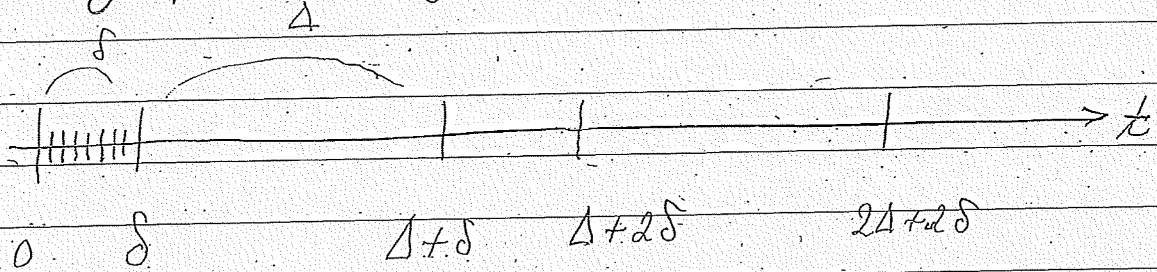
$$\boxed{\mathbb{E}_\omega\left[p_\omega^{(\infty)}(\nu | \cdot)\right] = p_\omega(\nu) = \omega_p(\Pi_\nu)}$$

Pf. Exercise, using (6) & (8'), p. 5.12.

§ 5.14

Quantum jumps & tracks

Here we consider a limiting regime where $\tau \to 0$; i.e., a very fast rate of shooting probes through cavity $\mathcal{P}$.



$$n = \frac{\delta}{\tau}$$

Limits: $\Delta > 0$ fixed.

(i) For fixed $\delta > 0$, let $\tau \to 0 \Rightarrow$

$$\boxed{n \to \infty}$$

(ii) Then let $\delta \to 0$!

Result.

(1) $[H_p, N] \neq 0$;

(2) Proj. measurements of probes lead to indirect measmt. of value of N, (photon number).

Then, in limiting regime #, we obtain a Markov chain, with

- state space $\mathcal{X} = \text{spec}\, N$
- transition function, Γ, given by

$$\boxed{\Gamma(\nu, \nu') := \left| \exp(-i\Delta H_p)_{\nu, \nu'} \right|^2}$$

→ Diffusion process on $\mathcal{X}$.

$\boxed{\text{Mott tracks}}$

A metaphor for the theory of *Indirect Measurements*

Plato's Allegory of the Cave – *'Politeia'*, in: Plato's *'Republic'*

As *Plato* was anticipating, more than 350 years BC, all we "prisoners of our senses" are able to perceive of the world are "shadows of reality" – *in the form of long streams of crude, uninteresting, directly perceptible signals ("projective observations") – from which the facts that give rise to the shadows can be reconstructed.*

According to *Socrates*, Philosophers (mathematicians & physicists) are "liberated prisoners" who are able to infer the fabric of reality from the shadows it creates on the wall of the cave. ($\nearrow$ Theory of perception!)

5.2 Solid state experiment, Mott tracks, etc.

I. The *Haroche-Raimond experiment:* $S = E \vee C$ (cavity)

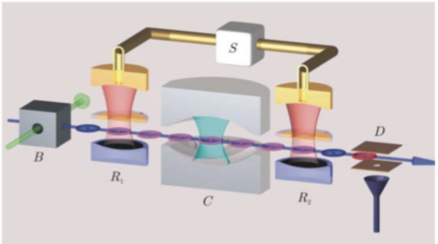
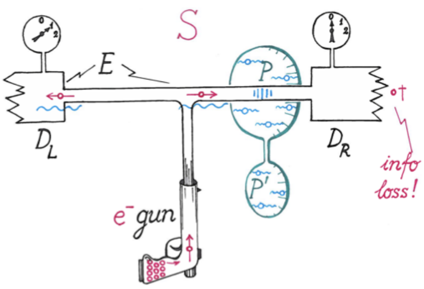


Fig. 4: Experimental setup to study microwave field states with the help of circular Rydberg atoms (see text).

B: atom/probe gun, R_1: State prep., C: Cavity, $\ldots$, D: Detector. Experiment measures # of *photons* in C indirectly, through probes.

II. A *solid-state (Gedanken-) experiment:*



Mott's Problem of Particle Tracks

In this example, one performs an indirect measurement of the number of electrons (charge) bound to the "quantum dot" P.

III. Appearance of *tracks* traced out by qm particles traversing laser light, or a bubble chamber, or a photographic plate; etc. – *Symmetry breaking by repeated projective measurements!*

Electrons and positrons produced simultaneously from individual gamma rays curl in opposite directions in the magnetic field of a bubble chamber. In the above example the gamma ray has lost some energy to an atomic electron, which leaves the long track, curling left. The gamma rays do not leave tracks in the chamber, as they have no electric charge.

Courtesy of the Lawrence Berkeley Laboratory, the University of California, Berkeley

Some details concerning the *solid-state experiment*

Isolated (open) system: $S = E \vee P$, where $P =$ quantum dot,
$E =$ "environment/equipment" consisting of:

(1) Probes: Independent electrons emitted by e^--gun, all prepared in the *same* initial state.
In the time interval $[(m-1)\tau, m\tau)$, the m^{th} electron, e^-, travels from the e^--gun through the T-shaped wire to either the detector D_L, or the detector D_R; and: $\tau =$ duration of a measurement cycle.

(2) Two electron detectors, D_L, D_R, serving to perform *projective observations* of final position of probes/electrons.

***

Explanation of how the *solid-state experiment* works:

- Physical quantities referring to the quantum dot P:

$$\mathcal{O}_P := \{\text{functions of } e^-\text{-number/charge operator } \mathcal{N}\}$$

- Physical quantities referring to the environment E:

$$\mathcal{O}_E = \{\mathbf{1}_P \otimes \mathbf{1}_{e_1^-} \otimes \cdots \otimes X_{e_m^-} \otimes \mathbf{1}_{e_{m+1}^-} \otimes \dots\}_{m=1,2,3,\dots},$$

Description of solid-state experiment

where the operator $X_{e_m^-}$ acts on the one-particle Hilbert space of the m^{th} probe (electron) traveling through the $T-$ shaped wires towards D_L, D_R, resp. It is given by

$$X_{e_m^-} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$

with infinitely degenerate eigenvalues $\xi = \pm 1$:

$$\xi = +1 \leftrightarrow e_m^- \text{ hits } D_L, \qquad \xi = -1 \leftrightarrow e_m^- \text{ hits } D_R.$$

The eigenprojection of $X_{e_m^-}$ corresp. to the eigenvalue ξ is denoted by π_ξ^m; $X_{e_m^-}$ observed around time $m\tau$.

State of S: Density Matrix Ω.

Our aim is to determine the *probability of the events* that, for $m = 1, 2, ..., k$, the m^{th} electron hits the detector D_{ξ_m}; $k = 1, 2, ...$

The LSW formula

For (strictly) independent electrons [3], this probability is given by a formula proposed by *Lüders, Schwinger* and *Wigner* (LSW):

$$\mu_\omega\big(\xi_1, \xi_2, \ldots, \xi_k\big) = \mathrm{tr}\big(\pi^k_{\xi_k} \cdots \pi^1_{\xi_1}\, \Omega\, \pi^1_{\xi_1} \cdots \pi^k_{\xi_k}\big) \qquad (1)$$

Since $\pi^k_1 + \pi^k_{-1} = \mathbf{1}, \forall k$, and because of the cyclicity of the trace,

$$\sum_{\xi_k} \mu_\omega(\xi_1, \xi_2, \ldots, \xi_{k-1}, \xi_k) = \mu_\omega(\xi_1, \xi_2, \ldots, \xi_{k-1}).$$

Thus, by a lemma due to *Kolmogorov*, μ_ω extends to a measure on the space, Ξ, of *"histories"* ($= \infty$ long measurement protocols $\underline{\xi} = \big(\xi_j\big)_{j=1}^{\infty}$; Ξ is equipped with the σ-algebra, Σ, gen. by cylinder sets).

First, we consider the situation where the passage of e^-'s from the electron gun through the $T-$ shaped wire to one of the detectors $D_\xi, \xi = \pm 1$, does *not* affect the charge, ν, of the quantum dot P, which is a conserved quantity $\rightarrow$ *"non-demolition measurements"*.

[3] the property of strict indep. of e^-'s is an instance of *"decoherence"*

Exchangeable probability measures

One can argue that the measure μ_ω is *exchangeable*, i.e.:

$$\mu_\omega\big(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)}\big) = \mu_\omega\big(\xi_1, \ldots, \xi_k\big), \tag{2}$$

for all permutations, σ, of $\{1, \ldots, k\}$, for arbitrary $k < \infty$. It then follows from *De Finetti's* Theorem that

$$\mu_\omega(\xi_1, \ldots, \xi_k) = \int_{\Xi_\infty} \mathrm{d}P_\omega(\nu) \prod_{m=1}^{k} p(\xi_m|\nu) \tag{3}$$

Here Ξ_∞ is the spectrum of the algebra generated by *tail events* (bounded functions on Ξ measurable at ∞[4]); i.e., it is the *"space of facts"* ("Dinge an sich") constituting *the true reality Plato has been talking about*; whereas the measurement protocols $\underline{\xi}_k := (\xi_m)_{m=1}^k$, $k < \infty$, are the *shadows on the wall of the cave that we prisoners are able to perceive* – as we shall now explain!

[4] equiv. classes (w.r. to a measure class determined by normal states of *S*) of functions on Ξ *not* dep. on any finite number of measurement outcomes

Interpretation of Ξ_∞ in the solid-state experiment

Suppose every electron traveling from the e^--gun to one of the detectors $D_{\pm 1}$ is prepared in the same one-particle state ϕ_0. Assuming that the charge operator, $\mathcal{N}$, of the quantum dot P is a conservation law, the time evolution of the state ϕ_0 until the e^- hits a detector is given by

$$U_\nu \phi_0,$$

where U_ν is a unitary operator on the one-electron Hilbert space dep. on the charge $\nu \leq N$ of P: The charge ($\propto$ nb. of e^-) bound by P creates a "Coulomb blockade" in the *right* arm of the $T-$ shaped wire. Thus, the larger ν, the more likely it is that an electron in the wire will be scattered onto the detector $D_1 \equiv D_L$.

The orthogonal projection onto the subspace of one-electron wave functions that vanish identically near $D_{-\xi}$ is denoted by π_ξ. The probability, $p(\xi|\nu)$, that an e^- hits D_ξ is given by *Born's Rule*

$$p(\xi|\nu) = \langle \phi_0, U_\nu^* \pi_\xi U_\nu \, \phi_0 \rangle, \tag{4}$$

and the "space of facts", Ξ_∞, is given by

$$\Xi_\infty = \text{spec}(\mathcal{N}) = \{0, 1, 2, \dots, N\}, N < \infty, \quad \mathcal{N} = \text{charge operator of } P.$$

Interlude:

Definition of a descending filtration of σ-algebras.

$\{\Sigma_n\}_{n=1}^{\infty}$ is a descending filtration of σ-algebras contained in the σ-algebra Σ of a measure space (Ξ, Σ) iff

$$\Sigma_n \subset \Sigma_m \subseteq \Sigma \equiv \Sigma_0, \qquad (i)$$

for all $n \geq m \geq 0$.

We set

$$\Sigma_\infty = \bigcap_{n \in N} \Sigma_n \qquad (ii)$$

We define $\mathcal{L}_n$ to be the algebra of all bounded, Σ_n-measurable functions on Ξ;

$$\mathcal{L}_\infty = \bigcap_n \mathcal{L}_n \qquad (iii)$$

is the algebra of bounded functions on Ξ measurable at ∞.

Let μ be a probability measure on (Ξ, Σ); let $\chi_\Delta = \chi_\Delta^2 > 0$ be a characteristic of a measurable set $\Delta \in \Sigma_\infty$; clearly χ_Δ is an orthogonal projection in $\mathcal{L}_\infty$, and every orthogonal projection in $\mathcal{L}_\infty$ is of this form. We define a probability measure, μ_∞, on (Ξ, Σ_∞) by setting

$$\mu_\infty(\Delta) := \mu(\Delta) = \int d\mu(\xi)\, \chi_\Delta(\xi), \qquad (iv)$$

$\Delta \in \Sigma_\infty$. Let $f \in \mathcal{L}_0 = L^\infty(\Xi)$ be a bounded Σ-measurable function on Ξ, and suppose that $f \geq 0$. Then

$$\mu_f(\Delta) := \int_\Xi d\mu(\xi)\, f(\xi)\, \chi_\Delta(\xi) \qquad (v)$$

defines a bounded measure on Σ_∞ with the property that

$$\mu_\infty(\Delta) = 0 \Rightarrow \mu_f(\Delta) = 0,$$

$\forall\, \Delta \in \Sigma_\infty$. Thus, μ_f is absolutely continuous with respect to μ_∞ (as measures on (Ξ, Σ_∞)). This implies that the Radon-Nikodym derivative, $\frac{d\mu_f}{d\mu_\infty}$, defines a positive function on the space $(\Xi_\infty, \Sigma_\infty)$, where Ξ_∞ is the spectrum of $\mathcal{L}_\infty$. We set

$$\frac{d\mu_f}{d\mu_\infty}(\eta) =: \mu(f|\eta), \quad \eta \in \Xi_\infty, \qquad (vi)$$

which is a non-negative function on Ξ_∞ μ_∞-almost everywhere, with

$$0 \leq \mu(f|\eta) \leq \|f\|_\infty \qquad (vii)$$

By construction, $\mu(f|\eta)$ is thus a bounded, positive, linear functional on $\mathcal{L}_0$ with $\mu(f \equiv 1|\eta) = 1$. We conclude that $\mu(f|\eta)$ determines a probabity measure, $d\mu(\xi|\eta)$, on (Ξ, Σ), for μ_∞-almost every $\eta \in \Xi_\infty$.

By construction, for $\Delta \in \Sigma_\infty$ and $0 \leq f \in \mathcal{L}_0$,

$$\int_\Delta d\mu_\infty(\eta) \left(\int_\Xi d\mu(\xi|\eta) f(\xi) \right) = \int_\Delta \mu(f|\eta)\, d\mu_\infty(\eta)$$

$$= \int_{\Delta} \frac{d\mu_f(\eta)}{d\mu_\infty(\eta)} \, d\mu_\infty(\eta)$$

$$= \mu_f(\Delta) = \int_{\Xi} d\mu(\xi) f(\xi) \chi_\Delta(\xi). \qquad (viii)$$

We assume that $\exists$ two Σ_∞ - measurable sets Δ, Δ', with $\mu(\Delta) > 0$, $\mu(\Delta') > 0$ and $\Delta \cap \Delta' = \phi$. Setting $f(\xi) = \chi_{\Delta'}(\xi)$, we find from (viii)

$$\int_{\Delta} d\mu_\infty(\eta) \, \mu(\Delta' | \eta) = \int_{\Xi} d\mu(\xi) \chi_{\Delta'}(\xi) \chi_\Delta(\xi)$$

$$= 0, \text{ because } \Delta \cap \Delta' = \phi \qquad (ix)$$

But

$$\int_{\Delta} d\mu_\infty(\eta) \, \mu(\Delta | \eta) = \int_{\Xi} d\mu(\xi) \chi_\Delta(\xi)$$

$$= \mu(\Delta) > 0 \qquad (x)$$

Likewise,

$$\int_{\Delta'} d\mu_\infty(\eta) \, \mu(\Delta' | \eta) = \mu(\Delta') > 0, \qquad (xi)$$

and
$$\int_{\Delta'} d\mu_\infty(\eta)\, \mu(\Delta/\eta) = 0 \qquad (xii)$$

We now define the measures $d\mu(\xi/\Delta)$ by

$$\int d\mu(\xi/\Delta) f(\xi) := \int_\Delta d\mu_\infty(\eta) \int d\mu(\xi/\eta) f(\xi),$$

for arbitrary $f \in \mathcal{L}_0$ and $\Delta \in \Sigma_\infty$.

By (viii),

$$f = f \cdot \chi_\Delta \quad \text{on } \operatorname{supp} d\mu(\cdot/\Delta) \qquad (xiii)$$

Thus, if $\tilde{\Delta}$ is an arbitrary Σ-measurable set with $\mu(\tilde{\Delta}/\Delta) > 0$, i.e.,

$$\begin{aligned} 0 < \mu(\tilde{\Delta}/\Delta) &= \int_{\tilde{\Delta}} d\mu(\xi/\Delta) \\ &= \int_\Delta d\mu_\infty(\eta) \int d\mu(\xi/\eta)\, \chi_{\tilde{\Delta}}(\xi) \\ &= \int d\mu(\xi)\, \chi_\Delta(\xi) \underbrace{\left(\chi_{\tilde{\Delta}}(\xi)\chi_\Delta(\xi)\right)}_{\leq \chi_\Delta(\xi)} \end{aligned}$$

We claim that if Δ and Δ' are two Σ_∞-measurable sets with $\Delta \cap \Delta' = \phi$ then

$d\mu(\cdot|\Delta)$ and $d\mu(\cdot|\Delta')$ are <u>mutually singular</u> measures.

<u>Proof of claim.</u> Suppose <u>not</u>. Then $\exists$ Σ-measurable set $\tilde{\Delta} \subset \operatorname{supp} d\mu(\cdot|\Delta)$, with $\mu(\tilde{\Delta}|\Delta) > 0$ such that $\mu(\tilde{\Delta}|\Delta') > 0$. Since $\tilde{\Delta} \subset \operatorname{supp} d\mu(\cdot|\Delta)$ we have that

$$\chi_{\tilde{\Delta}} = \chi_{\tilde{\Delta}} \cdot \chi_{\Delta} \leq \chi_{\Delta}.$$

Thus,

$$
\begin{aligned}
0 < \mu(\tilde{\Delta}|\Delta') &= \int_{\Delta'} d\mu_\infty(\eta) \int d\mu(\xi|\eta)\, \chi_{\tilde{\Delta}}(\xi) \\
&\leq \int_{\Delta'} d\mu_\infty(\eta) \int d\mu(\xi|\eta)\, \chi_{\Delta}(\xi) \\
&= \int d\mu(\xi)\, \chi_{\Delta}(\xi)\, \chi_{\Delta'}(\xi) \\
&= 0,
\end{aligned}
$$

a contradiction!

An example.

We choose $\Xi := \overset{\infty}{\underset{i=1}{\times}} \mathcal{X}_i$, where

$$\mathcal{X}_i = \mathcal{X} = \{1, \dots, M\}, \quad \forall i$$

Σ is the σ-algebra of cylinder sets in Ξ. Points in Ξ are sequences

$$\underline{\xi} = \{\xi_j\}_{j=1}^{\infty},$$

and we set $\underline{\xi}_n := \{\xi_1, \dots, \xi_n\}$, $\underline{\xi}_{>n} := \{\xi_j\}_{j=n+1}^{\infty}$.

The algebra $\mathcal{L}_n$ is the algebra of all bounded Σ-measurable functions on Ξ that only depend on $\underline{\xi}_{>n}$, and Σ_n is the smallest σ-algebra with the property that all functions in $\mathcal{L}_n$ are Σ_n-measurable.

Let μ be an exchangeable probability measure on (Ξ, Σ). Then, by De Finetti,

$$\mu(\underline{\xi}_n, \Xi_{>n}) = \int_{\Xi_\infty} d\mu_\infty(\eta) \prod_{j=1}^{n} p(\xi_j | \eta), \qquad (xiv)$$

$\Xi_{>n} = \{ \underline{\xi}_{>n} \mid \xi_j \in \mathcal{X}, \ \forall j \geq n+1 \}$, $\forall n = 1, 2, 3, \dots,$

for some non-negative measurable functions $p(\xi|\eta)$, $\xi = 1, \ldots, M$, on Ξ_∞, with $\sum_{\xi=1}^{M} p(\xi|\eta) = 1$, for almost every η.

In this example,

$$\mu(\underline{\xi}_n, \Xi_{>n} | \eta) \doteq \prod_{j=1}^{n} p(\xi_j | \eta) \qquad (xv)$$

For every $\xi \in \mathcal{X}$, we define a function

$$f_\xi(\underline{\xi}) := \lim_{k\to\infty} \frac{1}{k} \left\{ \sum_{j=1}^{k} \delta_{\xi \xi_j} \right\} \qquad (xvi)$$

the "frequency" of ξ. Frequencies have the following properties:

(1) The limit on the R.S. of (xvi) exists, for all $\underline{\xi} \in \operatorname{supp} \mu(\cdot|\eta)$;

(2) $f_\xi \in \mathcal{L}_\infty$, meaning that f_ξ does not depend on $\underline{\xi}_n$, for any $n < \infty$.

(3) On $\operatorname{supp}\mu(\cdot|\eta)$,

$$f_\xi(\underline{\xi}) = p(\xi|\eta), \qquad (xvii)$$

(i.e., f_ξ is a constant on $\operatorname{supp}\mu(\cdot|\eta)$!)

which is nothing but the "Law of Large Numbers".

Let us suppose that the functions

$$\{p(\xi|\eta) \mid \xi = 1, \cdots, M\}$$

separate points on Ξ_∞. Then, for any pair of points η, η' in Ξ_∞, with $\eta \neq \eta'$, $\exists\, \xi$ such that $p(\xi|\eta) \neq p(\xi|\eta')$. Hence the value of f_ξ on $\operatorname{supp} \mu(\cdot|\eta)$ is different from its value on $\operatorname{supp} \mu(\cdot|\eta')$, which proves again that $\mu(\cdot|\eta)$ and $\mu(\cdot|\eta')$ are mutually singular. (For, if $\Delta \subset \Xi$ were a measurable set with $\mu(\Delta|\eta), \mu(\Delta|\eta')$ strictly positive then, for $\underline{\xi} \in \Delta$,

$$p(\xi|\eta) = f_\xi(\underline{\xi}) = p(\xi|\eta'),$$

which contradicts $p(\xi|\eta) \neq p(\xi|\eta')$!)

Besides the Law of Large Numbers, we might also want to apply the Central Limit Theorem.

$\longrightarrow$ $\boxed{\text{Pass to Sect. 5.3!}}$

3. Indirect Non-Demolition Measurements: Basic Assumptions and General Results

Thinking of the solid-state experiment, we will henceforth assume:[5]

(i) The measures μ_ω are exchangeable (non-demolition measurements using independent e^-!), so that

$$\mu(\underline{\xi}_k|\nu) = \prod_{m=1}^{k} p(\xi_m|\nu), \quad \xi_m \in \mathcal{X}_S := \sigma(\hat{X}), \forall m, \ \nu \in \Xi_\infty.$$

(ii) The "space of facts" is a finite set of points ("charge values"):

$$\Xi_\infty = \{0, 1, 2, \ldots, N\}, \qquad \text{for some } N < \infty. \tag{6}$$

(iii) We also assume that $p(\xi|\cdot)$ separates points of Ξ_∞: There exists $\kappa > 0$ such that

$$\min_{\nu_1 \neq \nu_2} |p(\xi|\nu_1) - p(\xi|\nu_2)| \geq \kappa > 0, \quad \text{for some } \xi \in \mathcal{X}_S. \tag{7}$$

---

[5] these assumptions can and have been generalized

Summary of main results

Equivalence classes of functions on the space Ξ of histories measurable at ∞ form an abelian algebra: the algebra of *"observables at ∞"* $=$ $\{$functions on the "space of facts" $\Xi_\infty\}$ isomorphic to $\mathrm{Diag}_{(N+1)}$.
An example of an *"observable at infinity"* is the *"asymptotic frequency"* of an event $\xi \in \mathcal{X}_S$: We define the frequencies

$$f_\xi^{(l,l+k)}(\underline{\xi}) := \frac{1}{k}\left(\sum_{m=l+1}^{l+k} \delta_{\xi,\xi_m}\right), \quad \text{with } \sum_\xi f_\xi^{(l,l+k)}(\underline{\xi}) = 1. \qquad (8)$$

Main results, given Assumptions (i), (ii), (iii)):

(I) *Law of Large Numbers for exchangeable measures:* For every $\underline{\xi} \in \Xi$, the asymptotic frequency satisfies

$$\lim_{k\to\infty} f_\xi^{(l,l+k)}(\underline{\xi}) =: p(\xi|\nu), \qquad (9)$$

for some "fact" $\nu \in \Xi_\infty$.

"q-hypothesis testing" /parameter estimation

With each $\nu \in \Xi_\infty$ we associate a subset

$$\Xi_\nu(l,k;\underline{\varepsilon}) := \{\underline{\xi} \mid |f_{\underline{\xi}}^{(l,l+k)}(\underline{\xi}) - p(\xi|\nu)| < \epsilon_k\} \tag{10}$$

(see blackboard!) where

$$\epsilon_k \to 0, \sqrt{k}\,\epsilon_k \to \infty, \quad \text{as } k \to \infty$$

(II) *Distinguishability:* It follows from Hyp. (7) and definition (8) that, for k so large that $\epsilon_k < \kappa/2$,

$$\Xi_{\nu_1}(l,k;\underline{\varepsilon}) \cap \Xi_{\nu_2}(l,k;\underline{\varepsilon}) = \emptyset, \quad \nu_1 \neq \nu_2.$$

(III) *Central Limit Theorem:* $\Rightarrow$ Under suitable hypotheses on the states ω, (see (i) through (iii)),

$$\mu_\omega\left(\bigcup_\nu \Xi_\nu(l,k;\underline{\varepsilon})\right) \to 1, \quad \text{as } k \to \infty.$$

hypothesis testing – ctd.

Thus, for large enough k, every protocol $\underline{\xi}_k = (\xi_m)_{m=1}^k$ determines, *with an error that tends to* 0, *as* $k \to \infty$, a unique point $\nu \in \Xi_\infty$. Moreover, *Born's Rule* holds:

$$\boxed{\mu_\omega\big(\Xi_\nu(I, k; \underline{\varepsilon})\big) \underset{k\to\infty}{\to} \omega(\delta_{\mathcal{N},\nu}) = P_\omega(\nu).}$$

(IV) *Theorem of Boltzmann-Sanov* $\Rightarrow$ If the measures μ_ω are exchangeable one has that

$$\mu\big(\Xi_{\nu_1}(I, k; \underline{\varepsilon})|\nu_2\big) \leq C\ e^{-k\sigma(\nu_1\|\nu_2)},$$

where σ is the relative entropy of the distribution $p(\cdot|\nu_1)$ given $p(\cdot|\nu_2)$.

(V) *Theorem of Maassen-Kümmerer & Bauer-Bernard* (see (III), (IV)!)

The state of S, restricted to $B(\mathcal{H}_P)$, approaches an eigenstate, ω^ν, of the operator $\mathcal{N}$ corresponding to some eigenvalue ν (i.e., to a fixed charge ν of the quantum dot P), as the number k of projective probe measurements tends to ∞: *"Purification"!*

Beyond non-demolition measurements

It should be mentioned that, under natural assumptions, the "extremal" measures $\mu(\cdot|\nu)$, $\nu \in \Xi_\infty$, come from *normal states*, ω^ν, on $\mathcal{E}$ via the LSW formula! Theory can be extended to operators $\mathcal{N}$ with continuous spectrum.

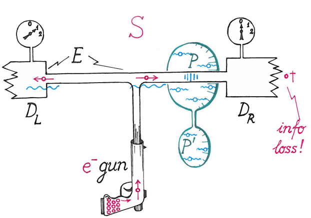
***

The theory of indirect measurements outlined so far only concerns measurements of time-independent "facts", which correspond to points in Ξ_∞: *non-demolition measurements!* The outcomes of such measurements only depend on the tails of histories (at arbitrarily late times). However, most interesting facts depend on time, i.e., are "events" appearing and disappearing, and $\Xi_\infty = \emptyset$! Thus, we must ask how one can infer or reconstruct information concerning *events* and their time evolution from finitely long recordings of projective observations of quantities associated with probes represented by operators that act on the Hilbert spaces of the probes.

4. Weak Measurements of Time-Dependent Quantities –

Markov Jump Processes on the Spectra of Observables & Mott Tracks

A more realistic version of the solid-state experiment:



The system S is the composition of the quantum dot P and the equipment E (consisting of the electron gun, the detectors D_L and D_R, and the "dumps" where the electrons get lost).
Monitor the electric charge bound to P, but *without* assuming that it is constant in time. In other words, we admit that P may exchange electric charge with its environment (P'), assuming that the total charge of P varies slowly in time as compared to the rate by which electrons travel through the T-shaped wire.

The formalism

As in part 2, we assume that every electron released by the e^- gun and traveling through the T-shaped wire is prepared in the same inital state, ϕ_0, and that the same "observable", $\hat{X}$, is measured for each electron, namely whether it reaches the dump on the left $\leftrightarrow$ eigenvalue $\xi = +1$, or the dump on the right $\leftrightarrow$ eigenvalue $\xi = -1$ of $\hat{X}$.
It is assumed that during the passage of one electron through the T-shaped wire the charge of P is *very nearly constant*, and that the probability of an electron to reach the dump labelled by ξ is given by

$$p(\xi|\nu) = \langle U_\nu \phi_0, \pi_\xi U_\nu \phi_0 \rangle,$$

where $\nu \in \{0, 1, \ldots, N\}$ is the charge of P, i.e., the eigenvalue of the charge operator $\mathcal{N}$, U_ν is a unitary "scattering operator" depending on ν, and π_ξ is the spectral projection of $\hat{X}$ corresponding to the eigenvalue ξ.
We still require assumption (iii) on $p(\xi|\nu)$, namely:

There exists $\kappa > 0$ such that

$$\min_{\nu_1 \neq \nu_2} |p(\xi|\nu_1) - p(\xi|\nu_2)| \geq \kappa > 0, \quad \text{for some } \xi \in \mathcal{X}_S \, (= \{-1, +1\})$$

Formalism – ctd.

However, we do not assume that the charge operator $\mathcal{N}$ *commutes with the time evolution of* S. Instead, we assume that

$$H_P = \varepsilon h_P, \text{ with } \|[h_P, \mathcal{N}]\| \leq 1, \; (\varepsilon > 0), \tag{11}$$

where H_P is the *Hamiltonian* of the dot P when it is *decoupled* from the equipment E, i.e., during periods when there isn't any electron traveling through the T-shaped wire.

The system S is *not* autonomous: Electrons are released by the e^- gun at random times; the j^{th} electron travels through the T-shaped wire at time t_j, with $t_j < t_{j+1}$ and

$$t_{j+1} - t_j \text{ is Poisson-distributed with rate } 1 \, , \; j = 1, 2, \ldots$$

Let $\Omega(t)$ denote the state of the system S at a time $t \in (t_j, t_{j+1})$, for some j. Then the density matrix, $\Omega_P(t)$, describing the state of the quantum dot P at time t is given by the partial trace

$$\Omega_P(t) = tr_E\big(\Omega(t)\big)$$

A recursion formula for the state of S

We assume that the dynamics of P during a period of time when there isn't any electron traveling through the T-shaped wire is generated by the Hamiltonian H_P, and that, at time t_j, P only interacts with the j^{th} electron released by the e^- gun.

Suppose the state, $\Omega_P(t')$, of P is known at a time $t' \in (t_{j-1}, t_j)$. Then we are able to calculate the state $\Omega_P(t)$ of P at time $t \in (t_j, t_{j+1})$. Here is a formula for $\Omega_P(t)$, given $\Omega_P(t')$:

$$\boxed{\Omega_P(t) = z_{\xi_j} \cdot e^{i(t-t_j)H_P} V_{\xi_j} e^{i(t_j-t')H_P} \Omega_P(t') e^{i(t'-t_j)H_P)} V_{\xi_j} e^{i(t_j-t)H_P},}$$

where z_{ξ_j} is a normalization factor, and $V_\xi \delta_{\mathcal{N},\nu} = \sqrt{p(\xi|\nu)}$.

The state, $\Omega_P(t)$, of P can be determined recursively from an initial state prepared at a time $t' < t_1$, for arbitrary time t! It is a random variable depending on ε (where ε, see (11), measures the rapidity of the time-evolution of the charge of P), on the *random times* $t_1, \ldots, t_j$, with $t_j < t < t_{j+1}$, and on the *measurement protocols* $(\xi_m)_{m=1}^{j}$.
We define probabilities, $\omega_{t,\varepsilon}(\nu)$, on the spectrum of the charge operator $\mathcal{N}$ as the diagonal elements of the density matrix $\Omega_P(t)$, and we denote by $\overline{\omega}_{t,\varepsilon}(\nu)$ the average of $\omega_{t,\varepsilon}(\nu)$ over all choices of random times and measurement protocols $\underline{\xi}$.

Main result

Consider the limiting regime given by $t = \varepsilon^{-2}\tau$, where $\tau \in \mathbb{R}$ is a rescaled time variable to be kept at an arbitrary, but fixed value. We are interested in the limiting regime

$$\varepsilon \searrow 0.$$

Let $Q(\nu, \nu')$ be the "Markov kernel" given by

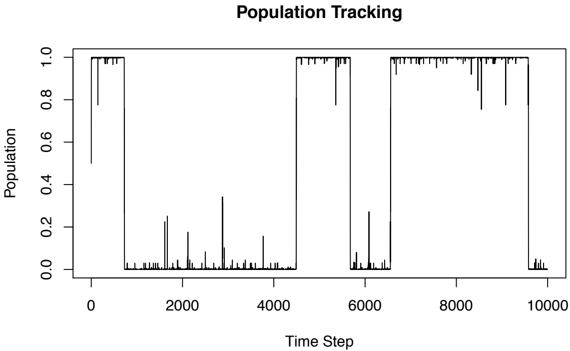
$$Q(\nu, \nu') = -\frac{|\langle \nu | h_P | \nu' \rangle|^2}{\sum_{\xi \in \mathcal{X}_S} V_\xi(\nu) V_\xi(\nu') - 1} + cc, \quad \nu \neq \nu',$$

with $Q(\nu, \nu) = \cdots \geq 0, \forall \nu$.
Then $\big(exp(-\tau Q)\big)_{\tau > 0}$ is the transition function of a Markov jump process on the spectrum, $\{0, 1, \ldots, N\}$, of the charge operator $\mathcal{N}$, and

$$\boxed{\lim_{\varepsilon \searrow 0} \overline{\omega}_{\varepsilon^{-2}\tau,\, \varepsilon}(\nu) = \sum_{\nu'=0}^{N} exp(-\tau Q)(\nu, \nu')\, \omega_{t=0}(\nu').} \tag{12}$$

A numerical simulation of the behavior of $\omega_{t,\varepsilon}(\nu)$

We consider an example where $N = 1$, with a quantum dot P whose state space is two-dimensional. The Hamiltonian $H_P = \varepsilon h_P$ describes exchange of electric charge between P and a charge reservoir tuning the average value of the electric charge of P to some fixed value $0 < \nu < 1$. The plot shows the behavior of $\omega_{t,\varepsilon}(\nu = 0)$, (the probability of populating the neutral state of P corresponding to $\nu = 0$), for small values of ε: *Quantum jumps*!



Perturbations of Non-Demolition Measurements

(Diffusive Evolution of States)

Isolated system: $S = P \vee E$,

P: subsystem of interest (e.g. cavity in H-R.)

E: environment/exp. equipment

Assume that E consists of independent probes, A_i, interacting, one after another, with P and being subject to a measurement afterwards; (A_i e.g. Rydberg atoms; ↗ H-R).

States of S: Density matrices, ρ_S, acting on $\mathcal{H}_S$, with

$$\mathcal{H}_S = \mathcal{H}_P \otimes \mathcal{H}_E \qquad (21)$$

States of P:

$$\rho_P = tr_E \, \rho_S \, , \qquad (22)$$

Hilbert space of a single probe A_i

$$\mathcal{H}_{A_i} \cong \mathbb{C}^k, \quad k < \infty, \quad \forall i.$$

Initial state of every probe is

$$\varphi_0 \in \mathbb{C}^k$$

Reference state in $\mathcal{H}_E$:

$$\bigotimes_{i=1}^{\infty} \varphi_0^{(i)}, \quad i \text{ indexes probes.}$$

$\mathcal{H}_E$ = completion of linear span of vectors

$$\bigotimes_{i=1}^{\infty} \psi^{(i)}, \qquad (23)$$

with $\psi^{(i)} = \varphi_0^{(i)}$, except for finitely many i.

$\exists$ single instrument, $\mathcal{I}$, given by a s.a. op. X on $\mathbb{C}^k$, with

$$X = \sum_{\xi \in \mathcal{X}} \xi \, \Pi_{\xi}, \qquad (24)$$

$\mathcal{X}$ = spec X assumed to be simple; i.e., $|\mathcal{X}| = k$; same instrument for all probes.

During every sufficiently small time interval $(s,t) \subset \mathbb{R}$, only (at most) one probe, A_i, interacts with P. At a time t_i, this interaction is interrupted and the observable X is measured for A_i, with result $\xi_i \in \mathcal{X}$.

The initial state of every probe A_i, before it starts to interact with P, is given by φ_0; i.e., by the density matrix

$$P_0^{(i)} = |\varphi_0\rangle\langle\varphi_0|, \qquad (25)$$

$i = 1, 2, 3, \cdots$.

Let $P^{(i-1)}_{P,+}$ be the state of P after it has interacted with $A_1, \dots, A_{i-1}$ and right before it starts to interact with A_i. The time-evolution of the state of P coupled to A_i from time t_{i-1} to time t_i, right before X is measured for A_i, is given by

$$P^{(i)}_{P,-} := U(t_i, t_{i-1})\, P^{(i-1)}_{P} \otimes P^{(i)}_{0}\, U(t_i, t_{i-1})^* \qquad (26)$$

At time t_i, X is measured for A_i, with resulting value $\xi_i \in \mathcal{X}$. After this measurement, the state of P is given by

$$P^{(i)}_{P,+} := \frac{tr_{\mathcal{H}_{A_i}}\left(P^{(i)}_{P,-}\ \mathbb{1} \otimes \Pi_{\xi_i}\right)}{tr_{\mathcal{H}_P \otimes \mathcal{H}_{A_i}}\left(P^{(i)}_{P,-}\ \mathbb{1} \otimes \Pi_{\xi_i}\right)} \qquad (27)$$

Assumption on time evolution:

Let N be a s.a. operator on $\mathcal{H}_p$ with (for simplicity) simple spectrum $\{0, 1, \cdots, N\}$; $\left(\mathcal{H}_p \cong \mathbb{C}^{N+1}, N < \infty\right)$.

Let P_ν be the orth. proj. onto the eigenspace of N corresponding to the eigenvalue $\nu \in \{0, 1, \cdots, N\}$.

Hypothesis.

$U(t_i, t_{i-1})$ acts trivially on

$$\bigotimes_{j \neq i} \mathcal{H}_{A_j},$$

while on the factor $\mathcal{H}_p \otimes \mathcal{H}_{A_i}$ it is generated by the Hamiltonian

$$H^{(i)} = H_p \otimes \mathbb{1}_{\mathcal{H}_{A_i}} + \sum_{\nu=0}^{N} P_\nu \otimes h_\nu, \qquad (28)$$

with

$$H_P = \varepsilon H, \qquad (29)$$

where H is an arbitrary s.a. matrix on $\mathcal{H}_P$; in particular, $[H, N]$ might be non-zero! Furthermore, $0 < \varepsilon \ll 1$.

The matrices h_ν act on $\mathcal{H}_{A_i} \cong \mathbb{C}^k$, are self-adjoint, but otherwise arbitrary.

To simplify our analysis, it is convenient to consider the following simplification:

$$U(t_i, t_{i-1}) := \left(\sum_{\nu=0}^{N} P_\nu \big|_{\mathcal{H}_P} \otimes U_\nu \big|_{\mathcal{H}_{A_i}} \right) \times$$

$$\times \exp\left(-i(t_i - t_{i-1}) H_S / \hbar\right)\big|_{\mathcal{H}_P} \otimes \mathbb{1}\big|_{\mathcal{H}_{A_i}}, \qquad (30)$$

where $\{U_\nu\}_{\nu=0,1,\dots,N}$ are arbitrary, but fixed unitaries on $\mathbb{C}^k$.

Rationale: A_i only briefly interacts with P, during a time interval $[t_{i-1}, t_{i-1}+\delta]$, δ indep. of i, $t_{i-1}+\delta < t_i$, $\forall i$. Its initial state φ_0 is then mapped to $U_\nu \varphi_0$, assuming that P is in an eigenstate of N corresp. to eigenvalue ν.

Let

$$V_\xi := \sum_\nu P_\nu \langle \psi_\xi, U_\nu \varphi_0 \rangle, \qquad (31)$$

where ψ_ξ is the eigenstate of X corresponding to the eigenvalue $\xi \in \mathcal{X}$.

Note that

$$[V_\xi, N] = 0, \ \forall \xi \in \mathcal{X}. \qquad (32)$$

We then find that

$$P^{(i)}_{P,+} = Z^{-1}_{\xi_i} V_{\xi_i} e^{-i(t_i - t_{i-1})H_S/\hbar} P^{(i-1)}_{P,+} \times$$

$$\times e^{i(t_i - t_{i-1})H_S/\hbar} V^*_{\xi_i} \qquad (33)$$

where Z_{ξ_i} is chosen such that

$$\underset{\mathcal{H}_P}{tr}\left(P^{(i)}_{P,+}\right) = 1. \qquad (34)$$

For concreteness, we now assume that measurement times, $\{t_i\}_{i=1,2,3,\dots}$, are random, with a Poisson distribution of rate $\gamma = 1$:

$$\sum_{\xi \in \mathcal{X}} V^*_\xi V_\xi = \sum_{\nu=0}^{N} P_\nu \sum_{\xi \in \mathcal{X}} \langle U_\nu \varphi_0, \psi_\xi \rangle \langle \psi_\xi, U_\nu \varphi_0 \rangle$$

$$= \sum_{\nu=0}^{N} P_\nu = \mathbb{1}\big|_{\mathcal{H}_P} \qquad (35)$$

Def.

$$\langle \nu' | V_\xi | \nu \rangle = \delta_{\nu'\nu} V_\xi(\nu) := \delta_{\nu'\nu} \langle \psi_\xi, U_\nu \varphi_0 \rangle \qquad (36)$$

From now on: $\rho^{(j)} := P^{(j)}_{P,+}$. (5.42)

Between two measurements, at times t_j and t_{j+1}, the state $\rho^{(j)}$ obtained after the j^{th} measurement at time t_j evolves according to Hamiltonian evolution:

$$\rho^{(j)} \longmapsto \rho^{(j)}_t := e^{-i(t-t_j)H_S} \rho^{(j)} e^{i(t-t_j)H_S},$$

$$t_j < t < t_{j+1}$$

Then, at time $t = t_{j+1}$, $\rho^{(j)}_{t_{j+1}}$ changes to

$$\rho^{(j+1)} := \frac{V_{\xi_{j+1}} \rho^{(j)}_{t_{j+1}} V^*_{\xi_{j+1}}}{tr\left(\rho^{(j)}_{t_{j+1}} V^*_{\xi_{j+1}} V_{\xi_{j+1}}\right)} \quad (37)$$

↳ Yields a trajectory of states:

$$\rho_t(\underline{\xi}_n, \underline{t}_n) = e^{-i(t-t_n)H_S} \rho^{(n)} e^{i(t-t_n)H_S} \quad (38)$$

" (22)

Theorem. (BCFFS I)

$$\mathbb{E}[\rho_t(\cdot,\cdot)] = e^{t\mathcal{L}} \rho,$$

ρ is initial condition, $\mathcal{L}$ is a Lindblad generator given by

$$\boxed{\mathcal{L}\rho = -\frac{i}{\hbar}\, ad_{H_S}(\rho) + \left(\sum_{\xi \in X} V_\xi \rho V_\xi^*\right) - \gamma\rho} \qquad (39)$$

(38) is called an "unravelling" of the Lindblad evolution.

We are interested in the situation where

$$\boxed{[V_\xi^\#, N] = 0,\ \forall \xi,\ \underline{\text{but}}\ [H_S, N] \neq 0} \qquad (40)$$

$$H_S := \varepsilon H,$$

ε small

$$\Phi_\xi(a) := V_\xi^* a V_\xi. \qquad (41)$$

Super - operator:

$$\tau_\varepsilon^s(\underset{\sim}{\xi}, t) := e^{i\varepsilon t_1 \mathrm{ad}_H} \Phi_{\xi_1} e^{i\varepsilon(t_2 - t_1)\mathrm{ad}_H} \Phi_{\xi_2} \cdots$$

$$\Phi_{\xi_{N_s}} e^{i\varepsilon(s - t_{N_s})\mathrm{ad}_H},$$

$\uparrow$

Poisson process

$\boxed{\rho_{\varepsilon^{-2}t}(\underset{-}{\xi}_n, t_n) \text{ converges to a jump process on } \operatorname{Spec} N}$

Jump process has transition fcts. generated by

$$\frac{2}{\gamma \hbar^2} \operatorname{Re} \frac{|\langle \nu' | H | \nu \rangle|^2}{1 - \sum_{\xi \in X} V_\xi(\nu') \overline{V_\xi(\nu)}} \qquad (\nu \neq \nu')$$

where

$$V_\xi(\nu) = \langle \nu | V_\xi | \nu \rangle$$

Generalizations and open problems

- More general models of probes and “cavities”; in particular:
- Weakly correlated probes; infinite-dimensional state spaces for cavity, P; operators $\mathcal{N}$ with continuous spectrum (!); ...
- More general models of indirect measurements of time-dependent quantities.
- *Important Example – to be carried out more fully:* Consider observables, $\vec{\mathcal{N}}$, with continuous spectrum, $\sigma(\vec{\mathcal{N}}) \simeq \mathbb{R}^d$. Then H_P may generate dynamics describing inertial motion on $\sigma(\vec{\mathcal{N}})$, and the full dynamics of P then describes tracks on $\sigma(\vec{\mathcal{N}})$ with “diffusive broadening”, (*“Mott tracks”* ...!).

Our conclusion: Quantum Mechanics and its foundations are well and alive. There are plenty of beautiful new experiments testing fundamental aspects of Quantum Mechanics, and there are plenty of interesting problems for theorists and mathematicians to worry about!

5.5. The appearance of particle tracks in pulsed electromagnetic fields

"Man kann die Welt mit dem p-Auge und man kann sie mit dem q-Auge sehen; aber wenn man beide Augen zugleich aufmachen will, wird man irre."

(Wolfgang Pauli an Werner Heisenberg, in: Brief vom 19. Oktober 1926)

Experimental set-up.

We study the quantum mechanics of a freely propagating particle bearing (a net electric charge or) an electric dipole moment. The particle is periodically illuminated by laser pulses: Every τ seconds, some lasers emit a very large number, N, of light pulses of wave length $\approx \lambda$. These pulses scatter off the particle in such a way that a change in the particle's velocity caused by light scattering is negligible. The mass, M of the partice is so large that the velocity uncertainty Δv satisfies

$$\Delta v \cdot \tau \approx \frac{\hbar}{M \cdot \lambda} \cdot \tau \lesssim \lambda. \qquad (1)$$

The scattered radiation, whose shape/state depends on the particle's approximate position, x, and velocity, v, is caught by a configuration of many detectors that perform direct measurements on all N scattered light waves. The pointers of the detectors have $k < \infty$ possible positions corresponding to spectral projections of some "detector observable" that is sensitive to certain features of the incoming light wave.

Since the pulses have a wave length λ, the accuracy of determining the particle's position, $x \in \mathbb{R}^3$, is limited by

$$|\Delta x| \gtrsim \lambda. \qquad (2)$$

Heisenberg's uncertainty relations then tell us that

$$\left(\frac{\lambda}{\tau} \overset{(1)}{>}\right) |\Delta v| \gtrsim \frac{\hbar}{M\lambda}, \qquad (3)$$

hence

$$\tau < \frac{M\lambda^2}{\hbar}. \qquad (4)$$

Condition (4) ensures that, after a position measurement, the future position of the particle, after a waiting time τ, can be predicted with an accuracy of order λ. In view of (2), a better accuracy of predicting the particle's position at the time of the next firing of the lasers would be useless.

Mathematical tools.

The classical phase space of the particle is given by

$$\Gamma = \mathbb{R}^3_x \times \mathbb{R}^3_p = \mathbb{R}^6 \qquad (5)$$

Points in Γ are pairs (x, p), where $p = Mv$ is the particle momentum. It is interesting to ask how precisely the quantum-mechanical position- and momentum operators, $\hat{x}$ and $\hat{v} = \frac{\hat{p}}{M}$, can be localized in a quantum-mechanical state of the particle, and whether there are orthonormal bases in the one-particle Hilbert space, $\mathcal{H}$, of uniformly small position- and velocity uncertainty.

For concreteness, we will work in the Schrödinger representation:

$$\left.\begin{aligned} \mathcal{H} &= L^2(\mathbb{R}^3, d^3x) \otimes \mathbb{C}^M, \quad \hat{x} = \text{multiplication by } x, \\ \hat{v} &= \frac{\hat{p}}{M} = -i\frac{\hbar}{M}\nabla_x, \end{aligned}\right\} \quad (6)$$

where $\mathbb{C}^M$ accommodates internal degrees of freedom (e.g., the states of a dipole moment), which are neglected in the following; (drop $\mathbb{C}^M$!).

We now review some known facts about phase-space localized orthonormal bases of $\mathcal{H}$.

Given a wave function $\psi \in \mathcal{H}$, with $\|\psi\|_2 = 1$, we define

$$\bar{x} \equiv \bar{x}_\psi := \langle \psi, \hat{x}\psi \rangle_{\mathcal{H}}, \quad (\Delta x)_\psi := \langle \psi, (\hat{x} - \bar{x}_\psi \mathbb{1})^2 \psi \rangle_{\mathcal{H}}^{1/2} \quad (7)$$

and

$$\bar{p} \equiv \bar{p}_\psi := \langle \psi, \hat{p}\psi \rangle_{\mathcal{H}}, \quad (\Delta p)_\psi := \langle \psi, (\hat{p} - \bar{p}_\psi \mathbb{1})^2 \psi \rangle_{\mathcal{H}}^{1/2} \quad (8)$$

Heisenberg's uncertainty relation says that

$$(\Delta x)_\psi \cdot (\Delta p)_\psi \geq \frac{\hbar}{2}. \qquad (9)$$

This relation is <u>invariant</u> under <u>scale transformations</u>, $x \to \vartheta x$, $0 < \vartheta < \infty$. The minimum in (9) is attained for the Gaussians

$$\psi_{\bar{x}}(x) = (2\pi)^{-3/4} (\Delta x)^{-3/2} e^{-\frac{(x-\bar{x})^2}{4(\Delta x)^2}}. \qquad (10)$$

(1) <u>Balian</u> has studied bases obtained by translations of a conveniently chosen, <u>fixed</u> wave function, ψ, over a lattice in phase space, Γ:

$$\psi_{\bar{x},\bar{p}}(x) := e^{ix\cdot\bar{p}} \psi(x-\bar{x}), \qquad (11)$$

with $\bar{x} \in a\mathbb{Z}^3$, $\bar{p} \in b\mathbb{Z}^3$, with a, b positive, and $a \cdot b = 2\pi\hbar$. Balian proves that if a basis of $\mathcal{H}$ obtained in this way is <u>orthonormal</u> then

$$(\Delta x)_\psi \cdot (\Delta p)_\psi = \infty. \qquad (12)$$

From now on, we work in units such that

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Heisenberg Proof of the Balian–Low Theorem★

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Abstract. We give an alternate proof of the fact that a function generating a basis of coherent states must have an infinitely long tail in either position space or momentum space. Our argument is a very natural one in which the Heisenberg Uncertainty Principle enters directly.

Phase space Wannier functions [1–3] have been of interest in electronic structure calculations [4] and signal analysis [5–7]. Unlike classical Wannier functions [8, 9], which are defined in terms of the eigenfunctions of periodic Hamiltonians, phase space Wannier functions have uniform tail lengths in both position space and momentum space. In this Letter we are concerned with a special class of such expansion functions, namely families of coherent states – i.e., expansions based on windowed Fourier transforms [10].

Let T_{mn} be the phase space translation operator in one dimension, i.e.

$$(T_{mn}\varphi)(x) = e^{i2\pi mx}\varphi(x+n). \tag{1}$$

DEFINITION. Let f be a square-integrable function on $\mathbb{R}$ such that $\{T_{mn}f\}$ is an orthonormal basis. Then $\{T_{mn}f\}$ is a *basis of coherent states* and f is said to *generate* it.

Let X and P be the position and momentum operators, respectively.

BALIAN–LOW THEOREM. *If* $f \in L^2(\mathbb{R})$ *generates a basis of coherent states then* Xf *and* Pf *cannot both be square-integrable.*

This is a no-go theorem for localizing phase space with a windowed Fourier transform. It states that such functions must have infinitely long tails in either position space or momentum space. Balian [2] and Low [3] independently established this theorem with an ingenious topological argument showing that if the function

$$\sum_l f(k+l)\, e^{i2\pi lk'} \tag{2}$$

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is continuous in both variables, then it must have a zero. Coifman and Semmes (see [10]) have more recently closed a technical loophole concerning what regularity properties for f will guarantee the continuity of (2).

We present here an elementary proof of the Balian–Low theorem in which the Heisenberg uncertainty principle enters directly.

New Proof. If Xf and Pf are both square-integrable, then

$$\begin{aligned}(Xf, Pf) &= \sum_{m,n} (Xf, T_{mn}f)(T_{mn}f, Pf) \\ &= \sum_{m,n} (T_{-m,-n}f, Xf)(Pf, T_{-m,-n}f) \end{aligned} \tag{3}$$

because

$$([T_{-m,-n}, X]f, f) = n(T_{-m,-n}f, f) = 0\,, \tag{4}$$

$$(f, [T_{-m,-n}, P]f) = 2\pi m(T_{m,n}f, f) = 0\,. \tag{5}$$

On the other hand,

$$\sum_{m,n} (T_{-m,-n}f, Xf)(Pf, T_{-m,-n}f) = (Pf, Xf)\,, \tag{6}$$

and so $([P,X]f, f) = 0$. This implies $f \equiv 0$. □

Remark. Actually, one has to be a little careful here because the premise of our argument does not guarantee the square-integrability of XPf and PXf. Let $\{\varphi_j\}$ be a sequence of smooth, compactly supported functions such that $\varphi_j \to f$, $X\varphi_j \to Xf$, and $P\varphi_j \to Pf$ in the *square integrable* sense of convergence. (Such an approximating sequence certainly exists.) Then

$$\begin{aligned} i\,\|\varphi_j\|^2 &= ([X, P]\varphi_j, \varphi_j) \\ &= (P\varphi_j, X\varphi_j) - (X\varphi_j, P\varphi_j) \\ &\to (Pf, Xf) - (Xf, Pf) = 0\,. \end{aligned} \tag{7}$$

But $\|\varphi_j\|^2 \to 0$ implies $\|\mathrm{f}\|^2 = 0$.

One may believe that sacrificing orthogonality is a reasonable way to beat this no-go theorem. However, when one expands an arbitrary function with respect to a basis that is not orthonormal, the coefficients are computed with respect to the dual basis. The bad news is that if an L^2-complete family of coherent states has good phase space localization, then the dual basis cannot.

THEOREM. *Let $\{T_{mn}f\}$ and $\{T_{mn}g\}$ be two families of coherent states which are complete in $L^2(\mathbb{R})$ but are bi-orthonormal, i.e.,*

$$(T_{mn}f, T_{m'n'}g) = \delta_{mm'}\,\delta_{nn'}\,, \tag{8}$$

If Xf is square-integrable, then Pg cannot be. If Pf is square-integrable, then Xg cannot be.

Proof. Look at the expansions

$$(Xf, Pg) = \sum_{m,n} (Xf, T_{mn}g)\,(T_{mn}f, Pg)\,, \tag{9}$$

$$(Pf, Xg) = \sum_{m,n} (Pf, T_{mn}g)\,(T_{mn}f, Xg) \tag{10}$$

and argue as before, using (8). Either of the two premises ($Xf, Pg \in L^2(\mathbb{R})$ for (9) and $Pf, Xg \in L^2(\mathbb{R})$ for (10)) will lead to the conclusion $(f, g) = 0$. □

In this context the von Neumann basis [3] easily comes to mind. However, this basis, which has exponential phase space localization, is a rather extreme example because the dual basis functions are not even square-integrable. Further, the basis is 'overcomplete'. It has been pointed out by Janssen [6] that if f is the standard Gaussian generating the von Neumann basis, then f lies in the closure of the linear span of $\{T_{mn}f\}_{(m,n)\neq(0,0)}$.

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References

1. Wilson, K., Generalized Wannier functions, Cornell University preprint.
2. Balian, R., *C. R. Acad. Sci. Paris* **292**, 1375 (1981).
3. Low, F., in *A Passion for Physics*, G. F. Chew Volume, World Scientific Press, Singapore, 1985.
4. Rehr, J., Sullivan, D., Wilkins, J., and Wilson, K., Phase space Wannier functions in electronic structure calculations, Cornell University preprint (1987).
5. Bacry, H., Grossmann, A., and Zak, J., *Phys. Rev.* **B12**, 1118 (1975).
6. Janssen, A., *J. Math. Phys.* **23**, 720 (1982).
7. Daubechies, I., Grossmann, A., and Meyer, Y., *J. Math. Phys.* **27**, 1271 (1986).
8. Wannier, G., *Phys. Rev.* **52**, 191 (1937).
9. des Cloizeaux, J., *Phys. Rev.* **135**, 3A, 698 (1964).
10. Daubechies, I., The wavelet transform, time-frequency localization, and signal analysis, AT&T Bell Laboratories preprint (1987).

$\hbar = 1$.

(2) Trying to construct a CONS in $\mathcal{H}$ of the form

$$\psi_j(x) = e^{i\bar{p}_j \cdot x} h_j(x - \bar{x}_j),$$

for conveniently chosen functions $h_j \in \mathcal{H}$, with $\|h_j\|_2 = 1$, $\langle h_j, \hat{x} h_j \rangle_{\mathcal{H}} = \langle h_j, \hat{p} h_j \rangle_{\mathcal{H}} = 0$, it has been proven, (see Steger, Gröchening & Malinnikova) that

$$\sup \left\{ \langle h_j, (\mathbb{1} + \hat{x}^2)^{1+\varepsilon} h_j \rangle_{\mathcal{H}} + \langle h_j, (\mathbb{1} + \hat{p}^2)^{1+\varepsilon} h_j \rangle_{\mathcal{H}} \right\} = \infty, \qquad (13)$$

for an arbitrary $\varepsilon > 0$.

(3) Jean Bourgain has proven the following remarkable

Theorem B. There exists a CONS (= complete orthonormal system) $\{\psi_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ with the property

$$(\Delta x)_{\psi_i} \leq K\lambda, \quad (\Delta p)_{\psi_i} \leq \frac{K}{\lambda} \qquad (14)$$

for any constant $K > 1/\sqrt{2}$. $\square$

This is a wonderful theorem. But Bourgain's construction of $\{\psi_i\}_{i=1}^{\infty}$ is not sufficiently explicit to be directly applicable to the problem of particle tracks. We will propose a reasonable looking conjecture that adds a little precision to Bourgain's theorem.

(4) Yves Meyer has constructed a CONS of "wavelets", $\{\psi^{\pm}_{n,j}\}_{n\in\mathbb{Z},\, j\in\mathbb{Z}}$, with the following properties:

$$\psi^{\pm}_{n,j}(x) = 2^{j/2}\,\psi^{\pm}(2^j x - n),$$

where

$$\operatorname{supp}\hat{\psi}^{+}(p) \subset (0,\infty), \quad \hat{\psi}^{-}(p) = \hat{\psi}^{+}(-p).$$

One can choose $\psi^{\pm}$ such that $\operatorname{supp}\hat{\psi}^{\pm}$ is compact and $\hat{\psi}^{\pm}$ is smooth; so that ψ^{+}, ψ^{-} belong to $\mathcal{S}(\mathbb{R})$. Analogous bases can also be constructed for $\mathcal{H} = L^2(\mathbb{R}^3)$.

Unfortunately, though,

$$\left\langle \psi^{\pm}_{n,j}, \hat{p}\, \psi^{\pm}_{n,j} \right\rangle \simeq (\Delta p)_{\psi^{\pm}_{n,j}} = O(2^{-j}) \qquad (15)$$

The fact that the momentum uncertainty $(\Delta p)_{\psi^{\pm}_{n,j}}$ has the same size as the expectation value of $\hat{p}$ in $\psi^{\pm}_{n,j}$ makes these bases useless for ~~our~~ purposes.

In the following it is convenient to work with dimensionless variables and operators:

$$X := \frac{\hat{x}}{\lambda}, \quad P := \hat{p}\cdot \lambda / \hbar, \qquad (16)$$

where λ is the wavelength of the light pulses emitted by the lasers.

Let $U(a) := e^{-ia\cdot P}$ be the unitary operator representing a space translation by the vector $\lambda a \in \mathbb{R}^3$ on $L^2(\mathbb{R}^3, d^3x)$. We propose a variant of Bourgain's Theorem B as a conjecture well adapted to our purposes, (which can probably be proven).

Conjecture PSL *)

There exists a family $\{Q_n\}_{n\in\mathbb{N}}$ of finite-dimensional orthogonal projections on $\mathcal{H}=L^2(\mathbb{R}^3, d^3x)$ with the following properties:

(i) $Q_n = Q_n^*$, $Q_n \cdot Q_m = \delta_{nm} Q_n$, $\forall n, m$, $\sum_{\nu\in\mathbb{N}} Q_n = \mathbb{1}$. (17)

We set $d_n := tr\, Q_n$ (= dimension of Q_n) and define

$$X_n := d_n^{-1} tr(Q_n X),\quad P_n := d_n^{-1} tr(Q_n P), \qquad (18)$$

and

$$\left.\begin{aligned} (\Delta X_n)^2 &:= d_n^{-1} tr\left(Q_n (X - X_n \mathbb{1})^2\right), \\ (\Delta P_n)^2 &:= d_n^{-1} tr\left(Q_n (P - P_n \mathbb{1})^2\right). \end{aligned}\right\} \qquad (19)$$

We set $R_n := \langle X_n \rangle \equiv \sqrt{X_n^2 + 1}$ (20)

(ii) There exists some $\varepsilon \in [0, \frac{1}{8}]$ such that

$$\left.\begin{aligned} & d_n \leq K^2 R_n^{2\varepsilon}, \\ \text{and} \quad & \\ & \sqrt{(\Delta X_n)^2 + (\Delta P_n)^2} \leq K R_n^{2\varepsilon}, \end{aligned}\right\} \qquad (21)$$

for some finite constant K.

---

*) "PSL" stands for "phase space localization"

For $(X,P) \in \Gamma = \mathbb{R}^6$, we set

$$R_k(X,P) := \sqrt{(X_k - X)^2 + (P_k - P)^2} \qquad (22)$$

We define projections

$$Q_{X,P}(\rho) := \sum_{k:\, R_k(X,P) \leq \rho} Q_k\,, \quad Q^{\perp}_{X,P}(\rho) := \mathbb{1} - Q_{X,P}(\rho). \qquad (23)$$

(iii) For ε as in (ii) and an arbitrary $a \in \mathbb{R}^3$, we set $\rho_{n,a} := L\left(\max\{\langle X_n + a\rangle, \langle X_n\rangle\}\right)^{1+\varepsilon}$. Then

$$\| Q^{\perp}_{X_n+a,P_n}(\rho_{n,a})\, U(a)\, Q_n\, U(-a)\, Q^{\perp}_{X_n+a,P_n}(\rho_{n,a}) \| \to 0,$$

as $L \to \infty$, uniformly in n. (24)

<u>Remarks.</u> (1) I suspect that there are many families $\{Q_n\}$ of projections with properties as described in Conjecture PSL, or better ones, (and that they can be constructed from spectral projections of suitably chosen random operators). They would enter the description of different devices serving to localize a charged particle in phase space with the help of light scattering.

(2) It is easy to construct <u>discrete models</u> $(\mathbb{R}^3_x \mapsto \mathbb{Z}^3, \mathbb{R}^3_p \mapsto \{1, \ldots, M\}$, for some $M < \infty)$ for which families of projections satisfying a modified form of Conjecture PSL can be constructed. $\square$

In an experiment where the wave length λ of the light pulses is large as compared to the Compton wave length of an electron and the energy of a pulse is tiny as compared to the rest energy of an electron interactions among photons can be neglected, and we can treat the different light pulses emitted in one operation of the lasers as <u>independent.</u> The pure states of the N pulses are unit rays in a Hilbert space

$$\mathcal{H}_\gamma^{(N)} = \mathcal{H}_\gamma^{\otimes N}, \qquad (25)$$

where $\mathcal{H}_\gamma$ is the Hilbert space of state vectors of a single light pulse.

We assume that the initial state of the j^{th} pulse is given by a vector $\gamma_j \in \mathcal{H}_\gamma$ in every operation of the lasers, for $j = 1, \dots, N$. The state of the pulse <u>after</u> scattering by the charged particle is assumed to be given by

$$S_n^j \gamma_j, \quad (S_n^j)^* = (S_n^j)^{-1}, \tag{26}$$

where S_n^j is a unitary operator, the <u>"scattering matrix"</u>, on $\mathcal{H}_\gamma$ that depends on the phase-space cell corresponding to the range of the projection Q_n where the charged particle happens to be localized.

When there are no light pulses hitting the particle it is assumed to propagate freely; i.e., its Hamiltonian is given by

$$H = \frac{P^2}{2\mu}, \tag{27}$$

with $\mu = M\lambda^2/\hbar^2$.

We are interested in tracking the particle prepared in a state given by a density matrix P acting on $Q_n \mathcal{H}$, for some $n \in \mathbb{N}$, i.e.,

$$P = Q_n P Q_n \qquad (28)$$

over a time period of duration τ before it is hit by a new series of N light pulses. After τ seconds, the state of the particle is given by

$$P_\tau = e^{-i\tau H} P e^{i\tau H}.$$

By (21), we have that

$$\begin{aligned} tr(PX) &= tr(PQ_n X Q_n) \\ &= tr(PQ_n(X - X_n \mathbb{1})Q_n) + X_n \end{aligned}$$

Note that

$$\begin{aligned} |tr(PQ_n(X - X_n\mathbb{1})Q_n)| &\leq \sqrt{d_n \, tr P^2} \cdot \Delta X_n \\ &\leq \sqrt{d_n} \cdot \Delta X_n . \end{aligned}$$

Thus

$$\left| tr(PX) - X_n \right| \overset{(21)}{\leq} K R_n^{\varepsilon} \Delta X_n \leq K^2 R_n^{3\varepsilon}, \qquad (29)$$

and

$$tr\left(P\left(X - X_n \mathbb{1}\right)^2\right) \leq d_n \left(\Delta X_n\right)^2$$

$$\leq K^2 R^{2\varepsilon} \left(\Delta X_n\right)^2 \qquad (30)$$

Similarly,

$$\left| tr(P\underline{P}) - \underline{P}_n \right| \leq \sqrt{d_n}\, \Delta \underline{P}_n \leq \cdots,$$

and

$$tr\left(P\left(\underline{P} - \underline{P}_n \mathbb{1}\right)^2\right) \leq d_n \left(\Delta \underline{P}_n\right)^2 \leq \cdots \qquad (31)$$

We set $\pi := \underline{P} - \underline{P}_n \mathbb{1}$ and introduce an auxiliary Hamiltonian

$$H_n := v_n \cdot \pi, \quad v_n := \frac{\underline{P}_n}{\mu}. \qquad (32)$$

Then

$$H = \frac{\underline{P}^2}{2\mu} = \frac{\pi^2}{2\mu} + H_n + \frac{\underline{P}_n^2}{2\mu}$$

Since $[H, \underline{P}] = [H, \pi] = [H_n, \underline{P}] = 0$, it follows that

$$\left\| P_\tau - e^{-i\tau H_n} P e^{i\tau H_n} \right\| \leq C \frac{\tau}{\mu} d_n \left(\Delta \underline{P}_n\right)^2, \qquad (33)$$

for some finite constant C.

Note that

$$e^{-i\tau H_n} Q_n \, e^{i\tau H_n} = U(\tau v_n)\, Q_n \, U(-\tau v_n).$$

By (23), (24), we have that

$$Q_{X_n+\tau v_n, P_n}(\rho_n, \tau v_n)\; U(\tau v_n)\, Q_n\, U(-\tau v_n)$$
$$= U(\tau v_n)\, Q_n\, U(-\tau v_n) + \mathcal{E}(X_n + \tau v_n, P_n), \quad (34)$$

where $\|\mathcal{E}(X_n + \tau v_n, P_n\|$ can be made as small as desired by choosing the constant L in the definition of $Q_{X,P}(\rho)$ (see (23)) large enough, uniformly in n.

We set $N_R := \{ n \mid R_n \equiv \langle X_n \rangle < R \}$. Combining (33) and (34) we conclude:

Given $R < \infty$ and $\delta > 0$, we can choose L large enough and $\frac{\tau}{\mu}$ small enough such that, for $P = Q_n P Q_n$ and $n \in N_R$,

$$\| P_\tau - Q_{X_n+\tau v_n, P_n}(\rho_n, \tau v_n)\, P_\tau\, Q_{X_n+\tau v_n, P_n}(\rho_n, \tau v_n) \| < \delta \quad (35)$$

This estimate shows that the localization of the state of the particle in phase space can be tracked quite accurately under the free time evolution, provides the time elapsed, $\frac{\tau}{\mu}$, is small enough and the initial position of the particle is not too far from the origin.

Next, we describe the evolution of the state of the particle and of the light pulses during a scattering event. The particle is prepared in an arbitrary initial density matrix, Ω, acting on $\mathcal{H} = L^2(\mathbb{R}^3, d^3x)$, the light pulses in states $\gamma_1, \dots, \gamma_N$, with $\gamma_j \in \mathcal{H}_\gamma$. We expand Ω according to a family, $\{Q_n\}_{n \in \mathcal{N}}$, of projections as introduced in Conjecture PSL:

$$\Omega = \sum_{n,m} \Omega_{nm}, \quad \Omega_{nm} := Q_n \Omega Q_m, \quad n, m \in \mathcal{N}. \quad (36)$$

The initial state of the total system, particle V laser pulses, is given by

$$\Psi := \sum_{n,m} \Omega_{nm} \otimes \Big(\bigotimes_{j=1}^{N} \underbrace{|\gamma_j\rangle\langle\gamma_j|}_{\equiv \text{orth. proj. onto } \gamma_j} \Big) \qquad (37)$$

We assume that the time of interaction between the particle and the laser pulses during one firing of the lasers is so short and that the mass μ is so large that Ω does not evolve during one scattering event between the particle and the light pulses.

The evolution of a vector

$$\varphi \otimes \chi \in \mathcal{H} \otimes \mathcal{H}_\gamma^{\otimes N}$$

during one scattering event is then given by

$$\varphi \otimes \chi \longmapsto U(\varphi \otimes \chi),$$

where

$$U(\varphi \otimes \chi) = \sum_n U(Q_n \varphi \otimes \chi) = \sum_n Q_n \varphi \otimes S_n \chi, \qquad (38)$$

The initial state of the total system, particle V laser pulses, is given by

$$\Psi := \sum_{n,m} \Omega_{nm} \otimes \Big(\bigotimes_{j=1}^{N} \underbrace{|\gamma_j\rangle\langle\gamma_j|}_{\equiv \text{orth. proj. onto } \gamma_j} \Big) \qquad (37)$$

We assume that the time of interaction between the particle and the laser pulses during one firing of the lasers is so short and that the mass μ is so large that Ω does not evolve during one scattering event between the particle and the light pulses.

The evolution of a vector

$$\varphi \otimes \chi \in \mathcal{H} \otimes \mathcal{H}_\gamma^{\otimes N}$$

during one scattering event is then given by

$$\varphi \otimes \chi \longmapsto U(\varphi \otimes \chi),$$

where

$$U(\varphi \otimes \chi) = \sum_n U(Q_n \varphi \otimes \chi) = \sum_n Q_n \varphi \otimes S_n \chi, \quad (38)$$

where
$$\mathcal{S}_n = \bigotimes_{j=1}^{N} S_n^j , \tag{39}$$

i.e.,
$$U = \sum_n Q_n \otimes \{ S_n^1 \otimes \cdots \otimes S_n^N \}, \tag{40}$$

and S_n^j is a unitary scattering matrix acting on $\mathcal{H}_\gamma$; (see (26)). Thus, during one scattering event, the state Ψ in (37) evolves to

$$\Psi \mapsto U \Psi U^* = \sum_{n,m} \Omega_{nm} \otimes \left\{ \bigotimes_{j=1}^{N} S_n^j |\gamma_j\rangle\langle\gamma_j| (S_m^j)^* \right\}$$
$$= \sum_{n,m} \Omega_{nm} \otimes \left\{ \bigotimes_{j=1}^{N} |S_n^j \gamma_j\rangle\langle S_m^j \gamma_j| \right\} \tag{41}$$

In passing we note that, when restricted to operators of the form

$$A/_{\mathcal{H}} \otimes \mathbb{1}/_{\mathcal{H}_\gamma^{(N)}} ,$$

we find that

$$tr\left(U \Psi U^* (A \otimes \mathbb{1})\right)$$
$$= \sum_{n,m} tr\left(\Omega_{nm} A\right) \prod_{j=1}^{N} \langle S_n^j \gamma_j , S_m^j \gamma_j \rangle_{\mathcal{H}_\gamma} \tag{42}$$

Since S^j_n is a unitary operator on $\mathcal{H}_\gamma$, $\forall j, n$, we have that

$$\left| \left\langle S^j_n \gamma_j, S^j_m \gamma_j \right\rangle_{\mathcal{H}_\gamma} \right| \leq 1 - \varepsilon^j_{nm},$$

where $\varepsilon^j_{nm} \geq 0$. We assume that the operators S^j_n depend properly on n in the sense that

$$\varepsilon^j_{nm} > 0, \quad \forall j, n, m, n \neq m. \quad (43)$$

If there is a uniform lower bound

$$\varepsilon^j_{nm} \geq \varepsilon_{nm} > 0, \quad n \neq m,$$

then

$$\prod_{j=1}^{N} \left\langle S^j_n \gamma_j, S^j_m \gamma_j \right\rangle \rightarrow 0, \text{ as } N \rightarrow \infty, \quad (44)$$

exponentially fast. If

$$tr\left(\Omega_{nm} A\right) \longrightarrow 0, \text{ as } |n|, |m| \rightarrow \infty,$$

we conclude that

$$\boxed{tr\left(U \Psi U^* (A \otimes \mathbb{1})\right) \longrightarrow \sum_n tr\left(\Omega_{nn} A\right)} \quad (45)$$

This is the phenomenon of "decoherence", which will play an important role in the next section.

We now suppose that every scattered laser pulse, $S^j_n \gamma_j$, triggers a detector. The detector is functional if its pointer position reaches a definite position $\xi \in \{1, \cdots, k\}$, for some $k < \infty$, corresponding to an orthogonal projection, $\pi^j_\xi = (\pi^j_\xi)^*$, on $\mathcal{H}_\gamma$, with

$$\pi^j_\xi \cdot \pi^j_\eta = \delta_{\xi\eta} \pi^j_\xi, \quad \sum_{\xi=1}^{k} \pi^j_\xi = \mathbb{1}, \qquad (46)$$

for all $j = 1, \cdots, N$. We define

$$p^j(\xi / n) := \langle S^j_n \gamma_j, \pi^j_\xi S^j_n \gamma_j \rangle, \qquad (47)$$

and

$$\tau^j(\xi / n, m) = \langle S^j_m \gamma_j, \pi^j_\xi S^j_n \gamma_j \rangle \qquad (48)$$

By the Schwarz inequality,

$$|\tau^j(\xi / n, m)| \leq \sqrt{p^j(\xi / n)\, p^j(\xi / m)} \qquad (49)$$

We then have that

$$tr\left(U\Psi U^* A\big|_{\mathcal{H}} \otimes \left\{ \bigotimes_{j=1}^{N} \pi^j_{\xi_j} \right\}\right)$$

$$= \sum_{n,m} tr_{\mathcal{H}}\left(\Omega_{nm} A\right) \prod_{j=1}^{N} \tau^j(\xi | n,m). \quad (50)$$

Furthermore,

$$prob_{\Omega}\{\xi_1, \dots, \xi_N\} := tr\left(U\Psi U^* \mathbb{1}_{\mathcal{H}} \otimes \left\{ \bigotimes_{j=1}^{N} \pi^j_{\xi_j} \right\}\right)$$

$$= \sum p_n \prod_{j=1}^{N} p^j(\xi_j | n), \quad (51)$$

where

$$p_n := tr(\Omega_{nn}) = tr(\Omega Q_n) \quad (52)$$

is the Born probability of finding the particle in the phase space cell corresponding to Q_n, given that its state is the density matrix Ω.

We define the probability measures

$$\mu^{(N)}(\underline{\xi} | n) := \prod_{j=1}^{N} p^j(\xi_j | n), \quad (53)$$

$\underline{\xi} = (\xi_1, \xi_2, \dots) \in \Xi$; compare to Eq. (4), §5.1.2.

In order to reach a situation that can be analyzed with mathematical precision, we henceforth require the following assumption:

Assumption ASY.*)

(i) $\lim_{N\to\infty} \mu^{(N)}(\cdot/n) =: \mu^{(\infty)}(\cdot/n)$ (54)

exists, with $\mu^{(\infty)}(\cdot/n)$ a probability measure on the space Ξ of ∞ long sequences $\underline{\xi} = (\xi_1, \xi_2, \xi_3, \dots)$; ($\sigma$-algebra of cylinder sets), for all n.

(ii) For $n \neq m$, the measures $\mu^{(\infty)}(\cdot/n)$ and $\mu^{(\infty)}(\cdot/m)$ are mutually singular. □

Remark. In Sects 5.1 and 5.3 we have seen that if $p^j(\xi/n) \equiv p(\xi/n)$, indep. of j, and if the functions $\{p(\xi/n)\}_{\xi=1}^{k}$ separate points on $\mathbb{Z}^6 \ni n$ then Assumption ASY holds.

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*) ASY stands for "asymptotics"

If Assumption ASY holds then the pointer positions $\xi_1, \dots, \xi_N$ of the detectors catching the N scattered light pulses <u>determine</u> a "<u>phase-space cell</u>" $n(\underline{\xi}_N)$, $\underline{\xi}_N = (\xi_1, \dots, \xi_N)$, with the property that the state of the particle belongs to the range of $Q_{n(\underline{\xi}_N)}$, with a probability approaching 1, as $N \to \infty$. The probability that $n(\underline{\xi}_N) = n$ is the Born probability $p_n = tr(\Omega\, Q_n)$, where Ω is the state of the particle <u>before</u> the light scattering event took place.

This fact combined with (35), for $P = p_n^{-1}\, \Omega_{nn}$, $n = n(\underline{\xi}_N)$, enables us to understand the formation of <u>particle tracks</u>, i.e., to solve Mott's problem. Bourgain's theorem is central for the success in this endeavor.

5.6. "Diminishing Potentialities", Entanglement, "Purification" and the emergence of Events in Quantum Mechanics – a simple model.

In this section we study a simple model that is a caricature of a static quantum-mechanical "antenna" interacting with some modes of the quantized electromagnetic field. In order to be able to perform explicit calculations, we make some drastic simplifications:

(1) Time is discrete: $\mathbb{R} \to \mathbb{Z}$.

(2) The state space of the antenna is finite-dimensional,

$$\mathcal{H}_A = \mathbb{C}^M \,, \quad M < \infty \qquad (55)$$

In the absence of the electromagnetic field, the evolution of a density matrix Ω on $\mathcal{H}_A$ by one time step is given by conjugation with a unitary matrix, V:

$$\Omega \longmapsto V\Omega V^*, \quad V \in U(M). \qquad (56)$$

(3) For every time $\tau \in \mathbb{Z}$, the algebra of our caricature of electromagnetic fields at time τ is given by

$$\mathcal{B}_\tau \simeq M_N(\mathbb{C}), \qquad (57)$$

i.e., by all $N \times N$ complex matrices. They act on the Hilbert space $\mathcal{H}_\tau \simeq \mathbb{C}^N$. The space $\mathcal{H}_\tau$ contains a special vector, φ_0, called "vacuum." The algebra of all functionals of the "electromagnetic fields" at times $\tau = \tau_0, \tau_0+1, \cdots, \tau_0+n =: \tau_1 \geq \tau_0$, $n \in \mathbb{Z}_+$, $\tau_0, \tau_1 \in \mathbb{Z}$, is then given by

$$\mathcal{B}_{[\tau_0,\tau_1]} := \bigotimes_{\tau \in [\tau_0,\tau_1]} \mathcal{B}_\tau \qquad (58)$$

This algebra acts on the Hilbert space

$$\text{"}\mathcal{H}_\infty := \bigotimes_{\tau \in \mathbb{Z}} \mathcal{H}_\tau\text{"} \qquad (59)$$

To render the definition of $\mathcal{H}_\infty$ precise, we introduce a CONS in $\mathcal{H}_\infty$, as follows:
To $\varphi_0 \in \mathcal{H}_\tau$ we add $N-1$ orthonormal vectors, $\varphi_1, \dots, \varphi_{N-1}$, such that $\{\varphi_k\}_{k=0}^{N-1}$ is a CONS in $\mathcal{H}_\tau \simeq \mathbb{C}^N$. We consider vectors

$$\Phi_{\underline{k}} := \bigotimes_{\tau \in \mathbb{Z}} \varphi_{k_\tau}, \qquad (60)$$

where $k_\tau = 0$, <u>for all but finitely many values of</u> τ. Let $\mathcal{D}$ denote the linear space of all <u>finite</u> linear combinations of the vectors $\Phi_{\underline{k}}$; $\mathcal{D}$ is equipped with a scalar product:

$$\langle \Phi_{\underline{k}}, \Phi_{\underline{k}'} \rangle = \prod_{\tau \in \mathbb{Z}} \delta_{k_\tau, k'_\tau}, \qquad (61)$$

extended anti-linearly in the first and linearly in the second argument. The space $\mathcal{H}_\infty$ is the completion of $\mathcal{D}$ in the norm given by $\|\Psi\| = \sqrt{\langle \psi, \psi \rangle}$, $\psi \in \mathcal{D}$.

An operator $A \in \mathcal{B}_\tau$ is called a "free E-field" at time τ. By (57) and (59), "E-fields" at different times commute with each other. This feature has a correspondence in the theory of the quantized free electromagnetic field on a Minkowski space-time of even dimension:

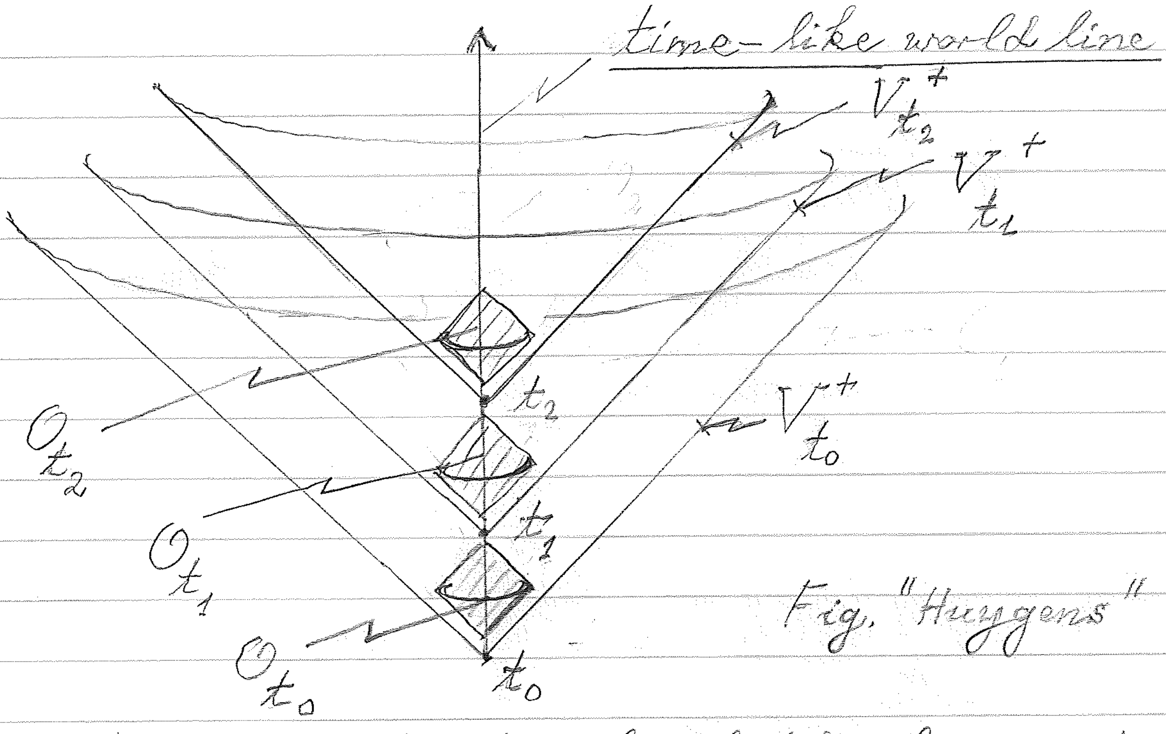


Fig. "Huygens"

Fact: Any functional of the free electromagnetic field localized in the "diamond" O_{t_0} commutes with any other functional of the free electromagnetic field localized in the forward light cone $V^+_{t_1}$

and in particular with any functional of the free e.m. field localized in $V^+_{t_1} \cap (V^+_{t_2})^c$, O_{t_1}, ... If $\mathcal{D}_\tau$ denotes the algebra of all bounded functionals of the free e.m. field localized in the diamond O_τ, ($\tau = t_0, t_1, t_2, \dots$) then the family $\{\mathcal{D}_\tau\}_{\tau \in \mathbb{Z}}$ of algebras has properties analogous to those of the family $\{\mathcal{B}_\tau\}_{\tau \in \mathbb{Z}}$ introduced above.

The fact stated above is a straightforward consequence of the support properties of the commutator distribution of the free e.m. field, which is a solution of the (massless!) wave equation and hence has a support conforming to Huygens' Principle; (in the form used by Buchholz).

Suppose that a massive observer moves

along the world line depicted in the above Figure who has a system of detectors capable of measuring the e.m. field in local regions of space-time, such as those "diamonds". Let us suppose the detectors were turned off (or not read) up to a time t. The observer can then only do measurements with detectors turned on somewhere in the forward light cone V_t^+; i.e., he has access to measurements of functionals of the e.m. field localized in V_t^+. Since the velocity of light is the limiting (maximal) velocity he does not have any means to do measurements of functionals of the electromagnetic field localized anywhere in his backward light cone V_t^-; i.e., he will not know about bursts of e.m.

radiation emerging from sources located in V_t^-. Thus the algebra of operators relevant to describe measurements or events concerning the e.m. field accessible to an observer moving along the worldline in the above Figure at times $\geq t$ is the algebra of functionals of the e.m. field localized inside V_t^+, (which equals $\overline{\bigvee_{\tau \geq t} \mathcal{D}_\tau}$).

If H_0 is the Hamiltonian of the free e.m. field then

$$e^{iH_0} \mathcal{D}_\tau \, e^{-iH_0} = \mathcal{D}_{\tau+1}. \qquad (62)$$

The model discussed in this section arises when we imagine to let the speed of light tend to ∞ in the above discussion of the e.m. field. Eq. (62) inspires us to construct the free time evolution of "E-fields" as follows: We define a unitary "shift operator", S, by by setting

$$\left.\begin{aligned}(S\underline{\Phi})_{\underline{k}} &:= \Phi_{\sigma(\underline{k})} = \bigotimes_{\tau\in\mathbb{Z}} \varphi_{\sigma(\underline{k})_\tau}, \\ \text{where} \quad \sigma(\underline{k})_\tau &:= k_{\tau-1}.\end{aligned}\right\} \tag{62}$$

The definition of S is extended to the domain $\mathcal{D}$ by linearity and to all of $\mathcal{H}_\infty$ by using the boundedness of S (actually, S is unitary) and the density of $\mathcal{D}$ in $\mathcal{H}_\infty$.

In (58), we have introduced the algebras $\mathcal{B}_{[\tau_0,\tau_1]}$, $\tau_1 \geq \tau_0 \geq 0$. We embed $\mathcal{B}_{[\tau_0,\tau_1]}$ into $B(\mathcal{H}_\infty)$ by setting

$$\gamma^{(0)}(E) := \mathbb{1}_0 \otimes \cdots \otimes \mathbb{1}_{\tau_0-1} \otimes E \otimes \Big[\bigotimes_{\tau>\tau_1} \mathbb{1}_\tau\Big] \tag{63}$$

for any $E \in \mathcal{B}_{[\tau_0,\tau_1]}$. This enables us to define algebras $\mathcal{E}^{(0)}_{\geq\tau}$, $\tau \geq 0$:

$$\mathcal{E}^{(0)}_{\geq\tau} := \overline{\bigvee_{\tau'\geq\tau} \mathcal{A} \otimes \gamma^{(0)}(\mathcal{B}_{[\tau,\tau']})}, \tag{64}$$

where, for concreteness, we imagine that the closure is taken in the weak topology on

$\mathcal{B}(\mathcal{H}_\infty)$; (so that the algebras $\mathcal{E}^{(0)}_{\geq \tau}$ are <u>von Neumann algebras</u>); and

$$\mathcal{A} := \mathcal{B}(\mathcal{H}_A) = \mathbb{M}_M(\mathbb{C}). \qquad (65)$$

Clearly
$$\mathcal{E}^{(0)}_{\geq \tau'} \subsetneqq \mathcal{E}^{(0)}_{\geq \tau}, \qquad (66)$$

for any $\tau' > \tau$; and

$$\mathcal{E}^{(0)}_{\geq \tau} \simeq \mathcal{A} \otimes \mathcal{B}_{\geq \tau} \simeq \mathcal{A} \otimes \mathcal{B}_{\geq 0}, \qquad (67)$$

for any $\tau \geq 0$, where

$$\mathcal{B}_{\geq \tau} := \bigvee_{\tau' > \tau} \mathcal{B}_{[\tau, \tau']} \otimes \Big[\bigotimes_{\tau'' > \tau'} \mathbb{1}_{\tau''} \Big]. \qquad (68)$$

The algebra $\mathcal{E}^{(0)}_{\geq \tau}$ describes <u>possible events</u> and <u>measurements</u> of physical quantities at times $\geq \tau$ in the <u>absence of interactions</u> between the antenna and the E-fields.

Next, we define the time evolution of operators in the Heisenberg picture, in the absence of interactions between antenna and E-fields:

This is accomplished by specifying how $\mathcal{E}^{(0)}_{\geq \tau+1}$ is imbedded into $\mathcal{E}^{(0)}_{\geq\tau}$. Recall that $\mathcal{E}^{(0)}_{\geqslant \tau+1} \simeq \mathfrak{A} \otimes \mathcal{B}_{\geq \tau+1}$. We define an <u>embedding</u> <u>*homomorphism</u> $\varepsilon^{(0)}: \mathcal{E}^{(0)}_{\geq\tau+1} \to \mathcal{E}^{(0)}_{\geq\tau}$ by setting

$$\varepsilon^{(0)}(A \otimes E) := \underbrace{VAV^*}_{\in \mathfrak{A}} \otimes \underbrace{\mathbb{1}_\tau \otimes E}_{\in \mathcal{B}_{\geq\tau}}, \qquad (69)$$

for arbitrary $A \in \mathfrak{A}$, $E \in \mathcal{B}_{\geq\tau+1}$. If ω_τ is a state on $\mathcal{E}^{(0)}_{\geq\tau} \simeq \mathfrak{A} \otimes \mathcal{B}_{\geq\tau}$ then its restriction, $\omega_{\tau+1}$, to $\mathcal{E}^{(0)}_{\geq\tau+1}$ is given by

$$\begin{aligned}\omega_{\tau+1}(A \otimes E) &:= \omega_\tau\left(\varepsilon^{(0)}(A \otimes E)\right) \\ &= \omega_\tau\left(VAV^* \otimes \mathbb{1}_\tau \otimes E\right) \qquad (70)\end{aligned}$$

Next, we introduce <u>interactions</u> between the antenna and the E-fields. For later purposes we introduce a shift operator, S, on $\mathcal{H}_\infty$ by setting

$$(S\Phi)_{\underline{k}} := \Phi_{\sigma(\underline{k})} = \bigotimes_{\tau\in\mathbb{Z}} \varphi_{\sigma(\underline{k})_\tau}, \qquad (72)$$

where $\sigma(\underline{k})_\tau := k_{\tau-1}$, and φ_k, $k=0,\dots,N-1$, and $\Phi_{\underline{k}}$ are as in (60). The definition of S is extended to the domain $\mathcal{D}$ (see (61)) by linearity and then to $\mathcal{H}_\infty$ by using that S is bounded – actually S is unitary – and the fact that $\mathcal{D}$ is dense in $\mathcal{H}_\infty$.

Dropping $\gamma^{(0)}$ in our notation, we then have that

$$S E S^{-1} \in \mathcal{B}_{\geq \tau+1}, \quad \forall E \in \mathcal{B}_{\geq \tau}, \qquad (73)$$

i.e., $S \mathcal{B}_{\geq\tau} S^{-1} = \mathcal{B}_{\geq \tau+1}$, for any τ.

Thus, if T is an operator in $\mathcal{B}_{[0]}$ then

$$T_\tau := S^\tau T (S^{-1})^\tau \in \mathcal{B}_{[\tau]} \qquad (74)$$

In order to describe interactions between the antenna and the E-fields, we introduce a partition of unity on $\mathcal{H}_A$ (see (55)) by orthogonal projections, $\{Q_n\}_{n=0}^{K-1}$, $K < M$, with

$$Q_n = Q_n^*, \quad Q_n \cdot Q_m = \delta_{nm} Q_n, \quad \forall n, m \qquad (75)$$

and
$$\sum_{n=0}^{K-1} Q_n = \mathbb{1}. \tag{76}$$

We define a unitary operator, W_0, on $\mathcal{H}_A \otimes \mathcal{H}_{\tau=0}$ by setting

$$W_0(\psi \otimes \varphi) := \sum_{n=0}^{K-1} Q_n \psi \otimes T^{(n)} \varphi, \tag{77}$$

where $T^{(n)}$ is a unitary operator on $\mathcal{H}_{\tau=0}$, for every n; (we may wish to suppose that $Q_0 = |\psi_0\rangle\langle\psi_0|$ projects onto the ground state, ψ_0, of the antenna, and that $T^{(0)} = \mathbb{1}$ – but we will not make use of these assumptions).

By (74)

$$\begin{aligned} W_\tau &:= (\mathbb{1} \otimes S)^\tau \, W_0 \, (\mathbb{1} \otimes S^{-1})^\tau \\ &= \sum_{n=0}^{K-1} Q_n \otimes T_\tau^{(n)} \end{aligned} \tag{78}$$

is then a <u>unitary operator</u> on $\mathcal{H}_A \otimes \mathcal{H}_\tau$.

We are interested in constructing a "co-filtration" (= descending filtration) of algebras $\mathcal{E}_{\geq \tau}$, with

$$\mathcal{E}_{\geqslant\tau} \simeq \mathcal{A} \otimes \mathcal{B}_{\geqslant\tau} \simeq \mathcal{A} \otimes S^{\tau} \mathcal{B}_{\geqslant 0} (S^{-1})^{\tau}, \quad (79)$$

and

$$\mathcal{E}_{\geqslant\tau'} \subsetneqq \mathcal{E}_{\geqslant\tau}, \quad \forall \tau' \geqslant \tau, \quad (80)$$

equipped with an explicit embedding map $\varepsilon : \mathcal{E}_{\geqslant\tau+1} \longrightarrow \mathcal{E}_{\geqslant\tau}, \ \forall \tau$.

The map ε is defined as follows:

For an operator $C \in \mathcal{E}_{\geqslant\tau+1}$,

$$\varepsilon(C) := W_{\tau}(V \otimes \mathbb{1}) C (V^* \otimes \mathbb{1}) W_{\tau}^* \in \mathcal{E}_{\geqslant\tau}. \quad (81)$$

Concretely, if $C = A \otimes E$, $A \in \mathcal{A}$, $E \in \mathcal{B}_{\geqslant\tau+1}$, then

$$\varepsilon(A \otimes E) =$$

$$= \sum_{n,m=0,\dots,K-1} Q_n V A V^* Q_m \otimes T_{\tau}^{(n)} \left(T_{\tau}^{(m)}\right)^* \otimes E$$

$$= \sum_{n,m=0,\dots,K-1} Q_n V A V^* Q_m \otimes S^{\tau} \Big(T^{(n)} \left(T^{(m)}\right)^* \otimes$$

$$\otimes S^{-\tau} E S^{\tau} \Big) S^{-\tau} \quad (82)$$

If ω_{τ} is a state on $\mathcal{E}_{\geqslant\tau}$ its restriction, $\omega_{\tau+1}$, to $\mathcal{E}_{\geqslant\tau+1}$ is given by

$$\omega_{\tau+1}(C) := \omega_{\tau}(\varepsilon(C)). \quad (83)$$

Since $\mathcal{E}_{\geq \tau+1} \simeq \mathcal{A} \otimes \mathcal{B}_{\geq \tau+1}$, we may choose C to be of the form $C = A \otimes E$, $A \in \mathcal{A}$, $E \in \mathcal{B}_{\geq \tau+1}$. Then (82) shows that

$$\omega_{\tau+1}(C) = \omega_\tau(\mathcal{E}(C)) =$$
$$= \sum_{n,m=0,\dots,K-1} \omega_\tau \Big(\underbrace{Q_n V A V^* Q_m}_{\in \mathcal{A}} \otimes \underbrace{\Big[T_\tau^{(n)} \big(T_\tau^{(m)}\big)^* \otimes E \Big]}_{\in \mathcal{B}_{\geq \tau}} \Big) \tag{84}$$

Let $\varphi^{(\underline{k})}$ to be the state on $\mathcal{B}_{\geq \tau}$ given by

$$\varphi^{(\underline{k})}(F) := \langle \Phi_{\underline{k}}, F \Phi_{\underline{k}} \rangle, \quad F \in \mathcal{B}_{\geq \tau}, \tag{85}$$

where $\Phi_{\underline{k}} \in \mathcal{H}_\infty$ is the vector defined in (60), ($k_{\tau'} = 0$, except for finitely many τ'), with τ arbitrary. Suppose that ω_τ is the state on $\mathcal{E}_{\geq \tau} \simeq \mathcal{A} \otimes \mathcal{B}_\tau$ given by

$$\omega_\tau(A \otimes F) = tr(\Omega A)\, \varphi^{(\underline{k})}(F), \tag{86}$$

where Ω is a <u>density matrix</u> on $\mathcal{H}_A$; see (55), (56). Then, for $C = A \otimes E \in \mathcal{A} \otimes \mathcal{B}_{\geq \tau+1} \simeq \mathcal{E}_{\geq \tau+1}$

we have that

$$\omega_{\tau+1}(C) = \omega_{\tau+1}(A\otimes E) =$$

$$= \sum_{n,m=0,\dots,K-1} tr\Big(\big(V^*\Omega_{mn}V\big)A\Big)\Big\langle \varphi_{k_\tau}, T^{(n)}\big(T^{(m)}\big)^*\varphi_{k_\tau}\Big\rangle \times$$

$$\times\, \varphi^{(\underline{k})}(E), \text{ with } \Omega_{mn} = Q_m \Omega Q_n. \quad (87)$$

We define operators $L^{(\alpha)}_{k_\tau}$, $\alpha = 0,\dots,N-1$, acting on $\mathcal{H}_A$ by

$$L^{(\alpha)}_{k_\tau} Q_n := \big\langle \varphi_{k_\tau}, T^{(n)}\varphi_\alpha\big\rangle_{\mathbb{C}^N} \cdot Q_n, \quad (88)$$

$n = 0,\dots,K-1$. Note that

$$\sum_\alpha L^{(\alpha)}_{k_\tau}\big(L^{(\alpha)}_{k_\tau}\big)^* = \sum_{n=0}^{k-1} Q_n = \mathbb{1}. \quad (89)$$

From (87), (88) we get

$$\omega_{\tau+1}(A\otimes E) = \sum_{\alpha=0}^{N-1} tr\Big(\Big(V^*\big(L^{(\alpha)}_{k_\tau}\big)^*\Omega L^{(\alpha)}_{k_\tau}V\Big)A\Big)\times$$

$$\times\, \varphi^{(\underline{k})}(E). \quad (90)$$

We conclude that $\omega_{\tau+1}$ has again the form

$$tr\big(\Omega'(\cdot)\big)\otimes\varphi^{(\underline{k})}(\cdot), \quad (91)$$

just like ω_τ, with

$$\Omega \mapsto \Omega' = \Psi_k(\Omega) := \\ = \sum_{\alpha=0}^{N-1} V^* (L_k^{(\alpha)})^* \Omega L_k^{(\alpha)} V, \qquad (92)$$

for $k = k_\tau$.

We have learned in Sect. 4.2 that the maps Ψ_k defined in (91) are trace-preserving (see (89), and use unitarity of V and cyclicity of the trace) and completely positive.

Apparently, before the effect of interactions on the dynamics of the E-fields would become visible their modes have already disappeared and become unobservable. Only the dynamics of the antenna exhibits observable effects of the interactions. We will analyze them more closely shortly.

The material discussed here is technically very simple, but conceptually quite subtle. It may therefore be useful to repeat the same story in the Schrödinger picture.

At every time $\tau \geq 0$, the algebra of operators describing possible events or measurements of physical quantities at times $\geq \tau$ has turned out to be isomorphic to $\mathcal{A} \otimes \mathcal{B}_{\geq \tau}$. The shift map, S, introduced in (73) shows that

$$\mathcal{A} \otimes \mathcal{B}_{\geq \tau} \simeq \mathcal{A} \otimes \mathcal{B}_{\geq \tau = 0}. \qquad (93)$$

In the Schrödinger picture, operators are considered to be time-independent, while states evolve in time. As an appropriate "algebra of observables" in the Schrödinger picture we will thus choose

$$\mathcal{K} := \mathcal{A} \otimes \mathcal{B}_{\geq 0} \equiv \mathcal{E}_{\geq 0} \qquad (94)$$

I will now first define the Schrödinger

evolution of state vectors in $\mathcal{H}_A \otimes \mathcal{H}_\infty$:

Let $\chi := \psi \otimes \Phi$, $\psi \in \mathcal{H}_A$, $\Phi \in \mathcal{H}_\infty$, be the state vector of the system at time $\tau = 0$. Then the state vector, χ_τ, of the system at time $\tau > 0$ is given in the Schrödinger picture by

$$\chi_\tau := \left[(V^* \otimes \mathbb{1})(\mathbb{1} \otimes S^{-1}) W_0^* \right]^\tau \chi$$

$$= \sum_{n_1, \ldots, n_\tau} \left[\prod_{j=1}^{\tau} V^* Q_{n_j} \right] \psi \otimes \left[\prod_{k=1}^{\tau} \left(T_{\tau=-k}^{(n_k)} \right)^* \right] S^{-\tau} \Phi, \tag{95}$$

which can be extended by linearity to all of $\mathcal{H}_A \otimes \mathcal{H}_\infty$. From (95) we easily derive the Schrödinger – Liouville evolution of density matrices on $\mathcal{H}_A \otimes \mathcal{H}_\infty$, i.e., mixed states of the system.

It is straightforward to relate the evolution (95) to the evolution we have previously studied in the Heisenberg picture:

Let $C := A \otimes S^{-1} E S \in \mathcal{K}$, with $E \in \mathcal{B}_{\geq 1}$.

Then

$$\langle \chi_{\tau=1}, C \chi_{\tau=1} \rangle$$

$$= \sum_{n,m=0,\dots,K-1} \langle Q_n \psi, V A V^* Q_m \psi \rangle \times$$

$$\times \left\langle (T_0^{(n)})^* \Phi, S(S^{-1} E S) S^{-1} (T_0^{(m)})^* \Phi \right\rangle$$

$$= \sum_{n,m=0,\dots,K-1} tr\left([V^* Q_m |\psi\rangle\langle\psi| Q_n V] A \right) \times$$

$$\times \langle \Phi, \underbrace{T_0^{(n)} (T_0^{(m)})^* \otimes E}_{\in \mathcal{B}_{\geq 0}} \Phi \rangle, \quad (96)$$

which reproduces (87) and hence (84), (for $\tau = 0$).

From (95) we see that the propagator in the Schrödinger picture is given by

$$U_\tau := [(V^* \otimes \mathbb{1})(\mathbb{1} \otimes S^{-1}) W_0^*]^\tau \quad (97)$$

Let us see what this amounts to when we

pass to the limit where the time step, dt, tends to 0, with

$$V \otimes \mathbb{1} = \mathbb{1} + i(H_A \otimes \mathbb{1})\, dt, \quad T^{(n)} = \mathbb{1} + i(\mathbb{1} \otimes v^{(n)})\, dt$$

and

$$S = \mathbb{1} + i(\mathbb{1} \otimes P)\, dt, \qquad (98)$$

choosing $\tau = \frac{t}{dt}$. We then find, using the <u>Trotter product formula</u>, the following expression for the propagator:

$$U(t) := \lim_{dt \to 0} U_{\tau = t/dt} = e^{-itH},$$

where

$$H = H_A \otimes \mathbb{1} + \mathbb{1} \otimes P + \sum_{n=0}^{K-1} Q_n \otimes v^{(n)}, \qquad (99)$$

with $v^{(n)}$ a functional of the E-fields at time $t=0$, for all n, and P the "<u>momentum operator</u>":

$$e^{itP} E_\alpha(s) e^{-itP} = E_\alpha(s+t). \qquad (100)$$

The E-fields, $E_\alpha(t)$, are now operator-valued distributions:

$$\langle \Phi, E_\alpha(t) E_\beta(s) \Phi \rangle = \delta(t-s) G_\Phi(\alpha, \beta; t), \quad (101)$$

where $G_\Phi(\cdot, \cdot; t)$ is the kernel of a positive semi-definite quadratic form, $\forall t$, that depends on the state vector, Φ, of the E-fields.

Expression (99) for the Hamiltonian H of the system in the limit where $dt \to 0$, time evolution of states of the system is generated by an operator whose spectrum covers the <u>entire real line</u>.

Next, we study the time evolution found in (87) – (92) in some more detail. We have seen that the dynamics of the E-fields is not very interesting: If, at some time τ, the antenna is in a state in the range of the projection Q_n the state of E-fields at time τ is transformed by the unitary operator

$(T^{(n)})^*$ acting on $\mathcal{H}_\tau$. The state of E-fields at future times $>\tau$ is not affected. Subsequently, at times $\tau+1, \tau+2, \dots$, E-fields at times $\leq \tau$ will never interact with the antenna, anymore, and can be forgotten.

Suppose that the density matrix Ω_τ describes the state of the antenna at time τ, that P_τ is a density matrix on $\mathcal{H}_\tau$ describing the state of the E-fields at time τ and that $\varphi^{(>\tau)}$ is a state on the E-fields at times $>\tau$. We also suppose that one does not carry out any measurements on E-fields at time τ. Then the state of the system at time $\tau+1$ is given by

$$\mathrm{tr}_{\mathcal{H}_A}\left(\Omega_{\tau+1}(\cdot)\right) \otimes \varphi^{(>\tau)}(\cdot), \qquad (102)$$

where, according to (87),

$$\Omega_{\tau+1} = \sum_{n,m=0,\ldots,K-1} g^{mn}(\tau)\, V^*(\Omega_\tau)_{mn} V, \qquad (103)$$

where

$$\left.\begin{aligned} g^{mn}(\tau) &:= tr_{\mathcal{H}_\tau}\left(\left(T^{(m)}\right)^* P_\tau T^{(n)}\right), \\ \text{and}\quad (\Omega_\tau)_{mn} &:= Q_m \Omega_\tau Q_n . \end{aligned}\right\} \qquad (104)$$

As shown in (88) – (92), the map $\Omega_\tau \mapsto \Omega_{\tau+1}$ in (103), (104) is <u>trace-preserving</u> and <u>completely positive</u>.

<u>Note that</u>

$$\begin{aligned} \overline{g^{mn}(\tau)} &= tr_{\mathcal{H}_\tau}\left(\left[\left(T^{(m)}\right)^* P_\tau T^{(n)}\right]^*\right) \\ &= tr_{\mathcal{H}_\tau}\left(\left(T^{(n)}\right)^* P_\tau T^{(m)}\right) \\ &= g^{nm}(\tau), \end{aligned} \qquad (105)$$

because $P_\tau^* = P_\tau$. Thus, the matrix

$$G(\tau) = \left(g^{mn}(\tau)\right)_{m,n=0,\ldots,K-1}$$

is selfadjoint and can be diagonalized by a unitary $K \times K$ matrix $D(\tau) = \left(d^{kj}(\tau)\right)_{k,j=0,\ldots,K-1}$:

$$\left.\begin{aligned} \gamma_k(\tau)\,\delta^{kl} &= \sum_{m,n} \overline{d^{mk}(\tau)}\, g^{mn}(\tau)\, d^{nl}(\tau), \\ \text{i.e.,} \quad \mathrm{diag}\big(\gamma(\tau)\big) &= D^*(\tau)\, G(\tau)\, D(\tau). \end{aligned}\right\} \quad (106)$$

The positive real numbers $\gamma_k(\tau)$ are the eigenvalues of $G(\tau)$. If $K > N = \dim \mathcal{H}_\tau$ then $K-N$ eigenvalues of $G(\tau)$ vanish if P_τ is a pure state. Let us suppose that

$$T^{(n)} = \mathbb{1} + O(\varepsilon), \quad 0 \leq \varepsilon \ll 1, \ \forall n \qquad (107)$$

Then we have that

$$\gamma_0(\tau) = K + O(\varepsilon), \ \gamma_k(\tau) = O(\varepsilon), \ k \geq 1, \quad (108)$$

and

$$d^{n0}(\tau) = \frac{1}{\sqrt{K}}\big(1 + O(\varepsilon)\big) = d^{0n}(\tau). \qquad (109)$$

We then find that

$$\Omega_{\tau+1} = \sum_{k=0}^{K-1} \gamma_k(\tau) \sum_{m,n=0,\dots,K-1} d^{km}(\tau)\, \overline{d^{kn}(\tau)}\, V^* (\Omega_\tau)_{mn} V. \qquad (110)$$

Recall that $\Omega_\tau = \sum_{m,n=0,\dots,K-1} (\Omega_\tau)_{mn}$. Thus, if Ω_τ is pure, $\Omega_\tau = |\psi\rangle\langle\psi|$, and (107) holds then

$$\Omega_{\tau+1} = V^* \Omega_\tau V + O(\varepsilon)$$
$$= |V^*\psi\rangle\langle V^*\psi| + O(\varepsilon). \qquad (111)$$

<u>Collapse Postulate</u>, (<u>temporary version</u>)

This is a postulate concerning the <u>dynamics of states in quantum mechanics</u>. It says:

<u>At every time $\tau = 0, 1, 2, \dots$, the state of the antenna is "purified"</u>: Thus if

$$\Omega_\tau = \left(tr(\Pi_\tau)\right)^{-1} \Pi_\tau, \quad \Pi_\tau = \Pi_\tau^* = \Pi_\tau^2 \qquad (112)$$

and $\Omega_{\tau+1}$ is given by (103), (110) then, by the spectral theorem

$$\Omega_{\tau+1} = \sum_\alpha \frac{p_\alpha(\tau+1)}{tr(\Pi_\alpha(\tau+1))} \Pi_\alpha(\tau+1), \qquad (113)$$

where $\Pi_\alpha(\tau+1) = \Pi_\alpha(\tau+1)^*$,

$\Pi_\alpha(\tau+1)\,\Pi_\beta(\tau+1) = \delta_{\alpha\beta}\,\Pi_\alpha(\tau+1)$, $\sum_\alpha \Pi_\alpha(\tau+1) = 1\!\!1_{\mathcal{H}_A}$

Then the state to be used to predict the future after time τ is one of the states

$$\Omega^{\alpha}_{\tau+1} := \frac{1}{tr\left(\Pi_{\alpha}(\tau+1)\right)} \Pi_{\alpha}(\tau+1), \qquad (114)$$

tensored with the state $\varphi^{(>\tau)}$ for the E-field; the probability of choosing the state

$$tr\left(\Omega^{\alpha}_{\tau+1}(\cdot)\right) \varphi^{(>\tau)}(\cdot)$$

is given by $p_{\alpha}(\tau+1)$. $\square$

Note that $p_{\alpha}(\tau+1) \geq 0$, $\sum_{\alpha} p_{\alpha}(\tau+1) = 1$!

According to the Collapse Postulate, the evolution of states is given by a stochastic branching process, which, according to (107) – (110), approaches the Schrödinger evolution (56) in the weak coupling limit, $\varepsilon \searrow 0$.

The "strong coupling limit":

It is not difficult, using massless E-fields (with ∞ many degrees of freedom, for every time τ) to construct simple models of interactions between the antenna and the E-fields that are quite "singular" in the infrared domain (E-fields with long wave lengths) with the property that

$$g^{mn}(\tau) = tr_{\mathcal{H}_\tau}\left(\left(T^{(m)}\right)^* P_\tau T^{(n)}\right)$$
$$= \delta^{mn}\left(1+O(\varepsilon)\right), \; |\varepsilon| \ll 1, \quad (115)$$

for a large class of states P_τ; e.g.,

$$P_\tau = |\varphi_0\rangle\langle\varphi_0| \quad (116)$$

Then, by (103),

$$\Omega_{\tau+1} = \sum_{n=0}^{K-1} V^* Q_n \Omega_\tau Q_n V + O(\varepsilon), \quad (117)$$

hence

$$\Omega_{\tau+2} = V^*\left\{\sum_{k,n=0,\dots,K-1} Q_k V^* Q_n \Omega_\tau Q_n V Q_k\right\} V \quad (118)$$
$$+ O(\varepsilon).$$

The map

$$\widetilde{\Omega}_\tau \longmapsto \sum_{k,n=0,\dots,K-1} Q_k V^* Q_n \widetilde{\Omega}_\tau Q_n V Q_k =: \widetilde{\Omega}_{\tau+1} \qquad (119)$$

defines a "quantum Markov chain", with a "transition operator", Γ, given by

$$\left.\begin{aligned} &\Gamma = (\Gamma_{nm}), \quad \Gamma_{nm} := Q_n V Q_m, \\ \text{and} \quad &\widetilde{\Omega}_\tau \longmapsto \widetilde{\Omega}_{\tau+1} = \sum_{n,m} \Gamma^*_{nm} \widetilde{\Omega}_\tau \Gamma_{nm}, \end{aligned}\right\} \qquad (120)$$

The maps (118) – (120) are completely positive and trace-preserving.

If $Q_n = |\psi_n\rangle\langle\psi_n|$, $\forall\, n = 0, \dots, K-1$, with $K = M = \dim \mathcal{H}_A$, $\{\psi_n\}_{n=0}^{M-1}$ a CONS in $\mathcal{H}_A$, then

$$Q_n V Q_k = |\psi_n\rangle\langle\psi_n, V\psi_k\rangle\langle\psi_k|.$$

Setting

$$\left.\begin{aligned} &P(n,k) := |\langle\psi_n, V\psi_k\rangle|^2, \\ \text{and} \\ &\mu_\tau(n) := \langle\psi_n, \widetilde{\Omega}_\tau \psi_n\rangle, \end{aligned}\right\} \qquad (121)$$

we find that

$$\mu_{\tau+1}(k) = \sum_n P(n,k)\, \mu_\tau(n) \qquad (122)$$

defines an ordinary Markov chain with state space given by

$$\mathcal{X} := \{0, \dots, M-1\} \qquad (123)$$

from which the quantum Markov chain in (119), (120) can be reconstructed. (Note that $P(n,k) \geq 0$, $\forall n,k$, and

$$\sum_k P(n,k) = \sum_n P(n,k) = 1 \,!)$$

The theory developed in Sects. 5.3 and 5.4 can be transferred to the present setting. Generalizing Sect. 5.4, we would choose $V = \mathbb{1} + O(\varepsilon)$, $g^{mn} = \langle \varphi_0, T^{(m)} (T^{(n)})^* \varphi_0 \rangle < 1$, for $m \neq n$, and we would follow the time evolution on a time scale of $O(\varepsilon^{-1})$. Then, if $Q_n = |\psi_n\rangle\langle\psi_n|$, for a CONS $\{\psi_n\}_{n=0}^{M-1}$ in

$\mathcal{H}_A$), we approach again a Markov chain, as $\varepsilon \searrow 0$, as discussed in Sect 5.4.

Mott tracks:

Combining the theory developed here with the results described in Sect. 5.5 (on the first 11 pages), trading the "antenna" for a charged particle, and choosing $Q_n = |\psi_n\rangle\langle\psi_n|$, where $\{\psi_n\}$ is a "Bourgain basis", as described in Theorem B, (Sect. 5.5, page 6), we arrive at a more reasonable theory of particle (Mott) tracks, provided

$$\left| g^{mn}(\tau) - \delta^{mn} \right| < \varepsilon \ll 1, \qquad (124)$$

uniformly in m, n and τ. The present theory works fine even if we do not (or cannot) monitor the E-fields!

Remark.

Let P be a density matrix on $\mathbb{C}^N \simeq \mathcal{H}_\tau$ that has $k \geq 2$ distinct, strictly positive eigenvalues. We choose the product state

$$\rho := \bigotimes_{\tau \in \mathbb{Z}} P_\tau , \quad P_\tau = P, \ \forall \tau \in \mathbb{Z}, \quad (125)$$

to be the reference state for the E-fields. Let π_ρ be the representation of the algebras $\mathcal{B}_{\geq \tau}$ introduced in (68) on the GNS Hilbert space associated with $\left(\rho, \bigvee_\tau \mathcal{B}_{\geq \tau}\right)$. Then the weak closure, $\overline{\mathcal{B}_{\geq \tau}}^{\rho}$, of the algebra $\mathcal{B}_{\geq \tau}$ in the representation π_ρ is a type III von Neumann algebra, as shown by Robert Powers, many years ago. This is reminiscent of the algebras $\mathcal{D}_\tau$ of bounded functionals of the free electromagnetic field localized in the diamond $\mathcal{O}_\tau$ (see below (61) and before

(62)) are type III_1 factors.

The fact that such "exotic" von Neumann algebras appear already in the analysis of simple toy models like the ones discussed in this chapter shows how stupid it is to despise somewhat advanced mathematical tools such as the theory of operator algebras.

Lecture, January 12, 2017

Short version of Chap. 4

1. Recap. from January 10: (Proj. measmts.)

Copenhagen interpretation:

If

$$X = X^* = \sum_{\xi} \xi \, \Pi_{\xi}, \quad \sum_{\xi} \Pi_{\xi} = 1 \qquad (1)$$

is measured at time t then state after measurement is an eigenstate of $X(t)$. If value of X not recorded state is an incoherent superposition of eigenstates of $X(t)$. Here

$$X(t) = U(s,t) X(s) U(t,s).$$

$$\rho \longmapsto \sum_{\xi} \rho\left(\Pi_{\xi}(t)(\cdot)\,\Pi_{\xi}(t)\right) \qquad (2)$$

This is not properly captured by LSW formula. But (2) and LSW formula are successful as <u>heuristic recipes.</u>

How can we "recover them" in a mathematically consistent formulation/extension of QM?

<u>Proposal:</u> $\boxed{\text{ETH approach to QM}}$

<u>Basic Input</u>: "Loss of access to info" in the form

$$\boxed{\mathcal{B}(\mathcal{H}_S) \supseteq \mathcal{E} \underset{\neq}{\supseteq} \overline{\mathcal{E}}_{\geq t} \underset{\neq}{\supseteq} \overline{\mathcal{E}}_{\geq t'} \supset \{\mathbb{C}\mathbb{1}\}} \quad t' > t \qquad (3)$$

"<u>Info loss</u>"

ρ: general state on $\mathcal{E}$, possibly <u>pure</u>

$$\rho_t := \rho|_{\mathcal{E}_{\geq t}} \qquad (4)$$

By (3), ρ_t is generally a <u>mixed</u> state on $\mathcal{E}_{\geq t}$

Suppose X, with $X(t) \in \mathcal{E}_{\geq t}$, is observed at time t. Followers of the Copenhagen interpretation would then say that

$$\rho_t(A) = \sum_{\xi} \rho_t(\pi_\xi(t)\, A\, \pi_\xi(t)) \qquad (5)$$

$$\Rightarrow \left\{ \begin{array}{l} \rho_t([\pi_\xi(t), A]) = 0, \ \forall \xi \\ \rho_t([X(t), A]) = 0, \end{array} \right\} \qquad (6)$$

$\forall A \in \bar{\mathcal{E}}_{\geq t}$.

$$\Rightarrow X \in \mathcal{C}_{\rho_t}(\bar{\mathcal{E}}_{\geq t}) \quad \text{("}\underline{\text{centralizer}}\text{" of } \rho_t) \qquad (7)$$

2. Centralizers of states on von Neumann algebras and their centers.

$\mathfrak{M}$: von Neumann alg.

ω : normal state on $\mathfrak{M}$.

C Centralizer of ω in $\mathfrak{M}$:

$$\mathcal{C}_\omega := \{ X \in \mathfrak{M} \mid \mathrm{ad}_X\, \omega = 0 \}, \qquad (8)$$

where

$$\mathrm{ad}_X\, \omega(y) := \omega([X, y]), \qquad (9)$$

$y \in \mathfrak{M}$.

C Lemma. (Exercise)

$\mathcal{C}_\omega$ is a von Neumann subalgebra of $\mathfrak{M}$, and ω is a normalized trace on $\mathcal{C}_\omega$.

See p. 4.21 (Notes)

<u>Cor.</u> $\mathcal{C}_\omega$ is a direct sum or integral of matrix algebras $M_n(\mathbb{C})$, $1 \leq n < \infty$, (including possibly <u>abelian</u> algebras: $\int^\oplus M_1(\mathbb{C})$) and of type II_1 - von Neumann algebras ($\longrightarrow$ Classification of v.N. algebras!)

$\longrightarrow$ p. 4.24; (Notes)

$$\boxed{\mathcal{Z}_\omega := \text{center of } \mathcal{C}_\omega} \qquad (10)$$

<u>Remark.</u> If $X \in \mathcal{C}_\omega$ then

$$Ad_{e^{i\lambda X}}\, \omega = \omega, \quad \forall \lambda,$$

i.e., ω <u>invariant</u> under $\{e^{i\lambda X}\}_{\lambda \in \mathbb{R}}$

3. Definition of Events.

Phys. system S, with data as above

$$\left(\mathcal{S}, \{\mathcal{E}_{\geq t}\}_{t \in \mathbb{R}} \xrightarrow{GNS} \mathcal{H}_S, \{\overline{\mathcal{E}}_{\geq t}\}_{t \in \mathbb{R}}, \text{etc.}\right)$$

stratum of states of phys. interest on C^*-alg $\mathcal{E}$

Remark.

For autonomous systems $\exists$ one-parameter group of *automorphisms, $\{\tau_t\}_{t\in\mathbb{R}_+}$ of $\mathcal{E}^-$ such that

$$\tau_t : \mathcal{E}_{\geq t'} \longrightarrow \mathcal{E}_{\geq t'+t}\,, \qquad (11)$$

$\forall\, t \geq 0$, $\forall\, t' \in \mathbb{R}$; (with props. ... ; see Buchholz & Roberts). $\Rightarrow \{\tau_t\}$ are endomorphisms, but not automorphisms, of $\mathcal{E}_{\geq t'}$, for any fixed $t' \in \mathbb{R}$.

Let ρ_t be state of S at time t, i.e., a state on $\mathcal{E}_{\geq t}$. We say that

$$\boxed{\begin{array}{c}\text{an event happens at time } t \text{ iff}\\ Z_{\rho_t} \neq \{\mathbb{C}\mathbb{1}\}.\end{array}} \qquad (12)$$

Then: $\boxed{\text{Event} = \text{some orthogonal proj. in } Z_{\rho_t}}$

Assume for simplicity that, for all $\rho \in \mathcal{S}$, $\forall t \in \mathbb{R}$,

$$\boxed{Z_{\rho_t} = \langle \Pi_\xi(t) \mid \xi \in \mathcal{X}_{\rho_t} \rangle,} \qquad (13)$$

$\mathcal{X}_t$ countable (or even finite).

3. The fundamental axiom of QM & ETH

Assume state of S at time t given by ρ_t on $\mathcal{E}_{\geq t}$; but that Z_{ρ_t} is non-trivial (as in (13)):

$$Z_{\rho_t} = \langle \Pi_\xi(t) \in \mathcal{E}_{\geq t} \mid \xi \in \mathcal{X}_{\rho_t} \rangle$$

Axiom. One of the events, e.g.,

$$\Pi_{\xi_*}(t), \quad \xi_* \in \mathcal{X}_{\rho_t}, \qquad (14)$$

actually happens at time $(\geq) t$.

<u>Born Rule:</u>

Probability that $\Pi_{\xi_*}(t)$, $\xi_* \in \mathcal{X}_{\rho_t}$, happens given by

$$\rho_t\big(\Pi_{\xi_*}(t)\big) \in (0,1).$$

If $\Pi_{\xi_*}(t)$ happens then state def. by

$$\boxed{\rho_{t,\xi_*}(\cdot) := \rho_t\big(\Pi_{\xi_*}(t)\big)^{-1}\,\rho_t\big(\Pi_{\xi_*}(t)(\cdot)\Pi_{\xi_*}(t)\big)} \tag{15}$$

should be used to predict future of S at times $> t$.

<u>The central enigma of quantum mechanics</u>

$$\left.\begin{array}{l}\text{In general, } \underline{\text{no}} \text{ relationship between} \\[1ex] \qquad Z_{\rho_{t'}} \quad \text{and} \quad \bigvee_{\xi \in \mathcal{X}_{\rho_t}} Z_{(\rho_{t,\xi})_{t'}} \\[1ex] \left(\text{where } \rho_{t'} \overset{t'>t}{:=} \rho_t\big|_{\mathcal{E}_{\geq t'}}\right).\end{array}\right\} \tag{16}$$

$\Rightarrow$ If we do not know true state of S at time t we cannot predict when the next event happens after time t.

But: Special cases; (e.g.,

$$\bigcap_{\rho \in \mathcal{S}' \subset \mathcal{S}} \mathcal{Z}_{\rho[t]} \neq \{\mathbb{C}\mathbb{1}\}, \dots)$$

all states triggering a "detector" at time $\approx t$

[t]: an interval around time t.

An example.

Ass.: All $\mathcal{E}^{-}_{\geq t}$ are type $I_{(\infty)}$. Then

$$\rho_t(A) = tr(P_t A), \quad A \in \mathcal{E}^{-}_{\geq t},$$

with $P_t \in \mathcal{E}^{-}_{\geq t}$

$$\Rightarrow \mathcal{Z}_{\rho_t} = \langle \text{bd. cont. fcs. of } P_t \rangle$$

$$= \langle \pi_i(t) \mid i = 1, 2, 3, \dots \rangle$$

spect. proj. of P_t, with e.v. $\neq 0$.

Fundamental axiom implies that, after time t, one of the states

$$\frac{\pi_i(t)}{tr(\pi_i(t))} =: P_{t,i}$$

survives.

Note $P_{t,i}^2 \propto P_{t,i}$, $[P_{t,i}, P_{t,j}] \neq 0$, $\forall i,j$.

$\rho_{i,t'}$: state obtained by restricting $P_{t,i}$ to $\mathcal{E}_{\geq t'}$, $(t' > t)$.

$\rho_{i,t'}$ given by density matrix $P_{i,t'}$

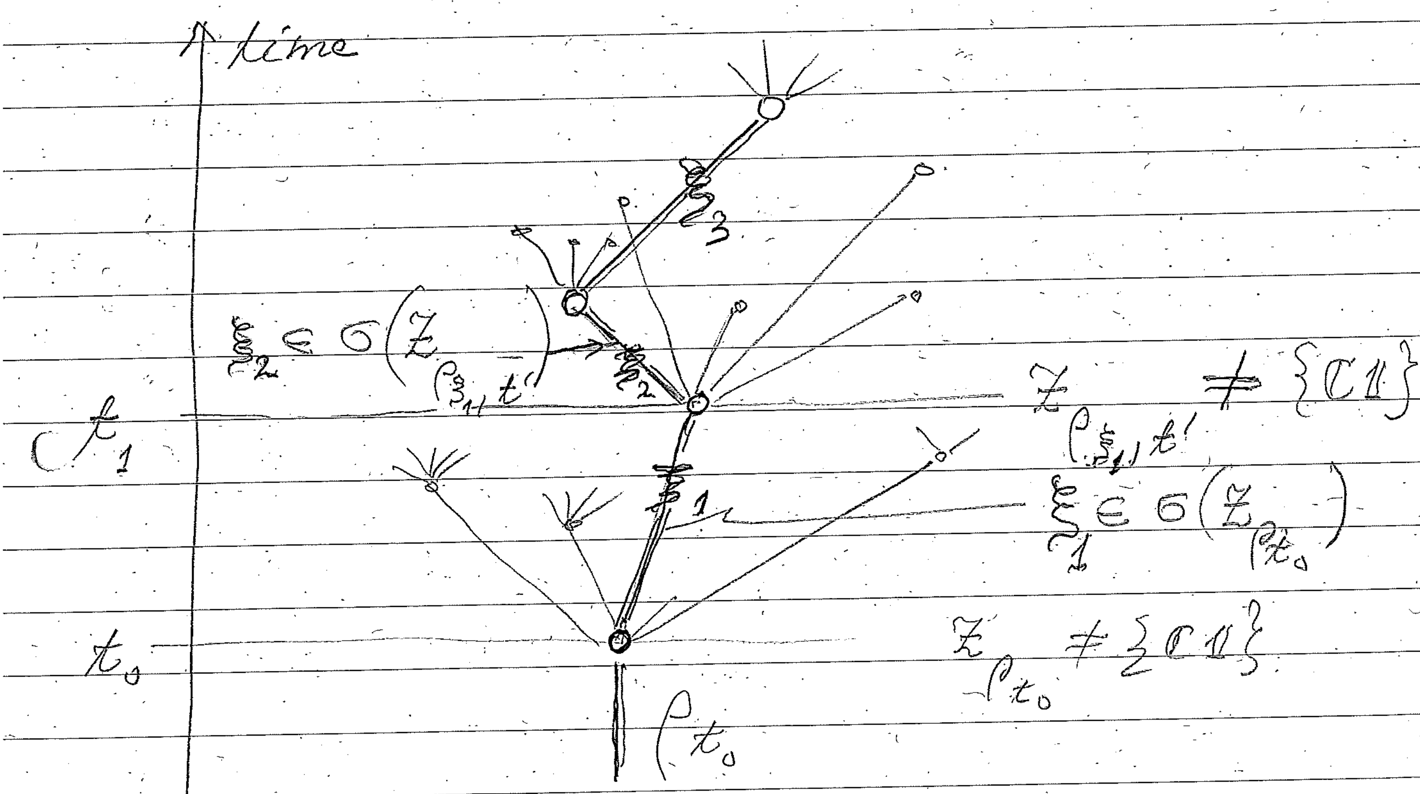
But, in general:

$$\left.\begin{array}{l} P_{i,t'}^2 \not\propto P_{i,t'} \\ [P_{i,t'}, P_{j,t'}] \neq 0, \end{array}\right\}$$

for some $i \neq j$

$\Rightarrow$ Spect. projections of $\{P_{i,t'}\}_{i=1,2,3,\dots}$ not commensurable with those of $P_{t'}$ (arising from $\rho_t \big|_{\mathcal{E}^-_{\geq t'}}$)!

$\hookrightarrow$ ETH picture of evolution of states



$\longrightarrow$ $\boxed{\text{"Events, Trees, Histories"}}$

Toy example: Lindblad evolution $\longrightarrow$ a new type of "stochastic branching process" with values in projections on $\mathcal{H}$; (only understood for discrete times).

---

4. Detection of Events using Instruments.

S: as above.

Definition. An instrument (to observe S) is an abelian C^*-algebra, $\mathcal{I}$, with the property that $\exists$ *representations, $\pi_t^{\mathcal{I}}$, of $\mathcal{I}$ with values in $\overline{\mathcal{E}}_{\geq t}$, $\forall t \in \mathbb{R}$.

For simplicity, we assume that all our instruments are finite-diml.:

$$\mathcal{Y} = \langle \Pi^{\mathcal{Y}}_{\xi} \mid \xi \in \mathfrak{X}_{\mathcal{Y}} = \sigma(\mathcal{Y}) \rangle \qquad (17)$$

Then $\mathcal{Y}$ is generated by all bd. functions of a <u>single</u> operator, $X_{\mathcal{Y}}$, and

$$X_{\mathcal{Y}}(t) := \pi^{\mathcal{Y}}_{t}(X_{\mathcal{Y}}) \in \mathcal{E}^{-}_{\geq t}, \qquad (18)$$

It could happen that

$$X_{\mathcal{Y}}(t) \in \mathcal{Z}_{\rho_t}. \qquad (19)$$

Then $\mathcal{Y}$ can detect <u>certain</u> events that might happen at time t (but possibly <u>not</u> all of them).

4.1 <u>Conditional expectations.</u>

$\mathcal{N}$ a von Neumann subalg. of $\mathcal{M}$.

<u>Def.</u> A linear map

$$\epsilon : \mathcal{M} \xrightarrow[\text{onto}]{} \mathcal{N} \qquad (20)$$

is a <u>conditional expectation</u> of $\mathfrak{M}$ onto $\mathfrak{N}$ <u>w.r. to a state</u> ω on $\mathfrak{M}$ iff

(i) $\|\in(X)\| \leq \|X\|, \quad \forall X \in \mathfrak{M}$

(ii) $\in(X) = X, \quad \forall X \in \mathfrak{N}$

(iii) $\omega = \omega \circ \in$

<u>Lemma</u> (Exercise)

- $\in(X^* X) \geq 0, \quad \forall X \in \mathfrak{M}$
- $\in(AXB) = A \in(X) B, \quad \forall A, B \in \mathfrak{N}, X \in \mathfrak{M}$
- $\in : \mathfrak{M} \to \mathfrak{N}$ is <u>completely positive</u>, with $\in(\mathbb{1}_{\mathfrak{M}}) = \mathbb{1}_{\mathfrak{N}}$

(see Takesaki, vol. II)

<u>Existence of conditional expectations</u>:

$\nearrow$ Takesaki; (vol. II, p. 211),

using <u>Tomita-Takesaki modular theory</u>.

4.2 Application to QM.

By Takesaki's results, $\exists$ conditional expectations

$$\overline{\in}_{\rho_t} : \mathcal{E}^-_{\geq t} \longrightarrow \mathcal{C}_{\rho_t},$$

and

$$\in_{\rho_t} : \mathcal{E}^-_{\geq t} \longrightarrow \mathcal{Z}_{\rho_t}$$

Given a "resolution" $\delta > 0$, an instrument $\mathcal{I}$ can detect an event at time t iff

$$\left\| \in_{\rho_t} \left(\pi^{\mathcal{I}}_{\xi}(t) \right) - \pi^{\mathcal{I}}_{\xi}(t) \right\| \leq \delta, \quad (21)$$

$\forall \xi \in \mathcal{X}_{\mathcal{I}}$.

<u>Lemma.</u> (Exercise) Suppose that

$$\left\langle \pi^{\mathcal{I}}_{\xi}(t) \,\middle|\, \xi \in \mathcal{X}_{\mathcal{I}} \right\rangle \simeq \mathcal{Z}_{\rho_t},$$

with resolution δ. ~~Then~~

Then

$$\left.\begin{aligned} \rho_t(A) = \sum_{\xi \in \mathcal{X}_{\mathcal{Y}}} \rho_t\left(\pi_\xi^{\mathcal{Y}}(t)\, A\, \pi_\xi^{\mathcal{Y}}(t)\right) \\ + O(\delta \|A\|), \end{aligned}\right\} \quad (22)$$

$\forall A \in \bar{\mathcal{E}}_{\geq t}$; (Pf.: See p. 4, 29 (Notes))

<u>Conjecture</u> (follows from Lemma if $\mathcal{Y}$ <u>finite</u>).

$\mathfrak{M}$, ω on $\mathfrak{M}$, $X \in \mathfrak{M}$, $\|X\| \leq 1$.

$$\|ad_X\, \omega\| < \varepsilon.$$

$\exists$ fn. $F : [0,1) \to \mathbb{R}_+$, $F(x) \to 0$, $x \to 0$,

op. $\tilde{X} \in \mathfrak{M}$, state $\tilde{\omega}$ on $\mathfrak{M}$ s.t.

$$ad_{\tilde{X}}\, \tilde{\omega} = 0$$

$$\|\tilde{X} - X\| < F(\varepsilon), \quad \|\omega - \tilde{\omega}\| < F(\varepsilon)$$

Remark. If $\mathfrak{M}$ is a type I factor, conjecture follows from thm. due to Halmos.

Thank you!

Important Remark.

Let $t' \underset{(=)}{\geq} t$, $\mathcal{Z}_{\rho_t} \neq \{\mathbb{C}\mathbb{1}\}$. Then:

$$\rho_t(A) = \sum_{\xi} \rho_t(\pi_\xi(t)\, A\, \pi_\xi(t)), \quad \forall A \in \overline{\mathcal{E}}_{\geq t'}$$

$\overset{\Uparrow}{\overline{\mathcal{E}}_{\geq t'}}$

Suppose $\exists\, X \in \mathcal{Z}_{\rho_{t'}}$ $(\rho_{t'} = \rho_t |_{\overline{\mathcal{E}}_{\geq t'}},\ t' > t)$. Then

$$0 = \rho_t([X, A]) = \sum_{\xi} \rho_t(\pi_\xi(t)\, [X, A]\, \pi_\xi(t)),$$

$\nwarrow$ (spect.) proj. in $\mathcal{Z}_{\rho_t}$.

for some $X \in \mathcal{C}_{\rho_{t'}}$, $\forall A \in \mathcal{E}_{\geq t'}$.

$\text{"}\sum_\eta \eta\, \pi^X_\eta(t')\text{"}$

$$\Rightarrow \quad \rho_t(A) = \sum_\eta \rho_t(\pi^X_\eta(t')\, A\, \pi^X_\eta(t'))$$

$$= \sum_\eta \sum_\xi \rho_t(\pi_\xi(t)\, \pi^X_\eta(t')\, A\, \pi^X_\eta(t')\, \pi_\xi(t))$$

$$= \sum_\xi \sum_\eta \text{ —"— }$$

$$\downarrow \ \text{"}$$

$$= \sum_\xi \rho_t(\pi_\xi(t)\, A\, \pi_\xi(t)), \quad \forall A \in \overline{\mathcal{E}}_{\geq t'}$$

If states $\rho_{t,\xi}$ can be "<u>distinguished</u>" by $\overline{\mathcal{E}}_{\geq t'}$ then $X \in \mathcal{Z}_{(\rho_{t,\xi})_{t'}}$, $\forall \xi$!

4. Information Loss, Events, Direct/Projective/von Neumann Measurements, ETH Ontology

This follows mostly:

- My slides
- Quantum Probability Theory, ... (with B. Schubnel)
- A "Garden of Forking Paths" – ... (with Ph. Blanchard & B. Schubnel)

Contents.

4.1. A formulation of Quantum Theory in terms of time-localized operator algebras

4.2. States of physical interest; passage to von Neumann algebras

4.3. Time evolution (Buchholz & Roberts); Information Loss

4.4. Instruments

(4.5. Digression on Tomita–Takesaki theory, and Conditional Expectations

4.6. Centralizers of states on von Neumann algebras and their
(centers – Events, and their observations using instruments.

$$\boxed{\mathcal{C}_{\rho_1 \oplus \rho_2} \neq \mathcal{C}_{\rho_1} \vee \mathcal{C}_{\rho_2}} \quad !$$

4.7. ETH ontology.

4.8. Toy examples of ETH ontology based on Lindblad evolutions and their unraveling.

4.1. A formulation of Quantum Theory in terms of time-localized operator algebras;

Consider NR quantum theory, and assume $\exists$ global time modeled by $\mathbb{R}$.

$\mathbb{R} \supset I \longrightarrow \mathcal{E}_I$; a C^*-algebra with $\mathbb{1}$.

Assumption;

$$I \subseteq I' \Rightarrow \mathcal{E}_I \subseteq \mathcal{E}_{I'} \qquad (4.1)$$

Def.

$$\mathcal{E}_{\geq t} := \overline{\bigvee_{I \subset [t,\infty)} \mathcal{E}_I}^{\|\cdot\|} \qquad (4.2)$$

$$\mathcal{E} := \overline{\bigvee_{t>-\infty} \mathcal{E}_{\geq t}}^{\|\cdot\|} \qquad (4.3)$$

Then

$$\mathcal{E} \supseteq \mathcal{E}_{\geq t} \supseteq \mathcal{E}_I , \qquad (4.4)$$

for arb $I \subseteq [t,\infty)$, arb. $t \in \mathbb{R}$.

[Phys. interpretation of $\mathcal{E}_I, \mathcal{E}_{\geq t}, \ldots$: As in older presentations!]

States = states on C^*-algebra $\mathcal{E}$.

4.2 States of physical interest, ...

$\mathcal{S}$: stratum of "states of physical interest" on $\mathcal{E}$.

Assume $\exists$ state $\omega_{ref} \in \mathcal{S}$ such that all other states in $\mathcal{S}$ are normal w.r. to ω_{ref}.

Apply GNS construction to $(\mathcal{E}, \omega_{ref})$:

$\xrightarrow[GNS]{}$ $\exists$ Hilbert space $\mathcal{H}$, *rep., π, of $\mathcal{E}$ on $\mathcal{H}$, unit ray $\Omega \in \mathcal{H}$ such that

$$\omega_{ref}(A) = \langle \Omega, \pi(A)\Omega \rangle_{\mathcal{H}}, \quad \forall A \in \mathcal{E}$$

From now on, will work on $\mathcal{H}$ and pass to von Neumann algebras

$$\overline{\mathcal{E}_I}, \ \overline{\mathcal{E}_{\geq t}}, \ \overline{\mathcal{E}}, \qquad (4.5)$$

where $^{-}$ denotes the closure in the (ultra-)weak topology on $B(\mathcal{H})$. We will occasionally omit "$-$".

4.3. Autonomous systems

System autonomous iff, for all $t \geq 0$,

$$\tau_t : \mathcal{E}_{\geq t'} \longrightarrow \mathcal{E}_{\geq t'+t} \subseteq \mathcal{E}_{\geq t'} \qquad (4.6)$$

$\{\tau_t\}_{t \geq 0}$: $*$ endomorphisms of $\mathcal{E}^{-}_{\geq t'}$, $\forall t'$.

Assume that

$$t \longmapsto \omega_{ref}\left(A^* \tau_t(B)\right) \text{ cont. in } t,$$

$\forall A, B \in \mathcal{E}_{\geq t'}$, all t'

[At zero temperature,

$$t \longmapsto \omega_0\left(A^* \tau_t(B)\right)$$

(vacuum)

extends to analytic function on

$\mathbb{C}_+ = \{t \mid \operatorname{Im} t > 0\}$, bd. by

$$\sqrt{\omega_0(A^*A)\, \omega_0(B^*B)},$$

$\forall A, B \in \mathcal{E}_{\geq t'}$, $\forall t'$.]

Theorem.

For autonomous systems we have the alternative.

(i) $\mathcal{E}^{-}_{\geq t}$ indep. of t and of type I

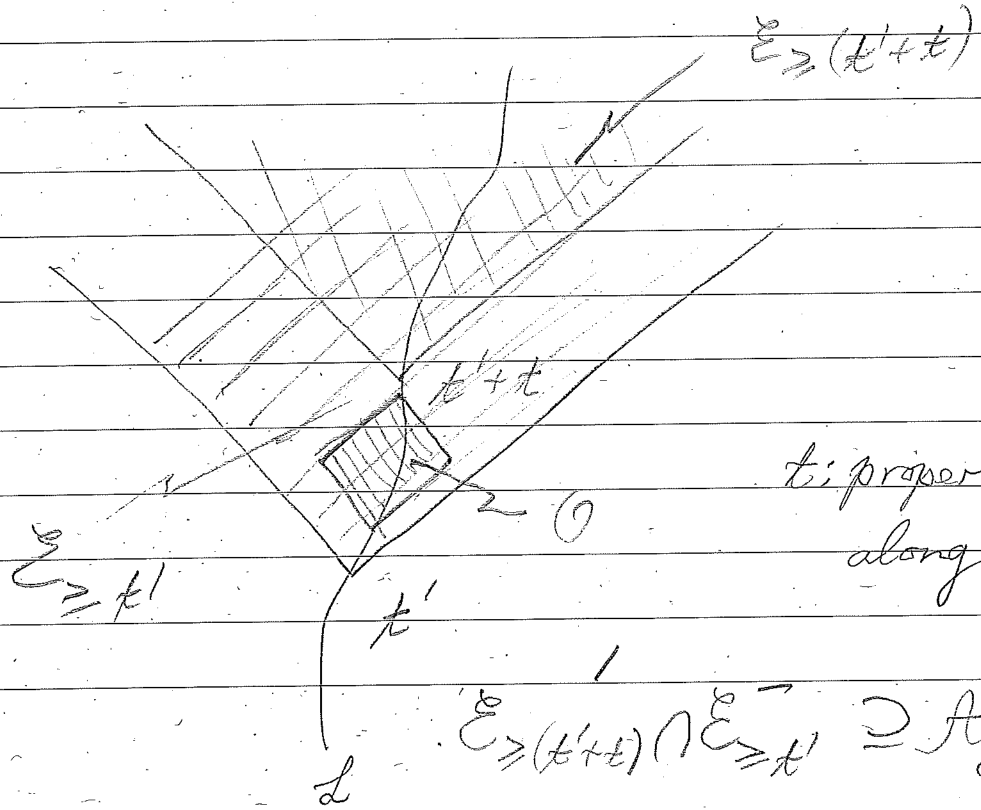
or

(ii) $\mathcal{E}^{-}_{\geq t} \subsetneq \mathcal{E}^{-}_{\geq (t+s)}$, $s>0$, and

then

$\mathcal{E}^{-}_{\geq t}$ of type III.

Information Loss in QED:



t: proper time along $\mathcal{L}$

$$\mathcal{E}'_{\geq (t'+t)} \cap \mathcal{E}^{-}_{\geq t'} \supseteq \mathcal{A}''_{asy}(\mathcal{O})$$

↪ Idea of Information Loss:

"Elemental states":

ω state on $\mathcal{E}_{\geq t}$ "elemental" iff $\overline{\mathcal{E}_{\geq t}}^{w}$ in GNS rep. assoc. with $(\mathcal{E}_{\geq t}, \omega)$ is a factor of type III_1.

↪ Decomposition of states of physical interest into classes corresp. to "elemental" states.

$$\boxed{\begin{array}{l}\text{Class of elemental state } \omega \\ \quad = \{\omega \circ \gamma \mid \gamma \in \mathrm{Inner}\, \mathcal{E}_{\geq t}\}\end{array}} \qquad (4.7)$$

Information Loss:

$$\boxed{\mathcal{E}^{-} \underset{\neq}{\supset} \mathcal{E}^{-}_{\geq t} \underset{\neq}{\supset} \mathcal{E}^{-}_{\geq s}, \quad s > t} \qquad (4.8)$$

Information Loss in non-autonomous systems.

$$\mathcal{E}_{\geq n} = \mathcal{E}_{P} \otimes \left(\bigotimes_{j \geq n} \mathcal{A}_j \right)$$

$$\mathcal{E}'_{\geq n} \cap \mathcal{E}_{\geq m} \underset{m<n}{=} \bigotimes_{j=m}^{n-1} \mathcal{A}_j \tag{4.9}$$

$\hookrightarrow$ $\boxed{\text{Info Loss}}$

4.4 Instruments

An instrument, $\mathcal{I}$, is an abelian C^*-algebra, $\mathcal{O}_{\mathcal{I}}$, w. $\mathbb{1}_{\mathcal{I}}$, along with a family $\{\pi_t\}_{t \in \mathbb{R}}$ of *homomorphisms,

$$\pi_t : \mathcal{O}_{\mathcal{I}} \longrightarrow \mathcal{E}_{\geq t}, \tag{4.10}$$

called reps. of $\mathcal{I}$ in $\mathcal{E}_{\geq t}$.

(Generalization: $\mathcal{E}_{\geq t} \longrightarrow \mathcal{E}_{I}$, $I \subset \mathbb{R}, \dots$)

$\longrightarrow$ Apply Gel'fand isomorphism to $\mathcal{O}_{\mathcal{I}}$!

To simplify matters, introduce

<u>Assumption.</u>

All instruments $\mathcal{J}$ are <u>finite</u>, meaning

$\mathcal{O}_{\mathcal{J}}$ is <u>finite-dim.</u> $\Longrightarrow$

$$\mathcal{O}_{\mathcal{J}} = \left\langle \Pi_{\xi} \mid \xi \in X, |X| < \infty \right\rangle, \quad (4.11)$$

$$\left.\begin{aligned} &\Pi_{\xi} \Pi_{\xi'} = \delta_{\xi\xi'} \Pi_{\xi'}, \quad \Pi_{\xi}^{*} = \Pi_{\xi}, \\ &\sum_{\xi \in X} \Pi_{\xi} = \mathbb{1}_{\mathcal{O}_{\mathcal{J}}}. \end{aligned}\right\} \quad (4.12)$$

Instruments will serve to describe

<u>measurements</u> and <u>observations</u>.

4.5. Digression on Tomita-Takesaki theory and Conditional Expectations

(see Bratteli & Robinson, vol. I!)

$\mathfrak{M}$: von Neumann algebra acting on $\mathcal{H}$

$\mathfrak{M}'$: commutant of $\mathfrak{M}$

$$\mathfrak{M}'' \equiv (\mathfrak{M}')' \underset{\uparrow}{=} \mathfrak{M}$$

for von Neumann alg.

$$Z_{\mathfrak{M}} := \text{center of } \mathfrak{M} = \mathfrak{M}' \cap \mathfrak{M}$$

Def.

(i) $\mathfrak{M}$ a factor iff $Z_{\mathfrak{M}} = \{\mathbb{C}\mathbb{1}\}$.

(ii) $\mathfrak{M}$ σ-finite iff every collection of mutually orthogonal projections in $\mathfrak{M}$ is finite or countably infinite.

(iii) $\psi \in \mathcal{H}$ separating iff $A\psi = 0$, $A \in \mathfrak{M}$, implies $A = 0$

$\psi \in \mathcal{H}$ cyclic iff $\{A\psi \mid A \in \mathfrak{M}\}$ is dense in $\mathcal{H}$.

Lemma.

$\left.\begin{array}{l}\psi \text{ cyclic for } \mathfrak{M} \\ \text{separating}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\psi \text{ separating for } \mathfrak{M}' \\ \text{cyclic}\end{array}\right.$ (4.13)

Proof: Obvious.

Generalization of separating: ω state on $\mathfrak{M}$

faithful iff $\omega(A) > 0$, for all $0 \neq A \in \mathfrak{M}_+$

Lemma.

$\mathfrak{M}$ is σ-finite $\Longleftrightarrow$ $\left\{\begin{array}{l}\exists \text{ a cyclic and separating} \\ \text{vector in a normal rep. } \pi \\ \text{of } \mathfrak{M}\end{array}\right.$

Def. A closed op. A on $\mathcal{H}$ affiliated w. $\mathfrak{M}$ ($A \eta \mathfrak{M}$) iff

$$\mathfrak{M}' D(A) \subseteq D(A) \quad \text{and}$$

$$AA' \supseteq A'A, \quad \forall A' \in \mathfrak{M}'$$

Here one may want to digress into Haag–Hugenholtz – Winnink!

Lemma. A closed, $A \eta \mathfrak{M}$.

$$A = U|A| \quad : \text{polar dec. of } A$$

Then U and spect. projections of $|A|$ belong to $\mathfrak{M}$.

The main characters of the drama.

Let $\Omega \in \mathcal{H}$ be cyclic and separating for $\mathfrak{M}$

$[\Leftrightarrow \Omega$ cyclic and separating for $\mathfrak{M}'$;

$\{A\Omega \mid A \in \mathfrak{M}\}$ dense in $\mathcal{H}$,

$A\Omega = 0,\ A \in \mathfrak{M} \Rightarrow A = 0.]$

Def.

$$\left.\begin{aligned} S_0 A \Omega &:= A^* \Omega, \quad A \in \mathfrak{M} \\ F_0 B \Omega &:= B^* \Omega, \quad B \in \mathfrak{M}' \end{aligned}\right\} \quad (4.14)$$

S_0, F_0 are densely defined anti-linear operators on $\mathcal{H}$.

Proposition.

- S_0 and F_0 are closable.
- $S_0^* = F_0^-$, $F_0^* = S_0^-$,

where the bar indicates closure.

- Given $\psi \in D(S_0^-)$, $\exists$ $Q \eta \mathfrak{M}$ such that

$$\psi = Q\Omega \quad , \quad S_0^- \psi = Q^* \Omega \ ;$$

and likewise for F_0^-.

Sketch of proof. $A \in \mathfrak{M}$, $B \in \mathfrak{M}'$

$$\langle B\Omega, S_0 A \Omega \rangle = \langle B\Omega, A^* \Omega \rangle$$

$$\underset{[A,B]=0}{=} \langle A\Omega, B^* \Omega \rangle$$

$$= \langle A\Omega, F_0 B \Omega \rangle$$

$\Rightarrow F_0 \subseteq S_0^* \Rightarrow S_0^*$ densely defined; S_0 closable.

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(B-R, vol. I, page 88)

Def. $S := \overline{S_0}$, $F = \overline{F_0}$.

Polar decomposition:

$$S = J\Delta^{1/2}, \qquad (4,15)$$

$\Delta = \Delta^* \geq 0$, J anti-unitary.

$\uparrow$ modular op. $\uparrow$ modular conjugation

assoc. w. $(\mathcal{M}, \Omega)$.

Proposition.

$$\Delta = S^*S = FS, \quad \Delta^{-1} = SF$$

$$S = J\Delta^{1/2}, \quad F = J\Delta^{-1/2}$$

$$J = J^*, \quad J^2 = \mathbb{1} \quad (\text{involution})$$

$$\Delta^{-1/2} = J\Delta^{1/2}J$$

Remarks. $S_0^2 A\Omega = S_0(S_0 A\Omega) = S_0(A^*\Omega) = A\Omega$

$$\Rightarrow S_0^2 = \mathbb{1} \Rightarrow S^2 = \mathbb{1}$$

$$\Rightarrow J\Delta^{1/2} = S = S^{-1} = \Delta^{-1/2}J^*$$

4.16

$$\Rightarrow \quad J^2 \Delta^{1/2} = \underbrace{J \Delta^{-1/2} J^*}_{\geq 0 \,!} \qquad *$$

Uniqueness of polar decomposition $\Rightarrow$

$$J^2 = \mathbb{1}$$

With $J^* J = \mathbb{1} = J^2 \Rightarrow J^* = J$

With $* \Rightarrow$ $$\boxed{\Delta^{\mp 1/2} = J \Delta^{\pm 1/2} J.} \qquad (4.16)$$

Next, for A, B, C in $\mathfrak{M}$,

$$(SAS) BC\Omega = SAC^* B^* \Omega$$

$$= BCA^* \Omega$$

and

$$B(SAS) C\Omega = BSAC^* \Omega$$

$$= BCA^* \Omega$$

Since $\overline{\{C\Omega \mid C \in \mathfrak{M}\}} = \mathcal{H}$,

$$[SAS, B] = 0, \quad \text{for arb. } B \in \mathfrak{M}$$

$$\Rightarrow \quad SAS \,\eta\, \mathfrak{M}', \quad \forall A \in \mathfrak{M}.$$

(4.17)

If Δ (and hence S, F, Δ^{-1}) were

bounded then

$$S\mathfrak{M}S \subseteq \mathfrak{M}', \quad F\mathfrak{M}'F \subseteq \mathfrak{M}$$

$$\Longrightarrow \Delta \mathfrak{M} \Delta^{-1} \subseteq \mathfrak{M}$$ (easy;

$$\Delta \mathfrak{M} \Delta^{-1} = \Delta^{1/2} J J \Delta^{1/2} \mathfrak{M} \Delta^{-1/2} J J \Delta^{-1/2}$$

$$= FS\mathfrak{M}SF \subseteq F\mathfrak{M}'F \subseteq \mathfrak{M}$$

Likewise for all powers of Δ

$$\Rightarrow \boxed{\Delta^{it} \mathfrak{M} \Delta^{-it} = \mathfrak{M}, \quad \forall t \in \mathbb{R}.} \qquad (4.17)$$

$$\Rightarrow J\mathfrak{M}J = J\underbrace{\Delta^{1/2} \mathfrak{M} \Delta^{-1/2}}_{=\mathfrak{M}} J$$

$$= S\mathfrak{M}S \subseteq \mathfrak{M}'$$

$$J\mathfrak{M}'J = J\underbrace{\Delta^{-1/2} \mathfrak{M}' \Delta^{1/2}}_{=\mathfrak{M}'} J = F\mathfrak{M}'F \subseteq \mathfrak{M}$$

$$\Rightarrow \boxed{J\mathfrak{M}J = \mathfrak{M}'} \qquad (4.18)$$

Now remove ass. # ! (See B-R, vol. I)

KMS condition.

ω : faithful, normal state on $\mathfrak{M}$.

$(\mathcal{H}_\omega, \pi_\omega, \Omega)$ obtained by GNS constr. from $(\mathfrak{M}, \omega)$.

Since ω faithful $\Rightarrow$ Ω (cyclic &) separating.

$$\sigma_t^\omega(A) := \pi_\omega^{-1}\left(\Delta^{it}\,\pi_\omega(A)\,\Delta^{-it}\right)$$

$\uparrow$

modular automorphism group assoc. w. $(\mathfrak{M}, \omega)$.

KMS condition w. $\beta = 1$:

$$\boxed{\begin{aligned}\omega\left(\sigma_i(A)\,B\right) &= \omega\left(\sigma_{i/2}(A)\,\sigma_{-i/2}(B)\right)\\ &= \omega(BA)\end{aligned}} \tag{4.19}$$

By calculation in GNS rep.:

$$\omega\left(\sigma_{i/2}(A)\,\sigma_{-i/2}(B)\right) = \langle \Delta^{1/2} A^*\Omega, \Delta^{1/2} B\,\Omega\rangle$$

$$= \langle J(J\Delta^{1/2}A^*)\Omega, J(J\Delta^{1/2}B)\Omega \rangle$$

$$= \langle JSA^*\Omega, JSB\Omega \rangle$$

$$= \langle JA\Omega, JB^*\Omega \rangle$$

$$= \langle B^*\Omega, A\Omega \rangle$$

$$= \langle \Omega, BA\Omega \rangle$$

$$= \omega(BA).$$

Conditional Expectations; (employs T-T th.!)

(see Takesaki, vol. I, p. 332; vol. II, p. 211)

Let ω be a faithful, normal state on a von Neumann algebra $\mathfrak{M}$. (A state is automatically a semi-finite weight! See p. 41, vol. II)

Let $\mathfrak{N} \subset \mathfrak{M}$ a von Neumann subalgebra of $\mathfrak{M}$.

Def.

A linear map $\mathcal{E} : \mathfrak{M} \xrightarrow[\text{onto}]{} \mathfrak{N}$

is called a conditional expectation of $\mathfrak{M}$ onto $\mathfrak{N}$ with respect to ω iff

(i) $\|\mathcal{E}(X)\| \leq \|X\|, \quad \forall X \in \mathfrak{M}$

(ii) $\mathcal{E}(X) = X, \quad \forall X \in \mathfrak{N}$

(iii) $\omega = \omega \circ \mathcal{E}$

Lemma. (See Takesaki, vol. II, p. 211; vol. III)

- $\mathcal{E}(X^*X) \geq 0, \quad \forall X \in \mathfrak{M}$
- $\mathcal{E}(AXB) = A\,\mathcal{E}(X)\,B, \quad \forall A, B \in \mathfrak{N}, \forall X \in \mathfrak{M}$
- $\mathcal{E}(X)^*\,\mathcal{E}(X) \leq \mathcal{E}(X^*X), \quad \forall X \in \mathfrak{M}$
- $\mathcal{E}$ is completely positive ($\rightarrow$ Stinespring)

Theorem. (See Takesaki, vol. II, p. 211)

$\mathfrak{M}, \mathfrak{N}, \omega$ as in above Def., Then

Existence of a conditional expectation
$\mathcal{E} : \mathcal{M} \to \mathcal{N}$ w. r. to ω
$\Longleftrightarrow \quad \sigma_t^{\omega}(\mathcal{N}) = \mathcal{N}, \quad \forall t \in \mathbb{R}$ (4.20)

modular automorphism group assoc. w. $(\mathcal{M}, \omega)$.

[$\mathcal{E}$ is normal and uniquely determined by ω.]

Digression on:

Classification of von Neumann algebras

Preliminaries. (See B-R, vol. I, Sect. 2.7)

1. Def. A von Neumann algebra $\mathfrak{M}$ is said to be semi-finite iff $\exists$ a faithful, normal semi-finite trace, τ, on $\mathfrak{M}$.

an example of a "weight".

2. Def. ω a weight on $\mathfrak{M}$; ($\nearrow$ Prop. 2.7.9, ff)

$$\mathfrak{M}_{\omega +} := \{ A \in \mathfrak{M}_+ \mid \omega(A) < \infty \}$$

$$\mathcal{L}_\omega := \{ A \in \mathfrak{M} \mid \omega(A^* A) < \infty \}$$

$\mathfrak{M}_\omega := \langle \mathfrak{M}_{\omega +} \rangle \equiv$ complex, linear span of $\mathfrak{M}_\omega$

Then $\mathfrak{M}_\omega = \mathcal{L}_\omega^* \mathcal{L}_\omega$

3. Def. A weight ω is called

- normal iff

$$\omega(A) = \sum_i \omega(A_i),$$

for every sequence $\{A_i\} \subset \mathfrak{M}_+$ with

$$\sum_i A_i =: A \in \mathfrak{M}_+ .$$

- ω is called faithful iff

$$A \in \mathfrak{M},\ \omega(A) = 0 \implies A = 0$$

- ω is called semi-finite iff

$\mathfrak{M}_\omega$ is σ-weakly dense in $\mathfrak{M}$.

Prop. Every von Neumann alg. admits a normal, faithful, semi-finite weight.

---

4. Connes' Radon-Nikodym Theorem

ω, ρ: normal, faithful, semi-finite weights (states) on $\mathfrak{M}$. Then there exists a cocycle $\Gamma_t \equiv (D\omega : D\rho)_t$ such that

$$(1)\quad \sigma_t^\omega(A) = \Gamma_t \sigma_t^\rho(A) \Gamma_t^*$$

$$(2)\quad \Gamma_{t+s} = \Gamma_t \sigma_t^\rho(\Gamma_s)$$

C3

(3) $\Gamma_t^* = (D\rho : D\omega)_t$,

$$(D\omega : D\rho)_t (D\varphi : D\varphi)_t = (D\omega : D\varphi)_t$$

(4) $\omega(A) = \rho(UAU^*)$, $\mathfrak{M} \ni U$ unitary

$$\Leftrightarrow \Gamma_t = (D\omega : D\rho)_t = U^* \sigma_t^\rho(U)$$

<u>Classification of factors;</u> (i.e., $\mathfrak{M}$ w. $Z(\mathfrak{M}) = \{\mathbb{C}\mathbb{1}\}$).

I. $\mathfrak{M}$ factor of type I $\Leftrightarrow$ $\mathfrak{M} = \mathcal{B}(\mathcal{H})$

dim $\mathcal{H}$ is then a complete invariant for $\mathfrak{M}$.

II $\mathfrak{M}$ factor of type II $\Leftrightarrow$ $\mathfrak{M}$ is semi finite,

but not of type I

Type II_1: Trace is <u>finite</u> $\Rightarrow$ i.e., a state

Type II_∞: " " <u>infinite</u>.

III $\mathfrak{M}$ is of type III if it is not semi-finite

Infinite von Neumann algebras are crossed products of semi-finite von Neumann algebras by a σ-weakly continuous one-parameter group, $\{\alpha_t\}_{t \in \mathbb{R}}$ of *automorphisms; if τ is the trace on the semi-finite v.N. algebra then

$$\tau \circ \alpha_t = e^{-t}\tau, \quad t \in \mathbb{R}.$$

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Def. If $\mathcal{M}$ is a factor define

$$S(\mathcal{M}) = \bigcap_{\omega} \sigma(\Delta_\omega),$$

ω: normal semifinite weight on $\mathcal{M}$.

$\sigma(A)$: spectrum of $A = A^*$.

Remark. Clearly $S(\mathcal{M}) \subseteq \mathbb{R}_+$.

<u>Theorem.</u> (Connes)

$$T_\omega := \{ t \in \mathbb{R} \mid \sigma_t^\omega \ \text{inner} \}$$

Then T_ω is an additive subgroup of $\mathbb{R}$ <u>indep.</u> of ω. If $\mathcal{M}$ is a factor define

$$\Gamma(\mathcal{M}) := \{ \lambda \in \mathbb{R} \mid t \in T \Rightarrow e^{i\lambda t} = 1 \}$$

Then

$$\Gamma(\mathcal{M}) = \log \left(S(\mathcal{M}) \setminus \{0\} \right)$$

$\Longrightarrow$ $S(\mathcal{M}) \setminus \{0\}$ is a <u>closed subgroup</u> of the multiplicative group of positive real numbers. $\Longrightarrow$

(i) $S(\mathcal{M}) = \{1\}$

(ii) $S(\mathcal{M}) = [0, \infty)$

(iii) $S(\mathcal{M}) = \{0\} \cup \{\lambda^n\}_{n \in \mathbb{Z}}$, $\lambda \in (0,1)$.

(iv) $S(\mathcal{M}) = \{0, 1\}$

$(i) \Longleftrightarrow \mathfrak{M}$ of type I or of type II

(namely $\mathfrak{M}$ semi finite)

$(ii)-(iv) \Longleftrightarrow \mathfrak{M}$ of type III

We say that

$\mathfrak{M}$ of type $\mathrm{III}_0 \Longleftrightarrow (iv)$

$\mathfrak{M}$ of type $\mathrm{III}_\lambda \Longleftrightarrow (iii)$

$\mathfrak{M}$ of type $\mathrm{III}_1 \Longleftrightarrow (ii)$

Type III_1 factor is crossed product of

type II_∞ factor by some $\{\alpha_t\}_{t\in\mathbb{R}}$.

A factor $\mathfrak{M}$ is called __hyperfinite__ iff

$\mathfrak{M}$ is generated by an ascending sequence

of finite-dimensional matrix algebras.

If $\mathfrak{M}$ is hyperfinite $\Longrightarrow \exists$ projection E

$E : \mathcal{B}(\mathcal{H}) \longrightarrow \mathfrak{M}$,

$E^2 = E, \quad E\,\mathcal{B}(\mathcal{H}) = \mathfrak{M}, \quad \|E\| = 1,$

which is a <u>conditional expectation</u>.

Converse is true, too; except possibly if $\mathfrak{M}$ is type III_1 factor.

Hyperfinite factors of type II_1, type II_∞, and type III_λ, $0 < \lambda \leq 1$ are <u>unique</u>.

Type III_0 classified by flow $\{Z_{\mathfrak{N}}, \alpha\}$, where

$$\mathfrak{M} = \mathfrak{N} \rtimes \alpha,$$

4.6. Centralizers of States on von Neumann Algebras and Their Centers — Events and Their Observations Using Instruments.

$\mathfrak{M}$; a von Neumann algebra

ω; a state on $\mathfrak{M}$

<u>Def.</u> The <u>centralizer</u> of ω in $\mathfrak{M}$ is the von Neumann subalgebra, $\mathcal{C}_\omega$, of $\mathfrak{M}$ defined by

$$\mathcal{C}_\omega := \{X \in \mathfrak{M} \mid ad_X\, \omega = 0\}, \qquad (4.21)$$

where $ad_X\, \omega$ is the bounded linear functional on $\mathfrak{M}$ defined by

$$ad_X\, \omega(Y) := \omega([X, Y]) \qquad (4.22)$$

<u>Lemma.</u> $\mathcal{C}_\omega$ is a von Neumann subalgebra of $\mathfrak{M}$.

Proof. (i) Clearly, if $X_1, X_2 \in \mathcal{C}_\omega$, $\lambda_1, \lambda_2 \in \mathbb{C}$ then $\lambda_1 X_1 + \lambda_2 X_2 \in \mathcal{C}_\omega$, by linearity of ω.

(ii) If $X \in \mathcal{C}_\omega$ then $X^* \in \mathcal{C}_\omega$:

$$\omega([X^*, y]) = \langle \Omega, (X^* y - y X^*) \Omega \rangle$$
$$= \overline{\langle (y^* X - X y^*) \Omega, \Omega \rangle}$$
$$= \langle \Omega, (y^* X - X y^*) \Omega \rangle$$
$$= -\omega([X, y^*])$$
$$= -\operatorname{ad}_X \omega(y^*)$$
$$= 0, \quad \forall y \in \mathfrak{M} \text{ if } X \in \mathcal{C}_\omega.$$

$$\Rightarrow \operatorname{ad}_{X^*} \omega = 0.$$

(iii) $X_1, X_2 \in \mathcal{C}_\omega \Rightarrow X_1 \cdot X_2 \in \mathcal{C}_\omega$:

$$\operatorname{ad}_{X_1 \cdot X_2} \omega(y) = \omega([X_1 X_2, y])$$
$$= \omega(X_1 (X_2 y) - (X_2 y) X_1)$$
$$+ \omega(X_2 (y X_1)) - \omega((y X_1) X_2)$$

$$= \mathrm{ad}_{X_1}\, \omega\,(X_2 y) + \mathrm{ad}_{X_2}\, \omega\,(y X_1)$$

$$= 0\,, \quad \forall y \in \mathcal{M}.$$

<u>Observation</u>. If $X \in \mathcal{C}_\omega$, $X = X^*$, then

$$\mathrm{Ad}_{e^{itX}}\, \omega = \omega\,, \quad \forall t$$

Here

$$\mathrm{Ad}_{e^{itX}}\, \omega\,(y) := \omega\left(e^{itX}\, y\, e^{-itX}\right);$$

and conversely.

<u>Theorem</u>. Let ω be <u>faithful</u> state on $\mathcal{M}$.

Let σ_t^ω be the modular automorphism group on $\mathcal{M}$ assoc. with $(\mathcal{M}, \omega)$.

Then

$$\mathcal{C}_\omega = \left\{ X \in \mathcal{M} \,\middle|\, \sigma_t^\omega(X) = X, \ \forall t \in \mathbb{R} \right\}$$

Proof. Exercise, using KMS condition (See, e.g., F-Schubnel, "Quantum Prob. Theory...")

Corollary. $\exists$ conditional expectation

$$\overline{E}_\omega : \mathfrak{M} \to \mathcal{C}_\omega ,$$

with $\omega = \omega \circ \overline{E}_\omega$.

Important observation:

$$\boxed{\omega \text{ is a normalized trace on } \mathcal{C}_\omega} \qquad (4.23)$$

$$\Rightarrow \mathcal{C}_\omega = \begin{cases} \text{abelian von Neumann alg.} \\ M_n(\mathbb{C}), \quad n \in \mathbb{N} \\ \text{type } \mathrm{II}_1 \text{ von Neumann alg.} \end{cases} \qquad (4.24)$$

or direct sum of such algebras

$\mathcal{C}_\omega = \bigoplus_i M_{n_i}(\mathbb{C}) \Leftrightarrow \mathfrak{M}$ type I v.N. alg.

$\mathfrak{M}$ type III $\Rightarrow \mathcal{C}_\omega$ type II_1.

If $\mathcal{C}_\omega$ is <u>not</u> a factor (e.g.,

$\mathcal{C}_\omega = \bigoplus_i M_{n_i}(\mathbb{C})$, non-trivial direct

sum of matrix algebras) then $\mathcal{C}_\omega$

contains a <u>non-trivial center</u>

$$\left.\begin{array}{l} Z_\omega := \text{center of } \mathcal{C}_\omega \\ E_\omega : \text{conditional expectation } \mathcal{M} \to Z_\omega \end{array}\right\}$$

Return to analysis of <u>isolated physical systems</u>: S a phys. system described by

- $\{\mathcal{E}_I\}_{I \subset \mathbb{R}}$: net of algebras indexed by intervals $I \subset \mathbb{R}$ of proper time
- $\mathcal{S}$: stratum of states on $\{\mathcal{E}_I\}_{I \subset \mathbb{R}}$ of physical interest
- $\mathcal{O}_S := \{\mathcal{J}_j \mid j \in \text{Inst.}\}$: family of instruments.

<u>Def.</u> <u>Events</u> happening at time t, given state, ρ_t, of Σ immediately before time t: (ρ_t: a normal state on $\mathcal{E}^{-}_{\geq t}$)

$\mathcal{C}_{\rho_t}$: centralizer of ρ_t on $\mathcal{E}^{-}_{\geq t}$.

$\mathcal{Z}_{\rho_t}$: center of $\mathcal{C}_{\rho_t}$.

An <u>event</u> happening at time t is an orthogonal projection, $\Pi = \Pi^* = \Pi^2$, belonging to $\mathcal{Z}_{\rho_t}$.

Assume that

$$\mathcal{Z}_{\rho_t} = \left\langle \Pi_\xi(t) \,\middle|\, \xi \in \mathcal{X}_t \right\rangle, \tag{4.25}$$

$\mathcal{X}_t$ countable ($|\mathcal{X}_t| < \infty$).

Observations (BFS – "Garden ...")

Consider von Neumann alg., $\mathfrak{M}$, and a normal state φ on $\mathfrak{M}$. Let $\mathcal{C}_\varphi \subseteq \mathfrak{M}$ be the centralizer of $\mathfrak{M}$. Let $X \in \mathcal{C}_\varphi$, with

$$X = \sum_{\xi \in \mathcal{X}} \xi \, \Pi_\xi \,, \tag{4.26}$$

where $\mathcal{X} \subset \mathbb{R}$, $|\mathcal{X}| =: N < \infty$, ($\mathcal{X}$ inherits order of $\mathbb{R}$).

Since $\mathcal{C}_\varphi$ is a von Neumann subalgebra of $\mathfrak{M}$, it follows that $\Pi_\xi \in \mathcal{C}_\varphi$, $\forall \xi \in \mathcal{X}$.

Thus, for an arbitrary $A \in \mathfrak{M}$,

$$\varphi(A) = \sum_{(\xi, \xi') \in \mathcal{X} \times \mathcal{X}} \varphi(\Pi_\xi A \Pi_{\xi'})$$

$$= \sum_{\xi, \xi'} \varphi(A \Pi_\xi \delta_{\xi\xi'})$$

$$= \sum_{\xi \in \mathcal{X}} \varphi(\Pi_\xi A \Pi_\xi) \tag{4.27}$$

Conversely, if, for X as in (4.26), (4.27) holds for all $A \in \mathfrak{M}$ then $X \in \mathcal{C}_\varphi$.

<u>Note.</u> For $X \in Z_\varphi$, (4.27) holds; (this is a special case!).

<u>Lemma.</u> (BFS)

Let X be as in (4.26), and let $\overline{E}_\varphi$ be the conditional expectation: $\mathfrak{M} \to \mathcal{C}_\varphi$.

Suppose that

$$\| \overline{E}_\varphi (\Pi_\xi) - \Pi_\xi \| < \delta ,$$

$\forall \xi \in \mathfrak{X}$. Then

$$\varphi(A) = \sum_{\xi \in \mathfrak{X}} \varphi(\Pi_\xi A \Pi_\xi) + \mathcal{O}(\delta \|A\|) \tag{4.28}$$

Proof.

$$\varphi(A) = \sum_{(\xi,\xi')\in\mathcal{X}\times\mathcal{X}} \Big\{ \Big[\varphi(\Pi_\xi A \Pi_{\xi'}) - \varphi(\overline{\epsilon}_\varphi(\Pi_\xi) A \Pi_{\xi'})\Big]$$

$$+ \Big[\varphi(A\Pi_{\xi'}, \overline{\epsilon}(\Pi_\xi)) - \varphi(A\Pi_{\xi'}, \Pi_\xi)\Big]\Big\}$$

$$+ \sum_{\xi\in\mathcal{X}} \Big\{ \Big[\varphi(A\Pi_\xi^2) - \varphi(A\Pi_\xi \overline{\epsilon}_\varphi(\Pi_\xi))\Big]$$

$$+ \Big[\varphi(\overline{\epsilon}_\varphi(\Pi_\xi) A \Pi_\xi) - \varphi(\Pi_\xi A \Pi_\xi)\Big]\Big\}$$

$$+ \sum_{\xi\in\mathcal{X}} \varphi(\Pi_\xi A \Pi_\xi) \tag{4.29}$$

The proof is completed by using that $\|\overline{\epsilon}_\varphi(\Pi_\xi) - \Pi_\xi\| < \delta$, $\forall \xi \in \mathcal{X}$ and that $\|\Pi_\xi\| = 1$. ∎

Corollary. In above lemma, we can replace $\mathcal{E}_\varphi$ by $\mathcal{Z}_\varphi$ and $\overline{\epsilon}_\varphi$ by ϵ_φ.

<u>Axiom.</u> Assume that, at time $t-$, S is in state ρ_t; and assume that Z_{ρ_t} is as in (4.25), page 4.26. Then

$$\rho_t(A) = \sum_{\xi \in \mathcal{X}_t} \rho_t\left(\Pi_\xi(t)\, A\, \Pi_\xi(t)\right), \qquad (4.30)$$

for all $A \in \bar{\mathcal{E}}_{\geq t}$. <u>The axiom says that one of the possible events</u> $\Pi_\xi(t)$, $\xi \in \mathcal{X}_t$, <u>actually happens</u>; say $\Pi_{\xi_*}(t)$. <u>Then one ought to use the state</u>

$$\rho_t\left(\Pi_{\xi_*}(t)\right)^{-1} \rho_t\left(\Pi_{\xi_*}(t)\,(\cdot)\,\Pi_{\xi_*}(t)\right) \qquad (4.31)$$

<u>to predict the future of S after time t.</u>

In (4.31),

$$\rho_t\left(\Pi_{\xi_*}(t)\right) \qquad (4.32)$$

is the <u>Born probability</u> of the event $\Pi_{\xi_*}(t)$ in the state ρ_t.

The state in (4.31) is henceforth denoted by

$$\boxed{\rho_{t,\xi_*}\,,\quad \xi_* \in \mathcal{X}_t} \tag{4.33}$$

The central feature of Quantum Mechanics

If $t' > t$ and $Z_{\rho_{t'}}$ non-trivial then

$$Z_{\rho_{t'}} \underset{\text{i.g.}}{\neq} \bigvee_{\xi \in \mathcal{X}_t} Z_{\rho_{t',\xi}} \tag{4.34}$$

Remark. Type I situation:

Suppose that $\rho_{t'} = \Phi(\rho_t)$, $(t' > t)$, where Φ is CP. Consider spect. dec. of ρ_t:

$$\rho_t = \sum_{i=1}^{\infty} p_i(t)\, P_i(t)$$

$$P_i(t) P_j(t) = \delta_{ij} P_i(t) = \delta_{ij} P_i(t)^*.$$

Then

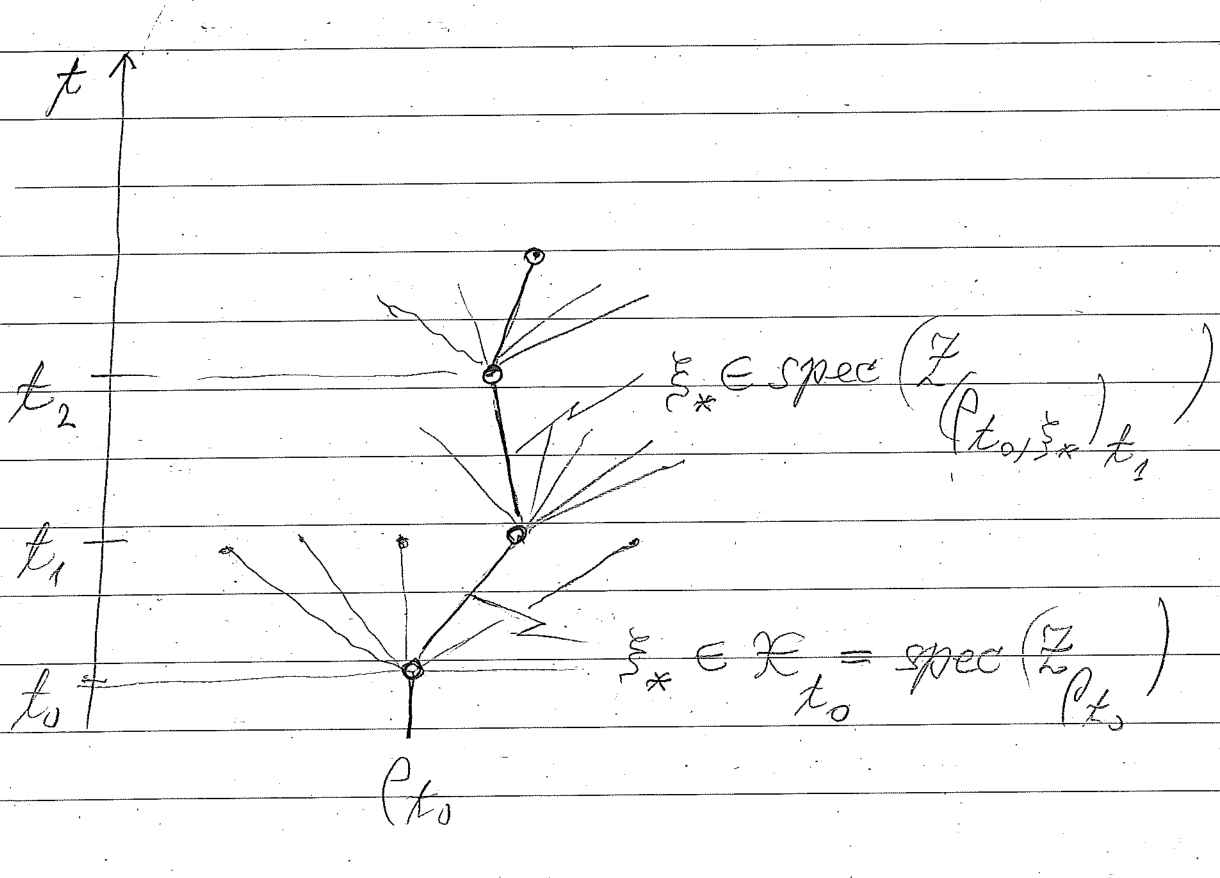
$$\Phi(P_i(t)), \quad i = 1, 2, 3, \dots,$$

are, in general, neither projections, nor orthogonal; and in general

$$[\Phi(P_i(t)), \Phi(P_j(t))] \neq 0,$$

$i \neq j$. $\Longrightarrow$ (4.34) in type I situation

Consequence: The ETH picture of time evolution of states: A NC stochastic branching process.



What is the mathematics appropriate to describe this?

Special case: S non-autonomous;

Z_{ρ_t} non-trivial only for discrete times.

$\mathcal{E}_{\geq t}^{-}$: type I, $\forall t$

Then (lemma):

$$Z_{\rho_t} = \langle \text{functions of } \rho_t \rangle$$

generated by spectral projections of density matrix ρ_t.

Mixed states generated by entanglement with "lost probes" (probes = degrees of freedom that, after some time, are no longer observable).

$\rightarrow$ ETH processes generated by Lindblad ops.; (see also Chap. 5)

Demystifying Quantum Mechanics - the ETH Approach

"I leave to several futures (not to all) my garden of forking paths"
J. L. Borges

Leysin, June 27, 2019[1]

[1] Jürg Fröhlich, ETH Zurich

Contents and credits

Contents:

Credits:

I am indebted to my last PhD student *Baptiste Schubnel* for enjoyable collaboration, to *Ph. Blanchard* and some others for cooperation, and to many colleagues, including some of the *"Bohmians"*, several colleagues at ETH Zurich, and *D. Buchholz* for useful discussions on QM.

1. What this lecture is about

This is about Foundations of Quantum Mechanics: I will explain the

"E T H - Approach to Quantum Mechanics"

where "E" stands for Events, "T" for Trees, and "H" for Histories. This approach enables us to introduce a precise notion of "events" into Quantum Mechanics (as advocated by the late R. Haag), explain what it means to observe an event by recording the value of an appropriate physical quantity, and to exhibit the stochastic dynamics of states of isolated open systems featuring events.

I will then focus on explaining how quantum theory might be reconciled with relativity theory, and what it may tell us about the fabric of space-time and its causal structure.

The "ETH - Approach to QM" results in a "Quantum Theory without observers". It does not provide any "extensions of Quantum Mechanics", all of which have remained unacceptably vague.

My goal – in my work on QM and for this lecture – is to help to remove the enormous confusion befuddling many people who claim to work on various aspects of QM and its applications.

Specific topics to be addressed; basic convictions

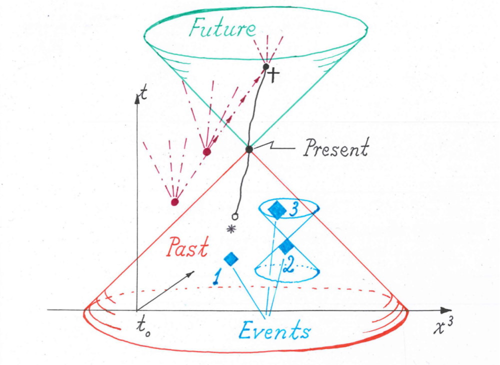
- Why are physical theories never fully predictive?
  Why is quantum theory *intrinsically probabilistic*?
  What are *"events"* in quantum theory & how do we record them?
  How do *states* of physical systems evolve according to *QM*?
- What is the role of *(Space-)Time* in quantum theory, and why and how does quantum theory distinguish between *past* and *future*?
- What is the fundamental significance of *"locality"* and *Einstein causality* in quantum theory?
  Could it be that a consistent *"Quantum Theory of Events"* must necessarily be *relativistic*; could it be that it explains why space-time is *even-dimensional* and why it is *curved*?

Etc. ...

1. Against "interpretations" of physical theories – in favor of solid "foundations"; (the example of electro-magnetism before STR).
2. . . . – in favor of *clear concepts and fundamental principles!*
3. Against denigration of precise mathematical tools.

2. What prevents theories from being (fully) predictive?

Space-time with an *event horizon*. (Observer sits at "Present"; is unaware of dangers lurking from outside his past light-cone; he might get killed at †. Events 1 & 2 are space-like separated; event 3 is in the future of 2)



t_0: time right after inflation $\rightarrow$ event horizon $\Rightarrow$ initial conditions not fully accessible!

Past = History of Events (Facts) / Future = Ensemble of Potentialities

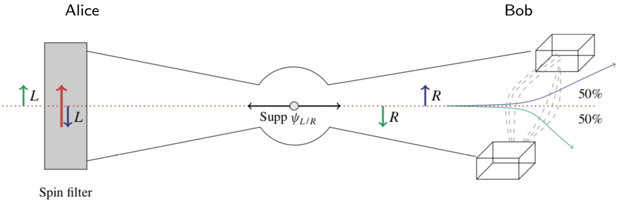
This fundamental dichotomy should be retained in Quantum Mechanics!

Quantum theory cannot be fully predictive, because ...

A Gedanken-Experiment ($\nearrow$ Faupin-F-Schubnel):

Two particles (electrons), P and P', prepared in a spin-singlet initial state, $\psi_{L/R}$,[2] with orbital wave functions chosen such that P propagates into the cone opening to the right, while P' propagates into the cone opening to the left and ending in the spin filter, (except for very tiny tails leaking beyond those cones).

Time evolution of P essentially *independent* of the one of Q, which includes P' and spin filter – consequence of cluster props. of propagator!



Q:={spin filter $\vee$ particle P'}   cone opening to R:= ess. supp of orbital wave function of P

[2] nowadays called a *Bell pair of Qbits*

... Quantum theory is fundamentally probabilistic –

in spite of the *deterministic nature of the Schrödinger Eq.!* –
Temporary assumptions (leading to a contradiction):

I. P and P': Spin-$\frac{1}{2}$ particles prepared in a *spin-singlet initial state*; spin filter prepared in a poorly known initial state not (necessarily) entangled with initial state of P' and P.

II. *Dynamics of state of total system fully determined by Schrödinger equation.* In particular, initial state of spin filter determines whether P' will pass through it or not, (given that the initial state of $P' \vee P$ is a spin-singlet state, with P' and P moving into opposite cones).

III. *Correlations between outcomes of spin measurements of P' and of P are as predicted by standard quantum mechanics*, (relying on the "Copenhagen interpretation") – "non-locality" of QM.

Fact: Assuming short-range interactions, Schrödinger evolution of state of system factorizes into free evolution of P tensored with complicated evolution of $Q := \{P' \vee \text{ spin filter}\}$, up to tiny errors. This follows from our choice of initial conditions & cluster properties of time evolution! Hence *spin of P is ess. conserved before measurement!* $\Rightarrow$

Time evolution of "observables" and of states in QM

If assumptions I. and II. held then:
Expectation value of spin of $P \approx 0, \forall$ times! $\Rightarrow$ State of spin of P' after interaction of P' with spin filter does **not** bias state of spin of P when measured, (e.g., in a Stern-Gerlach exp.)!

This consequence of the first two assumptions ***contradicts*** *the third (last) assumption stated above!*

Thus, if the usual *correlations* between two "independent" measurements (here of z-comp. of spins of P' and of P), predicted on the basis of the projection postulate of "Copenhagen", are observed[3] then it follows that the Schrödinger equation *cannot* describe the evolution of states of systems, and hence that qm dynamics is *fundamentally stochastic*.

It turns out that one may safely assume the validity of the *Heisenberg-picture evolution* of "observables" for *isolated systems* (define!), which is perfectly *deterministic*. But, in Quantum Mechanics, the *evolution of states is stochastic*. $\Rightarrow$ Equivalence of the *Heisenberg picture* and the *Schrödinger picture* is an erroneous claim!

[3] as suggested by the experiments of *Aspect, Gisin*, and others

3. "Non-locality" of QM versus "Einstein causality"

It is possible that the measurements of (components of) the spin of P and of the spin of P' are made in *space-like separated regions* of space-time, so that the localization regions of the corresp. projection operators $\Pi^{P'}_{\sigma',\vec{e}_z}$ and $\Pi^{P}_{\sigma,\vec{n}}$, with $\sigma = \pm, \sigma' = \pm$, are space-like separated. The order in which these two measurements occur then depends on the rest frame of the observer who records the data of both measurements. This implies that the operators $\Pi^{P'}_{\sigma',\vec{e}_z} \cdot \Pi^{P}_{\sigma,\vec{n}}$ and $\Pi^{P}_{\sigma,\vec{n}} \cdot \Pi^{P'}_{\sigma',\vec{e}_z}$ must have the *same effect* when applied on the state of the system. ... The most general way in which this can be guaranteed is to require that

$$\boxed{\Pi^{P'}_{\sigma',\vec{e}_z} \cdot \Pi^{P}_{\sigma,\vec{n}} = \Pi^{P}_{\sigma,\vec{n}} \cdot \Pi^{P'}_{\sigma',\vec{e}_z}} \tag{1}$$

This is *locality* (in the sense of RQFT) or *"Einstein causality"* !

—

[*Remark:* It might suffice to require a weaker form of locality by only requiring Eq. (1) to hold on all those states that actually admit measurements of the spins of P' and P in the prescribed local regions of space-time; (*"weak locality"* – compare to Jost's proof of the CPT theorem!).]

4. The "*ETH* Approach" to Quantum Theory

Next, we address the question of what is meant by *"events"* featured by *isolated systems*, and of how they can be recorded (in *direct/projective measurements/observations*). I sketch what I call the "*ETH* Approach" to QM. I first consider *non-relativistic QM*:

Let S be an isolated physical system. Pure states of S are given by unit rays in a separable Hilbert space $\mathcal{H}_S$; general states by density operators, ω, acting on $\mathcal{H}_S$, with $\omega(A) := Tr(\omega \cdot A)$, for any bd. operator A on $\mathcal{H}_S$.

Time is a fundamental quantity in n.r. physics. The time axis is given by $\mathbb{R}$. Let's suppose the *present time* is t_0, and let I be an arbitrary interval of *future times*, i.e., $I \subset [t_0, \infty)$. (Use Heisenberg picture!)

Definition: Let S be an isolated physical system. *"Potential future events"* in S – "*potentialities*" – are described by certain *orthogonal projections* acting on $\mathcal{H}_S$ associated with time intervals. The *algebra generated by all "potential future events" assoc. with a future interval, I, of times is denoted by $\mathcal{E}_I$, and we define

$$\mathcal{E}_{\geq t} := \overline{\bigvee_{I \subset [t,\infty)} \mathcal{E}_I}, \quad \text{and} \quad \mathcal{E} := \overline{\bigvee_{t \in \mathbb{R}} \mathcal{E}_{\geq t}}^{\|\cdot\|}, \tag{2}$$

"Principle of Diminishing Potentialities"

where the algebras $\mathcal{E}_{\geq t}, t \in \mathbb{R}$, are assumed to be *weakly* closed![4]

By definition,

$$\mathcal{E}_I \supseteq \mathcal{E}_{I'} \quad \text{if} \quad I \supseteq I' , \qquad \mathcal{E}_{\geq t} \supseteq \mathcal{E}_{\geq t'} \quad \text{if} \quad t' > t .$$

An *isolated open system* S can now be *defined* in terms of a filtration $\{\mathcal{E}_{\geq t} | t \in \mathbb{R}\}$ of algebras of potential future events (potentialities). The *"Principle of Diminishing Potentialities" (PDP)* is the statement that

$$\boxed{\mathcal{E}_{\geq t} \underset{\neq}{\supset} \mathcal{E}_{\geq t'} , \quad \text{whenever} \quad t' > t \geq t_0} \tag{3}$$

This principle can be proven to hold in simple models (with a Hamiltonian incorporating a "time operator"). Given a state ω of S, we set

$$\omega_t := \omega|_{\mathcal{E}_{\geq t}} , \quad \text{i.e.,} \quad \omega_t(A) = \omega(A) , \forall A \in \mathcal{E}_{\geq t} . \tag{4}$$

[4] Passing to von Neumann algebras is convenient, because the spectral projections of any element of the algebra will then also belong to the algebra!

Events

Note that ω might be a *pure* state on $\mathcal{E}$. But, since $\mathcal{E}_{\geq t} \subsetneq \mathcal{E}$, $\forall t < \infty$, ω_t will generally be a *mixed* state on $\mathcal{E}_{\geq t}$; (entanglement!). This observation opens the door towards a clear notion of what might be meant by *"events"* and to a theory of direct/projective measurements and observations (of "events").

To render the *Definition* (of pot. future events) more precise, we say that a *"potential future event"* is given by a family, $\{\pi_\xi | \xi \in \mathcal{X}\}$, of disjoint orthogonal projections contained in an algebra $\mathcal{E}_{\geq t}$, for some $t \geq t_0$, ($t_0 =$ time of "present"), with $\sum_{\xi \in \mathcal{X}} \pi_\xi = \mathbf{1}$.

In accordance with the "Copenhagen interpretation" of QM, it appears natural to say that a potential future event $\{\pi_\xi | \xi \in \mathcal{X}\} \subset \mathcal{E}_{\geq t}$ *actually happens* in the interval $[t, \infty)$ of times iff

$$\boxed{\omega_t(A) = \sum_{\xi \in \mathcal{X}} \omega(\pi_\xi \, A \, \pi_\xi), \quad \forall A \in \mathcal{E}_{\geq t} \,,} \tag{5}$$

i.e., no off-diagonal elements appear on the R.S. of (5)!

The centralizer of a state and its center

Next, we render the meaning of Eq. (5) more precise.

Let $\mathcal{M}$ be a von Neumann algebra, and let ω be a state on $\mathcal{M}$. Given an operator $X \in \mathcal{M}$, we set

$$ad_X(\omega)(A) := \omega([A, X]), \ \forall A \in \mathcal{M}.$$

We define the *centralizer* of a state ω on $\mathcal{M}$ by

$$\mathcal{C}_\omega(\mathcal{M}) := \{X \in \mathcal{M} | ad_X(\omega) = 0\}$$

Note that ω is a normalized trace on $\mathcal{C}_\omega(\mathcal{M})$ ... ! The *center*, $\mathcal{Z}_\omega(\mathcal{M})$, of $\mathcal{C}_\omega(\mathcal{M})$ is defined by

$$\mathcal{Z}_\omega(\mathcal{M}) := \{X \in \mathcal{C}_\omega(\mathcal{M}) | [X, A] = 0, \forall A \in \mathcal{C}_\omega(\mathcal{M})\}. \tag{6}$$

We are now prepared to introduce the notion of *(actual)* *"events"*.

Events happening at time $\geq t$

Let S be an isolated physical system. We set $\mathcal{M} := \mathcal{E}_{\geq t}$, $\omega := \omega_t$.

Definition: Given that ω_t is the state of S on the algebra $\mathcal{E}_{\geq t}$, an *"event"* is happening at time t iff $\mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t})$ contains at least two non-zero orothogonal projections, $\pi^{(1)}, \pi^{(2)}$, which are disjoint, i.e., $\pi^{(1)} \cdot \pi^{(2)} = 0$, and have non-vanishing *"Born probabilities"*, i.e.,

$$0 < \omega_t(\pi^{(i)}) < 1\,, \qquad \text{for } \; i = 1, 2\,. \qquad \square$$

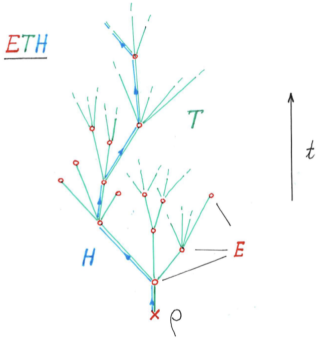
Let us suppose for simplicity that $\mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t})$ is generated by a family of orthogonal projections $\{\pi_\xi | \xi \in \mathcal{X}_{\omega_t}\}$, where $\mathcal{X}_{\omega_t} = \text{spec}[\mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t})]$ is a countable set.

"Axiom": If an *event* happens at time t, (i.e., $\text{card}(\mathcal{X}_{\omega_t}) \geq 2$, and $\omega_t(\pi_\xi) \neq 0$, for *at least two* different $\xi \in \mathcal{X}_{\omega_t}$), then the state ω_t must be replaced by one of the states $\omega_{t,\xi} := \omega_t(\pi_\xi)^{-1} \cdot \omega_t(\pi_\xi(\cdot)\pi_\xi)$, for some $\xi \in \mathcal{X}_{\omega_t}$ with $\omega_t(\pi_\xi) \neq 0$. The probability, $prob_t(\xi)$, for the state $\omega_{t,\xi}$ to be selected as the state of S right *after* time t is given by

$$prob_t(\xi) = \omega_t(\pi_\xi) \quad - \quad Born's Rule \qquad \square \qquad (7)$$

A metaphoric picture of the time evolution of states in QM – according to "*ETH*"

Apparently, the time-evolution of *states* of a phys. system *S* is described by a *stochastic branching process*, with branching rules as determined by the above *"Axiom"*. (This is mathematically precise if time is discrete.)
Illustration:



E: "Events", *T*: "Trees" of possible states, *H*: "Histories" of states.

This is different from and supercedes the "decoherence mumbo-jumbo"!

5. Events and their detection

We have characterised an *isolated open system* S in terms of a filtration of algebras

$$\{\mathcal{E}_{\geq t}\}_{t \in \mathbb{R}} ,$$

with

$$\boxed{\mathcal{E}_{\geq t} \underset{\neq}{\supset} \mathcal{E}_{\geq t'} , \qquad \text{whenever} \quad t' > t} \tag{8}$$

The flow of time in S, (i.e., the time evolution of S in the Heisenberg picture) is encoded in the *proper* embeddings (8), which, in an *autonomous* system S, are completely determined by its *Hamiltonian*.

However, the characterisation of S given in (8) is incomplete! To retrieve physical information from (8) and from our definition of *events*, we must specify operators that represent *"physical quantities"* characteristic of S and – when observed/measured – may *signal the occurrence of events*. Let

$$\mathcal{O}_S := \{\hat{X}_\iota | \iota \in \mathcal{I}_S\} \tag{9}$$

be a list/set of abstract linear operators representing physical quantities characteristic of S; (usually, $\mathcal{O}_S$ is not a linear space, let alone an alg.).

Measurements of physical quantities

For any operator $\hat{Y} \in \mathcal{O}_S$ and any time t, we specify a concrete self-adjoint operator $Y(t) \in \mathcal{E}_{\geq t}$ representing $\hat{Y}$ at time t; (i.e., $\exists$ a repr. of $\mathcal{O}_S$ by operators on $\mathcal{H}_S$, $\forall t \in \mathbb{R}$). For an *autonomous* system S, the operators $Y(t)$ and $Y(t')$ are conjugated to one another by the *propagator* of S .

Suppose that, at some time t, an *event* happens; i.e., $\exists$ a partition of unity, $\{\pi_\xi | \xi \in \mathcal{X}_{\omega_t}\} \subseteq \mathcal{Z}_{\omega_t} \subset \mathcal{E}_{\geq t}$, by disjoint (commuting) orthogonal projections, as above, containing ≥ 2 elements with strictly positive Born probabilities representing *possible events* (one of which *actually happens*). Let $\hat{Y} \in \mathcal{O}_S$, and let $Y(t) = \sum_{\eta \in spec(\hat{X})} \eta \, \Pi_\eta(t)$ (spectral dec. of Y(t)) be the operator epresenting $\hat{Y}$ at time t. If the "distance"[5]

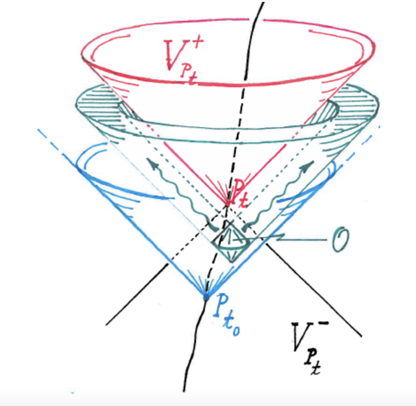
$$\mathrm{dist}\big(\Pi_\eta(t), \langle \pi_\xi | \xi \in \mathcal{X}_{\rho_t} \rangle\big) \text{ is ``very small''} \;, \forall \eta \in spec(\hat{Y}) \,, \tag{10}$$

then we say that the *physical quantity* $\hat{Y} \in \mathcal{O}_S$ is *observed/measured* after time t, because the state of S just after time t is then an approximate eigenstate of $Y(t)$. The measurement of $\hat{Y}$ is a *signal of an event happening* at time t. ...

[5] defined in terms of conditional expectations

6. Local Relativistic Quantum Theory

I temporarily assume that space-time is Minkowski space, $\mathbb{M}^4$, and, immodestly, that my own proper time is the time of the Universe.



Worldline of JF ↑

A Theorem of Buchholz

Theorem

In an RQFT in dim. $2n, n \geq 2$ with massless particles, such as photons and/or gravitons, the algebra, $\mathcal{E}_{\geq P_t}$, of all events potentially happening in the future of the space-time point P_t is "∞-dimensional" and does not admit any pure states; and the relative commutant $\mathcal{E}'_{\geq P_t} \cap \mathcal{E}_{\geq P_{t_0}}$ is "∞-dimensional", too, for arbitrary times $t_0 < t$. □

This result is a consequence of *"Huygens' Principle"* (in the jargon of Buchholz): Photons from the region $\mathcal{O}$ will asymptotically escape along lightcones in the future, $V^+_{P_{t_0}}$, of P_{t_0} but *below* $V^+_{P_t}$. We cannot catch up with them, anymore, if we have missed them just after they have been emitted. Thus, the *"Principle of Diminishing Potentialities" (PDP)* holds in the form proposed in Eq. (3) of the last Section:

$$\mathcal{E}_{\geq P_{t_0}} \underset{\neq}{\supset} \mathcal{E}_{\geq P_t}, \qquad \text{for } \; t > t_0 \,, \tag{11}$$

and we could now follow the arguments outlined in Sect. 5. However, I don't like to be in the center of the Universe; so, let's take JF out of the picture! Before knowing better I propose a formulation of *relativistic local Quantum Theory* with roughly the following features:

A tentative formulation of relativistic local quantum theory

Let $\mathcal{M}$ be some (Hausdorff) topological space. We consider a *fibre bundle*, ${}^{qm}\mathcal{F}$, with base space given by $\mathcal{M}$ and fibre above a point $P \in \mathcal{M}$ given by an "∞-dimensional" von Neumann algebra $\mathcal{E}_{\geq P}$. All the algebras $\{\mathcal{E}_{\geq P}\}_{P \in \mathcal{M}}$ are assumed to be *isomorphic* to a "universal" algebra $\mathcal{N}$,[6] (and one has to impose certain "compatibility conditions").

Definition:
We say that a point $P_0 \in \mathcal{M}$ is in the *past* of a point $P \in \mathcal{M}$, written as $P_0 \prec P$, iff $\mathcal{E}_{\geq P_0} \supseteq \mathcal{E}_{\geq P}$, and

$$\left(\mathcal{E}_{\geq P}\right)' \cap \mathcal{E}_{\geq P_0}$$

is an inifinite-dimensional (non-commutative) algebra. □

The relation $\prec$ introduces a *partial order* on $\mathcal{M}$. If $P_0 \not\prec P$ and $P \not\prec P_0$ then we say that P_0 and P are *space-like separated*, written as $P_0 \times P$. The relations "$\prec$" and "$\times$" determine a *"causal structure"* on $\mathcal{M}$.

[6]This framework could be generalized by first considering C^*-algebras, rather than von Neumann algebras, and introducing sheaves of algebras

What are *"events"*?

Let ω_Σ be a state defined on all the algebras $\mathcal{E}_{\geq P'}, \forall P' \in \Sigma$, where $\Sigma \in \mathcal{M}$ is a (region in a) space-like hypersurface containing a point $P \in \mathcal{M}$.

Definition: We say that an *"event"* happens in P iff the center $\mathcal{Z}_{\omega_\Sigma}(\mathcal{E}_{\geq P}) \equiv \mathcal{Z}^P_{\omega_\Sigma}$ of the centralizer, $\mathcal{C}_{\omega_\Sigma}(\mathcal{E}_{\geq P})$, is *non-trivial* and contains at least *two* projections, Π^P_1 and Π^P_2 with strictly positive *"Born probs."*

$$0 < \omega_\Sigma(\Pi^P_i) < 1, \qquad \text{for } 1 = 1, 2\,. \qquad \square$$

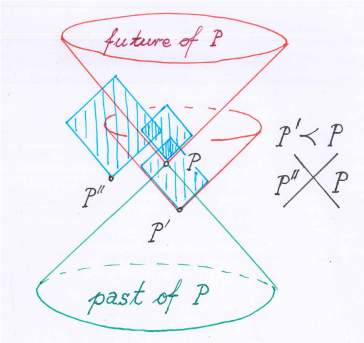
Let $\mathcal{X}^P_{\omega_\Sigma}$ denote the *spectrum* of $\mathcal{Z}^P_{\omega_\Sigma}$.

"Axiom" *(compatibility – locality)*: If two points, P and P'', of $\mathcal{M}$ are *space-like separated*, and "events", Π^P_ξ and $\Pi^{P''}_\eta$, actually happen in P and P'' then

$$[\Pi^P_\xi, \Pi^{P''}_\eta] = 0, \quad \forall\, \xi \in \mathcal{X}^P_{\omega_\Sigma} \text{ and all } \eta \in \mathcal{X}^{P''}_{\omega_\Sigma}\,. \qquad \square \qquad (12)$$

Causal structure on $\mathcal{M}$

$\rightarrow$ introduces *"causal (geometrical) structure"* on $\mathcal{M}$!

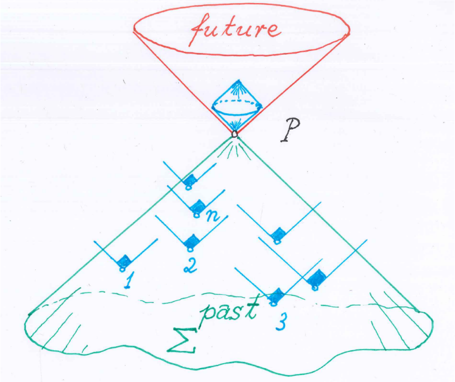


Graphical illustration of the axiom

However, projections describing *events* happening in P' and P do *not* commute in general, since P' is in the *past* of P.

Next, we describe *histories of events*. We choose a (region in a) space-like surface $\Sigma \in \mathcal{M}$ with the property that some compact region in Σ lies in the past of a point $P \in \mathcal{M}$, as shown in the following figure:

Histories of events



We suppose that a state ω_Σ associated with a space-like surface Σ is prescribed; (choice of "initial conditions"). Our task is to find out whether **all** events in the *past* of P but in the future of Σ (so-called "histories"), together with the state ω_Σ, uniquely determine a state, ω_P, on the algebra $\mathcal{E}_{\geq P}$ and, given ω_P, to find out whether an *event* happens at P.

Probabilities of histories of events

Inductive hypothesis: Let $P_1, P_2, \dots,$ be all points in the past of P but *not* in the past of *any* point on Σ with the property that, given initial conditions corresponding to ω_Σ, an event has happened at $P_i, i = 1, 2, \dots$. With any of these points we can then associate an orthogonal projection $\Pi^{P_i}_{\xi_i}$, $\xi_i \in \mathcal{X}^{P_i}_{\omega_{P_i}} = \text{spec}\big(\mathcal{Z}^{P_i}_{\omega_{P_i}}\big)$. We define *"history operators"*

$$H(P|\omega_\Sigma) := \overrightarrow{\prod_{i=1,2,\dots}} \Pi^{P_i}_{\xi_i}\,, \tag{10}$$

where P_i is either in the past of P_{i+1}, or P_i and P_{i+1} are space-like, $\forall i = 1, 2, \dots$ Thanks to the *compatibility-locality axiom* the operator $H(P|\omega_\Sigma)$ is well-defined! We then set

$$\omega_P(A) := prob\big(H(P|\omega_\Sigma)\big)^{-1} \omega_\Sigma(H(P|\omega_\Sigma)\, A\, H(P|\omega_\Sigma)^*)\,, \tag{11}$$

$\forall A \in \mathcal{E}_{\geq P}$, where

Events and the fabric of space-time

$$\boxed{prob\big(H(P|\omega_\Sigma)\big) := \omega_\Sigma\big(H(P|\omega_\Sigma)\cdot H(P|\omega_\Sigma)^*\big)}$$

Generalized Born Rule

Induction step: We are now able to answer the question whether an event happens in the space-time point P:
An event happens in P iff the center $\mathcal{Z}_{\omega_P}(\mathcal{E}_{\geq P})$ of the centraliser of ω_P is non-trivial and contains ≥ 2 disjoint orthogonal projections with strictly positive Born probabilities in ω_P.

Note: The *compatibility-locality axiom* is expected to yield non-trivial constraints on the geometry of space-time in the vicinity of two space-like separated points, P and P'', if it is known that $\exists$ events in P and P'' localised in explicitly known regions in the future of P and of P'', respectively, which are represented by projections commuting with one another; ... But these matters remain to be investigated more thoroughly in the future.

7. Summary and conclusions

- The "*ETH*-Approach" to Quantum Mechanics provides a logically coherent theory of events, of their recordings, and of measurements. It has resemblences to *"Many Worlds"*, *"GRW"*, ... ; yet, it supersedes these imprecise formalisms; and it describes but **one World**!
- As in the genesis of Special Relativity, *massless modes* (the e.m. field, gravity) – and (possibly) the even-dimensionality of space-time – play key roles in the genesis of a Quantum Theory solving the "measurement problem". *Has not been properly appreciated, so far!*
- The non-commutative nature of Quantum Theory and the *"compatibility-locality axiom"* governing relations between events determine a *"causal structure"* on space-time. *Events weave the fabric of space-time!* ($\nearrow$ "Emergent gravity")
- Thanks to the *"Principle of Diminishing Potentialities" (PDP)* and the natural presence of an *"arrow of time"* in the *"ETH*-Approach to Quantum Theory", the *"Information Paradox"* and the *"Unitarity Paradox"* appear to dissolve. ... ($\nearrow$ "Passive States")

I thank you for your attention !

A Brief Review of the "ETH- Approach to Quantum Mechanics"

Jürg Fröhlich

May 16, 2019

Abstract

To begin with, some of the conundrums concerning Quantum Mechanics and its interpretation(s) are recalled. Subsequently, a sketch of the "ETH-Approach to Quantum Mechanics" is presented. This approach yields a logically coherent quantum theory of "events" featured by physical systems and of direct or projective measurements of physical quantities, without the need to invoke "observers". It enables one to determine the stochastic time evolution of states of physical systems. We also briefly comment on the quantum theory of indirect or weak measurements, which is much easier to understand and more highly developed than the theory of direct (projective) measurements. A relativistic form of the ETH-Approach will be presented in a separate paper.

Contents

1 Introduction – comments on the foundations of Quantum Mechanics, and purpose of paper

Let me start with a few general remarks: I consider it to be an intellectual scandal that, nearly one hundred years after the discovery of matrix mechanics by *Heisenberg, Born, Jordan* and *Dirac*, many or most professional physicists – experimentalists and theorists alike – admit to be confused about the deeper meaning of Quantum Mechaincs (QM), or are trying to evade taking a clear standpoint by resorting to agnosticism or to overly abstract formulations of QM that often only add to the confusion. Attempts to replace QM by some alternative deterministic theory, one that does not have a "measurement problem", yet reproduces important predictions of QM, do not appear to have been very successful, so far. Unfortunately, most physicists have prejudices preventing them from taking a fresh, unbiased look at the subject, and discussions of the foundations of QM tend to be surprisingly emotional. *I feel it is time to change this situation!*

My own interests in the foundations of Quantum Mechanics were aroused in courses on QM taught by *Klaus Hepp* and *Markus Fierz* in the late sixties of the past century, which I took as an undergraduate student. I suppose that most serious students of Physics develop such interests during their first courses on QM. But I felt that the subject had better remain a hobby until later in my career. Not least because of the appearance of partly contradictory novel *"interpretations of QM"*, all of which left me unsatisfied, (see, e.g., [1, 2], and [3] for a brief survey), my views of the foundations of QM actually remained quite confused until a little more than ten years ago; (which did not prevent me from giving talks about the subject – some with modest impact – in numerous places). But when I was approaching mandatory retirement I felt an urge to clarify my understanding of some of the subjects I had had to teach to my students for thirty years – thermodynamics, effective dynamics (in particular Brownian motion), and, foremost, the foundations of QM; see [4, 5, 6, 7] and references given there, the last two papers having some relevance for the foundations of QM.[1] At the beginning of 2012, my interests in this subject became more serious, and I pursued them in joint efforts with my last PhD student, *Baptiste Schubnel.* Later, some further colleagues got interested in our efforts, including *M. Ballesteros, Ph. Blanchard, N. Crawford, J. Faupin* and *M. Fraas*, who collaborated with us in changing configurations. At this point, I wish to thank my collaborators for their support in this endeavor, as well as quite a few colleagues – too many to mention all of them – who were willing to listen to me and discuss ideas on basic questions concerning the foundations of QM with me. *D. Dürr* and *S. Goldstein* deserve my thanks for the encouragement and understanding they have provided.

In this paper, I present a sketch of the "ETH-Approach to Quantum Mechanics" [8, 9, 10]. The ETH-Approach is supposed to lay the foundations of a logically coherent quantum theory of *"events"* [14] and of *direct* or *projective measurements* of physical quantities (serving to record "events") that does not require invoking any "deos ex machina", such as "observers";

[1] I think it is more appropriate to speak of the "foundations of QM", rather than "interpretations of QM". We have to understand what QM tells us about Nature, *what it means* - once this is accomplished, the correct interpretation of the theory will come almost automatically.

(see also [2]). I have given quite a few talks about this new approach. Technical details have been presented in a short course taught at Les Diablerets, in January of 2017 [11], and in [12, 13]. Our work has profited from ideas proposed by the late *Rudolf Haag* [14], from a paper of *D. Buchholz* and the late *J. E. Roberts* [15], and from discussions with Buchholz. A form of the ETH-Approach apparently compatible with Einstein causality and Relativity Theory is described in [16]. But a comprehensive review of our work has not been released, yet.

Wide-spread recent interest in foundational problems surrounding QM has been triggered by problems in quantum information theory and by the 2012 Nobel Prize in Physics awarded to *S. Haroche* [17] and *D. Wineland*. Their discoveries, as well as results described in [18, 19], and references given there, have influenced some of our own work on the theory of indirect measurements in QM, which has appeared in [20, 21, 22] and is briefly sketched at the end of this paper. The theory of indirect ("non-demolition-" and "weak-") measurements is quite well developed and clear, *assuming* one understands what "events" and "direct measurements and observations" are, specifically *direct* observations of "probes" used to *indirectly* retrieve information on physical systems. The theory of "events" and of "direct (projective) measurements" actually constitutes the deep and controversial part of the foundations of QM, and it is a novel approach to this theory that I intend to outline in this paper.

2 Standard formulation of Quantum Mechanics and its shortcomings

In our courses on Quantum Mechanics, physical systems, S, are often described as pairs, $(\mathcal{H}, U)$, of a Hilbert space, $\mathcal{H}$, of pure state vectors and a propagator, U, consisting of unitary operators $\big(U(t,t')\big)_{t,t'\in\mathbb{R}}$, acting on $\mathcal{H}$ *seemingly* describing the time-evolution of state vectors in $\mathcal{H}$ from time t' to time t. The state space $\mathcal{H}$ of physically realistic systems tends to be infinite-dimensional (but separable). Alas, all infinite-dimensional separable Hilbert spaces are isomorphic, and the data invariantly encoded in the pair $(\mathcal{H}, U)$ do not tell us anything interesting about the physics of S, beyond spectral properties of the operators $U(t,t')$, (i.e., "energy levels"); and they lead one to the mistaken impression that QM might be a *linear* and *deterministic* theory – alas, one that is entirely inadequate to describe events and the outcome of observations and measurements.

We must therefore clarify what should be added to the formalism of QM in order to capture its fundamentally probabilistic nature and to arrive at a mathematical structure that enables one to describe **physical phenomena** ("events") in *isolated* open systems S, without a need to appeal to the intervention of "observers" with "free will" – as is done in the conventional

"Copenhagen Interpretation of QM" – or to assume that other "ghosts" not intrinsic to the theory come to our rescue.

Incidentally, an *isolated system* S is one that, for all practical purposes, does not have any interactions with its complement, i.e., with the rest of the Universe; meaning that, for long periods of time, interactions between the degrees of freedom of S and those of its complement can be neglected in the description of the Heisenberg-picture time evolution of operators. This does, however, *not* exclude that the *state* of S is entangled with the state of its complement. The special role played by isolated systems in discussions of the foundations of QM stems from the fact that, *only for an isolated system,* S, the time evolution in the **Heisenberg picture** of arbitrary operators acting on $\mathcal{H}$ is given by conjugation with its unitary propagator, U, which is determined by the *Hamiltonian* of S.

Physical quantities characteristic of a system S are described by certain self-adjoint linear operators, $X = X^*$, acting on $\mathcal{H}$. This feature is common to *all physical theories* used at present.[2] The Copenhagen Interpretation of Quantum Mechanics then stipulates that there are *"observers"* with "free will" who can decide to measure such physical quantities arbitrarily quickly, at arbitrary times, and at an arbitrary rate. It is argued that the time evolution of physcial states of S is determined by its unitary propagator U, which solves a (deterministic) *Schrödinger equation*, *except* when a measurement of a physical quantity represented by an operator $X = X^*$ is made: Immediately after the measurement of X the state of S, according to the Copenhagen Interpretation, is in an eigenstate of X corresponding to the measured value of X. If this value is not recorded, one is advised to use a density matrix describing an *incoherent* superposition of eigenstates of X, chosen in accordance with *Born's Rule*, to describe the future evolution of S.

For a variety of reasons, this is not a satisfactory recipe for how to apply QM to describe physical phenomena! One might want to view the evolution of states in the presence of measurements, as described in the Copenhagen Interpretation of QM, as some kind of stochastic process. But the problem is that one is dealing with a stochastic process that does *not* have a classical state space, and that it is *transition amplitudes*, rather than *transition probabilities*, that are given by matrix elements of an operator (the propagator U) satisfying a group composition law, i.e., a kind of Chapman-Kolomogorov equation.[3] According to the Copenhagen Inter-

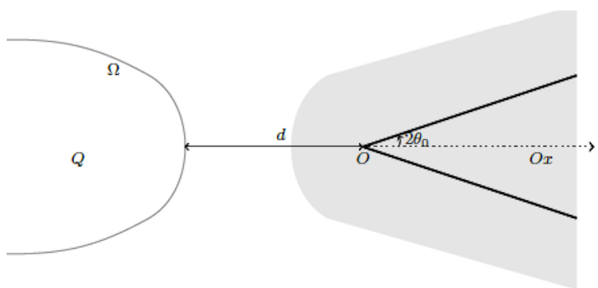
[2] In classical theories, these operators generate an *abelian* (C^*-) algebra, and time evolution is given by a *-automorphism group of this algebra generated by a vector field on its spectrum; while, in QM, the algebra generated by operators representing physical quantities (and events) is *non-commutative*, and time evolution is given by a *-automorphism group of such an algebra *only* if the system is *isolated*.

[3] It is advocated by certain groups of people that the problem arising from this fact can be remedied by invoking the phenomenon of "decoherence" and appealing to the

pretation, predicting/determining the transition probabilities describing the stochastic time-evolution of states of S in the presence of repeated measurements would apparently require knowing what kind of physical quantities are measured by the intervention of "observers", and at what times these measurements are made. For, any intermediate intervention of an "observer" destroys "interference effects"; and hence it seemingly affects the value of the transition probability between an initial state of S in the past and a target state in the future, *even* if a sum over *all possible* outcomes of the intermediate intervention is taken.[4] Without complete information on all intermediate measurements performed on S, which, in the Copenhagen Interpretation, is *not* provided by the theory, reliable predictions of future states of the system and of future expectation values of physical quantities become impossible. As a result, the Copenhagen Interpretation renders QM nearly "unpredictive" – even though, by experience, it is a heuristic framework supplementing QM that works well for many or most "practical purposes", because, much of the time (in particular when using a scattering matrix), one is interested in predicting the outcome of only *a single* measurement. The situation is hardly improved in a definitive way by resorting to concepts such as "decoherence" and interpretations such as "consistent histories" [1], "many worlds", etc.. (See [23, 2] for further information.)

Before proceeding to describe the "ETH-Approach", I recall an argument, presented in detail in [12], that shows that the Schrödinger equation does *not* describe the time evolution of *states* of systems in the presence of "events" or "measurements", *assuming* that the usual correlations between the outcomes of Bell-type measurements, claimed to be confirmed in many experiments, hold.

We consider the following Gedanken-Experiment [12], which, ultimately, will show that *time evolution of states in QM is intrinsically stochastic, in spite of the deterministic nature of the Schrödinger equation.*



Q = sub-system "confined" to Ω Particle P propagating into shaded cone

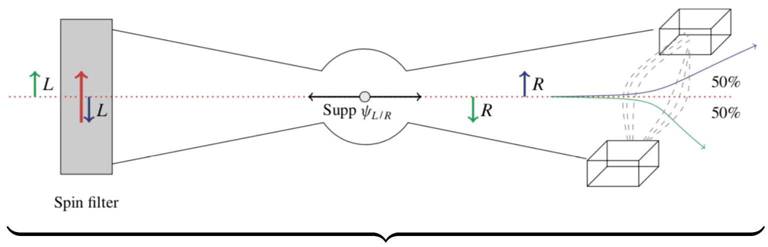
Figure 1

"consistency" of histories of events [1]. But I find the arguments supporting this point of view unconvincing.

[4]This is the case unless perfect "decoherence" holds.

We prepare the system $Q \vee P$ in a state (choosing an appropriate orbital wave function for P) with the property that particle P propagates into the shaded cone opening to the right, except for tiny tails leaking beyond this region, while the degrees of freedom of Q remain confined to a vicinity of the region Ω in the complement of the shaded cone, except for tiny tails. Thanks to cluster properties, expectation values of the Heisenberg-picture time evolution of physical quantities, such as spin, momentum, etc. referring to P in this state then turn out to be essentially *independent* of the time evolution of the degrees of freedom of Q. In other words, interaction terms in the Hamiltonian of the system coupling P to Q can be neglected. This is discussed in much detail in [12].

More concretely, we study the following system:



Q:={spin filter $\vee$ particle P'}   cone opening to right := ess. supp of orbital wave function of P;

(P will undergo a Stern-Gerlach spin measurement)

Figure 2

Temporary assumptions (leading to a contradiction):

- P and P': Two spin-$\frac{1}{2}$ particles prepared in a spin-singlet initial state localized, initially, in the central region shown in Figure 2; the orbital wave function of P is chosen such that P propagates into the cone opening to the right (except for very tiny tails), while the orbital wave function of P', an electron, is chosen such that this particle propagates into the cone opening to the left, with only very tiny tails leaking beyond this cone into the half-space to the right of the spin filter. (One may assume, for simplicity, that there are no terms in the total Hamiltonian of the system describing direct interations between P and P'.) The spin filter (e.g., a spontaneously magnetized metallic film) is prepared in a poorly known initial state.

- *The dynamics of the state of the total system is assumed to be fully determined by a Schrödinger equation given by a concrete self-adjoint Hamiltonian containing only short-range interaction terms.* In particular, the initial state of the total system (consisting of the spin filter, the two particles and possibly some Stern-Gerlach equipment serving to measure a component of the spin of particle P) is assumed to determine whether particle P' will pass through the spin filter, or not, (given that the initial state of $P' \vee P$ is a spin-singlet state, with P' and P moving into opposite cones). Since it is assumed that a Schrödinger equation determines the evolution of *states* of this system, the Schrödinger picture and the Heisenberg picture are equivalent.

- *Correlations between the outcomes of spin measurements of P' and of P are assumed to be those predicted by standard quantum mechanics*, (relying on the "Copenhagen interpretation" and apparently confirmed in many experiments): We first note that if P' passes through the spin filter then its spin is "up", (i.e., aligned with the majority spin of electrons in the spin filter), if it does not pass through the filter, (i.e., if it hops into a vacant state localised inside the spin filter), its spin is "down". The second assumption stated above then says that, whether P' passes through the filter, or not, is determined by the *inital state* of the total system and by solving a *deterministic Schrödinger equation*. In addition to the two assumptions already stated, we also assume that if the spin of P' is measured to be "up" the spin of P is measured to be "down" (for example, in a Stern-Gerlach experiment involving a magnetic field parallel to the majority spin of the spin filter), and if the spin of P' is "down" then the spin of P is "up".

Next, we recall the
Fact: Expectation values of observables (such as spin, momemntum, etc.) referring to particle P in the state of the system described above are *independent* of the degrees of freedom of $Q := \{P' \vee \text{ spin filter}\}$, for arbitrarily long times, up to very tiny corrections. Thus, to a very good approximation, their evolution can be assumed to be given by free-particle dynamics. This is a consequence of our choice of an initial state (propagation properties of the orbital wave functions of P and P') and of cluster properties of the time evolution – as shown in [12].
It follows that, to a very good approximation, the *spin of P is conserved before it is measured* $\Rightarrow$

$$\underline{\text{Expectation value of spin of } P \approx 0, \forall \text{ times before measurement time,}}$$
$$\underline{\textit{independently} \text{ of the evolution of } Q = \{P' \vee \text{ spin filter}\}!}$$

But *this* **contradicts** *the third (last) assumption stated above:* The first two assumptions imply that the values of the z-component of the spin of P' measured with the help of the spin filter do apparently *not* introduce any bias in the outcomes of measurements of the z-component of the spin of P. In other words, the second assumption stated above is incompatible with the *Bell-type "non-locality"* of Quantum Mechanics, as expressed in the third assumption stated above.
This argument is robust, in the sense that it suffices to assume that correlations between measurements of a component of the spin of P' and a

component of the spin of P are fairly close to those predicted by the Bell-type non-locality described in the third assumption.

Conclusion: If the third assumption holds true then the quantum-mechanical time evolution of *states* of physical systems in the presence of measurements (or "events") is *not* given by a deterministic Schrödinger equation, and the equivalence of the Heisenberg picture and the Schrödinger picture apparently fails. Quantum Mechanics appears to be intrinsically probabilistic (and "non-local", in the sense of Bell-type correlations – which does, however, *not* invalidate locality in the sense of "Einstein causality")! These conclusions agree with ones reached by studying gedankenexperiments such as *"Wigner's friend"* and other related ones, e.g., one recently proposed in [24].

Our task is thus to find out what one has to add to a minimal formulation of Quantum Mechanics in order to be able to describe the *stochastic dynamics of states* of physical systems in the presence of "events" and their recordings (in projective measurements), in such a way that correlations between the outcomes of measurements agree with the Bell-type "non-locality" of Quantum Mechanics – without the need to assume that "observers" intervene. The results reviewed in the next section are intended to represent some progress in this direction.

3 Summary of the "ETH-Approach"

In this section I briefly describe the so-called *"ETH-Approach to Quantum Mechanics"* [8, 9, 10, 11, 12, 13], which is designed to retain attractive features of the Copenhagen Interpretation but eliminates its fatal weaknesses; and I note that "E" stands for "Events", "T" for "Trees", and "H" for "Histories". In the following, I attempt to explain what these terms mean, and why the concepts underlying the "ETH-Approach" are important for an understanding of the foundations of Quantum Mechanics (QM). The basic premises and contentions of this approach are as follows:

I. Potential Events. In the ETH-Approach to QM, *Time*, denoted by t, is taken as an irreducible concept. It is described by the real line, $\mathbb{R}$, with its usual order relation.[5] But in order to make the following discussion mathematically watertight it is advisable to sometimes assume that time is *discretized*, $t \in \mathbb{Z}$. An important idea underlying the ETH-Approach is that time is not merely a parameter, but that it can be monitored by recording "events" happening in an isolated open system. (The precise meaning of this idea will become clearer later

[5] The role of *space*-time in a relativistic version of the "ETH-Approach" is discussed in [16]

on.)
Let $t_0 \in \mathbb{R}$ be the time of the present. We consider an isolated physical system S and we denote by $\mathcal{H}$ the Hilbert space of pure state vectors of S. Our first task is to clarify what is meant by *"potential events"* in S that may happen at some future time $t > t_0$, or later: Potential events are described by families, $\{\pi_\xi, \xi \in \mathcal{X}\}$ of orthogonal projections acting on $\mathcal{H}$, with the properties that

$$\begin{aligned} \pi_\xi \cdot \pi_\eta &= \delta_{\xi\eta}\, \pi_\xi, \ \forall \xi, \eta \text{ in } \mathcal{X}, \quad \text{(disjointeness)} \\ \sum_{\xi \in \mathcal{X}} \pi_\xi &= \mathbb{1}, \quad \text{(partition of unity).} \end{aligned} \tag{1}$$

For simplicity we henceforth assume that the sets $\mathcal{X}$ labelling the projections that represent potential events are countable, discrete sets. (This merely serves to avoid technical complications in our exposition; of course, continuous spectra occur, too.) The concrete projection operators acting on the Hilbert space $\mathcal{H}$ of S representing a *specific* potential event, e.g., the click of a detector belonging to S when it is hit by a particle in S, depend on the future time at which the event might happen. In an autonomous system, the projection operators representing such a specific event potentially happening either at a time $t > t_0$ or at another time $t' > t_0$ are conjugated to one another by the propagator $U(t, t')$ of the system; (Heisenberg-picture evolution of operators). All potential events that may happen at a time $t > t_0$, or later, generate a *-algebra denoted by $\mathcal{E}_{\geq t}$. It immediately follows from the definition that

$$\mathcal{E}_{\geq t'} \subseteq \mathcal{E}_{\geq t}, \quad \text{if } t' > t.$$

For simplicity we assume that *all physically relevant states* of S can be described by density matrices acting on $\mathcal{H}$, and that the algebras $\mathcal{E}_{\geq t}$ are closed in the weak topology of the algebra, $B(\mathcal{H})$, of all bounded operators acting on $\mathcal{H}$. Typically, all the algebras $\mathcal{E}_{\geq t}$ are then isomorphic to one *universal* (von Neumann) algebra[6] $\mathcal{N}$, i.e.,

$$\mathcal{E}_{\geq t} \simeq \mathcal{N}, \quad \forall t \in \mathbb{R}. \tag{2}$$

The algebra, $\mathcal{E}$, of all potential events that may happen in the course of history is defined by

$$B(\mathcal{H}) \supseteq \mathcal{E} := \overline{\bigvee_{t \in \mathbb{R}} \mathcal{E}_{\geq t}}\,, \tag{3}$$

(where the closure is taken in the operator norm of $B(\mathcal{H})$).

[6]In local relativistic quantum theories with massless particles, the algebra $\mathcal{N}$ tends to be a von Neumann algebra of type III; see [15]

II. The Principle of Diminishing Potentialities. In the quantum theory of (autonomous) systems with finitely many degrees of freedom – as treated in our introductory courses on QM – the algebras $\mathcal{E}_{\geq t}$ turn out to be *independent* of time t; and usually $\mathcal{E}_{\geq t} = B(\mathcal{H})$. For such systems, one *cannot* develop a sensible quantum theory of events, and it is impossible to come up with a logically coherent, intrinsically quantum-mechanical description of the retrieval of information on such systems, i.e., of measurements, *without* adding further quantum systems with infinitely many degrees of freedom that serve to "measure" the former systems; (or without resorting to something like "Copenhagen"). In this respect, quantum systems with finitely many degrees of freedom are as "interesting" as the causal outside of a black hole: no information leaks out! In order to encounter non-trivial dependence of the algebras $\mathcal{E}_{\geq t}$ on time t, we must consider *isolated (open) systems with infinitely many degrees of freedom* and with the property that the propagator U of S is generated by a Hamiltonian whose spectrum is *absolutely continuous* and (if time is continuous) *unbounded above and below* or is *semi-bounded, but without any spectral gaps*; i.e., we must assume that there exist massless modes.

Our contention is that a basic property of a quantum theory of isolated open systems, S, enabling one to describe *events* and their *recording* in projective measurements of physical quantities is captured in the following **"Principle of Diminishing Potentialities"** (PDP):

$$\boxed{\mathcal{E}_{\geq t'} \underset{\neq}{\subseteq} \mathcal{E}_{\geq t} \underset{\neq}{\subseteq} \mathcal{E}, \quad \text{whenever} \quad t' > t.} \tag{4}$$

To be more precise, one expects that if time is continuous the relative commutant

$$(\mathcal{E}_{\geq t'})' \cap \mathcal{E}_{\geq t}, \quad \text{with} \quad t' > t,$$

is an infinite-dimensional, non-commutative algebra. (If time is discrete this relative commutant can, however, be a finite-dimensional algebra.) Examples of non-relativistic and relativistic systems satisfying property (4) will be discussed elsewhere, (see also [11]).[7] Here I just mention that (PDP), in the sense of a relativistic variant of Eq. (4), is a *theorem* in local relativistic quantum field theories with massless particles in four space-time dimensions.[8] This follows from important results in [15] and is used in [16].

Definition 1. *Isolated open systems S* (featuring events) are henceforth *defined* in terms of a filtration, $\{\mathcal{E}_{\geq t}\}_{t\in\mathbb{R}}$ (or, for the sake of

[7] I sometimes fear that unrealistically simple examples advanced with the intention to clarify aspects of the foundations of QM have had the opposite effect: They have contributed to clouding our views.

[8] and the algebras $\mathcal{E}_{\geq t}, t \in \mathbb{R}$, are von Neumann algebras of type III.

simplicity and precision, $\{\mathcal{E}_{\geq t}\}_{t\in\mathbb{Z}}$), of (von Neumann) algebras satisfying the *"Principle of Diminishing Potentialities"* (4), all represented on a common Hilbert space $\mathcal{H}$, whose projections describe potential events. $\square$

If Ω denotes the density matrix on $\mathcal{H}$ representing the actual state of a system S we use the notation

$$\omega(X) := tr(\Omega\, X), \qquad \forall X \in B(\mathcal{H}),$$

to denote the expectation value of the operator X in the state ω determined by Ω. We define

$$\omega_t(X) := \omega(X), \qquad \forall X \in \mathcal{E}_{\geq t}, \tag{5}$$

i.e., ω_t is the *restriction* of the state ω to the algebra $\mathcal{E}_{\geq t}$.
Note that, as a cconsequence of (PDP) and of *entanglement*, the restriction, ω_t, of a state ω on the algebra $\mathcal{E}$ to a subalgebra $\mathcal{E}_{\geq t} \subset \mathcal{E}$ will usually be **mixed** *even* if ω is a **pure** state on $\mathcal{E}$.

III. Actual Events. Henceforth we only study isolated open systems S for which (PDP), in the form of Eq. (4), holds. Let $\{\pi_\xi, \xi \in \mathcal{X}\} \subset \mathcal{E}_{\geq t}$ be a potential event that might start to happen at some time t, with $\{\pi_\xi, \xi \in \mathcal{X}\}$ *not* contained in $\mathcal{E}_{\geq t'}$, for $t' > t$. Tentatively, we say that this potential event **actually starts to happen** at time t iff

$$\omega_t(X) = \sum_{\xi\in\mathcal{X}} \omega_t\big(\pi_\xi\, X\, \pi_\xi\big), \qquad \forall X \in \mathcal{E}_{\geq t}, \tag{6}$$

meaning that ω_t is an incoherent superposition of states labelled by the points $\xi \in \mathcal{X}$; in other words, off-diagonal expectations, $\omega_t\big(\pi_\xi\, X\, \pi_\eta\big)$, $\xi \neq \eta$, do *not* contribute to the right side of (6). Equation (6) is equaivalent to saying that the projections $\pi_\xi, \xi \in \mathcal{X}$, belong to the *centralizer* of the state ω_t.

Given a *-algebra $\mathcal{M}$ and a state ω on $\mathcal{M}$, the centralizer, $\mathcal{C}_\omega(\mathcal{M})$, of the state ω is defined to be the subalgebra of $\mathcal{M}$ spanned by all operators, Y, in $\mathcal{M}$ with the property that

$$\omega([Y, X]) = 0, \qquad \forall X \in \mathcal{M}.$$

The *center* of the centralizer, denoted by $\mathcal{Z}_\omega(\mathcal{M})$, is the abelian subalgebra of the centralizer consisting of all operators in $\mathcal{C}_\omega(\mathcal{M})$ commuting with all other operators in $\mathcal{C}_\omega(\mathcal{M})$.
We note that the center, $\mathcal{Z}(\mathcal{M})$, of the algebra $\mathcal{M}$ is contained in

$\mathcal{Z}_\omega(\mathcal{M})$, *for all* states ω.

Definition 2. A potential event $\{\pi_\xi, \xi \in \mathcal{X}\} \subset \mathcal{E}_{\geq t}$, with $\{\pi_\xi, \xi \in \mathcal{X}\}$ not contained in $\mathcal{E}_{\geq t'}$, for $t' > t$, *actually starts to happen* at time t iff $\mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t})$ is *non-trivial*,

$$\{\pi_\xi, \xi \in \mathcal{X}\} \text{ generates } \mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t}), \tag{7}$$

and

$$\omega_t(\pi_{\xi_j}) \text{ is } \textit{strictly positive}, \ \xi_j \in \mathcal{X}, \ j = 1, 2, \ldots, n\,, \tag{8}$$

for some $n \geq 2$. □

IV. The fundamental Axiom. We are now in a position to describe the evolution of states in the ETH-Approach to QM. Let ω_t be the state of an isolated system S right before time t. Let us suppose that an event $\{\pi_\xi, \xi \in \mathcal{X}\}$ generating $\mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t})$ starts to happen at time t, in the sense of Definition 2.

Axiom. The actual state of the system S right *after* time t when the event $\{\pi_\xi, \xi \in \mathcal{X}\}$ has started to happen is given by one of the states

$$\omega_{t,\xi_*}(\cdot) := [\omega_t(\pi_{\xi_*})]^{-1}\, \omega_t\big(\pi_{\xi_*}(\cdot)\pi_{\xi_*}\big)\,, \tag{9}$$

for some $\xi_* \in \mathcal{X}$ with $\omega_t(\pi_{\xi_*}) > 0$, ("**state-collapse** postulate"[9]). The probability for the system S to be found in the state ω_{t,ξ_*} right after time t when the event $\{\pi_\xi, \xi \in \mathcal{X}\}$ has started to happen is given by **Born's Rule**, i.e., by

$$prob\{\xi_*, t\} = \omega_t(\pi_{\xi_*}). \qquad \square \tag{10}$$

Remarks:
(1) The projection π_{ξ_*} selecting the actual state ω_{t,ξ_*} of S (and sometimes also the point $\xi_* \in \mathcal{X}$) is called the *"actual event"* happening at time t.
(2) The contents and meaning of this Axiom are clear and mathematically watertight as long as time is discrete. (If time is continuous further precision ought to be provided.)

This Axiom, Eqs. (9) and (10), conveys the following picture of quantum dynamics: In Quantum Mechanics, the evolution of states of an isolated open system S featuring events, in the sense of Definitions 1 and 2 proposed above, is given by a (rather unusual novel type of) stochastic branching process,

[9] a rather unfortunate name!

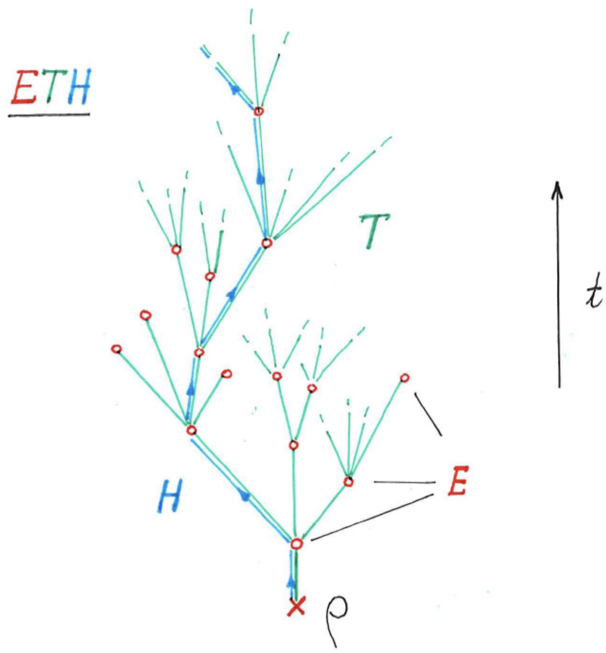
whose state space is what I call the *"non-commutative spectrum"*, $\mathfrak{Z}_S$, of S. Assuming that Eq. (2) holds, the non-commutative spectrum of S is defined by

$$\mathfrak{Z}_S := \bigcup_{\omega} \mathcal{Z}_\omega(\mathcal{N})\,, \quad \text{with} \quad \mathfrak{X}_S := \bigcup_{\omega} \text{spec}\Big(\mathcal{Z}_\omega(\mathcal{N})\Big), \tag{11}$$

where the union over ω is a disjoint union, and ω ranges over *all* physical states of S.[10] Eq. (7) and **Born's Rule**, Eq. (10), specify the *branching probabilities* of the process.

This picture of the stochastic time evolution of states of an isolated open system S is illustrated, metaphorically (for discrete time), in Figure 3, below. It differs substantially from and supercedes the *"decoherence mumbo-jumbo"*.

Let us suppose, for the sake of simplicity and mathematical precision, that time is discrete, ($t \in \mathbb{Z}$). It is important to note that, in general, the events (described by orthogonal projections in $\mathcal{E}_{\geq t'}$) predicted to happen at a later time $t' > t$ on the basis of the states $\omega_{t,\xi}, \xi \in \mathcal{X}$, where $\{\pi_\xi, \xi \in \mathcal{X}\}$ generates $\mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t})$, are *different* from the events one would predict to happen at time t' on the basis of the state $\omega_t|_{\mathcal{E}_{\geq t'}}$, used when the actual event



Time evolution of a state of S with initial condition $\omega := \rho$
E: "Events", T: "Tree" of possible future states, H: "History" of actual events/states.
Figure 3

[10]The set $\mathfrak{X}_S$ can also be defined in terms of a certain "flag manifold" associated with the Hilbert space $\mathcal{H}$

happening at time t is not known (i.e., has not been recorded); and the projections representing these different sets of events usually do *not* commute with one another. Furthermore, for $t' > t$, the operators in $\mathcal{Z}_{\omega_{t,\xi}}(\mathcal{E}_{\geq t'})$ and in $\mathcal{Z}_{\omega_{t,\eta}}(\mathcal{E}_{\geq t'}), \xi, \eta \in \mathcal{X}$, (with $\omega_t(\pi_\xi)$, $\omega_t(\pi_\eta)$ strictly positive), but $\xi \neq \eta$, do *not* in general commute with each other. This is a *fundamental difference* between the *"non-commutative branching processes"*, described here, and classical stochastic branching processes.

The discussion above is mathematically sound if time is discrete, but requires more precision if time is taken to be continuous.

To be on the safe side, we temporarily choose time to be discrete ($t \in \mathbb{Z}$). Let H be the Hamiltonian of an isolated open system, and suppose that

$$\|e^{iH} - \mathbf{1}\| \ll 1\,. \tag{12}$$

Let us suppose that $\{\pi_{t,\xi}, \xi \in \mathcal{X}_t\}$ is an event that starts to happen at time t, provided the state of S at time t is given by ω_t; (i.e., $\{\pi_{t,\xi}, \xi \in \mathcal{X}_t\}$ generates $\mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t})$). Let ξ_* be the element of $\mathcal{X}_t$ with the property that, in accordance with the Axiom stated in IV., above, the state of S right after time t is given by

$$\omega_{t,\xi_*}(\cdot) := [\omega_t(\pi_{t,\xi_*})]^{-1}\, \omega_t\big(\pi_{t,\xi_*}(\cdot)\pi_{t,\xi_*}\big)\,,$$

with $\omega_t\big(\pi_{t,\xi_*}\big) > 0$. Let $t' = t+1$ be the time following t, and let $\{\pi_{t',\xi}, \xi \in \mathcal{X}_{t'}\}$ be the event happening at time t', *provided* that the state of S at time t' is given by ω_{t,ξ_*}. Then assumption (12) suggests that there exists an element $\xi_\natural \in \mathcal{X}_{t'}$ with the property that

$$\begin{aligned} \omega_{t,\xi_*}\big(\pi_{t',\xi_\natural}\big) &\approx 1, \quad \text{but} \\ \omega_{t,\xi_*}\big(\,\pi_{t',\xi}\,\big) &\ll 1, \ \forall\, \xi \neq \xi_\natural\,,\ \xi \in \mathcal{X}_{t'}\,. \end{aligned} \tag{13}$$

According to the Axiom in IV., in particular **Born's Rule**, the actual state of S right after time t' is then very likely given by

$$\omega_{t,\xi_*,t',\xi_\natural}(\cdot) := [\omega_{t,\xi_*}(\pi_{t',\xi_\natural})]^{-1}\omega_{t,\xi_*}\big(\pi_{t',\xi_\natural}(\cdot)\pi_{t',\xi_\natural}\big) \approx \omega_{t,\xi_*}(\cdot)\,.$$

The state $\omega_{t,\xi_*,t',\xi_\natural}$ is close to the one that would commonly be used in quantum mechanics in the absence of any "measurements" (invoking the equivalence of the Schrödinger- and the Heisenberg picture), namely the state $\omega_{t,\xi_*}(\cdot)$.

However, for purely statistical (*entropic* !) reasons, every once in a while, i.e., at rare times t', an event $\pi_{t',\xi}$ is realised that has a *very small Born probability*, $\omega_{t'}(\pi_{t',\xi}) \ll 1, \quad \xi \in \mathcal{X}_{t'}$.

Digression on "Missing Information" associated with an event:[11] Given the event $\{\pi_{t,\xi}, \xi \in \mathcal{X}_t\}$ happening at time t, assuming that ω_t is the actual state of S right before time t, we define the *"missing information"* (or *"entropy production"*), $\sigma(\omega_t, \mathcal{X}_t)$, associated with this event by

$$\sigma(\omega_t, \mathcal{X}_t) := -\sum_{\xi \in \mathcal{X}_t} \omega_t(\pi_{t,\xi}) \cdot \ell n\big(\omega_t(\pi_{t,\xi})\big) \tag{14}$$

Assuming that (12) holds, the "missing information" associated with most events that ever happen is very small. If the "missing information" associated with *all* events were tiny then taking the state of S in the Heisenberg picture to be constant in time would be a good approximation to its stochastic evolution. However, every once in a while, events corresponding to a *large* "missing information" (entropy production) may be encountered, and these are the events that will most likely be noticed and recorded, because they trigger a substantial change of the state of S. (Some people will want to call them "measurements".)

Let t_0 be the time at which the system S has been prepared in a state ω, (as discussed in [13]), and $t_j := t_0 + j \in \mathbb{Z}$; further, let π_{t_j,ξ_j} be the *actual event* happening at time t_j, given the initial state ω of S and earlier actual events $\pi_{t_\ell,\xi_\ell}, \ell < j$, $j = 1, 2, \ldots, n$; (see Definition 2 and Axiom). We define

$$\mu_\omega\big(\xi_1, \xi_2, \ldots, \xi_n | X\big) := \omega\Big(\prod_{j=1}^{n} \pi_{t_j,\xi_j} \cdot X \cdot X^* \cdot (\prod_{j=1}^{n} \pi_{t_j,\xi_j})^*\Big)\,, \tag{15}$$

where X is an arbitrary non-zero operator in $\mathcal{E}_{\geq t}$, for some $t > t_n$, with $\omega\big(X \cdot X^*\big) > 0$. Then $\mu_\omega(\ldots|X)$ is a positive measure on the Cartesian product $\bigtimes_{j=1}^{n} \mathcal{X}_{t_j}$. *Note that the space $\mathcal{X}_{t_{k+1}}$ depends on the choice of ω and on* **all** *the actual events $\pi_{t_1,\xi_1}, \ldots, \pi_{t_k,\xi_k}$ that happened at times $t_1 < \cdots < t_k$,* **before** *t_{k+1}; with $k = 1, 2, \ldots, n-1$.* For any m, with $0 < m < n$, we set

$$X(\underline{\xi}^{(m,n)}) := \prod_{j=m+1}^{n} \pi_{t_j,\xi_j} \cdot X\,,$$

and $X(\underline{\xi}^{(n,n)}) := X$. Then

$$\mu_\omega\big(\xi_1, \ldots, \xi_n | X\big) = \mu_\omega\big(\xi_1, \ldots, \xi_m | X(\underline{\xi}^{(m,n)})\big)\,.$$

The measure $\mu_\omega\big(\ldots|X\big)$ has the (possibly somewhat perplexing) property that

[11] This digression can be omitted at first reading, and the reader is invited to proceed to point V., below.

$$\sum_{\xi_{k+1},\cdots,\xi_m} \mu_\omega\big(\xi_1,\ldots,\xi_k,\xi_{k+1},\ldots,\xi_m|X(\underline{\xi}^{(m,n)})\big) =$$

$$= \mu_\omega(\xi_1,\ldots,\xi_k|X(\underline{\xi}^{(m,n)})\big)\,, \tag{16}$$

for arbitrary k, with $1 \leq k \leq m \leq n$, as one easily verifies. (Identity (16) may look familiar to the reader from a similar one satisfied by the "Lüders-Schwinger-Wigner formula" [25] for the probability of a sequence of outcomes of measurements, assuming perfect *decoherence*. However, it actually has quite a different origin!) It is sometimes convenient to define $\mu_\omega\big(\ldots|X\big)$ as a measure on the space

$$\mathfrak{X}_n := \big(\mathfrak{X}_S\big)^{\times n}\,,$$

where $\mathfrak{X}_S$ has been defined in Eq. (11), with the convention that

$$\pi_{t_k,\xi} = 0, \quad \text{unless} \;\; \xi \in \mathcal{X}_{t_k} \subset \mathfrak{X}_S\,.$$

For $X = \mathbb{1}$, $\mu_\omega(\ldots|\mathbb{1})$ is a *probability measure* on $\mathfrak{X}_n$. If arbitrarily long sequences of events are considered it is useful to introduce the "path space"

$$\mathfrak{X}_\infty := \varinjlim_{n\to\infty} \mathfrak{X}_n\,.$$

Thanks to property (16), the measures $\mu_\omega(\ldots|\mathbb{1})$ determine a unique probability measure on $\mathfrak{X}_\infty$. This follows from a well known lemma due to *Kolmogorov.*
Next, we define the *"missing information per event"* of a sequence of events, as follows:

$$\sigma_n(\mu_\omega) := -\frac{1}{n}\sum_{\xi_1,\cdots,\xi_n} \mu_\omega(\xi_1,\ldots,\xi_n|\mathbb{1})\cdot \ell n\big(\mu_\omega(\xi_1,\ldots,\xi_n|\mathbb{1})\big)\,,$$

and

$$\sigma(\mu_\omega) := \text{limsup}_{n\to\infty}\sigma_n(\mu_\omega) \tag{17}$$

If events happening at times $t_1,\ldots,t_n$ are not recorded then $\sigma_n(\mu_\omega)$ is a measure of how much the state of the system at time $t > t_n$ deviates from the (initial) state ω used in the Heisenberg picture of standard QM. Of particular interest is the so-called *relative entropy*

$$\begin{aligned} S_n\big(\mu_\omega\|\mu_\omega^{opp}\big) \;\; &:= \sum_{\xi_1,\cdots,\xi_n} \mu_\omega(\xi_1,\ldots,\xi_n|\mathbb{1})\times \\ &\times \;\; \Big(\ell n\,\mu_\omega(\xi_1,\ldots,\xi_n|\mathbf{1}) - \ell n\,\mu_\omega^{opp}(\xi_1,\ldots,\xi_n|\mathbb{1})\Big)\,, \end{aligned} \tag{18}$$

where

$$\mu_\omega^{opp}(\xi_1,\ldots,\xi_n|\mathbb{1}) := \omega\Big(\big(\prod_{j=1}^n \pi_{t_j,\xi_j}\big)^*\cdot\prod_{j=1}^n \pi_{t_j,\xi_j}\Big)$$

is the measure obtained when the order of the events is (time-)reversed. The relative entropy $S_n\big(\mu_\omega \| \mu_\omega^{opp}\big)$ is *non-negative*, and its growth in n, as $n \to \infty$, is a measure of the *irreversibility* of histories of events featured by the system and reflects the "arrow of time".

End of Digression.

V. <u>Recording events by "projective measurements" of physical quantities.</u>
We consider an isolated open system S described in terms of a filtration $\{\mathcal{E}_{\geq t}\}_{t \in \mathbb{R}}$ of algebras represented on its Hilbert space $\mathcal{H}$ of pure state vectors, as described in Defnition 1, (paragraph I.). We propose to clarify how events happening in S can be recorded by projectively (directly) measuring *"physical quantities"* characteristic of S. (Time may be taken to be continuous; but, for the sake of simplicity and mathematical precison, the reader is invited to continue to assume that $t \in \mathbb{Z}$.)

<u>Definition 3</u>. A *"physical quantitiy"* characteristic of S is an abelian (C^*-) algebra, $\mathcal{Q}$, with the property that, for each time t, there exists a representation, $\sigma_t^{\mathcal{Q}}$, of $\mathcal{Q}$ on $\mathcal{H}$ as a subalgebra of $\mathcal{E}_{\geq t}$. □

For autonomous systems, the representations $\sigma_t^{\mathcal{Q}}$ and $\sigma_{t'}^{\mathcal{Q}}$ are unitarily equivalent, with

$$\sigma_t^{\mathcal{Q}}(A) = U(t', t)\, \sigma_{t'}^{\mathcal{Q}}(A)\, U(t, t'), \quad \forall A \in \mathcal{Q}\,,$$

where $U(t', t) = \exp\big(i(t - t')H\big)$ is the propagator of S, with t, t' arbitrary times; (Heisenberg-picture dynamics).

For simplicity, we assume that the physical quantities $\mathcal{Q}$ available to identify properties of S or record events all have discrete spectrum; i.e.,

$$\mathcal{Q} = \langle \Pi_\eta^{\mathcal{Q}} | \eta \in \mathcal{Y}^{\mathcal{Q}} =: \mathrm{spec}(\mathcal{Q}) \rangle, \tag{19}$$

where $\mathcal{Y}^{\mathcal{Q}}$ is a discrete set, which we view as a subset of the real line, and the operators $\Pi_\eta^{\mathcal{Q}}$ are disjoint orthogonal projections. (Of course, continuous spectra also arise. But in order to avoid technical complications, we ignore them here.) We can then describe $\mathcal{Q}$ as the algebra given by all functions of a single self-adjoint operator, $\widehat{Y}$, with discrete spectrum, $\mathrm{spec}(\widehat{Y}) \simeq \mathcal{Y}^{\mathcal{Q}}$, and spectral projections $\Pi_\eta^{\mathcal{Q}}$. For every time t, there exists a self-adjoint operator, $Y(t) = \sigma_t^{\mathcal{Q}}(\widehat{Y})$, acting on $\mathcal{H}$ that represents $\widehat{Y}$ at time t.

It is interesting to ask whether physical quantities can serve to detect or record events happening in S. For a restricted set

$$\mathcal{O}_S = \{\mathcal{Q}_j\}_{j \in \mathfrak{J}}$$

of physical quantities characteristic of S, it is arbitrarily unlikely that one of the algebras $\sigma_t^{\mathcal{Q}_j}(\mathcal{Q}_j), j \in \mathfrak{J}$, has a non-trivial intersection with (e.g., contains or is contained in) an algebra $\mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t})$ describing the event happening at time t, for some state ω_t. To cope with this problem, we have to understand how well $\mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t})$ can be approximated by the algebra generated by a family, $\{Q_\alpha(t)\}_{\alpha=0}^N$, of disjoint orthogonal projections contained in (or equal to) an algebra $\sigma_t^{\mathcal{Q}}(\mathcal{Q})$, for some $\mathcal{Q} \in \mathcal{O}_S$.

There are different ways of quantifying how well the algebra generated by $\{Q_\alpha(t)\}_{\alpha=0}^N$ approximates the event described by $\mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t})$. To keep our discussion brief, it is convenient to introduce *"conditional expectations"* of algebras:

Definition 4.
Let $\mathcal{N}$ be a (von Neumann) subalgebra of a (von Neumann) algebra $\mathcal{M}$. A linear map

$$\epsilon_\omega : \mathcal{M} \underset{\text{onto}}{\to} \mathcal{N} \tag{20}$$

is a *conditional expectation* from $\mathcal{M}$ onto $\mathcal{N}$ with respect to a normal state ω on $\mathcal{M}$ iff

(i) $\|\epsilon_\omega(X)\| \leq \|X\|, \quad \forall X \in \mathcal{M}$
(ii) $\epsilon_\omega(X) = X, \quad \forall X \in \mathcal{N}$
(iii) $\omega \circ \epsilon_\omega = \omega$ □

Conditional expectations have the following properties:

(iv) $\epsilon_\omega(X^*X) \geq 0, \quad \forall\, X \in \mathcal{M}$
(v) $\epsilon_\omega(AXB) = A\epsilon_\omega(X)B, \quad \forall\, A, B, \in \mathcal{N}, \forall\, X \in \mathcal{M}$
(vi) $\epsilon_\omega : \mathcal{M} \to \mathcal{N}$ is completey positive, and $\epsilon_\omega(\mathbb{1}_{\mathcal{M}}) = \mathbb{1}_{\mathcal{N}}$

See, e.g., [26] for an exposition of the theoy of conditional expectations. Under very general assumptions, there exist conditional expectations

$$\epsilon_{\omega_t} : \mathcal{E}_{\geq t} \to \mathcal{Z}_{\omega_t}\big(\mathcal{E}_{\geq t}\big)\,, \tag{21}$$

for arbitrary times t.

Let ω_t be the state of a system S right before an event $\{\pi_\xi, \xi \in \mathcal{X}_t\}$ generating $\mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t})$ starts to happen. I propose to clarify in which way a physical quantity $\mathcal{Q} \in \mathcal{O}_S$ can be used to record this event, and how precisely the value of this quantity identifies the *actual* event, $\xi_* \in \mathcal{X}_t$, happening at time t.
We assume that there exists a physical quantity $\mathcal{Q}$ and a family of disjoint orthogonal projections $\{\widehat{Q}_\alpha\}_{\alpha=0}^N \subset \mathcal{Q}$, $N \geq 2$, with the following properties:

(a) $\sum_{\alpha=0}^{N} Q_\alpha(t) = \mathbb{1}$, where $Q_\alpha(t) = \sigma_t^{\mathcal{Q}}(\widehat{Q}_\alpha)$, $\alpha = 1, \ldots, N$, $\forall t$;

(b) there exists a positive number $\delta \ll 1$ such that

$$\omega_t\Big(\sum_{\alpha=1}^{N} Q_\alpha(t)\Big) \geq 1 - \delta \quad \Leftrightarrow \quad \omega_t\big(Q_0(t)\big) \leq \delta\,;$$

(c) Given an operator $X \in \mathcal{E}_{\geq t}$, we define

$$\mathrm{dist}\big(X, \mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t})\big) := \|X - \epsilon_{\omega_t}(X)\|.$$

We assume that

$$\mathrm{dist}\big(Q_\alpha(t), \mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t})\big) < \delta\,, \quad \text{for } \alpha = 1, \ldots, N\,. \tag{22}$$

In the following, we use the notation $\mathcal{O}(\varepsilon)$ to denote any real number whose absolute value is bounded above by *const.* ε, where *const.* is a *uniformly bounded* positive constant. Properties (a) through (c) of $\{\widehat{Q}_\alpha\}_{\alpha=0}^{N}$ can be used to derive the following equations:

For an arbitrary operator $X \in \mathcal{E}_{\geq t}$,

$$\begin{aligned}
\omega_t(X) &= \sum_{\alpha=1}^{N} \omega_t\big(Q_\alpha(t)\, X\big) + \mathcal{O}(\delta\|X\|) \\
&= \sum_{\alpha=1}^{N} \omega_t\big(Q_\alpha(t)[Q_\alpha(t)X]\big) + \mathcal{O}(\delta\|X\|) \\
&= \sum_{\alpha=1}^{N} \omega_t\big(\epsilon_{\omega_t}(Q_\alpha(t))[Q_\alpha(t)X]\big) + \mathcal{O}(\delta\, N\|X\|) \\
&= \sum_{\alpha=1}^{N} \omega_t\big(Q_\alpha(t)X\, \epsilon_{\omega_t}(Q_\alpha(t))\big) + \mathcal{O}(\delta\, N\|X\|) \\
&= \sum_{\alpha=1}^{N} \omega_t\big(Q_\alpha(t)X\, Q_\alpha(t)\big) + \mathcal{O}(\delta\, N\|X\|).
\end{aligned} \tag{23}$$

Apparently, if $\delta\, N \ll 1$ then, to a good approximation, the state ω_t is an incoherent superposition of eigenstates of the disjoint projections $Q_\alpha(t)$, $\alpha = 1, \ldots, N$. We then say that, at approximately time t, "a projective (direct) measurement of $\mathcal{Q}$ takes place".

__**Definition 5**__. *(Resolution of $\mathcal{Q}$ in recording an event)*
Assuming that $\mathcal{X}_t$ is a countable set, then, for any $\delta \in (0,1)$, there exists a subset $\mathcal{X}_t^{(M)} \subseteq \mathcal{X}_t$ whose cardinality is given by a finite integer M such that

$$\omega_t\Big(\sum_{\xi \in \mathcal{X}_t^{(M)}} \pi_{t,\xi}\Big) \geq 1 - \delta\,.$$

Then, for an arbitrary operator $X \in \mathcal{E}_{\geq t}$,

$$\omega_t(X) = \sum_{\xi \in \mathcal{X}_t^{(M)}} \omega_t\big(\pi_{t,\xi}\, X\, \pi_{t,\xi}\big) + \mathcal{O}(\delta\, \|X\|)\,.$$

The *"resolution"* of $\{Q_\alpha(t)\}_{\alpha=0}^{N} \subset \mathcal{Q}$ in recording the event $\{\pi_{t,\xi}, \xi \in \mathcal{X}_t\}$ starting to happen at time t is defined by

$$\mathfrak{R} := \frac{N}{M} \cdot (1-\delta)\,, \text{ for } 2 \leq N \leq M\,, \;\; (\mathfrak{R} = 0, \text{ for } N = 1)\,. \qquad \square \tag{24}$$

It turns out that property (c), Eq. (22), above, implies that, given an orthogonal projection $Q_\alpha(t) \in \sigma_t^{\mathcal{Q}}(\mathcal{Q})$, there exists an orthogonal projection $P_\alpha \in \mathcal{Z}_{\omega\, t}\big(\mathcal{E}_{\geq t}\big)$ such that

$$\|Q_\alpha(t) - P_\alpha\| < \mathcal{O}(\delta)\,. \tag{25}$$

A proof of this simple lemma can be found in the appendix of [3].

Since the projections $\pi_{t,\xi}, \xi \in \mathcal{X}_t$ generate the abelian algebra $\mathcal{Z}_{\omega\, t}\big(\mathcal{E}_{\geq t}\big)$, we have that

$$\pi_{t,\xi} \cdot P = \pi_{t,\xi}, \;\text{ or }\; \pi_{t,\xi} \cdot P = 0, \quad \forall \xi \in \mathcal{X}_t\,, \tag{26}$$

for any orthogonal projection $P \in \mathcal{Z}_{\omega\, t}\big(\mathcal{E}_{\geq t}\big)$. Equations (25) and (26) then imply the

Result. For any $\alpha = 1, \dots, N$, and for all $\xi \in \mathcal{X}_t$,

$$\boxed{\|\pi_{t,\xi}\, Q_\alpha(t) - \pi_{t,\xi}\| < \mathcal{O}(\delta)\,, \text{ or } \|\pi_{t,\xi} Q_\alpha(t)\| < \mathcal{O}(\delta)\,.}$$

Thus, if the physical quantity $\mathcal{Q}$ is measured to have a value corresponding to an eigenstate of the projection $Q_\alpha(t)$ right after the event at time t has started to happen, then we know that the state of S right after time t is given by

$$[\omega_t(\pi_{t,\xi_\flat})]^{-1} \omega_t\big(\pi_{t,\xi_\flat}(\cdot)\pi_{t,\xi_\flat}\big)\,,$$

for some $\xi_\flat \in \mathcal{X}_t$ for which

$$\|\pi_{t,\xi_\flat} Q_\alpha(t) - \pi_{t,\xi_\flat}\| < \mathcal{O}(\delta)\,. \tag{27}$$

Furthermore:
The higher the resolution, $\mathfrak{R}$, of $\mathcal{Q}$ in recording the event $\{\pi_{t,\xi}, \xi \in \mathcal{X}_t\}$, the more precise the information provided by a measurement of $\mathcal{Q}$ is; if $N = M$ and δ is sufficiently small then every $\widehat{Q}_\alpha$ determines a unique point $\xi_\flat \in \mathcal{X}_t$ with the property that $\|Q_\alpha(t) - \pi_{t,\xi_\flat}\| < \mathcal{O}(\delta)$. (In the

limit where $\delta \to 0$ the information on the event that starts to happen at time t becomes totally accurate.)

Remarks:
(1) The main results of this paragraph are Eq. (23), the **Result** stated above, and Eq. (27).
(2) The concepts presented in paragraph V. and results cloosely related to the ones described above can be obtained without ever using the theory of conditional expectations. However, their use appears to render the presentation more elegant.

This completes our review of the "*ETH*-Approach to Quantum Mechanics" in a non-relativistic setting. Some idealized models fitting into this framework are discussed elsewhere, [11]. A relativistic form of this approach can be found in [16]. The material in [16] leads one to speculate that a logically coherent quantum theory of events, measurements and observations in *realistic* autonomous isolated (open) systems – not involving the intervention of "observers" – can only be developed in the realm of *local relativistic quantum theories* with *massless* particles, and for even-dimensional space-times.

4 Scattered remarks about indirect measurements, conclusions

I start this section with a few comments on "indirect measurements" (see [27, 19] for important early results) and then sketch some conclusions.

Let S be an isolated open system, as discussed in Sections 2 and 3. I assume that the system has been prepared in such a way that there is a specific physical quantity, $\mathcal{Q}$, characteristic of S that repeatedly records events featured by S (i.e., is "measured projectively"), at times $t_1 < t_2 < \dots t_n, n \in \mathbb{N}$, as discussed in paragraph V. of Section 4, Eqs. (23) and (27). Let us assume that the spectrum of $\mathcal{Q}$ is a finite set $\mathcal{Y}^{\mathcal{Q}} = \{0, 1, \dots, k\}$, so that $\mathcal{Q}$ is generated by a single operator, $\widehat{Y}$, with eigenvalues $0, 1, 2, \dots, k$. Let

$$\underline{\eta}^{(n)} := \{\eta_1, \eta_2, \dots, \eta_n\}, \quad \eta_j \in \mathcal{Y}^{\mathcal{Q}}\,,\ j = 1, 2, \dots, n\,, \tag{28}$$

be the sequence of values of $\widehat{Y}$ measured at times $t_1, t_2, \dots, t_n$, as explained in paragraph V. of Section 4. This means that the state of S right after time t_j is in an approximate eigenstate corresponding to the eigenvalue η_j of the operator $Y(t_j)$ representing $\widehat{Y}$ at time t_j, for $j = 1, 2, \dots, n$, as expressed in Eq. (23). The sequence $\underline{\eta}^{(n)}$ is called a "measurement protocol" of length n. As an example, $\widehat{Y}$ may describe the functioning of k different detectors that click when a certain type of particle (e.g., a photon or an atom), called a *"probe"*, belonging to S impacts them, with the following meaning of its eigenvalues:

$$\eta = 0 \leftrightarrow \text{ none of the detectors clicks }, \eta = \ell \leftrightarrow \text{ detector } \ell \text{ has clicked}, \quad \ell = 1, \dots, k.$$

Given a measurement protocol $\underline{\eta}^{(n)}$ of length n, we define the frequency (of occurrence) of the value $\eta \in \mathcal{Y}^{\mathcal{Q}}$ by

$$f_\eta\big(\underline{\eta}^{(n)}\big) := \frac{1}{n}\Big(\sum_{j=1}^{n} \delta_{\eta\,\eta_j}\Big). \tag{29}$$

Note that

$$f_\eta\big(\underline{\eta}^{(n)}\big) \geq 0, \quad \text{and} \quad \sum_{\eta=1}^{k} f_\eta\big(\underline{\eta}^{(n)}\big) = 1.$$

Of particular interest is the asymptotics of $f_\eta\big(\underline{\eta}^{(n)}\big)$, as $n \to \infty$. Let us temporarily suppose that, $\forall \eta = 1, \dots, k$, the limit of $f_\eta\big(\underline{\eta}^{(n)}\big)$, as $n \to \infty$, exists whenever a copy of S prepared in a fixed state is subjected to very many repeated measurements of $\widehat{Y}$, with

$$\lim_{n\to\infty} f_\eta\big(\underline{\eta}^{(n)}\big) \in \{p(\eta|\alpha)\}_{\alpha=1}^{N}, \tag{30}$$

for some $N < \infty$; (this is a *"Law of Large Numbers"*, see [20]). In (30),

$$p(\eta|\alpha) \geq 0, \;\text{ and } \sum_{\eta=1}^{k} p(\eta|\alpha) = 1, \tag{31}$$

for all $\alpha = 1, \dots, N$, for some $N < \infty$. Apparently, the probability measures $p(\cdot|\alpha), \alpha = 1, \dots, N$, describe all possible limiting values the frequencies $f_{(\cdot)}(\underline{\eta}^{(n)})$ may converge to. We propose to interpret the parameter α as follows: α characterizes a *time-independent* property of S, i.e., it is an eigenvalue of a self-adjoint operator, A, on $\mathcal{H}$ representing a physical quantity of S that commutes with the operators $Y(t_j), j = 1, 2, \dots$, and is a *conservation law*, meaning that A is time-independent (under the Heisenberg time evolution of operators on $\mathcal{H}$). Such an indirect measurement of A is called a *"non-demolition measurement"*. One expects that conservation laws are elements of

$$\mathcal{E}_\infty := \bigwedge_{t\in\mathbb{R}} \mathcal{E}_{\geq t},$$

where $\mathcal{E}_\infty$ is an algebra in the center of the algebra $\mathcal{E}$ defined in (3), ("asymptotic abelianness" in time). Under suitable hypotheses this expectation can actually be proven.

Thus, if the frequencies $f_\eta\big(\underline{\eta}^{(n)}\big)$ are seen to converge to the value $p(\eta|\alpha_*)$, as $n \to \infty$, $\eta \in \mathcal{Y}^{\mathcal{Q}}$, for some $\alpha_* \in \text{spec}(A)$, and if the measures $p(\cdot|\alpha)$ separate points in the spectrum, spec(A), of A, then we *know* that, asymptotically, as $t \to \infty$, the value of the conservation law A approaches α_*. (The fact that the measures $p(\cdot|\alpha)$ may depend on α in a non-trivial way, at all, is a consequence of *"entanglement"*; see [19, 18, 20].)

Evidently, one would like to prove (30) and to predict the probability of indirectly measuring a value α_* for A, assuming one knows the initial state of S. However, this can only be done if the events encoded by the values $\eta_1, \eta_2, \dots,$ of the physical quantity $\hat{Y}$, which is measured at times $t_1, t_2, \dots,$ are the *only* events happening in S. For a limited class of systems (see [18, 20]), one can prove that if this is the case then (30) holds, the state of S approaches an eigenstate of A corresponding to some eigenvalue $\alpha_* \in \text{spec}(A)$, as time $t \to \infty$, (*"purification"*), and the probability of measuring the value α_* is given by *Born's Rule* applied to the initial state of S and the operator A, see [20].

Usually, operators on $\mathcal{H}$ representing physical quantities of S are *not* time-independent. If the rate of change in time of a physical quantity, A, of S that one attempts to measure *indirectly*, as described above, is very *small*, as compared to the rate of repeated projective measurements of the physical quantity $\hat{Y}$ used to determine the value of A,[12] then it turns out that, to good accuracy, the dynamics of the state of the system S is described by a *Markov jump process* on the set of eigenspaces of the operator (A). The sample paths of this process describe **"quantum jumps"** of (the state of) S from one approximate eigenstates of A to another one. This picture has been given a precise meaning in [20, 22], in the framework of some simple models.

Concluding Remarks:

(1) The ETH-Approach to QM sketched in this paper is a "Quantum Mechanics without observers". It introduces a precise notion of "events" into the quantum formalism; and it furnishes quantum theory with a clear "ontology".

(2) The ETH-Approach establishes a precise formalism to describe the *stochastic time evolution of states* of isolated (open) systems featuring events. As I have tried to explain, while, for an *isolated* system, the Heisenberg-picture time evolution of operators, in particular of physical quantities characteristic of such a system, determined by the unitary propagator of the system is perfectly adequate, the time evolution of its *states* is described by a novel kind of *stochastic branching process* with a "non-commutative state space". This is described in some detail in paragraph IV. of Section 3. The analysis presented there shows that it is simply *not true* – in any naive sense – that the "Heisenberg picture" and the "Schrödinger picture" are equivalent.

[12] One speaks of a "weak measurement" of A

(3) It is explained in paragraph V. of Section 3 what a "physical quantity" characteristic of an isolated open system is, what it means to measure such a quantity "projectively", and how "projective measurements" of physical quantities can be used to record events. This also lays a basis for a precise *theory of indirect measurements.*

(4) It is important to note that, in the ETH-Approach to QM, the expected value of a *conservation law* represented by a self-adjoint operator A in the actual state of an isolated open system featuring events is **not** constant in time, (as it would be if states evolved according to the Schrödinger equation).

(5) A *"passive state"* of an isolated open system S prepared at some time t_0 is a state ω for which $\mathcal{Z}_{\omega_t}(\mathcal{E}_{\geq t}) = \{\mathbb{C}\,\mathbb{1}\}$, for all times $t > t_0$. We expect that it often happens that states of S approach "passive states" asymptotically, as $t \to \infty$, (with $\sigma(\mu_\omega) = 0$, see (17)). Thermal equilibrium states are "passive states".

(6) Clearly, the ETH-Approach to QM is so general that, for the time being, it is very hard to use it to carry out explicit calculations for *realistic* model systems and to show in which way its predictions *differ* – usually (hopefully) only ever so slightly – from those made on the basis of, for example, the Copenhagen Interpretation of QM, or *Bohmian Mechanics.* I emphasize, however, that *differences in the predictions of the ETH-Approach and other versions of QM – however small they may be – really exist!*

References

[1] R. B. Griffiths, *Consistent Histories and the Interpretation of Quantum Mechanics*, J. Stat. Phys. **36** (1984), 219-272;
M. Gell-Mann and J. B. Hartle, *Classical Equations for Quantum Systems*, Phys. Rev. D**47** (1993), 3345-3382

[2] D. Dürr and S. Teufel, *Bohmian Mechanics – The Physics and Mathematics of Quantum Theory*, Springer-Verlag, Berlin-Heidelberg-New York 2009

[3] J. Fröhlich and B. Schubnel, *Do We Understand Quantum Mechanics – Finally?*, arXiv:1203.3678, in: Proceedings of conference in memory of Erwin Schrödinger, Vienna, January 2011, publi. in 2012

[4] W.K. Abou Salem and J. Fröhlich, *Status of the Fundamental Laws of Thermodynamics*, J. Stat. Phys. **126** (2007), 1045–1068

[5] W. De Roeck and J. Fröhlich, *Diffusion of a Massive Quantum Particle Coupled to a Quasi-Free Thermal Medium*, Commun. Math. Phys. **303** (2011), 613-707

[6] J. Fröhlich, Zhou Gang and A. Soffer, *Friction in a Model of Hamiltonian Dynamics*, Commun. Math. Phys. **315** (2012), 401-444;
J. Fröhlich and Zhou Gang, *Emission of Cherenkov Radiation as a Mechanism for Hamiltonian Friction*, Adv. Math. **264** (2014), 183-235

[7] R. Bauerschmidt, W. De Roeck and J. Fröhlich, *Fluctuations in a Kinetic Transport Model for Quantum Friction*, arXiv:1403.5790, J. Physics A: Math. Theor. **47** (2014), 275 003

[8] J. Fröhlich and B. Schubnel, *Quantum Probability Theory and the Foundations of Quantum Mechanics*, arXiv:1310.1484, in: *The Message of Quantum Science – Attempts Towards a Synthesis*, Ph. Blanchard and J. Fröhlich (eds.), Springer-Verlag, Berlin-Heidelberg-New York 2015

[9] Ph. Blanchard, J. Fröhlich and B. Schubnel, *A 'Garden of Forking Paths' – the Quantum Mechanics of Histories of Events*, Nucl. Phys. B**912** (2016), 463-484

[10] B. Schubnel, *Mathematical Results on the Foundations of Quantum Mechanics*, PhD thesis 2014, available at https://doi.org/10.3929/ethz-a-010428944

[11] J. Fröhlich, *'ETH' in Quantum Mechanics*, Notes of Lectures on the Foundations of Quantum Mechanics, Les Diablerets, January 9 - January 14, 2017

[12] J. Faupin, J. Fröhlich and B. Schubnel, *On the Probabilistic Nature of Quantum Mechanics and the Notion of 'Closed' Systems*, Ann. Henri Poincaré **17** (2016), 689-731

[13] J. Fröhlich and B. Schubnel, *The Preparation of States in Quantum Mechanics*, J. Math. Phys. **57** (2016), 042 101

[14] R. Haag, *Fundamental Irreversibility and the Concept of Events*, Commun. Math. Phys. **132** (1990), 245-251;
R. Haag, *Events, Histories, Irreversibility*, in: *Quantum Control and Measurement*, Proc. ISQM, ARL Hitachi, H. Ezawa and Y. Murayama (eds.), North Holland, Amsterdam 1993;

Ph. Blanchard and A. Jadczyk, *Event-Enhanced Quantum Theory and Piecewise Deterministic Dynamics*, Annalen der Physik **4** (1995), 583-599

[15] D. Buchholz and J. E. Roberts, *New Light on Infrared Problems: Sectors, Statistics, Symmetries and Spectrum*, Commun. Math. Phys. **330** (2014), 935-972;
D. Buchholz, *Collision Theory for Massless Bosons*, Commun. Math. Phys. **52** (1977), 147-173

[16] J. Fröhlich, *Quantum Theory and Causality*, Talks at the University of Leipzig (2018), TU-Stuttgart (2019), IHES (2019) and at Vietri sul Mare (Italy) (2019); paper in preparation

[17] C. Guerlin, J. Bernu, S. Deléglise, C. Sayrin, S. Gleyzes, S. Kuhr, M. Brune, J. M. Raimond and S. Haroche, *Progressive Field-State Collapse and Quantum Non-Demolition Photon Counting*, Nature **448** (7156) (2007), 889-893;
S. Haroche, *Controlling Photons in a Box and Exploring the Quantum to Classical Boundary*, Nobel Lecture, December 8, 2012, in: *The Nobel Prizes*

[18] M. Bauer and D. Bernard, *Convergence of Repeated Quantum Non-Demolition Measurements and Wave-Function Collapse*, Phys. Rev. A**84** (2011), 044 103:1-4

[19] H. Maassen and B. Kümmerer, *Purification of Quantum Trajectories*, Springer-Verlag, Lecture Notes - Monograph Series **48** (2006), 252-261

[20] M. Ballesteros, M. Fraas, J. Fröhlich and B. Schubnel, *Indirect Retrieval of Information and the Emergence of Facts in Quantum Mechanics*, arXiv:1506.01213, J. Stat. Phys. **162** (2016), 924-958

[21] M. Ballesteros, N. Crawford, M. Fraas, J. Fröhlich and B. Schubnel, *Non-Demolition Measurements of Observables with General Spectra*, arXiv:1706.09584, Proceedings publ. in 2018

[22] M. Ballesteros, N. Crawford, M. Fraas, J. Fröhlich and B. Schubnel, *Perturbation Theory for Weak Measurements in Quantum Mechanics, I - Systems with Finite-Dimensional States Space*, arXiv1700.03149, Ann. Henri Poincaré **20** (2019), 299-335

[23] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics*, Cambridge University Press, Cambridge (UK) 1987.

See also:
J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement*, Princeton University Press, Princeton NJ, 1983;
K. Hepp, *Quantum Theory of Measurement and Macroscopic Observables*, Helv. Phys. Acta **45** (1972), 237-248;
H. Primas, *Asymptotically Disjoint Quantum States*, in: *Decoherence: Theoretical, experimental and Conceptual Problems*, pp 161-178, Ph. Blanchard, D. Giulini, E. Joos, C. Kiefer and I.-O. Dtamatescu (eds.), Springer-Verlag, Berlin 2000

[24] D. Frauchiger and R. Renner, *Quantum Theory Cannot Consistently Describe the Use of Itself*, Nature Communications **9** (2018), # 3711

[25] G. Lüders, *Über die Zustandsänderung durch den Messprozess*, Annalen der Physik (Leipzig) **443** (1950), 322-328;
J. Schwinger, *The Algebra of Microscopic Measurement*, Proc. Natl. Acad. Sci. (USA) **45** (1959), 1542-1553;
E. P. Wigner, *The Collected Works of Eugene Paul Wigner, Part A: The Scientific Papers*, B. R. Judd, G. W. Mackey (eds.), Springer-Verlag, New York 1993

[26] M. Takesaki, *Conditional Expectations in von Neumann Algebras*, J. Funct. Anal. **9** (1972), 306-321;
F. Combes, *Poids et Espérances Conditionnelles dans les Algèbres de von Neumann*, Bull. Soc. Math. France **99** (1971), 73-112

[27] K. Kraus, *States, Effects and Operations*, Springer-Verlag, Berlin-Heidelberg-New York 1983

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A "garden of forking paths" – The quantum mechanics of histories of events

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Dedicated to the memory of Raymond Stora

Abstract

This is a survey of a novel approach, called "*ETH approach*", to the quantum theory of events happening in isolated physical systems and to the effective time evolution of states of systems featuring events. In particular, we attempt to present a clear explanation of what is meant by an "event" in quantum mechanics and of the significance of this notion. We then outline a theory of direct (projective) and indirect observations or recordings of physical quantities and events. Some key ideas underlying our general theory are illustrated by studying a simple quantum-mechanical model of a mesoscopic system.

"I leave to several futures (not to all) my garden of forking paths"[1]

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[1] Jorge Luis Borges, "*El jardín de senderos que se bifurcan*," Editorial Sur, 1941.

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1. Introduction – walking out of the quantum maze

1.1. Introductory remarks

Recent years have seen enormous progress in setting up beautiful experiments that successfully test fundamental features of quantum mechanics. Furthermore, there have been substantial new developments in the areas of quantum information theory and its practical uses and of quantum computing. These advances have made renewed studies of the foundations of quantum theory commendable and, perhaps, even somewhat fashionable – after a long period during which such studies were facing suspicion.

Unfortunately, the success of recent efforts to clarify the message and interpretation of quantum mechanics and to formulate this theory in a logically coherent way is rather limited. Much confusion and disorientation still surround its foundations, even among professional physicists – so much so that many mathematicians do not want to think about it. There are many wrong or misleading prejudices. To mention one example, we tend to teach to our students that, in the Schrödinger picture, the quantum-mechanical time evolution of states of physical systems is described by the Schrödinger equation for a wave function (or the Liouville equation for a density matrix), and that the Schrödinger picture and the Heisenberg picture are equivalent. Well, when stated in this generality and in case we wish to describe the time evolution of systems featuring *events* (amenable to observation), *nothing could be farther from the truth*; see subsect. 2.4!

Given that quantum mechanics was discovered ninety years ago, the present rather low level of understanding of its deeper meaning may be seen to represent some kind of intellectual scandal. We would like to help, in modest ways, to alleviate some of the confusion blurring this most important theory.

Fairly shortly before his death, our unforgettable mentor and friend *Raymond Stora* developed a lively interest in questions concerning the foundations of quantum mechanics. We feel that it is fitting to dedicate a paper on this subject to his memory.

1.2. Some fundamental questions and problems

In our courses on quantum mechanics, we tend to describe physical systems, S, as pairs of a Hilbert space, $\mathcal{H}$, of pure state vectors, and a unitary propagator, $(U(t,s))_{t,s\in\mathbb{R}}$, describing the time evolution of states (from time s to time t). Unfortunately, these data encode hardly any interesting invariant data about S, besides spectral properties of the unitary operators $U(t,s)$, which would enable one to draw conclusions about physical properties of S. Moreover, they give the erroneous impression that quantum mechanics might be a deterministic theory, because the Schrödinger equation is a deterministic evolution equation. These observations give rise to the following

Fundamental questions and problems:

- What does one have to add to the data described above to arrive at a mathematical structure that – through interpretation – can be given physical meaning *without the intervention of "observers" or addition of ad-hoc postulates concerning "measurements" to the theory?*
- What is the origin of the *intrinsic randomness* in quantum theory, given the deterministic character of the Schrödinger equation? In which way does quantum randomness differ from classical randomness?

- How are *"states", "observables"* and *"events"* defined in quantum mechanics; what is the meaning of these notions? Do we understand the *time-evolution* of "states" and of "observables" of physical systems in quantum mechanics, and what does it have to do with solving the Schrödinger equation?
- What is meant by an *"isolated system"* in quantum mechanics, and why is this an important notion? (*Answer:* Because only for isolated systems we understand, in a general way, how to describe the *time evolution of "observables"*.) Is it understood, in quantum mechanics, how to prepare a system S in a prescribed initial state? Etc.

Answers to these important questions, except to the last one, are sketched in the following sections. (A fairly detailed discussion of the last question can be found in [1,2].)

1.3. Purpose of analysis

Besides addressing the questions raised in the last subsection, the main purpose of this paper is to sketch or review arguments in favor of some of the following basic claims.

- In quantum mechanics, the "state" of a system – as conventionally defined – does *not* describe "what is" or "will be"; it does not have an ontological status. Rather it is a mathematical device enabling us to make bets about the most likely events seen to happen in the future. (The "ontology" lies in *time-ordered sequences of events*, sometimes called *"histories"*, not in "states".)
- The success of a quantum theory of "events" (that can be detected through observations or measurements) hinges on our ability to update the state of a system in time in accordance with events observed in the past, i.e., on a description of the time evolution of states in the presence of events, which observers can, in principle, record with the purpose to optimize their predictions of future events.
- Our description of the time evolution of states of systems exhibiting events exploits a fundamental mechanism of *"loss of access to information"* (for short, "information loss") and of *entanglement with degrees of freedom carrying away "inaccessible (lost) information"*. This mechanism allows for the evolution of pure states into mixtures.
- There is no reason to expect that there are *"information- or unitarity paradoxes"* in quantum mechanics. In fact, the quantum-mechanical time evolution of states of physical systems exhibiting information loss and featuring events that can be recorded is *never unitary*; (see subsect. 2.4).
- Somewhat advanced mathematical concepts, such as functional analysis, in particular operator algebras (including type III_1 – von Neumann algebras), functional integration, stochastic processes, elements of statistics, etc. have been invented to be *used* in the study of Quantum Theory – they do not represent a superfluous luxury.

In the following sections, we sketch arguments in favor of some of these claims; (for a more detailed presentation we refer to [3–5]). In particular, we outline a novel theory of events, observations and measurements in quantum mechanics based on two basic concepts:

1. Fundamental "loss of access to information" and entanglement with degrees of freedom that are no longer observable, i.e., carry away lost information.

2. Specification of a list of physical quantities characterizing possible events that can, in principle, happen, (depending on the state the system has been prepared in) and be recorded directly.

2. Information loss and events in quantum mechanics

We start this section with a somewhat pedestrian definition of physical systems in quantum mechanics, (subsect. 2.1). Afterwards, we introduce the concept of "loss of access to information", (subsect. 2.2). This will guide us towards a novel quantum-mechanical theory of events amenable to observation, (subsect. 2.3). Finally, in subsect. 2.4, we describe the time evolution of states in physical systems featuring events that can be recorded.

It should be emphasized that what we are trying to understand in this paper is *Quantum Mechanics* – pure and simple; we are *not* trying to extend or generalize this theory.

2.1. Definition of physical systems

Definition N-R: In non-relativistic quantum mechanics, an isolated system, S, is characterized by the following data, (items 1 through 3):

- 1. A pair,

$$(\mathcal{H}, \{U(t,s)\}_{t,s\in\mathbb{R}}), \tag{1}$$

of a Hilbert space $\mathcal{H}$ of pure state vectors and a unitary propagator U with the usual properties: $U(t,s)$ is a unitary operator on $\mathcal{H}$, for all pairs of times (t,s), and

$$U(t,t) = \mathbf{1}, \quad U(t,s)\cdot U(s,r) = U(t,r), \quad \forall t,s,r \text{ in } \mathbb{R}.$$

- 2. A list,

$$\mathcal{O}_S = \{\hat{X}_i\}_{i\in I_S}, \tag{2}$$

of bounded self-adjoint operators $\hat{X}_i$ representing physical quantities of S that could be recorded directly. We assume that $\mathcal{O}_S$ contains an identity element, $\mathbf{1}$, and that if f is an arbitrary real-valued, bounded, continuous function on $\mathbb{R}$ and $\hat{X}$ is an arbitrary operator in $\mathcal{O}_S$ then $f(\hat{X})$ also belongs to $\mathcal{O}_S$.

Remarks. (i) In this paper, *"physical quantities of a system S"* are always represented by self-adjoint (bounded) linear operators.[2] If during a certain interval, $\mathcal{I}$, of time it is possible to unambiguously assign an objective value to a physical quantity of S represented by an operator $\hat{X} \in \mathcal{O}_S$ we say that, during the time interval $\mathcal{I}$, an "event" is happening; namely the event that $\hat{X}$ has an objective value that could, in principle, be observed directly. What this means mathematically will be explained below.

(ii) Note that, in general, $\mathcal{O}_S$ is not an algebra; it is not even a linear space! Typically, $\mathcal{O}_S$ may be generated by just a few (possibly only finitely many) operators. Let $\mathcal{A}_S$ denote the algebra generated by $\mathcal{O}_S$ (closed in a C^*-norm). In simple examples of physical systems (see Eq. (8) and Sect. 3 for a concrete model system), the operators in $\mathcal{O}_S$ all commute among themselves. We

[2] This is actually a feature common to all physical theories known to us – quantum and classical.

can then identify $\mathcal{A}_S$ with $\mathcal{O}_S$; and it is a well known theorem due to I.M. Gel'fand that, under this assumption,

$$\mathcal{O}_S \simeq \{\text{continuous functions on a compact Hausdorff space } \mathcal{X}_S\} =: C(\mathcal{X}_S) \tag{3}$$

The topological space $\mathcal{X}_S$ is called the *spectrum* of $\mathcal{O}_S$.

Given an algebra $\mathcal{A}$ of operators, a maximal abelian subalgebra of $\mathcal{A}$ is a commutative subalgebra, $\mathcal{M} \subseteq \mathcal{A}$, with the property that the subalgebra of operators in $\mathcal{A}$ that commute with *all* operators in $\mathcal{M}$ is equal to $\mathcal{M}$. In order to keep this paper reasonably short and easy to read, we introduce the following

Simplifying Assumption:
Every maximal abelian algebra, $\mathcal{M}$, contained in $\mathcal{A}_S$ is generated by a *finite* family of commuting orthogonal projections, $\{\Pi_{\xi_1}, \dots, \Pi_{\xi_N}\}$. Then $\mathcal{M} = C(\mathcal{X})$, where $\mathcal{X} = \{\xi_1, \dots, \xi_N\}$ is the spectrum of $\mathcal{M}$. It is assumed that there is *at least one* maximal abelian subalgebra, $\mathcal{M}_S$, in $\mathcal{A}_S$ with the property that *all* self-adjoint elements of $\mathcal{M}_S$ belong to $\mathcal{O}_S$. However, there may be several such maximal abelian algebras, $\{\mathcal{M}_S^{(i)}\}_{i \in I_S}$, not commuting with each other.[3] The orthogonal projections contained in an algebra $\mathcal{M}_S^{(i)}$, $i \in I_S$, are called *"possible events"*; any real linear combination of the orthogonal projections generating $\mathcal{M}_S^{(i)}$ is then a physical quantity, $\hat{X}_i$, belonging to $\mathcal{O}_S$.[4]

(iii) The occurrence of events in a system S does *not* depend on the presence of "observers"; i.e., our formulation of quantum mechanics does *not* invoke "observers" who decide to measure some quantity (and may then disagree on exactly which quantity they would like to measure and when). But, of course, any useful physical theory must talk about objects and phenomena that intelligent beings can observe if they choose to do so, and it should help them to cope with the challenges of a changing world by enabling them to agree among themselves whether some events have happened and to make useful and plausible predictions about future events. – That much about "physical quantities" ("observables"), "(possible) events", and philosophy!

- 3. At every time t, there exists a representation

  $$\mathcal{A}_S \ni \hat{X} \mapsto X(t)$$

  of the algebra $\mathcal{A}_S$ by operators, $X(t)$, acting on the Hilbert space $\mathcal{H}$ with the property that $\hat{X}^*$ is represented by the operator $X(t)^*$; in particular, if $\hat{X}$ is self-adjoint then $X(t)$ is a self-adjoint operator on $\mathcal{H}$. The operators $X(t)$ and $X(s)$ are unitarily conjugated to each other by the propagator of S, i.e.,

  $$X(t) = U(s,t)X(s)U(t,s), \qquad \text{for times } s, t \in \mathbb{R}, \hat{X} \in \mathcal{A}_S.$$

By $\mathcal{A}_S(t)$ we denote the algebra $\{X(t) | \hat{X} \in \mathcal{A}_S\} \subseteq B(\mathcal{H})$, where, as usual, $B(\mathcal{H})$ denotes the *algebra of all bounded operators on the Hilbert space* $\mathcal{H}$.

[3] Example: $\mathcal{M}_S^{(1)}$ = {all bounded continuous functions of the position of a particle, P}, and $\mathcal{M}_S^{(2)}$ = {all bounded $continuousfunctionsofthemomentumofP$}.

[4] A more general analysis of the role of maximal abelian subalgebras of $\mathcal{A}_S$ in our formulation of quantum theory, *not* assuming that they are generated by finitely many projections, will be presented elsewhere.

Possible events observable at times $\geq t$ generate an algebra $\mathcal{E}_{\geq t}$:

$$\mathcal{E}_{\geq t} := \{\text{linear combinations of } \prod_i X_i(t_i) | \hat{X}_i \in \mathcal{O}_S, t_i \geq t\}^-, \tag{4}$$

with $\mathcal{E} := \overline{\mathcal{E}_{>-\infty}}$. For concreteness, we assume that the closure is taken in the weak operator topology on $B(\mathcal{H})$.[5]

With a view towards an extension of our formalism to *relativistic quantum (field) theory*, we briefly outline a somewhat more general notion of physical systems.

Definition R: In quantum theory, a general isolated physical system S is characterized by the following data:

- 1. A list,

  $$\mathcal{O}_S = \{\hat{X}_i\}_{i \in I_S},$$

  of bounded self-adjoint operators $\hat{X}_i$ representing physical quantities of S. As before, we let $\mathcal{A}_S$ denote the C^*-algebra generated by $\mathcal{O}_S$, and we continue to impose the simplifying assumption formulated in Remark (ii) after item 2, above, etc.
- 2. A net $(\mathcal{E}_\mathcal{I})_{\mathcal{I} \subset \mathbb{R}}$ of (von Neumann) algebras, $\mathcal{E}_\mathcal{I}$, indexed by time intervals $\mathcal{I}$, with the interpretation that $\mathcal{E}_\mathcal{I}$ is generated by possible events localized in the time interval $\mathcal{I}$. This net is assumed to have the property that if $\mathcal{I} \subset \mathcal{I}'$ then $\mathcal{E}_\mathcal{I} \subset \mathcal{E}_{\mathcal{I}'}$. We define

  $$\mathcal{E}_{\geq t} := \overline{\bigvee_{\mathcal{I} \subseteq [t,\infty)} \mathcal{E}_\mathcal{I}}, \qquad \mathcal{E} := \overline{\mathcal{E}_{>-\infty}}. \tag{5}$$

  In (5), the closure is taken in the weak operator topology on $B(\mathcal{H})$.
- 3. For every time $t \in \mathbb{R}$ there is a *representation

  $$\mathcal{A}_S \ni \hat{X} \mapsto X(t) \in \mathcal{E}_{\geq t} \tag{6}$$

  of the algebra $\mathcal{A}_S$ by operators in $\mathcal{E}_{\geq t}$. The representations of $\mathcal{A}_S$ corresponding to different times are unitarily equivalent.

  It is assumed, furthermore, that, for every $\hat{X} \in \mathcal{O}_S$ and every $\varepsilon > 0$, there exist a finite duration $\tau = \tau(\hat{X}, \varepsilon) < \infty$ and an operator $X_\varepsilon(t) \in \mathcal{E}_{[t,t+\tau]}$ such that

  $$\|X(t) - X_\varepsilon(t)\| < \varepsilon.$$

Remark: In Definition R, "time" refers to the proper time of an observer, and the net $\{\mathcal{E}_\mathcal{I}\}_{\mathcal{I} \subset \mathbb{R}}$ depends on the worldline of that observer; see Fig. 1, below. This does *not* mean that the theory becomes "observer-dependent". But it does mean that one has to find out how one and the same sequence of events is seen by different observers, i.e., how to map the data concerning a sequence of events recorded by one observer to the data recorded by another observer. Luckily, for the purposes of the analysis presented here we do not need to address this problem, which lies somewhat beyond the scope of this paper.

The analysis presented in the following sections is based on *Definition R*; (but no attempt is made to present an analysis that takes into account the laws of relativity theory).

[5] A sequence, or net, $(A_i)_{i \in I}$ of bounded operators on $\mathcal{H}$ is said to converge weakly iff $(\langle \psi, A_i \varphi \rangle)_{i \in I}$ converges, for arbitrary vectors ψ and φ in $\mathcal{H}$. The algebras $\mathcal{E}_{\geq t}$ and $\mathcal{E}$ are von Neumann algebras, because they are closed under weak convergence. In the following, it is convenient to work with von Neumann algebras. But the reader is kindly asked not to worry about this technicality.

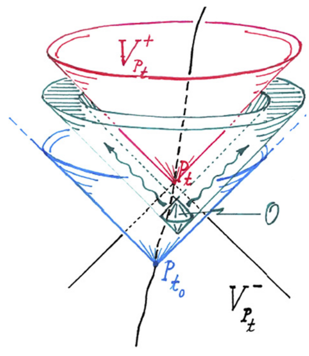

Fig. 1. An illustration of Property (∗).

2.2. Information loss

The idea of "information loss" or, more precisely, "loss of access to information" is encapsulated in the following general assumption concerning the algebras $\mathcal{E}_{\geq t}$, $t \in \mathbb{R}$:

$$B(\mathcal{H}) \supseteq \mathcal{E} \supseteq \mathcal{E}_{\geq t} \underset{\neq}{\supset} \mathcal{E}_{\geq s} \supseteq \mathcal{A}_S(s), \qquad s > t. \tag{7}$$

Information Loss!

A precise formulation of "Information Loss" is to assume that if $s > t$ then

$$\mathcal{E}'_{\geq s} \cap \mathcal{E}_{\geq t} \neq \emptyset, \tag{$*$}$$

where, for an algebra $\mathcal{A}$ of bounded operators acting on $\mathcal{H}$, $\mathcal{A}'$ is the algebra of all bounded operators on $\mathcal{H}$ commuting with all operators in $\mathcal{A}$. In fact, one expects that $\mathcal{E}'_{\geq s} \cap \mathcal{E}_{\geq t}$ is typically an infinite-dimensional algebra (at least for some $s > t$), an expectation extracted from the analysis of examples; see [6,7].

Property (∗) is far from obvious and appears to only hold in theories of systems with infinitely many degrees of freedom including *massless ones*, such as photons or phonons. D. Buchholz and the late J.E. Roberts have presented a deep analysis of Property (∗) in quantum electrodynamics, formulated in the framework of algebraic quantum field theory; see [6]. In their work, the analogue of the algebra $\mathcal{E}_{\geq t}$ is played by an algebra of bounded functions of the electromagnetic field smeared out with test functions with support in the forward light cone $\overline{V}^+_{P_t}$ erected over a space-time point P_t at proper time t that belongs to the worldline of an observer. They show that Property (∗) follows from Huyghens' Principle for the electromagnetic field and the existence of asymptotic electromagnetic field operators; see Fig. 1.

Fig. 1 indicates that $\mathcal{E}_{\geq t_0}$ properly contains $\mathcal{E}_{\geq t}$, for $t > t_0$, and that, asymptotically, flashes of light emitted from region $\mathcal{O}$ belong to $\mathcal{E}'_{\geq t} \cap \mathcal{E}_{\geq t_0}$.

Information Loss, in the sense of Eq. (7) (with Property (∗) valid for *some* $s > t$), holds in many models of *non-autonomous* systems describing a small system (e.g., an n-level atom) alternatingly coupled to various mutually independent dispersive media (e.g., the quantized electromagnetic field, or the phonons of a dynamical crystal lattice) during finite intervals of time; see [7]. Here we briefly sketch the example of a mesoscopic system consisting of a T-shaped conducting wire ending in three reservoirs denoted by D_L, D_R and "e^- gun"; see Fig. 2.

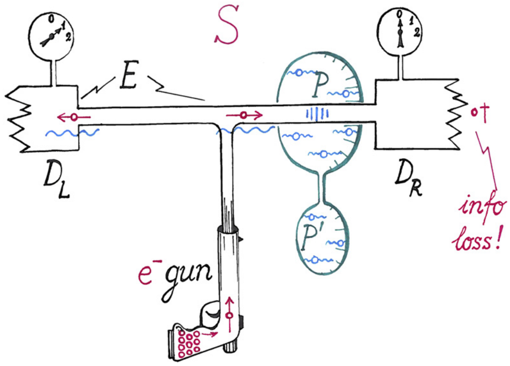

Fig. 2. A mesoscopic system.

The reservoir "e^- gun" has a higher chemical potential than the reservoirs D_L and D_R. Hence "e^- gun" emits electrons at a certain rate that move through the T-shaped wire until they dive into one of the reservoirs D_L or D_R where they disappear for ever. Before they disappear they trigger detectors that emit a signal (flash of light or sound wave) whenever an electron has arrived at D_L or D_R, respectively. In this example, the system S is the composition of the equipment E with a quantum dot $P \vee P'$ in a semi-conductor matrix. The electric charge localized inside the component P of the quantum dot, which can fluctuate by electron exchange between the components P and P', determines the a-priori probability by which an electron traveling through the T-shaped wire will dive into D_R. The equipment E consists of the three reservoirs, "e^- gun", D_L and D_R, the T-shaped wire, and the detectors at the entrance gates to D_L and D_R. The *only* physical quantity of S that can be observed directly is the flash of light or sound emitted by the detector on the left or the right whenever an electron dives into D_L or D_R, respectively. Mathematically, this quantity can be represented by the operator

$$\hat{X} = \mathbf{1}_{P\vee P'} \otimes \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{8}$$

which has the (infinitely degenerate) eigenvalues $\xi = \pm 1$, with

$$\xi = +1 \leftrightarrow D_L \text{ clicks}, \qquad \xi = -1 \leftrightarrow D_R \text{ clicks}.$$

The family $\mathcal{O}_S$ of operators consists of all bounded functions of the operator $\hat{X}$; its spectrum, $\mathcal{X}_S$, consists of two points, $\{-1, +1\}$. Access to the "information" represented by an electron that travels through the T-shaped wire is *lost*, as soon as that electron has dived into one of the reservoirs D_L or D_R. (To make this precise one must assume that the detectors have infinitely many degrees of freedom.)

This example is discussed in much detail in [5]. It illustrates how properties of a physical system S – in the example, the charge of the dot P – can be determined *indirectly* through a long sequence of repeated observations of physical quantities represented by operators in $\mathcal{O}_S$. Results from [5] concerning this example are summarized in section 3. (Our efforts have been stimulated by the experiments described in [8]; see also [9].)

2.3. Direct detection of events – projective recordings of physical quantities

Let $\{\mathcal{M}_S^{(i)}\}_{i\in I_S}$ denote those maximal abelian subalgebras of $\mathcal{A}_S$ that belong to $\mathcal{O}_S$. In this subsection, we clarify what it means that a physical quantity of a system S represented by an operator $\hat{X} \in \mathcal{M}_S^{(i)}$, for some $i \in I_S$, is recorded or measured directly (or "projectively") around some time t, i.e., that $\hat{X}$ has an objective value at or around time t. We will explain the roles played by "information loss", in the sense of Eq. (7), and of entanglement of observable degrees of freedom of S with inaccessible ("lost") degrees of freedom.

Let $\xi_1 < \cdots < \xi_N$ denote the eigenvalues of the operator $\hat{X}$, and let $\Pi_{\xi_1}, \ldots, \Pi_{\xi_N}$ be the corresponding spectral projections, with $\Pi_{\xi_j} \in \mathcal{M}_S^{(i)}, \forall j = 1, \ldots, N$. These projections have the interpretation of "possible events"; (Π_{ξ_i} corresponds to the possible event that the physical quantity represented by the operator $\hat{X}$ is observed, at some time t, to have the value ξ_j). For the mesoscopic system considered in subsection 2.2 (see Fig. 2), $\mathcal{O}_S \equiv \mathcal{M}_S$ is generated by a single operator, $\hat{X}$, with only two eigenvalues $\xi = \pm 1$.

Let $X(t) \in \mathcal{E}_{\geq t}$ be the operator on $\mathcal{H}$ representing $\hat{X}$. Then

$$X(t) = \sum_{j=1}^{N} \xi_j \Pi_{\xi_j}(t), \tag{9}$$

where $\Pi_{\xi_j}(t)$ is the spectral projection of $X(t)$ corresponding to the eigenvalue ξ_j; (the eigenvalue ξ_j is independent of time t, while the projections $\Pi_{\xi_j}(t)$ representing the projection $\Pi_{\xi_j} \in \mathcal{M}_S^{(i)}$ depend on t, but are all unitarily conjugate to one another, for every $j = 1, \ldots, N$).

It is compatible with the "Copenhagen interpretation" of quantum mechanics (whatever this interpretation may be, in more precise terms) to say that if the physical quantity corresponding to the operator $\hat{X} \in \mathcal{M}_S^{(i)}$ has an objective value around some time t, then the *state* of S,

$$\rho(\cdot) = Tr_{\mathcal{H}}(P \cdot), \text{ where } P \text{ is a density matrix on } B(\mathcal{H}), \tag{10}$$

when restricted to the algebra $\mathcal{E}_{\geq t}$, is indistinguishable from an *incoherent superposition* of eigenstates of the operator $X(t)$, in the following precise sense: Let

$$\rho_t := \rho|_{\mathcal{E}_{\geq t}},$$

then

$$\rho_t(A) = \sum_{j=1}^{N} \rho_t(\Pi_{\xi_j}(t) A \Pi_{\xi_j}(t)) + O(\delta \|A\|), \qquad \forall A \in \mathcal{E}_{\geq t}, \tag{11}$$

for some constant δ, with

$$\delta \ll \min_{1 \leq i < j \leq N} |\rho_t\big(\Pi_{\xi_i}(t) - \Pi_{\xi_j}(t)\big)|.$$

Eq. (7) and entanglement with inaccessible degrees of freedom imply that the state ρ_t is, in general, a *mixed* state on $\mathcal{E}_{\geq t}$, *even* if the state ρ may be a *pure* state on $B(\mathcal{H})$, so that Eq. (11) is by no means inconsistent.

Given a state φ on a von Neumann algebra $\mathcal{M}$, we define the *centralizer* (or stabilizer), $\mathcal{C}_\varphi$, of φ to be the subalgebra of $\mathcal{M}$ defined by

$$\mathcal{C}_\varphi := \{A \in \mathcal{M} | \mathrm{ad}_A(\varphi) = 0\}, \tag{12}$$

where

$$\mathrm{ad}_A(\varphi)(B) := \varphi([A, B]), \qquad \text{for arbitrary } B \in \mathcal{M};$$

see the Appendix for further details. For $\mathcal{M} = \mathcal{E}_{\geq t}$ and $\varphi = \rho_t$, the centralizer is henceforth denoted by $\mathcal{C}_{\rho_t}$.

Let us assume that either the algebra $\mathcal{M}$ is isomorphic to a direct sum $\bigoplus_i B(\mathcal{H}_i)$, where $\mathcal{H}_i$, $i = 1, 2, \ldots$, are Hilbert spaces, (i.e., that $\mathcal{M}$ is of type I), or that φ is a separating state on $\mathcal{M}$ (meaning that $\varphi(A^*A) = 0$, for some $A \in \mathcal{M}$, implies that $A = 0$).[6] Then there exists a linear map $E_\varphi : \mathcal{M} \to \mathcal{C}_\varphi$, called a *conditional expectation* from $\mathcal{M}$ to $\mathcal{C}_\varphi$, with the following properties:

$$E_\varphi(XAY) = XE_\varphi(A)Y, \quad \forall X, Y \in \mathcal{C}_\varphi, \forall A \in \mathcal{M},$$
$$0 \le E_\varphi(A)^* E_\varphi(A) \le E_\varphi(A^*A), \quad \forall A \in \mathcal{M}.$$

Let $\mathcal{Z}_\varphi$ denote the *center* of $\mathcal{C}_\varphi$, (i.e., the algebra of all operators in $\mathcal{C}_\varphi$ commuting with all operators in $\mathcal{C}_\varphi$). Under the same assumptions, there also exists a conditional expectation e_φ from $\mathcal{M}$ to $\mathcal{Z}_\varphi$ with the same properties as those of E_φ. (The general theory of conditional expectations in von Neumann algebras is developed in [10,11,13]; applications to the centralizer of a von Neumann algebra can be found in [14,15] and in references quoted therein.) The conditional expectations from $\mathcal{E}_{\ge t}$ to $\mathcal{C}_{\rho_t}$ and from $\mathcal{E}_{\ge t}$ to $\mathcal{Z}_{\rho_t}$, the center of $\mathcal{C}_{\rho_t}$, are denoted by E_{ρ_t} and e_{ρ_t}, respectively.

Let $X(t)$, ξ_j and $\Pi_{\xi_j}(t)$, $j = 1, \ldots, N$, be as in Eq. (9). It is not hard to show that

$$\text{Eq. (11)} \Leftarrow \|E_{\rho_t}(\Pi_{\xi_j}(t)) - \Pi_{\xi_j}(t)\| \le \delta', \quad \forall j = 1, \ldots, N, \tag{13}$$

for some $\delta' = O(\delta/N)$. This and the next claim are proven in the Appendix.

Obviously, Eq. (11) also holds if

$$\|e_{\rho_t}(\Pi_{\xi_j}(t)) - \Pi_{\xi_j}(t)\| \le \delta', \quad \forall j = 1, \ldots, N. \tag{14}$$

We are now prepared to formulate the

Fundamental axiom of events in quantum mechanics:

Let $\mathcal{P} := \{\Pi_{\xi_1}, \ldots, \Pi_{\xi_N}\}$ be a partition of unity in $\mathcal{A}_S$, (i.e., $\sum_{j=1}^N \Pi_{\xi_j} = \mathbf{1}|_{\mathcal{A}_S}$) consisting of commuting orthogonal projections that are contained in some maximal abelian subalgebra $\mathcal{M}_S^{(i)} \subseteq \mathcal{O}_S, i \in I_S$. These projections have the physical interpretation of *"possible events"*, and any real linear combination of them is an operator, $\hat{X} \in \mathcal{O}_S$, representing a *physical quantity* of S. Given a state ρ (on the algebra $\mathcal{E}$) which the system S has been prepared in, we propose to define what it means that *one* out of these N possible events actually happens (or materializes) around some later time t.

We fix a "threshold, Δ_t, for detection (of an event) at time t" satisfying

$$0 < \Delta_t \ll \min_{i \ne j = 1, \ldots, N} |\rho_t(\Pi_{\xi_i}(t) - \Pi_{\xi_j}(t))|, \tag{15}$$

where $\rho_t = \rho|_{\mathcal{E}_{\ge t}}$, and $\Pi_{\xi_j}(t) \in \mathcal{A}_S(t)$ is the orthogonal projection on the Hilbert space $\mathcal{H}$ representing the projection $\Pi_{\xi_j} \in \mathcal{P}$. Let $\mathcal{P}(t) := \{\Pi_{\xi_1}(t), \ldots, \Pi_{\xi_N}(t)\}$.

The fundamental axiom has two parts:

- 1. *Occurrence of Events in Quantum Mechanics:*
  One of the (possible) events $\Pi_{\xi_1}, \ldots, \Pi_{\xi_N}$ happens (materializes) around time t – put differently, the physical quantity $\hat{X} = \sum_j \xi_j \Pi_{\xi_j}$ has an *objective value* around time t – iff

[6] If the state φ is separating on the von Neumann algebra $\mathcal{M}$ then $\mathcal{C}_\varphi$ is seen to be the subalgebra of operators in $\mathcal{M}$ invariant under the modular automorphism group, $(\sigma_t^\varphi)_{t \in \mathbb{R}}$, corresponding to $(\mathcal{M}, \varphi)$ [12]; see, e.g., [11,3] and references given there.

$$\mathrm{dist}(\mathcal{P}(t), \mathcal{Z}_{\rho_t}) := \max_{j=1,\dots,N} \| e_{\rho_t}(\Pi_{\xi_j}(t)) - \Pi_{\xi_j}(t) \| \leq \Delta_t / N. \tag{16}$$

Remarks. (i) *Time of occurrence of events:* Obviously, Eq. (16) implies Eq. (11). The earliest time when a possible event in $\mathcal{P}$ can materialize is the smallest time $t = t_{\min}$ at which inequality (16) holds, after the preparation of S in state ρ. Let $\mathcal{I}_{t_{\min}}$ be the largest interval of time containing $t_{\min}$ such that inequality (16) holds for all $t \in \mathcal{I}_{t_{\min}}$. Then one of the possible events $\{\Pi_{\xi_j}\}_{j=1}^N$ happens in $\mathcal{I}_{t_{\min}}$. Most likely it happens around the time, t_*, minimizing the function $\mathrm{dist}(\mathcal{P}(t), \mathcal{Z}_{\rho_t})$ defined in Eq. (16).

(ii) *Duration of events:* Let $\tau = \tau(\mathcal{P}(t_*))$ be such that there are self-adjoint operators

$$\{\Pi_j(t_*, \tau)\}_{j=1}^N \subset \mathcal{E}_{[t_*, t_*+\tau]}, \qquad \text{with} \quad \| \Pi_{\xi_j}(t_*) - \Pi_j(t_*, \tau) \| \leq \Delta_{t_*}/N,$$

see item 3 of *Definition R*, subsect. 2.1. Then the *duration of the event* happening around time t_* is given by τ.

(iii) *A simple special case:* If the algebra $\mathcal{E}_{\geq t}$ is of type I (which, alas, it usually won't be!) then the state ρ_t can be represented by a density matrix, $P_t \in \mathcal{E}_{\geq t}$. Let

$$P_t = \sum_{j=1}^N p_j(t) \pi_j(t),$$

be the spectral decomposition of P_t, where the operators $\pi_j(t)$ are the spectral projections of P_t, $p_j(t) > 0, \forall j$, and

$$\sum_{j=1}^N p_j(t)\, \dim\, (\pi_j(t)) = 1.$$

Then *one* of the possible events $\Pi_{\xi_1}, \dots, \Pi_{\xi_N}$ happens around time t iff

$$\max_{j=1,\dots,N} \| \Pi_{\xi_j}(t) - \pi_j(t) \| \leq \Delta_t / N, \quad \text{with} \quad \Delta_t < \min_{j=2,\dots,N} \big(p_j(t) - p_{j-1}(t) \big).$$

- 2. *Randomness in Quantum Mechanics:*
  Under the condition that (16) holds at some time $t = t_* \in \mathcal{I}_{t_{\min}}$, the probability that the possible event $\Pi_\xi \equiv \Pi_{\xi_j} \in \mathcal{P}$ actually materializes at time t_* is given by

$$p_\xi(t_*) = \rho(\Pi_\xi(t_*)) \tag{17}$$

Born's Rule

If the event corresponding to the projection $\Pi_\xi \in \mathcal{P}$ is detected to have happened at time t_* then the state

$$\rho_{\xi, t_*}(\cdot) := p_\xi(t_*)^{-1} \rho_{t_*}\big(\Pi_\xi(t_*) \cdot \Pi_\xi(t_*)\big) \tag{18}$$

must be used for improved predictions of future events at times $> t_*$; i.e., the state of S on the algebra, $\mathcal{E}_{\geq t_*}$, of possible events after time t_*, *conditioned on the event corresponding to* Π_ξ *to have materialized at time* t_*, is given by ρ_{ξ, t_*}.

"Projection-, or Collapse Postulate"

Remarks, ctd.: (iv) Apparently, if it is known that an isolated system S was prepared in a state ρ before the earliest event has happened, then the quantum theory of *S predicts* at or around what time t_* the first event will occur, for what duration, τ, the event will last, and to *which family, $\mathcal{P}$, of possible events* that event belongs to. (We recall that $\mathcal{P}$ is contained in a maximal abelian subalgebra $\mathcal{M}_S^{(i)} \subseteq \mathcal{O}_S$ of $\mathcal{A}_S$.) But *which event* from the family $\mathcal{P}$ materializes at time t_* *cannot be predicted with certainty* – Quantum Mechanics only enables us to calculate the "frequency" or probability by which a specific element $\Pi_\xi \in \mathcal{P}$ corresponds to the event materializing around time t_*, and this probability is given by *Born's Rule*. In colloquial language, one may say that if one knows the state in which an isolated physical system S was prepared before the first event occurs then one can predict (using quantum mechanics) "which pointer (of an instrument) will start to turn first, at approximately what time it will start to turn, and for how long it will turn before it will come to rest; but its final position cannot be predicted."

(v) Note that many or most quantum-mechanical models of isolated systems that we discuss in our courses and books, such as models of systems of finitely many oscillators or of atoms treated according to Schrödinger's wave mechanics and *not* coupled to the quantized radiation field, *do not describe any events* (in the sense this notion has been given above)! The reason is that they give rise to algebras $\mathcal{E}_{\geq t}$ that are *independent* of t; i.e., that they do not exhibit any "loss of access to information", in the sense of Eq. (7). Before one incorporates equipment (with infinitely many degrees of freedom), such as detectors, etc., which the degrees of freedom of interest (e.g., the ones describing an atom) interact with, into the quantum-mechanical description it is *impossible* to formulate a logically coherent theory of events and observations.

Furthermore, one must expect that most systems have states, called "passive states", with the property that there won't be any events happening *even if* there is "loss of access to information", in the sense of Eq. (7). The reason is that the centers $\mathcal{Z}_{\rho_t}$ of the centralizers $\mathcal{C}_{\rho_t}$ of the algebras $\mathcal{E}_{\geq t}$ may turn out to be trivial, for all times t, (or be independent of t), for *certain* states ρ (called "passive"). One may even expect that, generically, a state is passive, and that equilibrium states at positive temperature are passive states.

(vi) It is conceivable that, in a more elaborate formulation of quantum mechanics, there is no need to specify the list $\mathcal{O}_S$ of physical quantities of an isolated physical system S that can, in principle, be detected directly. Rather, one can imagine that the algebras

$$\{\mathcal{Z}_{\rho_t} | t \in \mathbb{R}, \rho \text{ an arbitrary state on } \mathcal{E} \text{ of physical interest}\}$$

will *determine* $\mathcal{O}_S$.

2.4. The effective time evolution of states of systems featuring events

Equations (17) and (18) clarify the nature of the time evolution of states of systems featuring events. It is illustrated in the following Fig. 3, where:

- E stands for "event" (meaning that an event corresponding to a projection Π_ξ from some family $\mathcal{P}$ belonging to a maximal abelian subalgebra $\mathcal{M}_S^{(i)} \in \mathcal{O}_S$, $i \in I_S$, materializes)
- T stands for "tree" (of states of S corresponding to possible events, according to Eq. (18)); and
- H stands for "history" (of observed events)

We thus speak of the *"ETH approach"* to the interpretation of quantum mechanics (describing the effective quantum-mechanical time evolution of states of systems that feature events).

Let us summarize some basic elements of the "ETH approach".

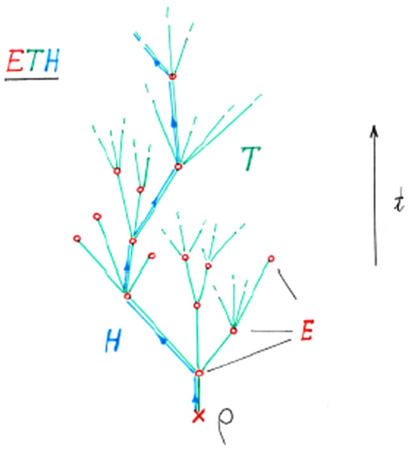

Fig. 3. "ETH approach" to quantum mechanics. (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

1. *"Observables at infinity":* Under rather general hypotheses, one can show that the algebra

$$\mathcal{E}_\infty := \bigcap_{t \in \mathbb{R}} \mathcal{E}_{\geq t}$$

is *abelian*, and that it is in the center of all the algebras $\mathcal{E}_{\geq t}, t \in \mathbb{R}$. Hence $\mathcal{E}_\infty$ is contained in the centers $\mathcal{Z}_{\rho_t}$ of the centralizers of all states $\rho_t = \rho|_{\mathcal{E}_{\geq t}}$, for an arbitrary state ρ on $\mathcal{E}$ and all times t. Thus, the states ρ_t can be decomposed over the spectrum, $\mathcal{X}_\infty$, of the algebra $\mathcal{E}_\infty$. Points in $\mathcal{X}_\infty$ are called *"facts"*, because they correspond to objective values of *time-independent* physical properties represented by operators in $\mathcal{E}_\infty$; see also [4].
2. In [16] and [17] the notion of "consistent histories" has been introduced and discussed, which, in conjunction with some understanding of the phenomenon of "decoherence" (see Sect. 4), is supposed to lead to a logically coherent interpretation of quantum mechanics. The problems with the approach in [16,17] are: (1) that there tend to exist many "consistent histories" that are *incompatible* with each other, hence mutually exclude one another; and (2) that, in the understanding of the theory presented in these papers, the propagator of a system and the choice of an initial state do *not* determine *which* physical quantities that give rise to consistent histories will actually be observed in the course of time. Given the time evolution of a system and its initial state, the choice of a sequence of physical quantities giving rise to consistent histories thus remains quite arbitrary, i.e., lies – like beauty – "in the eye of the beholder".
This problem is avoided in the "ETH approach", as we now briefly explain. Suppose that, at some time t_0, a physical system S is prepared in a state $\rho = \rho^0$ (on the algebra $\mathcal{E}_{\geq t_0}$). Our formalism then enables us to *predict* a time, t_1, around which the first event after the preparation of S in state ρ_0 materializes and a family, $\mathcal{P}^1 \subset \mathcal{O}_S$, of possible events, $\Pi^1_\xi \in \mathcal{P}^1$, to which the event that materializes at time t_1 belongs; see Eqs. (11) and (16). Suppose now that the event happening at time t_1 corresponds to the projection $\Pi^1_{\xi_1} \in \mathcal{P}^1$. The "fundamental axiom" (see item 2, (18)) then instructs us that, in order to improve our predictions of the future after time t_1, we should use the state

$$\rho^1(\cdot) := \rho^0(\Pi^1_{\xi_1}(t_1))^{-1} \rho^0(\Pi^1_{\xi_1}(t_1)(\cdot)\Pi^1_{\xi_1}(t_1)),$$

on the algebra $\mathcal{E}_{\geq t_1}$, where $\Pi^1_{\xi_1}(t_1)$, (with $\Pi_{\xi_1} \in \mathcal{P}^1$), is the orthogonal projection on $\mathcal{H}$ describing the event happening at time t_1. Given the state ρ^1, one can now *predict* a time t_2 and a family $\mathcal{P}^2 \subset \mathcal{O}_S$ of orthogonal projections with the property that an event corresponding to some element of $\mathcal{P}^2$ happens around time t_2; etc.

Suppose that the state of the system after the kth event has happened around time t_k is given by ρ^k, which is a state on the algebra $\mathcal{E}_{\geq t_k}$, $k = 1, 2, 3, \ldots$. This state determines a time $t_{k+1} > t_k$ and a family, $\mathcal{P}^{k+1}$, of orthogonal projections describing possible events that might materialize at time t_{k+1}. (It can happen that ρ^k is a "passive state", in which case $t_{k+1} = \infty$.) Suppose that the event happening at time t_{k+1} is detected to be given by the operator $\Pi^{k+1}_{\xi_{k+1}}(t_{k+1})$ representing the projection $\Pi^{k+1}_{\xi_{k+1}} \in \mathcal{P}^{k+1}$. According to Eq. (18), the state on the algebra $\mathcal{E}_{\geq t_{k+1}}$ to be used to predict the future at times $> t_{k+1}$ is then given by

$$\rho^{k+1}(\cdot) := \rho^k(\Pi^{k+1}_{\xi_{k+1}}(t_{k+1}))^{-1}\rho^k(\Pi^{k+1}_{\xi_{k+1}}(t_{k+1})(\cdot)\Pi_{\xi_{k+1}}(t_{k+1})). \tag{19}$$

If, however, the event happening at time t_{k+1} (representing some element of $\mathcal{P}^{k+1}$) is *not* recorded then the state on $\mathcal{E}_{\geq t_{k+1}}$ to be used to predict the future after time t_{k+1} is given by

$$\rho^{k+1}(\cdot) := \rho^k(\cdot)|_{\mathcal{E}_{\geq t_{k+1}}} \simeq \sum_{\Pi^{k+1}_{\xi} \in \mathcal{P}^{k+1}} \rho^k(\Pi^{k+1}_{\xi}(t_{k+1})(\cdot)\Pi^{k+1}_{\xi}(t_{k+1})). \tag{20}$$

Recall that the distance between $\mathcal{P}^{k+1}(t_{k+1})$ and $\mathcal{Z}_{\rho^k_{t_{k+1}}}$ $(\subset \mathcal{E}_{\geq t_{k+1}})$ is tiny!

In the "ETH approach", a *history* consists of a sequence, $\big(t_k, \Pi^k_{\xi_k}(t_k)\big)_{k=1,2,3,\ldots}$, where $t_1 < t_2 < t_3 < \ldots$ are times, $\Pi^k_{\xi_k}(t_k)$ (with $\Pi^k_{\xi_k} \in \mathcal{P}^k \subseteq \mathcal{O}_S$) is the orthogonal projection on $\mathcal{H}$ describing the event happening at approximately time t_k, with t_k and $\mathcal{P}^k$ determined by the state ρ^{k-1} corresponding to the event that happened at time t_{k-1}, according to Eq. (18).

Such a history is denoted for short by

$$\{(\xi_k, t_k)|k = 1, 2, 3, \ldots\} \tag{21}$$

Events that have materialized at some time, but have not been recorded can be omitted from the list (21) – as follows from Eq. (20).

Quantum mechanics, as understood in the "ETH approach", predicts the probabilities of histories. In fact, these probabilities are given by a well-known formula, which we call *"LSW-formula"* (for "Lüders–Schwinger–Wigner", see [18–20]). It is the unique generalization of Born's Rule ($k = 1$) to all values of k. Here it is:

$$\mathrm{Prob}\{((\xi_k, t_k)|k = 1, 2, 3, \ldots\} := \rho^0\Big(\prod_{k=1,2,3,\ldots} \Pi^k_{\xi_k}(t_k) \cdot (\prod_{k=1,2,3,\ldots} \Pi^k_{\xi_k}(t_k))^*\Big) \tag{22}$$

Some applications of this formula will be sketched in Sect. 3.

3. It should be emphasized that a physical quantity represented by an operator $\hat{X} \in \mathcal{O}_S$ that, for a suitably chosen initial state, has an objective value around some time t – meaning that the spectral projections of $X(t)$ belong to a family $\mathcal{P}(t)$ of possible events happening at time t – will usually *not* have an objective value at an earlier or later time, because the quantity in question evolves in time; i.e., the operators $X(t)$ representing that quantity depend on time t. In fact, for typical choices of an element $\hat{X} \in \mathcal{O}_S$, the operators $X(t)$ do *not* commute with operators describing "conserved quantities", such as energy, momentum or angular momentum, etc. (They *do* however commute with operators representing "Super-Selection Rules".

But energy and momentum are, of course, not Super-Selection Rules.) It then follows that the detection of the value of the physical quantity represented by an operator $\hat{X} \in \mathcal{O}_S$, with the property that $X(t)$ depends on t, *violates* energy- (and possibly angular momentum- ...) conservation, in the sense that the distribution of energies (and angular momentum, i.e., the energy- and angular-momentum fluctuations) in the states before and after the observation of the value of $\hat{X}$, see Eq. (18), are *different* from each other.
The relation between the *duration* of the event corresponding to the recording of the value of a physical quantity and the *amount of energy fluctuation* accompanying this event is given by the usual *time-energy uncertainty relations*; see, e.g., [21].

3. Indirect observation/reconstruction of properties of physical systems

In this section, we present a brief outline of the theory of indirect non-demolition measurements, as originally developed in [22]; see also [23,9,5] and references given there. Our discussion is limited to the analysis of a simple example, which is inspired, in part, by the beautiful experiments described in [8].

3.1. The example of a mesoscopic system

A *property*, P, of a physical system S is the value of a *time-independent* physical quantity. Examples of properties of S that can be recorded directly are "observables at infinity", as described in item 1 of subsect. 2.4. Our purpose, in this section, is to present a sketchy outline of how properties of a system S can be determined *indirectly* from recordings of long sequences of events, (i.e., from recordings of the values of physical quantities represented by operators belonging to $\mathcal{O}_S$), as discussed in the last section. Such an indirect observation of a property P of S is sometimes called a *"non-demolition measurement"*. A presentation of the general theory of non-demolition measurements is beyond the scope of this paper; but see [22,23,9,5]. In the following, we therefore focus our attention on the concrete example of a mesoscopic system S sketched at the end of subsection 2.2. In this example, the list $\mathcal{O}_S$ of physical quantities whose values can be recorded directly consists of all bounded functions of a single operator, namely the operator $\hat{X}$ defined in Eq. (8) of subsect. 2.2, which has only two eigenvalues ± 1 corresponding to projections $\Pi_{\pm 1}$.

We imagine that a history, $(\xi_k, t_k)_{k=1,2,3,\ldots}$, of events corresponding to values $\xi_k = \pm 1$ of the physical quantity represented by the operator $\hat{X}$ has been recorded. In our example, the recording of the value $\xi_k = 1$ at time t_k means that the detector near D_L has clicked around time t_k, (i.e., an electron traveling through the T-channel has entered the reservoir at the end of the left arm of the T-channel), while the recording of $\xi_k = -1$ at time t_k means that the detector near D_R has clicked around time t_k.

For the following discussion, the values of the times $t_1 < t_2 < t_3 < \ldots$ at which events (i.e., clicks of a detector) are happening are unimportant. We therefore omit reference to these times in our notations, denoting histories by $\underline{\xi} = (\xi_k)_{k=1,2,3,\ldots}$, with $\xi_k = \pm 1$, $\forall k = 1, 2, 3, \ldots$. By Ξ we denote the space of all arbitrarily long histories. A sequence, $\underline{\xi}^n := (\xi_k)_{k=1}^n$, of n recorded detector clicks belonging to a history $\underline{\xi}$ is called a *"measurement protocol"* of length n.

Given an initial state ρ of S, the "LSW formula", Eq. (22) of subsect. 2.4, determines a probability measure μ_ρ on the space Ξ: The probability of a measurement protocol $\underline{\xi}^n = (\xi_1, \ldots, \xi_n)$ of length n is given by

$$\mu_\rho(\xi_1, \ldots, \xi_n) := \rho\Big(\Pi_{\xi_1}(t_1) \cdots \Pi_{\xi_n}(t_n) \cdots \Pi_{\xi_1}(t_1)\Big), \tag{23}$$

see (22). We note that

$$\sum_{\xi_n} \mu_\rho(\xi_1, \ldots, \xi_{n-1}, \xi_n) = \mu_\rho(\xi_1, \ldots, \xi_{n-1}), \quad \text{and} \quad \sum_{\underline{\xi}^n} \mu_\rho(\underline{\xi}^n) = 1.$$

By a lemma due to Kolmogorov, these properties imply that μ_ρ, as defined by (23), extends to a probability measure on the space Ξ of histories.

We suppose that the chemical potential of the reservoir "e^- gun" is only very slightly higher than the chemical potentials of the reservoirs D_L and D_R, so that the rate, τ, at which "e^- gun" releases an electron into the T-channel is so slow that, at any given moment, there is typically only at most one electron traveling through the T-channel, and that, after an electron has entered D_ℓ, the state of this reservoir and of the detector near it relaxes to the original state in a time much shorter than τ, for $\ell = L, R$. These assumptions can be interpreted as saying that the electrons traveling through the T-channel – to get lost in one of the reservoirs, D_L or D_R, at the end of the horizontal arms of the T-channel – and their successive detections are all *independent* of each other. This implies that the measures μ_ρ are *"exchangeable"*, i.e.,

$$\mu_\rho(\xi_1, \ldots, \xi_n) = \mu_\rho(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(n)}), \quad \forall \text{ permutations } \sigma \text{ of } \{1, \ldots, n\}, \tag{24}$$

for all $n = 1, 2, 3, \ldots$ and all states ρ of the system whose restriction to the three reservoirs have the desired properties, (in particular, the prescribed chemical potentials).

By de Finetti's theorem, Eq. (24) implies that μ_ρ is a convex combination of *product measures*. For simplicity, we suppose that it is a *finite* convex combination of product measures:

$$\mu_\rho(\xi_1, \ldots, \xi_n) = \sum_{\nu=0}^{N} \pi_\rho(\nu) \prod_{i=1}^{n} p(\xi_i|\nu), \tag{25}$$

where

$$p(\xi|\nu) \geq 0, \quad \forall \xi, \nu, \quad \text{and} \quad \sum_{\xi=\pm 1} p(\xi|\nu) = 1, \quad \forall \nu = 0, 1, \ldots, N,$$

and

$$0 \leq \pi_\rho(\nu) < 1, \quad \forall \nu, \quad \text{with} \sum_{\nu=0}^{N} \pi_\rho(\nu) = 1.$$

The physical interpretation of these quantities is as follows:

- ν is the number of electrons bound by the quantum dot P. Because of possible electron exchange between P and P', the state ρ of S is, in general, *not* an eigenstate of the electron number operator of P; i.e., ν does usually *not* have a sharp value in the state ρ. It is assumed, however, that ν is a *static* quantity, i.e., that the electron number operator of P commutes with the Hamiltonian of the system.
- $p(\xi|\nu)$ is the a-priori probability that an electron traveling through the T-channel reaches the detector near D_L ($\leftrightarrow \xi = 1$) or the one near D_R ($\leftrightarrow \xi = -1$), respectively. This probability clearly depends on the number ν of electrons bound to the dot P, because these electrons create a "Coulomb blockade" in the arm of the T-channel above P and extending to the right, towards D_R.

- $\pi_\rho(\nu)$ is the Born probability (in the state ρ) for the number of electrons bound to the dot P to be equal to ν, with $\nu = 0, \dots, N$.

3.2. Summary of results on indirect measurements

In this last subsection, we summarize some recent results on the system described above. We omit the proofs, which the reader may find in [5].

We define the frequency, $f_\xi^{(n)}$, of the value ξ in a measurement protocol $\underline{\xi}^n$ of length n (with $\xi = 1 \leftrightarrow$ electron reaches D_L, $\xi = -1 \leftrightarrow$ electron reaches D_R) as follows:

$$f_\xi^{(n)}(\underline{\xi}) := \frac{1}{n}\Big(\sum_{k=1}^{n} \delta_{\xi\xi_k}\Big), \quad \text{with} \sum_{\xi=\pm 1} f_\xi^{(n)}(\underline{\xi}) = 1, \quad \forall n. \tag{26}$$

The following results have been established in [5].

1. Law of Large Numbers

   For every history $\underline{\xi}$,

   $$\lim_{n\to\infty} f_\xi^{(n)}(\underline{\xi}) = p(\xi|\nu), \tag{27}$$

   for some $\nu = 0, 1, \dots, N$. $\square$

   For simplicity, we assume that

   $$\min_{\nu_1 \neq \nu_2} |p(1|\nu_1) - p(1|\nu_2)| \geq \kappa > 0. \tag{28}$$

   With each $\nu = 0, 1, \dots, N$ we associate a subset, Ξ_ν, of Ξ defined by

   $$\Xi_\nu(n, \underline{\varepsilon}) := \{\underline{\xi} \in \Xi \,||\, f_\xi^{(n)}(\underline{\xi}) - p(\xi|\nu)| < \varepsilon_n\}, \tag{29}$$

   where

   $$\varepsilon_n \to 0, \sqrt{n}\varepsilon_n \to \infty, \quad \text{as } n \to \infty.$$

2. Disjointness

   It follows from assumption (28) and definition (29) that, for n so large that $\varepsilon_n < \frac{\kappa}{2}$,

   $$\Xi_{\nu_1}(n, \underline{\varepsilon}) \cap \Xi_{\nu_2}(n, \underline{\varepsilon}) = \emptyset, \quad \nu_1 \neq \nu_2. \quad \square \tag{30}$$

3. Born's Rule and Central limit Theorem

   Under appropriate hypotheses on the state ρ (see [9,5]),

   $$\lim_{n\to\infty} \mu_\rho(\Xi_\nu(n, \underline{\varepsilon})) = \pi_\rho(\nu) \tag{31}$$

   Born's Rule

   Furthermore,

   $$\mu_\rho\Big(\bigcup_\nu \Xi_\nu(n, \underline{\varepsilon})\Big) \to 1, \quad \text{as } n \to \infty. \quad \square \tag{32}$$

4. Theorem of Boltzmann–Sanov

   Defining the relative entropy $\sigma(\nu_1\|\nu_2)$ by

$$\sigma(\nu_1\|\nu_2) := \sum_{\xi=\pm1} p(\xi|\nu_1)\big(\log_2 p(\xi|\nu_1) - \log_2 p(\xi|\nu_2)\big),$$

one has that

$$\mu\big(\Xi_{\nu_1}(n,\underline{\varepsilon})|\nu_2\big) \leq Ce^{-n\sigma(\nu_1\|\nu_2)}, \tag{33}$$

where $\mu(\cdot|\nu)$ is the product measure determined by $p(\xi|\nu)$.

Remark: Results 1 through 3 hold in much greater generality; see [5]. Concerning Eq. (25) and Result 4, we remark that there is a general theory of how to decompose measures μ_ρ into "extremal measures", $\mu(\cdot|\nu)$, $\nu \in \Xi_\infty$, where Ξ_∞ is the spectrum of the algebra of functions on Ξ measurable at ∞. (Functions on Ξ measurable at ∞ take the same values on any two histories $\underline{\xi}$ and $\underline{\eta}$ with $\xi_k = \eta_k$, except for finitely many k.) One can show that, under suitable assumptions, extremal measures are determined again by states of the system via the "LSW formula".

We pause to interpret Results 1 through 4. It follows from (32) that if n is very large then the set $\bigcup_\nu \Xi_\nu(n,\underline{\varepsilon})$ has apparently nearly full measure with respect to μ_ρ. By (30), it then follows that, for very large n, essentially every history $\underline{\xi}$ belongs to exactly one of the sets $\Xi_\nu(n,\underline{\varepsilon})$, and hence a measurement protocol $\underline{\xi}^n$ of length n determines the number ν of electrons bound to the dot P nearly unambiguously, with an error margin that tends to 0, as n tends to ∞. In the limit $n \to \infty$, measurement protocols determine the number ν of electrons in the dot P precisely, which implies that this number becomes sharp (i.e., does not exhibit any fluctuations, anymore), as n tends to ∞. This is the phenomenon of *"purification"* first studied in [23]. Furthermore, the empirical probability of a history $\underline{\xi} \in \Xi_\nu(n,\underline{\varepsilon})$ tends to $\pi(\nu)$, as n tends to ∞, which establishes Born's Rule for non-demolition measurements.

Finally, by Result 4 (Boltzmann–Sanov), the *time*, T, it takes to indirectly determine the number ν of electrons bound to the dot P is given, approximately, by

$$T = \tau/\sigma, \tag{34}$$

where τ is the rate at which "e^- gun" shoots electrons into the T-channel (i.e., the time elapsing between two consecutive electrons traveling through the T-channel, or two consecutive clicks of detectors), and

$$\sigma := \min_{\nu_1\neq\nu_2} \sigma(\nu_1\|\nu_2).$$

It should be emphasized that most indirect measurements are *not* non-demolition measurements. In the example of the mesoscopic system studied above, it is an idealization to assume that the number of electrons in the dot P is static (i.e., that the electron number operator counting the number of electrons bound to P commutes with the Hamiltonian of the system). It is therefore important to generalize the theory of indirect measurements sketched here to situations where properties of a system S change in time, albeit much more slowly than the rate at which direct observations of physical quantities in $\mathcal{O}_S$ are made. A beginning of such a theory has been described in [5].

4. Some hints to the literature, conclusions

There are many precursors of some of the ideas described in this paper, and it is quite impossible to do justice to all authors who have contributed (more and less) important pieces to the mosaic. The puzzling features of quantum mechanics and the problems surrounding its interpretations have been discussed by many people, including Schrödinger [24] and, later on, Bell;

see [25]. A general reference where many of the (older) interpretations of quantum mechanics are described is [26].[7] By no means are we attempting to distribute credit to various schools of thought, and we offer our apologies to all those colleagues who feel that their work should be cited here, but isn't. However, in all modesty, we feel that we have developed a novel approach to a "quantum theory of events and experiments", and we hope that the reader may have profited from reading this summary of some of the key ideas underlying our approach. (More details concerning the approach summarized in this paper can be found in [1,3].)

One might say that "loss of access to information", in the sense of Eq. (7), is a special form of what is called *"decoherence"*. The concept of decoherence was introduced and discussed in [30–32] and, in a very clear way, in [33] and further developed in [34,35], and references given there. There cannot be any doubt that "decoherence" is a basic building block in a quantum theory of events and experiments. Attempts to arrive at a logically coherent theory of observations and measurements based on the concepts of "consistent histories" and "decoherence" have been presented in [16,17,36].[8] The crucial concept of an "event" was introduced and discussed, and its importance in understanding the deeper meaning of quantum mechanics emphasized, in [37]. This thread of thoughts has been taken up in [38,39], where a formulation of an "event-enhanced quantum theory" inspired by [37] and [40] has been proposed.

In our approach, the concept of an "event" is given a clear meaning, and it plays a fundamental role; see subsects. 2.3 and 2.4. Our formulation of the quantum theory of systems exhibiting "loss of access to information", in the sense of Eq. (7), subsect. 2.2, and "events", as defined in the "fundamental axiom" of subsect. 2.3, introduces a clear distinction between the past and the future: the past is factual – it consists of events that have materialized –, while the future consists of potentialities, namely of "possible events" that might happen, but need not happen.

When analyzing a problem such as the deeper meaning of Quantum Mechanics one must fear that not all readers will find one's approach to the problem entirely convincing. Most certainly, our analysis is no exception. Moreover, we realize that various rather interesting and important technical issues concerning our approach remain open; (although there is no reason why one should not be able to settle them). We therefore conclude our report with a famous quote:

> *"Wir stehen selbst enttäuscht und sehn betroffen*
> *den Vorhang zu und alle Fragen offen. ...*
>
> *Verehrtes Publikum, los, such dir selbst den Schluss!*
> *Es muss ein guter da sein, muss, muss, muss!"*
>
> (Bertolt Brecht, in: *"Der gute Mensch von Sezuan"*)

This paper is a token of our deep gratitude for the wisdom Raymond Stora has shared with us and the friendship he has bestowed upon us.

[7] The Bohmian point of view, which is of definite interest, but is not relevant for the material in this paper, is presented in detail in [27], and references given there. For the many-worlds interpretation of quantum mechanics, see [28,29].

[8] We refrain from discussing the merits and success of various attempts to interpret quantum mechanics – which does not mean that we do not have opinions about them.

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Appendix. Proofs of Eqs. (13) and (14)

We first briefly explain the notion of a "centralizer", $\mathcal{C}_\varphi$, of a state, φ, on a von Neumann algebra, $\mathcal{M}$. We recall that

$$\mathcal{C}_\varphi := \{X \in \mathcal{M} | \mathrm{ad}_X(\varphi)(\cdot) := \varphi([X,\cdot]) = 0\}.$$

It follows from this definition that $\mathcal{C}_\varphi$ is a *subalgebra* of $\mathcal{M}$: If X and Y are elements of $\mathcal{C}_\varphi$ then, obviously, any linear combination of X and Y belongs to $\mathcal{C}_\varphi$, too. Furthermore, for arbitrary $A \in \mathcal{M}$,

$$\varphi\big(XYA\big) = \varphi\big((YA)X\big) = \varphi\big(Y(AX)\big) = \varphi\big((AX)Y\big) = \varphi\big(AXY\big),$$

i.e., XY belongs to $\mathcal{C}_\varphi$, too.

Next, let $X = X^*$ belong to $\mathcal{C}_\varphi$, with

$$X = \sum_{j=1}^{N} \xi_j \Pi_{\xi_j}$$

the spectral decomposition of X, where $\xi_1 < \cdots < \xi_N$ are the eigenvalues of X and $\Pi_{\xi_1}, \ldots, \Pi_{\xi_N}$ its spectral projections. Since any polynomial in X belongs to $\mathcal{C}_\varphi$, too, it follows that $\Pi_{\xi_j} \in \mathcal{C}_\varphi$, for any $j = 1, \ldots, N$. Thus, for an arbitrary operator $A \in \mathcal{M}$,

$$\begin{aligned}\varphi(A) &= \sum_{i,j=1,\ldots,N} \varphi\big(\Pi_{\xi_i} A \Pi_{\xi_j}\big) \\ &= \sum_{i,j=1,\ldots,N} \varphi\big(A \Pi_{\xi_j} \delta_{ij}\big) \\ &= \sum_{i=1}^{N} \varphi\big(\Pi_{\xi_i} A \Pi_{\xi_i}\big). \end{aligned} \tag{35}$$

Conversely, if Eq. (35) holds for arbitrary $A \in \mathcal{M}$ then it obviously follows that X belongs to $\mathcal{C}_\varphi$. It is obvious that Eq. (35) also holds if X belongs to the center, $\mathcal{Z}_\varphi$, of $\mathcal{C}_\varphi$.

Application: Proofs of Eqs. (13) and (14)

Let $X(t) = \sum_{j=1}^{N} \xi_j \Pi_{\xi_j}(t)$ be as in (9), and let A be an arbitrary operator in $\mathcal{E}_{\geq t}$. We rewrite $\rho_t(A)$ as follows:

$$\begin{aligned}\rho_t(A) = \sum_{i,j=1}^{N} &\{[\rho_t\big(\Pi_{\xi_i}(t)A\Pi_{\xi_j}(t)\big) - \rho_t\big(E_{\rho_t}(\Pi_{\xi_i}(t))A\Pi_{\xi_j}(t)\big)] \\
&+ [\rho_t\big(A\Pi_{\xi_j}(t)E_{\rho_t}(\Pi_{\xi_i}(t))\big) - \rho_t\big(A\Pi_{\xi_j}(t)\Pi_{\xi_i}(t))\big)]\} \\
&+ \sum_{i=1}^{N}\{\rho_t\big(A\Pi_{\xi_i}(t)^2\big) - \rho_t\big(A\Pi_{\xi_i}(t)E_{\rho_t}(\Pi_{\xi_i}(t))\big)] \\
&+ [\rho_t\big(E_{\rho_t}(\Pi_{\xi_i}(t))A\Pi_{\xi_i}(t)\big) - \rho_t\big(\Pi_{\xi_i}(t)A\Pi_{\xi_i}(t)\big)]\} \\
&+ \sum_{i=1}^{N} \rho_t\big(\Pi_{\xi_i}A\Pi_{\xi_i}(t)\big). \end{aligned} \tag{36}$$

In the second and in the fourth line we have used Eq. (12), which is legitimate, because $E_{\rho_t}(\Pi_{\xi_i}(t))$ belongs to the centralizer $\mathcal{C}_{\rho_t}$ of ρ_t. Since we are assuming that $\|E_{\rho_t}(\Pi_{\xi_j}(t)) - \Pi_{\xi_j}(t)\| \leq \delta'$, $\forall j = 1, \dots, N$, it follows that the absolute values of the four terms on the right side of line 1 and on lines 2, 3 and 4, respectively, are bounded above by $N\delta' \|A\|$. This implies (11), and hence (13) is proven.

To prove (14), all we have to do is to repeat the above argument with $E_{\rho_t}(\Pi_{\xi_i}(t))$ replaced by $e_{\rho_t}(\Pi_{\xi_i}(t))$, which belongs to $\mathcal{Z}_{\rho_t} \subseteq \mathcal{C}_{\rho_t}$.

References

[1] J. Faupin, J. Fröhlich, B. Schubnel, On the probabilistic nature of quantum mechanics and the notion of closed systems, Ann. Henri Poincaré 17 (2016) 689–731.

[2] J. Fröhlich, B. Schubnel, The preparation of states in quantum mechanics, J. Math. Phys. 57 (2016), in press.

[3] J. Fröhlich, B. Schubnel, Quantum probability theory and the foundations of quantum mechanics, in: Ph. Blanchard, J. Fröhlich (Eds.), The Message of Quantum Science – Attempts Towards a Synthesis, in: Lecture Notes in Physics, vol. 899, Springer-Verlag, Berlin, Heidelberg, New York, 2015.

[4] Ph. Blanchard, R. Olkiewicz, Decoherence induced transition from quantum to classical dynamics, Rev. Math. Phys. 15 (2003) 217–243;
Ph. Blanchard, M. Hellmich, P. Lukiewicz, R. Olkiewicz, Theory of decoherence effects in finite and infinite open quantum systems using the algebraic approach, in: The Message of Quantum Science – Attempts Towards a Synthesis, in: Lecture Notes in Physics, vol. 899, Springer-Verlag, Berlin, Heidelberg, New York, 2015, & references given there.

[5] M. Ballesteros, M. Fraas, J. Fröhlich, B. Schubnel, Indirect retrieval of information and the emergence of facts in quantum mechanics, arXiv:1506.01213, 3 June 2015 J. Stat. Phys. (2016), in press.

[6] D. Buchholz, J.E. Roberts, New light on infrared problems: sectors, statistics, symmetries and spectrum, Commun. Math. Phys. 330 (2014) 935–972;
see also: D. Buchholz, Collision theory for massless bosons, Commun. Math. Phys. 52 (1977) 147–173.

[7] B. Schubnel, Mathematical results on the foundations of quantum mechanics, PhD thesis, ETH, Zurich, December 2014.

[8] C. Guerlin, J. Bernu, S. Deleglise, C. Sayrin, S. Gleyzes, S. Kuhr, M. Brune, J.M. Raimond, S. Haroche, Progressive field-state collapse and quantum non-demolition photon counting, Nature 448 (7156) (2007) 889–893.

[9] M. Bauer, D. Bernard, Convergence of repeated quantum non-demolition measurements and wave-function collapse, Phys. Rev. A 84 (2011) 044103.

[10] J. Dixmier, Formes linéaire sur un anneau d'opérateurs, Bull. Soc. Math. Fr. 81 (1953) 9–39;
H. Umegaki, Conditional expectation in operator algebras, I, Tohoku Math. J. 6 (1954) 177–181;
H. Umegaki, Conditional expectation in operator algebras, II, Tohoku Math. J. 8 (1956) 86–100;
H. Umegaki, Conditional expectation in operator algebras, III, Kodai Math. Semin. Rep. 11 (1959) 51–64;
H. Umegaki, Conditional expectation in operator algebras, IV, Kodai Math. Semin. Rep. 14 (1962) 59–85.

[11] F. Combes, Poids et espérances conditionnelles dans les algèbres de von Neumann, Bull. Soc. Math. Fr. 99 (1971) 73–112.

[12] O. Bratteli, D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics, I, Springer-Verlag, New York, 1979;
O. Bratteli, D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics, II, Springer-Verlag, New York, 1981.
[13] M. Takesaki, Conditional expectations in von Neumann algebras, J. Funct. Anal. 9 (1972) 306–321.
[14] R.H. Herman, Centralizers and an ordering for faithful, normal states, J. Funct. Anal. 13 (1973) 317–323.
[15] U. Haagerup, Connes' bicentralizer problem and uniqueness of the injective factor of type III_1, 1985.
[16] R.B. Griffiths, Consistent histories and the interpretation of quantum mechanics, J. Stat. Phys. 36 (1984) 219–272.
[17] M. Gell-Mann, J.B. Hartle, Classical equations for quantum systems, Phys. Rev. D 47 (8) (1993) 3345–3382.
[18] G. Lüders, Über die Zustandsänderung durch den Messprozess, Ann. Phys. (Leipz.) 443 (1950) 322–328.
[19] J. Schwinger, The algebra of microscopic measurement, Proc. Natl. Acad. Sci. USA 45 (1959) 1542–1553.
[20] E.P. Wigner, in: B.R. Judd, G.W. Mackey (Eds.), The Collected Works of Eugene Paul Wigner, Part A: The Scientific Papers, Springer-Verlag, New York, 1993.
[21] P. Pfeifer, J. Fröhlich, Generalized time-energy uncertainty relations and bounds on life times of resonances, Rev. Mod. Phys. 67 (4) (1995) 759–779.
[22] K. Kraus, States, Effects and Operations, Springer-Verlag, Berlin, Heidelberg, New York, 1983.
[23] H. Maassen, B. Kümmerer, Purification of Quantum Trajectories, Lecture Notes-Monograph Series, vol. 48, Springer-Verlag, 2006, pp. 252–261.
[24] E. Schrödinger, The philosophy of experiment, Il Nuovo Cimento 1 (1955) 5–15.
[25] J.S. Bell, Speakable and Unspeakable in Quantum Mechanics: Collected Papers on Quantum Philosophy, Cambridge University Press, Cambridge (UK), 1987.
[26] J.A. Wheeler, W.H. Zurek, Quantum Theory and Measurement, Princeton University Press, Princeton, NJ, 1983.
[27] D. Dürr, S. Teufel, Bohmian Mechanics – The Physics and Mathematics of Quantum Theory, Springer-Verlag, Berlin, Heidelberg, New York, 2009.
[28] H. Everett, Relative state formulation of quantum mechanics, Rev. Mod. Phys. 29 (1957) 454–462.
[29] B.S.M. De Witt, Quantum mechanics and reality, Phys. Today 23 (1970) 30–35.
[30] G. Ludwig, Die Grundlagen der Quantenmechanik, Grundlehren der mathematischen Wissenschaften, vol. LXX, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1954.
[31] D. Zeh, in: E. Joos, C. Kiefer, D. Giulini, J. Kupsch, I.-O. Stamatescu, D. Zeh (Eds.), Decoherence and the Appearance of a Classical World in Quantum Theory, Springer-Verlag, Heidelberg, 1996, and references given there.
[32] H. Primas, Asymptotically disjoint quantum states, in: Ph. Blanchard, D. Giulini, E. Joos, C. Kiefer, I.-O. Stamatescu (Eds.), Decoherence: Theoretical, Experimental, and Conceptual Problems, Springer-Verlag, Berlin, 2000, pp. 161–178, and references given there.
[33] K. Hepp, Quantum theory of measurement and macroscopic observables, Helv. Phys. Acta 45 (1972) 237–248.
[34] W.H. Zurek, Decoherence, einselection, and the quantum origins of the classical, Rev. Mod. Phys. 75 (2003) 715–775.
[35] Ph. Blanchard, M. Hellmich, Decoherence in infinite quantum systems, in: Quantum Africa 2010, AIP Conf. Proc. 1469, 2012, pp. 2–15;
Ph. Blanchard, R. Olkiewicz, Decoherence induced continuous pointer states, Phys. Rev. Lett. 90 (2003) 010403;
Ph. Blanchard, P. Lugiewicz, R. Olkiewicz, From quantum to quantum via decoherence, Phys. Lett. A 314 (2003) 29–36;
Ph. Blanchard, R. Olkiewicz, Decoherence in Heisenberg's representation, Int. J. Mod. Phys. B 18 (2004) 501, See also Ref. [4].
[36] R. Omnès, Consistent interpretations of quantum mechanics, Rev. Mod. Phys. 64 (1992) 339–382.
[37] R. Haag, Fundamental irreversibility and the concept of events, Commun. Math. Phys. 132 (1990) 245–251;
R. Haag, Events, histories, irreversibility, in: H. Ezawa, Y. Murayama (Eds.), Quantum Control and Measurement, Proc. ISQM, ARL Hitachi, North Holland, Amsterdam, 1993;
see also: H.P. Stapp, Theory of Reality, Found. Phys. 7 (1977) 313–323.
[38] Ph. Blanchard, A. Jadczyk, Event-enhanced quantum theory and piecewise deterministic dynamics, Ann. Phys. 4 (1995) 583–599.
[39] A. Jadczyk, Particle tracks, events and quantum theory, Prog. Theor. Phys. 93 (1995) 631–646.
[40] G.C. Ghirardi, A. Rimini, T. Weber, Unified dynamics for microscopic and macroscopic systems, Phys. Rev. D 34 (1986) 470.

PHASE TRANSITIONS AND CONTINUOUS SYMMETRY BREAKING

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Summary

A short introduction to the subject of phase transitions and (continuous) symmetry breaking is presented. The Goldstone theorem is proven. Some details of the method of reflection positivity and infrared bounds are exhibited. Applications to QFT and statistical mechanics are outlined.

1. Introduction to the subject of phase transitions and symmetry breaking

Historically, phase transitions in systems of condensed matter have been studied quite systematically since the 19th Century, using the general formalism of thermo-dynamics, by Clausius, Clapeyron, Maxwell, Gibbs and others. Their findings are taught in any decent course on the "Theory of Heat".

In the second half of the 19th Century and at the beginning of the 20th Century, Statistical Mechanics was created by Maxwell, Boltzmann, Gibbs and Einstein.

The first example of a system for which a phase transition accompanied by a broken symmetry has been exhibited, using the methods of equilibrium statistical mechanics, is the ideal Bose gas. In 1925, Albert Einstein proved that this system exhibits what is called Bose-Einstein condensation, as the temperature is lowered at fixed, positive density. This phenomenon turned out to be accompanied by the spontaneous breaking of a $U(1)$ symmetry; (gauge trsfs. of the 1st kind).

1.1 Bose-Einstein condensation in the ideal Bose gas

We consider an ideal gas of non-relativistic atoms confined to a cubical region Λ in phys. space $\mathbb{E}^3$ at a positive temperature T and density $\rho > 0$. (It is assumed that the system is in a state of thermal equilibrium.)

For simplicity, we impose periodic b.c. on the wave functions of the atoms, and we describe the system in the <u>grand-canonical ensemble</u>. (!)

The one-atom Hamiltonian is given by

$$h_\Lambda := -\frac{\hbar^2}{2m}\Delta_\Lambda \qquad (1)$$

with periodic b.c. at $\partial\Lambda$. The eigenfunctions of h_Λ are

$$\psi_{\vec{k},\sigma} = u_\sigma \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{L^3}}, \qquad (2)$$

where $\{u_\sigma\}_{\sigma=1}^{S}$ is a CONS in $\mathbb{C}^{2S+1}$, S is the spin of an atom, and L is the length of an edge of Λ;

$$\vec{k} = \frac{2\pi}{L}\vec{n}, \quad \vec{n} \in \mathbb{Z}^3. \qquad (3)$$

Eigenvalue of h_Λ on $\psi_{\vec{k},\sigma}$ is

$$\varepsilon_{\vec{k}} = \frac{\hbar^2\vec{k}^2}{2m}, \quad \text{indep. of } \sigma.$$

Hilbert space of system: <u>Fock space</u>

$$\mathcal{F}_\Lambda := \bigoplus_{n=0}^{\infty} L^2(\Lambda, d^3x)^{\otimes_s n}$$

Hamiltonian

$$H_\Lambda := \bigoplus_{n=0}^{\infty} H_\Lambda^{(n)},$$

$$H_\Lambda^{(0)} = 0, \quad H_\Lambda^{(1)} = h_\Lambda, \quad H_\Lambda^{(n)} = \sum_{j=1}^{n} h_{\Lambda,j}, \ \dots$$

N: particle # operator

μ: chemical potential

Mention 2nd quantization; a^*-, a-ops.!

<u>Grand partition function:</u>

$$\Xi_\Lambda(\beta,\mu) = \mathrm{Tr}_{\mathcal{F}_\Lambda}\left(e^{-\beta(H_\Lambda - \mu N)}\right)$$

$$= \prod_{\vec{k} \in \frac{2\pi}{L}\mathbb{Z}^3} \left(1 - e^{-\beta(\varepsilon_{\vec{k}} - \mu)}\right)^{-(2S+1)} \qquad (4)$$

summing geometric series.

<u>Pressure:</u>

$$p_\Lambda(\beta,\mu) = \frac{1}{\beta|\Lambda|} \log \Xi_\Lambda(\beta,\mu)$$

$$- \frac{2S+1}{\beta(2\pi)^3} \sum_{\vec{k}} \left(\frac{2\pi}{L}\right)^3 \log\left(1 - z e^{-\beta \varepsilon_{\vec{k}}}\right) \qquad (5)$$

$$z = e^{\beta\mu} \quad (\underline{\text{fugacity}})$$

R.S. is Riemann-sum approximation to

$$-\frac{2S+1}{\beta(2\pi)^3}\int d^3k \log\left(1-z e^{-\beta\hbar^2\vec{k}^2/2m}\right)$$

As $L\to\infty$ (TD limit), we find that (polar coos.)

$$p(\beta,\mu) = -\frac{2S+1}{\beta}\frac{1}{2\pi^2\hbar^3}\left(\frac{2m}{\beta}\right)^{3/2}\int_0^\infty \log\left(1-z e^{-x^2}\right)x^2\,dx$$

$$= \frac{2}{3}\frac{2S+1}{\beta}\frac{1}{2\pi^2\hbar^3}\left(\frac{2m}{\beta}\right)^{3/2}\sum_{n=1}^\infty\int_0^\infty dx\, x^4\left(z e^{-x^2}\right)^n$$

$$= \frac{2}{3}\frac{2S+1}{\beta}\frac{1}{2\pi^2\hbar^3}\left(\frac{2m}{\beta}\right)^{3/2}\frac{3\sqrt{\pi}}{8}\left(\sum_{n=1}^\infty\frac{z^n}{n^{5/2}}\right)$$

<u>Def.</u> (ξ-function)

$$\xi_r(z) := \sum_{n=1}^\infty \frac{z^n}{n^r} \tag{6}$$

Then

$$\boxed{p(\beta,\mu) = \frac{2S+1}{\beta}\frac{1}{\lambda^3}\xi_{5/2}(z),} \tag{7}$$

where

$$\lambda := \left(\frac{2\pi\hbar^2}{mkT}\right)^{1/2}$$ (<u>thermal wave length</u>)

$$\left(\beta = \frac{1}{kT}\right)$$

From (4) and (5):

$$\rho = \frac{\partial p}{\partial \mu} = z \frac{\partial(\beta p)}{\partial z} = \frac{2S+1}{\lambda^3} \zeta_{3/2}(z), \quad (8)$$

$$u = \lim_{L \to \infty} \frac{U_\Lambda}{|\Lambda|} = \frac{\langle H_\Lambda \rangle_{\beta,\mu}}{L^3}$$

$$= -\frac{\partial(\beta p)}{\partial \beta} \underset{\substack{\uparrow \\ (7),\ \text{def. of } \lambda.}}{=} \frac{3}{2} p \quad (9)$$

<u>Classical limit</u>: $\rho \lambda^3 \ll 1$, (i.e., $|z| \ll 1$, or T large; see (6), (7)). Then

$$p(\beta,\mu) \simeq kT(2S+1)\lambda^{-3} z,$$
$$\rho \simeq (2S+1)\lambda^{-3} z, \quad (10)$$

i.e.

$$p(\beta,\mu) \simeq kT\rho, \quad u = \frac{3}{2} kT\rho.$$
(ideal gas)

For ρ large, $p(\beta,\mu) < p_{cl.}(\beta,\mu)$!

- <u>BEC</u>

Eq. (5) meaningful, as long as

$$z\, e^{-\beta \min \varepsilon_{\vec{k}}} = z < 1, \tag{11}$$

i.e.,

$$\mu < \varepsilon_{\vec{k}=0} = 0.$$

For $\mu < 0$, we have (7), (8) and (9)!

Let us consider <u>low temperatures</u> at <u>fixed</u> ρ:

By (6), $\zeta_r(z)$ well def. for $|z| < 1$, monotone increasing for $z \in [0,1)$. For $r > 1$,

$$\zeta_r(z=1) < \infty, \tag{12}$$

$\zeta_{3/2}(1) \simeq 2.612$, $\zeta_{5/2}(1) \simeq 1.342$.

By (7),

$$p(\beta,\mu) \underset{\mu \nearrow 0}{\nearrow} p^*(\beta) = kT\, \frac{2S+1}{\lambda^3}\, \zeta_{5/2}(1) \tag{13}$$

$$\rho(\beta,\mu) \underset{\mu \nearrow 0}{\nearrow} \rho_c(T) = \frac{2S+1}{\lambda^3}\, \zeta_{3/2}(1) \tag{14}$$

$\Longrightarrow$ Fixing $\rho > 0$, $\exists$ crit. temperature $T_c = T_c(\rho)$ such that $z \nearrow 1$, as $T \searrow T_c$!

By (14), Eq. for T_c given by

$$\rho \simeq \underbrace{2.612}_{\zeta_{3/2}(1)} (2S+1) \left(\frac{m k T_c}{2\pi \hbar^2} \right)^{3/2}, \quad \text{i.e.,}$$

$$T_c \simeq \frac{2\pi \hbar^2}{m k} \left(\frac{\rho}{2.612(2S+1)} \right)^{2/3} \tag{15}$$

<u>What happens for $T < T_c$?</u>

When $z \nearrow 1$, the Riemann sum in (5) <u>cannot</u> be replaced by the integral

$$\int d^3k \log\left(1 - z e^{-\beta \varepsilon_{\vec{k}}}\right),$$

anymore, as $L \to \infty$. The occupation # $n_{\vec{k}}$ of orbital $\psi_{\vec{k},\sigma}$ has expectation value (see (4))

$$\langle n_{\vec{k}} \rangle_{\beta,\mu} = \frac{2S+1}{z^{-1} e^{\beta \varepsilon_{\vec{k}}} - 1}, \tag{16}$$

which <u>diverges</u>, for $\vec{k} = 0$ as $z \nearrow 1$!

For $z \lesssim 1$ and $L < \infty$,

$$\underset{|\vec{k}| \neq 0}{\langle n_{\vec{k}} \rangle_{\beta,\mu}} \ll \langle n_{\vec{k}=0} \rangle_{\beta,\mu}$$

By (16),

$$\beta\mu = -\ln\left(1+\frac{1}{\langle n_{\vec{0}}\rangle_{\beta,\mu}}\right) \simeq -\frac{1}{\langle n_{\vec{0}}\rangle_{\beta\mu}}$$

Thus, for $L<\infty$, $\langle n_{\vec{0}}\rangle_{\beta\mu} \propto \rho L^3$ is "<u>macroscopically occupied</u>", when $T<T_c(\rho)$, with $\langle n_{\vec{0}}\rangle_{\beta,\mu} \propto N = \rho L^3$. In the TD limit, at fixed T but variable ρ

$$\frac{\langle n_{\vec{0}}\rangle_{\beta,\mu}}{L^3} \xrightarrow[L\to\infty]{} \rho-\rho_c > 0, \qquad (17)$$

for $\rho>\rho_c(T)$, (with $\rho_c(T)$ given by (14); after $T_c\to T$). Thus we find for the occupation density $\rho_L(\vec{k}) := \frac{\langle n_{\vec{k}}\rangle_{\beta\mu}}{L^3}$ in the thermodynamic limit, at $z=1$:

$$\rho(\vec{k}) := \lim_{L\to\infty} \rho_L(\vec{k}) = \frac{2S+1}{e^{\beta\hbar^2\vec{k}^2/2m}-1} + (\rho-\rho_c)\,\delta^{(3)}(\vec{k}) \qquad (18)$$

and hence

$$\rho = \int \rho(\vec{k})\, d^3k = \underbrace{(2S+1)\lambda^{-3} \xi_{3/2}(1)}_{\rho_c = \rho_c(T)} + \underbrace{\rho_{\vec{k}=0}}_{= \rho - \rho_c}$$

With $\lambda = \left(\frac{2\pi \hbar^2}{m k T}\right)^{1/2}$ it follows that,

for $T < T_c = T_c(\rho)$, (as in (15)!),

$$\rho_{\vec{k}=0} = \rho \left(1 - \left(\frac{T}{T_c}\right)^{3/2}\right) \qquad (19)$$

0 – mode does not ~~contri~~ contribute to $p(\beta, \mu)$, nor to u!

$$u = \frac{3}{2} p = \frac{3}{2} (2S+1) kT \lambda^{-3} \xi_{5/2}(z)$$

$$\underset{z \nearrow 1}{=} \frac{3}{2} kT \rho_c \frac{\xi_{5/2}(1)}{\xi_{3/2}(1)}$$

$$\simeq \left(\frac{3}{2} \times 0.513\right) \cdot kT \rho \left(\frac{T}{T_c}\right)^{3/2} \qquad (20)$$

$$\Rightarrow c_v \propto T^{3/2} \quad (T \text{ small})$$
$$\propto \frac{3}{2} k \quad (T \text{ large})$$
$\Big\}\rightarrow$ 3rd law valid.

$\frac{dc_v}{dT}$ has discontinuity at $T = T_c$!

$p - v$ diagram.

Phase boundary for BEC, in terms of pressure:

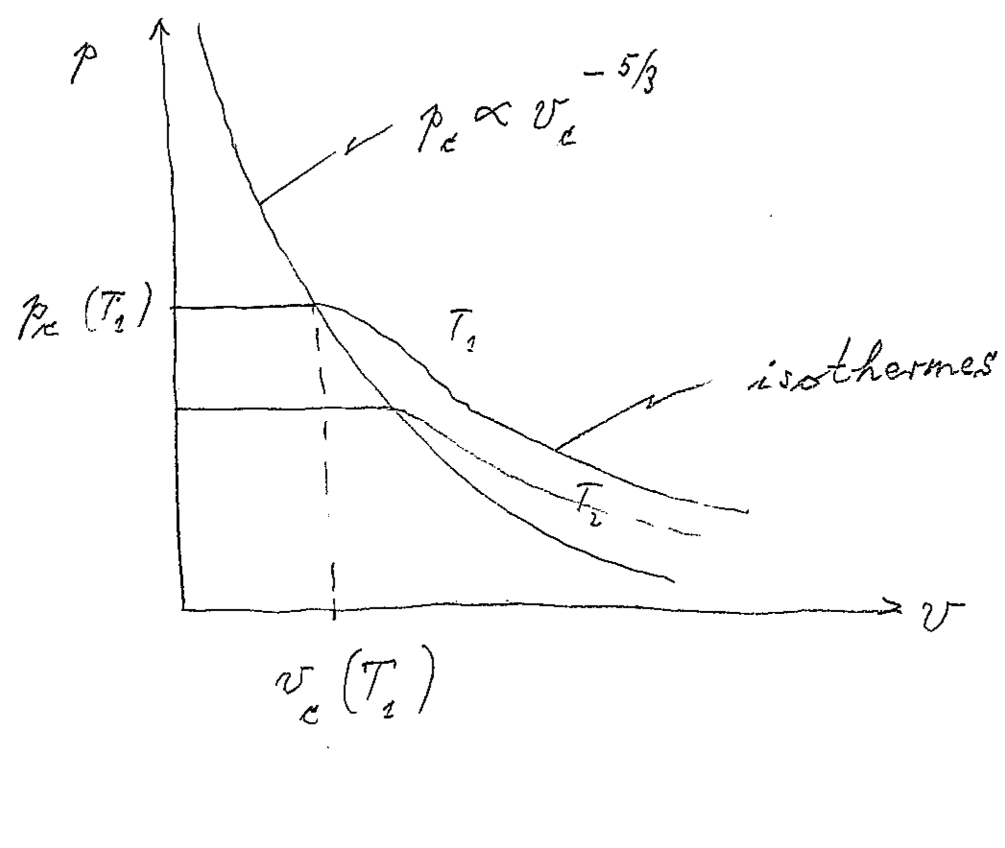
$$p_c(T) \underset{\mu \nearrow 0}{=} (2S+1)\, kT \left(\frac{mkT}{2\pi \hbar^2}\right)^{3/2} \xi_{5/2}(1)$$

$$\rho_c(T) = (2S+1) \left(\frac{mkT}{2\pi \hbar^2}\right)^{3/2} \xi_{3/2}(1)$$

$$\equiv v_c(T)^{-1}$$

$$\Rightarrow \quad p_c \propto \rho_c^{5/3} = v_c^{-5/3} \tag{21}$$

p constant for $\rho \geq \rho_c$!



<u>Symmetry breaking:</u>

Creation- & annihilation ops.

$$\frac{1}{|\Lambda|^2}\int d^3x\, d^3y\, \langle a^*(x)\, a(y)\rangle^{(\Lambda)}_{\beta,\mu=0}$$

$$\xrightarrow[L\to\infty]{} \rho-\rho_c > 0\,, \quad \text{for } T<T_c$$

(at fixed $\rho > 0$). <u>Thus, LRO!</u>

$\Rightarrow$ For symmetry breaking b.c.,

$$\langle a^{\#}(x)\rangle_{\beta,\,b.c.} \neq 0 \tag{22}$$

$\Rightarrow$ $\boxed{\text{Symmetry } a^{\#}(x) \to e^{\pm i\theta} a^{\#}(x),\ \theta\in[0,2\pi) \text{ spontaneously broken!}}$

Bose gases in 1 and 2 dimensions

The function $\rho(\vec{k}) := \lim_{L\to\infty} \rho_L(\vec{k}) = \frac{\langle n_{\vec{k}} \rangle_{\beta,\mu}}{L^3}$

$$= \frac{2S+1}{e^{\beta\hbar^2\vec{k}^2/2m} - 1} \qquad (\vec{k} \neq 0)$$

has a singularity at $\vec{k} = 0$ that, in $d = 1, 2$, is <u>non-integrable</u>.

$$\rho(\beta,\mu) \propto T\,\zeta_1(z) \underset{z \nearrow 1}{\nearrow} \infty \qquad (23)$$

Eq. (23) has soln. $z = z(\rho) < 1$, for <u>arb. large</u> ρ;

$\Longrightarrow$ $\rho - \rho_c = 0$, $\forall \rho$, i.e., <u>no BEC</u>, and

$$\rho(\vec{k}) = \frac{2S+1}{z(\rho)^{-1} e^{\beta\hbar^2\vec{k}^2/2m} - 1}, \quad \forall \vec{k},$$

with $z(\rho) < 1$, i.e., $\rho(0) < \infty$!

1.2 Bloch's theory of Heisenberg's ferromagnets

$$\vec{S} = (S^1, S^2, S^3), \quad (\hbar = 1, \dots)$$

<u>$S = 1/2$:</u> $S^1 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$, $S^2 = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}$, $S^3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$

$$\Gamma \overset{e.g.}{=} \mathbb{Z}^3 \ni i \mapsto \vec{S}_i \quad (\text{spin of iron ion at } i)$$

<u>Heisenberg:</u>

$$H_\Lambda = - \sum_{\langle i,j \rangle} J \vec{S}_i \cdot \vec{S}_j - \sum_j h S_j^3 ; \quad (J > 0, \; h > 0)$$

exchange int.

Breaks $SU(2)$ symm.

$\mathbb{Z}^3 \supset \Lambda$ (cube) with <u>periodic b.c.</u>

$$\mathcal{H}_\Lambda = \bigotimes_{j \in \Lambda} \mathbb{C}^2_{(j)} ,$$

Groundstate of H_Λ:

$$\Omega_\Lambda = \bigotimes_j \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{(j)}$$

$$S_j^\pm := S_j^1 \pm i S_j^2$$

Then

$$S_j^+ \Omega_\Lambda = 0, \quad \forall j$$

$$S_j^3 \Omega_\Lambda = \tfrac{1}{2} \Omega_\Lambda$$

$$H_\Lambda \Omega_\Lambda = E_\Lambda \Omega_\Lambda , \quad \text{where}$$

3 bonds per site in $d=3$

$$E_\Lambda = -|\Lambda| \left(\frac{3}{4} J + \frac{1}{2} h\right) \qquad (d=3)$$

For,

$$H_\Lambda = -\frac{1}{2} \sum_{\langle i,j \rangle} J \left(S_i^+ S_j^- + S_i^- S_j^+ + 2 S_i^3 S_j^3\right) - \sum_j h S_j^3 .$$

<u>Excited states:</u> <u>Magnons</u> (<u>spin waves</u>)

$$\varphi_\Lambda(\vec{k}) = \frac{1}{\sqrt{|\Lambda|}} \sum_j e^{i\vec{k}\cdot\vec{j}} S_j^- \Omega_\Lambda$$

Then a simple calculation shows that

$$H_\Lambda \varphi_\Lambda(\vec{k}) = \left(E_\Lambda + \varepsilon(\vec{k})\right) \Omega_\Lambda ,$$

where
$$\varepsilon(\vec{k}) = \frac{1}{2}\left(\hat{J}(0) - \hat{J}(k)\right) + \underline{h} \simeq \underset{\substack{\uparrow \\ \text{NR kinematics}}}{\text{const.}\,\vec{k}^2} + \underline{h} \qquad (24)$$

with
$$\hat{J}(k) = \sum_{|j|=1} J e^{i\vec{k}\cdot\vec{j}} = 2J\left(\sum_{\alpha=1}^{3} \cos k_\alpha\right)$$

Multi-magnon states <u>interact,</u> because of

$$-\sum_{\langle i,j \rangle} J S_i^3 S_j^3 \longrightarrow \text{bound states of magnons.}$$

Magnon scattering at $T=0$: <u>Dyson, Hepp</u>

Bloch's approximation:

For T very small, may treat magnons as an ideal Bose gas. Use canonical ensemble. Then:

$$Z_\Lambda = Tr\left(e^{-\beta H_\Lambda}\right)$$
$$\underset{\text{Bloch}}{\simeq} \sum_{\underline{n}} e^{-\beta\left(E_\Lambda + \sum_{\vec{k}} n_{\vec{k}}\, \varepsilon(\vec{k})\right)}$$

$\Rightarrow$ Free energy:

$$F_\Lambda = E_\Lambda + \frac{1}{\beta} \sum_{\vec{k} \in B_\Lambda} \ln\left(1 - e^{-\beta \varepsilon(\vec{k})}\right) \qquad (25)$$

B_Λ: Brillouin zone $(= \Lambda^*)$

$\Rightarrow$ Magnetization:

$$M_\Lambda = -\frac{\partial F_\Lambda}{\partial h} = |\Lambda| - \sum_{\vec{k} \in B_\Lambda} \frac{\partial \varepsilon(\vec{k})}{\partial h}\, n_\beta(\vec{k}), \qquad (26)$$

where
$$n_\beta(\vec{k}) = \frac{e^{-\beta \varepsilon(\vec{k})}}{1 - e^{-\beta \varepsilon(\vec{k})}}$$

But
$$\frac{\partial \varepsilon(\vec{k})}{\partial h} = 1 \quad ; \text{see } (24)$$

Thus,

$$m(h) = \lim_{L \to \infty} \frac{M_\Lambda}{L^3} \simeq 1 - \frac{1}{(2\pi)^3} \int_B n_\beta(\vec{k})\, d^3k$$

<u>Spontaneous magnetization ($h \searrow 0$):</u>

$$m_0 = \lim_{h \searrow 0} m(h)$$

$$\simeq 1 - \frac{1}{(2\pi)^3} \int_B \underbrace{\frac{e^{-\beta \varepsilon_0(\vec{k})}}{1 - e^{-\beta \varepsilon_0(\vec{k})}}}_{\propto \text{ const. } |\vec{k}|^{-2} \ (|\vec{k}| \to 0)} d^3k$$

$$I(\beta) := \int \frac{e^{-\beta \varepsilon_0(\vec{k})}}{1 - e^{-\beta \varepsilon_0(\vec{k})}}\, d^3k$$

$$\simeq \beta^{-3/2}\, I(1), \quad \text{for } \beta \text{ large.}$$

$$\Longrightarrow \boxed{m_0 \simeq \left(1 - K\, T^{3/2}\right)} \quad (T \text{ small}) \qquad (27)$$

<u>$T^{3/2}$-law of Bloch.</u>

<u>Note:</u> In $d = 1, 2$,

$$\frac{e^{-\beta \varepsilon_0(\vec{k})}}{1 - e^{-\beta \varepsilon_0(\vec{k})}} \sim \frac{1}{\beta} |\vec{k}|^{-2}$$

is <u>non-integrable.</u> $\Rightarrow$ <u>No spontaneous magnetization!</u>

1.3 Theorem of Mermin & Wagner.

$$[S^1, S^2] = i\, S^3 \quad + \text{ cyclic}$$

$$[S^+, S^-] = 2 S^3$$

$$[S^3, S^\pm] = \pm\, S^\pm$$

Let
$$\langle A \rangle := \frac{Tr(e^{-\beta H} A)}{Tr(e^{-\beta H})} \qquad (28)$$

<u>Bogoliubov's inequality</u>

<u>For arb. operators</u> A <u>and</u> B,

$$\frac{\beta}{2} \langle A A^* + A^* A \rangle \langle [[B,H], B^*] \rangle \geq |\langle [A,B] \rangle|^2 \qquad (29)$$

<u>Proof.</u> Let $Z_\beta = Tr(e^{-\beta H})$. Then (Duhamel 2-pt. fu.)

$$(A,B) := \frac{1}{Z_\beta} \int_0^\beta d\tau\, Tr\left(e^{-(\beta-\tau)H} A^* e^{-\tau H} B\right). \qquad (30)$$

This defines a <u>scalar product</u> (if $\dim \mathcal{H} < \infty$): <u>cyclicity of the trace!</u>

Def. $F_A(\tau) := \frac{1}{Z_\beta} \mathrm{Tr}\left(e^{-(\beta-\tau)H} A^* e^{-\tau H} A\right)$

By cyclicity of trace,

$$\frac{d^2 F_A(\tau)}{d\tau^2} = \frac{1}{Z_\beta} \mathrm{Tr}\left(e^{-(\beta-\tau)H}[H, A^*] e^{-\tau H}[H, A^*]^*\right)$$

$$> 0$$

$\Rightarrow$ $F_A(\tau)$ is convex. $\Longrightarrow$

$$\int_0^\beta d\tau\, F_A(\tau) < \frac{\beta}{2}\left(F_A(0) + F_A(\beta)\right)$$
$$= \frac{\beta}{2}\left(\langle A^* A\rangle + \langle A A^*\rangle\right) \qquad (31)$$

Next,

$$\mathrm{Tr}\left(e^{-\beta H}[A, B]\right) = \mathrm{Tr}\left(B e^{-\beta H} A - e^{-\beta H} B A\right)$$

and

$$B e^{-\beta H} = \int_0^\beta e^{-(\beta-\tau)H} \beta [H, B] e^{-\tau H} d\tau + e^{-\beta H} B$$

$$\left(\text{easy}; \quad e^{-\beta H} = \lim_{n\to\infty}\left(1 - \frac{\beta H}{n}\right)^n !\right)$$

$$\Rightarrow \langle [A, B]\rangle = \beta\left([H, B]^*, A\right) \qquad (32)$$

Now, we use the Schwarz inequality for $(\cdot\,,\,\cdot)$:

$$\left([H,B]^*, A\right)^2 \leq \left([H,B]^*, [H,B]^*\right)\left(A,A\right) \qquad (33)$$

If we choose $A := [H,B]^*$ in (32), above, then

$$\left\langle [[H,B]^*, B] \right\rangle \overset{(32)}{=} \beta \left([H,B]^*, [H,B]^*\right) \qquad (34)$$

Thus, the first factor on the R. S. of (33) can be written as $\beta^{-1} \left\langle [[H,B]^*, B] \right\rangle$

Combining (32), (33) and (34), we get

$$|\langle [A,B] \rangle|^2 \leq \left\langle [[H,B]^*, B] \right\rangle \cdot (A,A)$$

$$\overset{(31)}{\leq} \frac{\beta}{2} \underbrace{\left\langle [[H,B]^*, B] \right\rangle}_{\langle [[B,H], B^*] \rangle} \left(\langle A^*A \rangle + \langle AA^* \rangle \right)$$

Q. E. D.

<u>Application.</u>

$$\left.\begin{aligned} \hat{S}^i(\vec{k}) &:= \sum_{j \in \Lambda} e^{i\vec{k}\cdot\vec{j}}\, S^i_{\vec{j}} \\ \hat{J}(\vec{k}) &:= \sum_{j \in \Lambda} e^{i\vec{k}\cdot\vec{j}}\, J_{0\vec{j}} \end{aligned}\right\} \qquad (35)$$

$$m_\Lambda(\beta) = \frac{1}{|\Lambda|} \left\langle \sum_{j \in \Lambda} S_j^3 \right\rangle \qquad (H = H_\lambda, \text{ as above})$$

In Bogoliubov's inequality, we set

$$A = \hat{S}^-(-\vec{k})$$
$$B = \hat{S}^+(\vec{k})$$

Then

$$\langle AA^* + A^*A \rangle = \left\langle \hat{S}^+(\vec{k})\, \hat{S}^-(-\vec{k}) + \hat{S}^-(-\vec{k})\, \hat{S}^+(\vec{k}) \right\rangle \tag{36}$$

$$\langle [B, A] \rangle = \sum_{j \in \Lambda} \sum_{j' \in \Lambda} e^{i\vec{k}\cdot(\vec{j}-\vec{j}')} \left\langle [S_j^+, S_{j'}^-] \right\rangle \tag{37}$$

$$= 2 \sum_{j \in \Lambda} \langle S_j^3 \rangle = 2|\Lambda|\, m_\Lambda(\beta).$$

<u>Furthermore:</u>

$$\langle [[B, H], B^*] \rangle = \sum_{j, j' \in \Lambda} \left(1 - e^{i\vec{k}\cdot(\vec{j}-\vec{j}')}\right) J_{\vec{j}\vec{j}'} \times$$

$$\times \left\langle \frac{1}{2} S_{\vec{j}'}^- S_{\vec{j}}^+ + S_{\vec{j}'}^3 S_{\vec{j}}^3 \right\rangle$$

$$+ h\, |\Lambda|\, m_\Lambda(\beta). \tag{38}$$

By Bogoliubov's <, $\langle [[B, H], B^*] \rangle > 0$

$\Longrightarrow$ Add same quantity to (38') with $\vec{k} \to -\vec{k}$.

$$\Longrightarrow \langle [[B, H], B^*] \rangle \leq 2 \sum_{j, j' \in \Lambda} \left(1 - \cos \vec{k}\cdot(\vec{j} - \vec{j}')\right) J_{\vec{j}\vec{j}'}$$

$$\times \underbrace{\left\langle \frac{1}{2}\left(S^-_{\vec{j}'} S^+_{\vec{j}} + S^+_{\vec{j}'} S^-_{\vec{j}}\right) + 2 S^3_{\vec{j}'} S^3_{\vec{j}} \right\rangle}_{\leq 2S(S+1), \text{ by Schwarz.}}$$

$$+ 2h |\Lambda| m_\Lambda(\beta),$$

where $\vec{S}\cdot\vec{S} = S(S+1)$ $(S = \text{spin})$.

Thus

$$\langle [[B, H], B^*] \rangle \leq \Big\{ 2 \sum_j \left(1 - \cos \vec{k}\cdot\vec{j}\right) J_{0\vec{j}} \cdot 2S(S+1) + 2h m_\Lambda(\beta) \Big\} |\Lambda| \quad (39)$$

By Bogoliubov, (36), (37) and (39), we find that

$$\langle \hat{S}^+(\vec{k}) \hat{S}^-(-\vec{k}) + \hat{S}^-(-\vec{k}) \hat{S}^+(\vec{k}) \rangle$$

$$\geq \frac{\beta^{-1} m_\Lambda(\beta)^2}{4S(S+1)\underbrace{\left(\hat{J}(0) - \hat{J}(\vec{k})\right)}_{\text{const.}|\vec{k}|^2,} + 2h m_\Lambda(\beta)} \cdot |\Lambda| \quad (40)$$

for n.n. couplings!

Summing over $\vec{k} \in B_\Lambda$, we then find that

$$\sum_{\vec{k}\in B_\Lambda} L.S. = |\Lambda| \sum_{j\in\Lambda} \langle S_j^+ S_j^- + S_j^- S_j^+ \rangle$$

$$\lesssim |\Lambda|^2 S(S+1),$$

so $(L \to \infty,\ m(\beta) \equiv m(\beta, h))$:

$$S(S+1) \geq \beta^{-1}\, m(\beta,h)^2 \times$$

$$\times (2\pi)^{-d} \int_B \frac{d^d k}{4S(S+1)\left(\hat{J}(0) - \hat{J}(\vec{k})\right) + 2hm(\beta,h)} \qquad (41)$$

Conclusions:

1) Upper bound for $m(\beta, h)$

2) For $d = 1, 2$, $m(\beta,h) \to 0$, as $h \to 0$

3) In $d \geq 3$, for $m(\beta, h=0) > 0$ (spont. magn.),

$$\langle \hat{S}^+(\vec{k})\hat{S}^-(-\vec{k}) + \hat{S}^-(-\vec{k})\hat{S}^+(\vec{k}) \rangle \propto |\vec{k}|^{-2},$$

as in Bloch's treatment.

Proof unenlightening, because it hides rôle of cont. symmetry!

1.4 Phase transitions and symmetry breaking in QFT

We consider a scalar QFT of a field $\varphi(x) = (\varphi^1(x), \dots, \varphi^N(x))$, $N \geq 1$, x a space-time point; Lagrange density

$$\mathcal{L}(\varphi(x), \partial_\mu \varphi(x)) = \frac{1}{2} \partial_\mu \varphi(x) \cdot \partial^\mu \varphi(x) - U(\varphi(x)),$$

where U is a semi-bounded (hence of even degree) polynomial, e.g.,

$$U(\varphi) = \frac{\lambda}{4!} |\varphi|^4 - \frac{\sigma}{2} |\varphi|^2 - \lambda^{-3/2} J \cdot \varphi$$

In axiomatic field theory and in renormalized perturbation theory, the vacuum expectation values

$$\left\langle \Omega, \prod_{i=1}^{n} \varphi^{\alpha_i}(\vec{x}_i, t_i)\, \Omega \right\rangle \qquad (42)$$

can be analytically continued in the time variables $t_1, \dots, t_n$ from real to purely imaginary times, $t_j = i\tau_j$. This is a conse–

quence of the <u>spectrum condition</u> $(H \geq 0)$, <u>locality</u> & of <u>Poincaré invariance</u>. Formally, we set

$$\phi(\vec{x}, \tau) := \sqrt{\lambda}\, \varphi(\vec{x}, i\tau)$$

Then (42) $\longrightarrow$

$$\left\langle \phi(x_1) \cdots \phi(x_n) \right\rangle_J = \frac{1}{Z} \int \mathcal{D}\phi \prod_{i=1}^{n} \phi(x_i)\, e^{-\frac{S_J(\phi)}{\lambda}}, \tag{43}$$

where

$$S_J(\phi) = \int d^d x \left[\frac{1}{2} |\nabla\phi|^2(x) + \right.$$
$$\left. + \frac{1}{4!} |\phi|^4(x) - \frac{\sigma}{2} |\phi|^2(x) - \lambda J \cdot \phi(x) \right] \tag{44}$$

$$\text{"}\mathcal{D}\phi = \prod_x d^N \phi(x)\text{"},$$

Z a normalization factor s.t. $\langle \mathbb{1} \rangle = 1$.

We propose to evaluate (43), with S_J as in (44), by the <u>saddle point method</u>:

fluctuation field

$$\phi(x) =: \phi_c + \chi(x), \quad \phi_c = \text{const.},$$

where ϕ_c is a minimum of $S_J(\phi)$.

Given some ϕ_c; we choose J such that

$$\boxed{\langle \phi(x) \rangle_J = \phi_c \,, \text{ hence } \langle \chi(x) \rangle_J = 0}$$

We expand $S(\phi)$ around ϕ_c to 2^{nd} order in $\chi(x)$ and calculate. In order to arrive at more or less meaningful expressions, we confine space - imaginary time to a cube Λ, with <u>periodic b.c. at $\partial\Lambda$</u>, and impose a <u>UV cutoff</u>:

$$Z_\Lambda(J) := \int \mathcal{D}\phi \, e^{-S_J(\phi)/\lambda}$$

$$\underset{\underline{\lambda \text{ small}}}{\simeq} \int \mathcal{D}\chi \exp - \frac{1}{\lambda} \int_\Lambda d^d x \Big[\frac{1}{2} (\nabla \chi)^2 (x) + \underset{J=0}{U}(\phi_c) + \frac{1}{2} U''(\phi_c) \chi(x)^2 \Big] \quad (45)$$

(ϕ_c chosen such that terms <u>linear</u> in $\chi(x)$ vanish; terms $O(\chi^3)$ neglected!)

The functional integral in (45) is Gaussian and can therefore be evaluated <u>explicitly</u>:

$$Z(J) = e^{-\frac{1}{\lambda} U(\phi_c)\,|\Lambda|} \Big[\underbrace{\det\left(-\Delta + U''(\phi_c)\right)}_{= \exp \operatorname{Tr} \ln \left(-\Delta + U''(\phi_c)\right),}\Big]^{1/2}$$

$\phi_c = \phi_c(J)$.

Taking logarithms, dividing by $|\Lambda|$ and letting $L \to \infty$, we find that

<u>eff. potential!</u>

$$V(\phi_c) := -\lim_{L\to\infty} \frac{\log Z_\Lambda\left(J = J(\phi_c)\right)}{|\Lambda|}$$

$$\simeq \lambda^{-1} U_{J=0}(\phi_c) + \underbrace{\frac{1}{2}\int \frac{d^d k}{(2\pi)^4} \ln\left(k^2 + U''(\phi_c)\right)}_{\approx V_{1\,loop}(\phi_c)} + \underline{c.t.} \qquad (46)$$

where c.t. stands for "<u>counter terms</u> (to order λ^0)" eliminating the divergences on the R.S. of (46).

Let us regularize the theory by imposing an <u>ultraviolet cutoff</u>:

$$|k| \leqslant \kappa < \infty, \qquad (47)$$

e.g., "<u>lattice approximation</u>;" (expand remark).

Introducing polar coordinates in the integral on the R.S. of (46) and changing variables to $\rho := k^2$, with $\rho \leq \kappa^2$, we find in <u>$d=4$</u>:

$$V_{1\,loop}(\phi_c) = \frac{\kappa^2}{32\pi^2} U''(\phi_c) + \frac{U''(\phi_c)^2}{64\pi^2} \times$$

$$\times \left(\ln\left(\frac{U''(\phi_c)}{\kappa^2}\right) - \frac{1}{2}\right) + C_\kappa ,$$

where $C_\kappa \overset{d=4}{\overset{\downarrow}{=}} O(\kappa^4 \ln \kappa)$ is a divergent constant.

We set

$$V_{0\,loop}(\phi_c) = \frac{1}{4!\,\lambda}|\phi_c|^4 - \frac{\sigma}{2\lambda}|\phi_c|^2$$

$$\underbrace{\frac{B_\kappa}{2}|\phi_c|^2 + \frac{C_\kappa}{4!}|\phi_c|^4}_{\equiv \text{ c.t. !}}$$

We choose B_κ and C_κ so as to cancel the divergent terms in $V_{1\,loop}(\phi_c)$. Then

$$V(\phi_c) = \frac{1}{\lambda} U_{J=0}(\phi_c) + \frac{U''(\phi_c)^2}{64\pi^2} \ln U''(\phi_c) \qquad \text{finite}, \rightarrow = 0$$

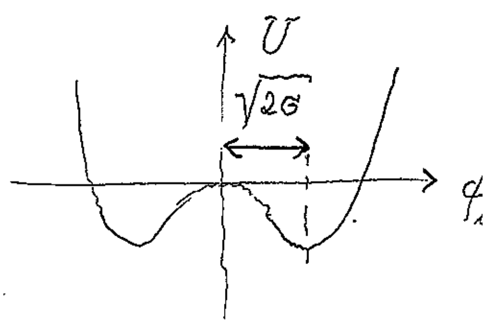
$$\left(+ \frac{b}{2}|\phi_c|^2 + \frac{c}{4!}|\phi_c|^4\right) \qquad (48)$$

$$+ \underbrace{\text{h.o. in } \lambda}$$

Choice of $b, c \leftrightarrow$ ren. conditions $\qquad$ <u>loop expansion!</u>

We observe that if we had chosen U to be a polynomial of order ≥ 6 then $(U''(\phi_c))^2$ would be of order $2 \deg U - 4 > \deg U$, and we would have to introduce counterterms of degree $> \deg U$.

$\Longrightarrow$ Theories with $\deg U \geq 6$ are <u>perturbatively non-renormalizable</u>.



For $\sigma > 0$,

$$U''(\phi_c) = \frac{1}{2}|\phi_c|^2 - \sigma$$

becomes <u>negative</u>, when $|\phi_c| < \sqrt{2\sigma}$; hence $\ln U''(\phi_c)$ has an <u>imaginary part</u>. In this region, (48) becomes <u>invalid</u>. The correct def. then is:

$$V(\phi_c) \equiv V(\phi_c^*),$$

$|\phi_c^*| = \sqrt{2\sigma}$, for $|\phi_c| < |\phi_c^*|$.

Then $V(\phi_c)$ is <u>flat</u> in ball $|\phi_c| \leq \sqrt{2\sigma}$.

$\longrightarrow$ Signal for <u>spontaneous symmetry breaking and phase trans.</u>!

Note that, given J, ϕ_c is chosen such that

$$\langle \phi(x) \rangle_J = \phi_c \;, \quad \text{i.e.,} \; \langle \chi(x) \rangle_J = 0 .$$

Define

$$W_\Lambda(J) := \frac{1}{|\Lambda|} \ln \int \mathcal{D}\phi \, e^{-S_J(\phi)/\lambda} = \frac{1}{|\Lambda|} \ln Z_\Lambda(J)$$

with S_J as in (44). Then, as $L \to \infty$,

$$\frac{\partial W(J)}{\partial J} = \langle \phi(x) \rangle_J \stackrel{!}{=} \phi_c \qquad (49)$$

Note that $W(J)$ is <u>convex</u> in J; we have chosen a definition of $V(\phi_c)$ that makes it <u>convex</u> in ϕ_c. Thus (49) implies that $V(\phi_c)$ is the <u>Legendre transform</u> of $W(J)$:

$$V(\phi_c) = \sup_J \{ \phi_c \cdot J - W(J) \}$$

Then $W(J)$ is the Legendre transform of $V(\phi_c)$

$$W(J) = \sup_{\phi_c} \{ \phi_c \cdot J - V(\phi_c) \} ,$$

i.e.,

$$\boxed{J = \frac{\partial V(\phi_c)}{\partial \phi_c}} \qquad (50)$$

If $V(\phi_c)$ has a <u>flat piece</u>, i.e., $|\phi_c| < |\phi_c^*|$ then

$$J = \left.\frac{\partial V(\phi_c)}{\partial \phi_c}\right|_{|\phi_c| < |\phi_c^*|} = \text{const.} \underset{\substack{\uparrow \\ \text{in our case}}}{=} 0$$

$\Rightarrow$ <u>Spontaneous symmetry breaking for $|\phi| \leq |\phi_c^*|$!</u>

Any ϕ_c with $|\phi_c| < |\phi_c^*|$ is the barycenter of a measure $d\mu(\xi)$ on the sphere, S^{N-1}, of radius $|\phi_c^*| = \sqrt{2\sigma}$ centered at O, i.e.,

$$\phi_c = \int \xi \, d\mu(\xi) \quad \text{describes phase mixture.}$$

Points $\xi \in S^{N-1} \longleftrightarrow$ "<u>pure phases</u>."

<u>Goldstone's Theorem.</u>

If a continuous symmetry $O(N)$, $N \geq 2$ is spontaneously broken, as discussed above, then the field theory describes $N-1$ <u>massless</u>, scalar particles, so called <u>Goldstone bosons.</u>

<u>Proof:</u> See below.

1.5 <u>Proof of spontaneous symmetry breaking in $\lambda|\vec{\phi}|^4$-theory in $d=3$ space-time dimensions.</u>

Lagrangian

$$\mathcal{L}(\vec{\varphi}, \partial_\mu \vec{\varphi}) = \frac{1}{2}(\partial_\mu \vec{\varphi}) \cdot (\partial^\mu \vec{\varphi}) - U(|\vec{\varphi}|),$$

$$U(|\vec{\varphi}|) = \frac{\lambda}{4!}\left(|\vec{\varphi}|^2 - \rho^2\right)^2; \quad \vec{\varphi} = (\varphi^1, \dots, \varphi^N).$$

In $d=3$ space-time dimensions, this theory makes perfect sense, <u>non-perturbatively</u>; (Glimm & Jaffe, Feldman & Osterwalder, Park). The construction proceeds by Euclidean region functional integrals, as outlined in Sect. 1.4.

It is convenient to Wick-order the potential $U(|\vec{\varphi}|)$ with respect to the <u>mass less</u> free field (which is only possible if $d \geq 3$!):

$$:|\vec{\varphi}|^2:(x) = \lim_{y \to x} \left\{ \vec{\varphi}(x) \cdot \vec{\varphi}(y) - \langle 0 | \vec{\varphi}(x) \cdot \vec{\varphi}(y) | 0 \rangle \right\}, \quad (3.112)$$

where

$$\langle 0 | \vec{\varphi}(x) \cdot \vec{\varphi}(y) | 0 \rangle = N(-\Delta)^{-1}_{xy} = \frac{N}{4\pi |x-y|}; \quad (3.113)$$

and similarly for $:|\vec{\varphi}|^4:$. We set

$$U_\Lambda(\rho^2; |\vec{\varphi}|) := \int_\Lambda d^3x \left\{ \frac{\lambda}{4!} :|\vec{\varphi}|^4:(x) + \delta m^2(\lambda) :|\vec{\varphi}|^2:(x) - \frac{\lambda}{12} \rho^2 :|\vec{\varphi}|^2:(x) \right\}$$

$$= U_\Lambda(0; |\vec{\varphi}|) - \frac{\lambda \rho^2}{12} \int_\Lambda d^3x :|\vec{\varphi}|^2:(x), \quad (3.114)$$

where Λ is a space-time cube with sides of length L, $\delta m^2(\lambda) = \text{const}\, \lambda^2$ is the logarithmically divergent coefficient of a mass counterterm (independent of ρ^2), and we omit the constant $\frac{\lambda}{4!} \rho^4$.

In $d=3$ dimensions, there are <u>no field-strength and coupling-constant renormalizations</u>!

We now define

$$Z_\Lambda(\rho^2) := \frac{1}{Z_\Lambda(0)} \int \mathcal{D}\vec{\varphi}\, e^{-\frac{1}{2}\int d^3x\, (\partial_\mu \vec{\varphi})(x)\cdot(\partial^\mu \vec{\varphi})(x)} \times$$
$$\times\, e^{-U_\Lambda(0;|\vec{\varphi}|)}\, e^{\frac{\lambda\rho^2}{12}\int_\Lambda d^3x\, :|\vec{\varphi}|^2:(x)}, \qquad (3.115)$$

and
$$F(\rho^2) := \lim_{L\to\infty} \frac{1}{L^3} \ln Z_\Lambda(\rho^2).$$

It is fairly easy to show that $F(\rho^2)$ is a <u>convex</u> function of ρ^2, with $F(\rho^2) \nearrow \infty$, as $\rho^2 \to \infty$.

Thus,
$$\frac{\partial F(\rho^2)}{\partial \rho^2} \overset{(3.115)}{=} \frac{\lambda}{12} \langle :|\vec{\varphi}|^2:(0)\rangle_{\rho^2} \nearrow \infty,$$

as $\rho^2 \to \infty$. We conclude that, given any finite constant $M^2 > 0$, there exists some $\rho_M^2 < \infty$ such that

$$\langle :|\vec{\varphi}|^2:(0)\rangle_{\rho^2} \geq M^2 > 0, \quad \text{for } \rho^2 > \rho_M^2 \qquad (3.116)$$

Next, we note that

$$\langle :|\vec{\varphi}|^2:(0)\rangle_{\rho^2} = \lim_{y\to 0} \langle \vec{\varphi}(y)\cdot\vec{\varphi}(0)\rangle_{\rho^2} - \frac{N}{4\pi|y|}$$
$$= \lim_{y\to 0} \langle \vec{\varphi}(y)\cdot\vec{\varphi}(0)\rangle^c_{\rho^2} - \underbrace{\langle \vec{\varphi}(0)\rangle_{\rho^2}^2}_{\phi_c^2} - \frac{N}{4\pi|y|}. \qquad (3.117)$$

The *Källen – Lehmann representation* of the two-point function (see QFT I) says that

$$\langle \vec{\varphi}(y)\cdot\vec{\varphi}(0)\rangle^c_{\rho^2} = N\int_0^\infty d\mu(a^2)\left(-\Delta+a^2\right)^{-1}_{y,0}$$

$$= N\int_0^\infty d\mu(a^2)\ \frac{e^{-a|y|}}{4\pi|y|} \quad (d=3) \qquad (3.118)$$

Since this theory is a *canonical* field theory (there is no field strength renormalization!), i.e.,

$$[\varphi_A(\vec{y},0),\ \pi_A(0,0)] = i\,\delta^{(2)}(\vec{y}),$$

it follows (exercises!) that

$$\int_0^\infty d\mu(a^2) = 1. \qquad (3.119)$$

Thus

$$\langle \vec{\varphi}(y)\cdot\vec{\varphi}(0)\rangle^c_{\rho^2} - \frac{N}{4\pi|y|} = N\int_0^\infty d\mu(a^2)\ \frac{e^{-a|y|}-1}{4\pi|y|}$$

$$\leq 0. \qquad (3.120)$$

Combining (3.116), (3.117) and (3.120), we conclude that

$$0 < M^2 \leq \langle \vec{\varphi}(0)\rangle^2_{\rho^2},\ \text{for } \rho^2 > \rho^2_M, \qquad (3.121)$$

i.e., $\left|\langle\vec{\varphi}(0)\rangle_{\rho^2}\right| > 0$. This proves that the $O(N)$-

symmetry $(\vec{\varphi} = (\varphi_1, \dots, \varphi_N),\ N \geq 1)$ is spontaneously broken for ρ^2 large enough. (Details can be found in Fröhlich, Simon and Spencer, Commun. Math. Phys. 50, (1976).) Only $N = 1, 2, 3$ are controlled rigorously!

These arguments fail in $d = 2$, because ~~Wick-ordering~~ with respect to a massless free field is meaningless, and in $d \geq 4$, because the $\lambda |\vec{\varphi}|^4$ – theory without ultraviolet cutoffs does not make sense. (If a suitable ultraviolet cutoff is introduced, the same conclusions — spontaneous symmetry breaking — hold for arbitrary $d \geq 3$.) In $d = 2$, only the $\varphi \to -\varphi$ symmetry of the one-component $\lambda\varphi^4$ theory can be broken spontaneously, for ρ^2 large enough; but an $O(N)$ – symmetry, with $N \geq 2$, cannot be broken (Mermin – Wagner theorem).

Peierls argument; (G-J-S).

Proof of the Goldstone Theorem in the Euclidian region (following K. Symanzik)

We consider a relativistic QFT of some scalar field φ with a continuous internal symmetry given by a compact Lie group G with Lie algebra $\mathfrak{g}$ and corresponding conserved Noether currents j^μ_X, $X \in \mathfrak{g}$. As an example, we may think of a Lagrangian field theory of an N-component, real scalar field $\vec{\phi}$ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \vec{\phi}\right)\cdot\left(\partial^\mu \vec{\phi}\right) + \frac{\mu^2}{2}\vec{\phi}^{\,2} - \frac{\lambda}{4!}\left(\vec{\phi}\cdot\vec{\phi}\right)^2, \quad (3.122)$$

with $G = O(N)$ and, for $X = E_{ij} - E_{ji}$, $E_{ij} =$ matrix unit, $i, j = 1, \dots, N$,

$$j^\mu_{ij} = \left(\partial^\mu \phi^i\right)\phi^j - \left(\partial^\mu \phi^j\right)\phi^i. \quad (3.123)$$

The field φ transforms under a representation

ρ of G. We set $\tilde{X} := d\rho(X)$, for $X \in \mathfrak{g}$.

Let

$$Q_X := \int_{t=\text{const}} d^{d-1}x \; j^0_X(\vec{x}, t) \tag{3.124}$$

denote the <u>conserved charge</u> corresponding to the conserved current j^μ_X. Then

$$[Q_X, \varphi(x)] = (\tilde{X}\varphi)(x). \tag{3.125}$$

Let Ω_{φ_c} be a <u>vacuum</u> of the theory that breaks the symmetry G of the theory spontaneously, in the sense that

$$\langle \Omega_{\varphi_c} | \varphi(x) | \Omega_{\varphi_c} \rangle = \varphi_c \neq 0. \tag{3.126}$$

Let $H_{\varphi_c} \subset G$ denote the isotropy subgroup of φ_c, and let $\mathfrak{h}_{\varphi_c} \equiv \mathfrak{h}$ denote its Lie algebra.

Then

$$\tilde{X}\varphi_c = 0, \quad \forall X \in \mathfrak{h}. \tag{3.127}$$

For any $Y \in \mathfrak{g} \ominus \mathfrak{h}$, we then have that

$$\tilde{Y}\varphi_c \neq 0. \tag{3.128}$$

Combining (3.125) and (3.128), we find that

$$\begin{aligned}\langle \Omega_{\varphi_c} | [Q_y, \varphi(0)] | \Omega_{\varphi_c} \rangle &= \langle \Omega_{\varphi_c} | (\tilde{y}\,\varphi)(0) | \Omega_{\varphi_c} \rangle \\ &= \tilde{y} \langle \Omega_{\varphi_c} | \varphi(0) | \Omega_{\varphi_c} \rangle = \tilde{y}\,\varphi_c \neq 0,\end{aligned} \quad (3.129)$$

for arbitrary $y \in \mathfrak{g} \ominus \mathfrak{h}$. On the left side of (3.129), we rewrite Q_y using (3.124) and then apply a formula ~~(3.19)~~ from Euclidian field th. ($H \geq 0!$):

$$\begin{aligned}\tilde{y}\,\varphi_c &= \langle \Omega_{\varphi_c} | [Q_y, \varphi(0)] | \Omega_{\varphi_c} \rangle \\ &= \int_{t=0} d^{d-1}x \, \langle \Omega_{\varphi_c} | [j^0_y(\vec{x},0), \varphi(0)] | \Omega_{\varphi_c} \rangle \\ &= \lim_{\varepsilon \searrow 0} \int d^{d-1}x \, \langle \Omega_{\varphi_c} | j^0_y(\vec{x},0) e^{-\varepsilon H} \varphi(0) \\ &\qquad - \varphi(0) e^{-\varepsilon H} j^0_y(\vec{x},0) | \Omega_{\varphi_c} \rangle \\ &\doteq \lim_{\varepsilon \searrow 0} \int d^{d-1}x \, \underbrace{\langle (j^0_y(\vec{x},\varepsilon) - j^0_y(\vec{x},-\varepsilon))\, \varphi(0) \rangle}_{\text{Schwinger fu.}},\end{aligned} \quad (3.130)$$

where the integrand on the R.S. of (3.130) is an imaginary time (Euclidian) Green- or Schwinger function, which, in a Lagrangian scalar field theory, such as that in (3.122), can be expressed by a Wick-rotated $\left(\theta = \frac{\pi}{2}\right)$ functional integral, see (3.16), (3.17). We set

$$W^{\mu}_{y}(x) := \left\langle j^{\mu}_{y}(\vec{x}, t)\, \varphi(0) \right\rangle \tag{3.131}$$

$$= \begin{cases} \left\langle \Omega_{\varphi_c} \middle| j^{\mu}_{y}(\vec{x}, it)\, \varphi(0) \middle| \Omega_{\varphi_c} \right\rangle, & t > 0 \\ \left\langle \Omega_{\varphi_c} \middle| \varphi(0)\, j^{\mu}_{y}(\vec{x}, it) \middle| \Omega_{\varphi_c} \right\rangle, & t < 0 \end{cases}$$

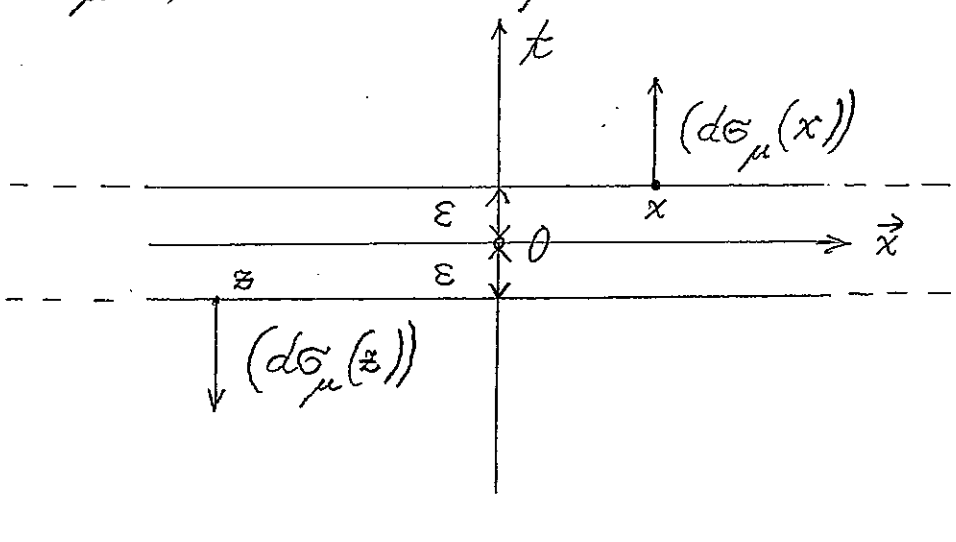
Then, combining (3.129) – (3.131), we find that

$$M_y := \tilde{y}\varphi_c = \lim_{\varepsilon \searrow 0} \int_{S_\varepsilon} d\sigma_\mu(x)\, W^{\mu}_{y}(x) \neq 0, \tag{3.132}$$

where S_ε is the slab

$$S_\varepsilon = \left\{ x \in \mathbb{R}^d \,\middle|\, x = (\vec{x}, t),\ t = \pm\varepsilon \right\}, \tag{3.133}$$

and $d\sigma_\mu(x)$ is the surface element on S_ε:



We now use the fact that j^μ_y is a <u>conserved current</u>, i.e.,

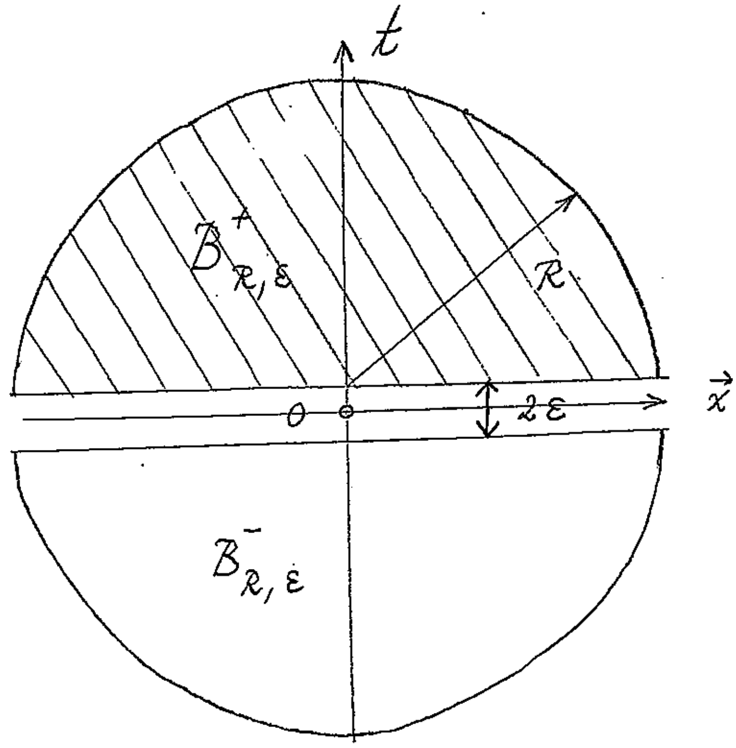
$$\partial_\mu j^\mu_y(x) = 0, \quad \forall y \in \mathfrak{g}; \qquad (3.134)$$

(Noether's theorem; see Chapter 5 of QFT I).

In the Euclidian (imaginary time, $\theta = \frac{\pi}{2}$) region, this implies that

$$\partial_\mu W^\mu_y(x) \overset{!}{=} 0, \text{ for } x \neq 0. \qquad (3.135)$$

Equation (3.135) permits us to apply <u>Gauss' theorem</u>.



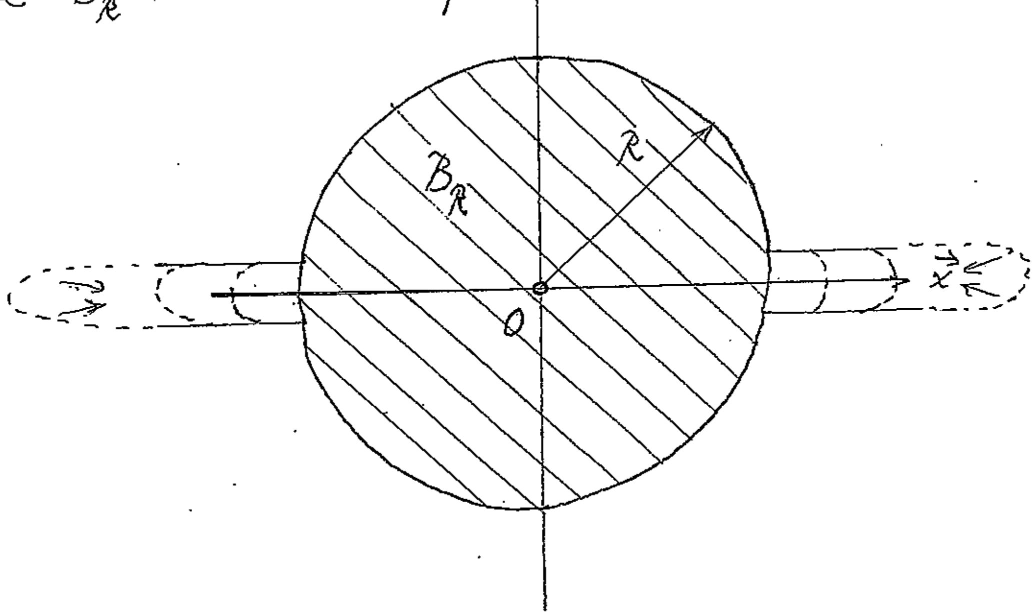
Note that, by (3.135), Gauss' theorem can be applied on $B^+_{R,\varepsilon} \cup B^-_{R,\varepsilon}$. It yields

$$0 \neq M_y = \lim_{\varepsilon \downarrow 0} \int_{S_\varepsilon} d\sigma_\mu(x)\, W_y^\mu(x)$$

$$+ \underbrace{\sum_{\delta = \pm} \int_{\partial B^\delta_{R,\varepsilon}} d\sigma_\mu(x)\, W_y^\mu(x)}_{=\,0,\ \text{by Gauss}}$$

$$= \int_{\partial B_R} d\sigma_\mu(x)\, W_y^\mu(x), \qquad (3.134)$$

for any $R > 0$, (using again Gauss' theorem), where B_R is a ball of radius R centered at 0



Because the vacuum Ω_{φ_c} is Poincaré-invariant, definition (3.131) implies that

$$R^\mu{}_\nu W_y^\nu(Rx) = W_y^\mu(x),$$

for arbitrary $R \in SO(d)$. Thus $x_\mu W_y^\mu(x)$ is

<u>invariant</u> under arbitrary rotations of Euclidian space-time, $\mathbb{E}^d$. Hence

$$W_y^\mu(x) = x^\mu f(|x|), \tag{3.135}$$

(with $|x| = \sqrt{x^2}$). By (3.134), we then find that

$$W_y^\mu(x) = \text{const.}\ \frac{x^\mu}{|x|^d}, \tag{3.136}$$

which also follows from (3.135) and the fact that $\partial_\mu W_y^\mu(x) = 0$, for $x \neq 0$.

Eqs. (3.131) and (3.136) show that φ and j_y^0 couple the vacuum Ω_{φ_c} to a <u>massless one-particle state</u> describing a "Goldstone boson".

In two space-time dimensions, there is no scalar relativistic field, φ, coupling the vacuum to a zero-mass one-particle state, and we therefore conclude that a continuous internal symmetry generated by charges associated with Poincaré-covariant conserved currents <u>cannot be broken spontaneously</u>; ("Mermin-Wagner theorem").

For more details, see S. Weinberg, "The Quantum Theory of Fields", Vol. II, pages 163-191.

1.6 <u>Pions as Goldstone bosons</u> (see also Weinberg)

<u>Isospin symmetry</u> (Heisenberg, Kemmer)

Interpret $\begin{pmatrix} p \\ n \end{pmatrix}$ as isospin $(SU(2))$ doublet, $(\leftarrow P=+)$

and $\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$ as isospin triplet. $(\leftarrow P=-)$

<u>Conjecture:</u> If electroweak int. were turned off then isospin $SU(2)$ would be an <u>exact global symm.</u> of the strong interactions.

<u>But:</u> π much lighter than p, n!

Could π be a <u>Goldstone boson</u> of a spont. broken approximate symm.?

This symm. <u>cannot</u> be isospin $SU(2)$, because this symm. does not appear to be spont. broken in limit where electroweak int. turned off.

In particle physics, we often encounter particle multiplets that are either <u>even</u> or <u>odd</u>

under parity, P, and form multiplets under some symm., such as isospin $SU(2)$. One might expect that $P=+$ and $P=-$ multiplets match. Then one could associate to a global symm. group, G, of the theory <u>two</u> families of

<u>Noether currents</u>:

$$\underbrace{V^{\mu}_{X}(\vec{x},t)}_{\substack{P=+ \\ \text{vector currents}}}, \quad \underbrace{A^{\mu}_{X}(\vec{x},t)}_{\substack{P=- \\ \text{axial vector currents}}}, \qquad X \in \mathfrak{g} = \mathrm{Lie}\, G,$$

with

$$\partial_\mu V^{\mu}_{X} = 0, \quad \partial_\mu A^{\mu}_{X} = 0 \tag{1}$$

and

$$\underbrace{\int d^3x\, A^{0}_{X}(\vec{x},t)}_{=: Q^{-}_{X}} \, |h\rangle_{P=\pm} = |\tilde{X} h\rangle_{P=\mp} \tag{2}$$

hadron G-multiplet.

In concrete models,

$$\partial_\mu A^{\mu}_{X} \propto \text{mass}^2 \neq 0 .$$

This is the PCAC hypothesis. In a theory without a natural mass scale, e.g. <u>QCD with massless quarks</u>, it follows that (1) holds.

<u>But</u>: (2) <u>not</u> observed in nature!

<u>Way out</u>: Symmetry generated by charges $\{Q_X^-\}_{X \in \mathfrak{g}}$ is spontaneously broken.

<u>Algebra of "conserved" charges</u>

$$Q_X^+ = \int d^3x \; V_X^0(\vec{x},t) \qquad (3)$$
$$Q_X^- = \int d^3x \; A_X^0(\vec{x},t).$$

Then

$$[Q_X^+, Q_Y^+] = Q^+_{[X,Y]}$$
$$[Q_X^+, Q_Y^-] = Q^-_{[X,Y]} \qquad (4)$$
$$[Q_X^-, Q_Y^-] = Q^+_{[X,Y]}$$

($\to$ "current algebra"!)

We may alternatively introduce <u>left</u> - and <u>right handed charges</u>:

$$Q^L_X := \tfrac{1}{2}(Q^+_X + Q^-_X)$$
$$Q^R_X := \tfrac{1}{2}(Q^+_X - Q^-_X). \qquad (5)$$

If all particles were <u>massless</u> <u>handedness</u> would be a good quantum number, and the symm. G would act separately on left-handed and right-handed multiplets, with

$$[Q^{L/R}_X, Q^{L/R}_Y] = Q^{L/R}_{[X,Y]}, \quad [Q^L_X, Q^R_Y] = 0, \qquad (6)$$

as follows from (4).

Yet, hadrons are <u>not</u> massless.

In QCD with <u>massless</u> u- and d-quarks, V^μ_X and A^μ_X are conserved, and (4), (6) hold. Yet, one expects that colour singlet hadronic states are <u>massive</u>, ("diml. transmutation").

This is compatible with (4), (6) only if the symmetry, $G^A \simeq SU(2)$, generated by $\{Q^-_X\}_{X \in \mathfrak{g} = su(2)}$ is <u>spontaneously broken</u>, while $G^V \simeq SU(2)$, generated by $\{Q^+_X\}_{X \in su(2)}$, is <u>unbroken</u>.

The Goldstone bosons associated to the broken $SU(2)_{axial}$ ($\underline{3}$ broken generators) are precisely the <u>pions</u>, which form indeed a <u>pseudo-scalar isospin triplet</u>.

Now, pions are not massless. This is due to the fact that the u- and the d-quarks are not strictly massless (thanks to the Higgs field of the weak interactions). Hence, $\partial_\mu A^\mu_X \propto \text{mass}^2 \neq 0$, and G^A is only an <u>approximate</u> global symmetry.

Let $|\pi_j\rangle$, $j = +, 0, -$ denote a one-pion state of 4-momentum p, ($p^2 = m_\pi^2$). Then, by the proof of Goldstone's theorem in Sect. 1.5,

$$\langle \Omega | A_i^\mu(x) | \pi_j \rangle$$

$$= F_\pi \, \delta_{ij} \, p^\mu \, \frac{e^{ip\cdot x}}{2(2\pi)^{3/2}\sqrt{2p_0}} \,, \qquad (7)$$

see (3.136). Experimentally,

$$F_\pi \simeq 184 \, MeV. \qquad (8)$$

F_π is related to the pion decay constant:

$$\underset{\substack{\uparrow \\ \text{decay rate}}}{\Gamma(\pi \to \mu + \nu)} = \frac{G_{weak}^2 \, F_\pi^2 \, m_\mu^2 \, (m_\pi^2 - m_\mu^2)^2}{16\pi \, m_\pi^3} \qquad (9)$$

$$G_{weak} \simeq 1.15 \times 10^{-5} \; GeV^{-2}$$

For a caricature of lattice QCD, Salmhofer and Seiler have proven the spontaneous breaking of symms. such as $SU(2)^A$ by showing that $\langle \Omega | \bar{\psi}\psi(x) | \Omega \rangle \neq 0$

$$\text{while} \qquad \langle \Omega | \bar{\psi}\gamma^5\psi(x) | \Omega \rangle = 0, \tag{9}$$

where $\psi \overset{\text{e.g.}}{=} \begin{pmatrix} u \\ d \end{pmatrix}$, $\bar{\psi} = \psi^* \gamma^0$, Ω: vacuum; $d \geq 4$

This is called <u>chiral symmetry breaking.</u> *)

It is the special case where $G = U(1)$. Their methods also work for $G = (S)U(N)$, $N \leq 4$, in dimension $d \geq 4$.

S. & S. used methods (RP & infrared bounds) that I am going to explain next.

<u>History:</u> Nambu, Nambu & Jona-Lasinio, Gell-Mann, Goldstone et al.; founders of QCD.

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*) $SU(2)_V \times SU(2)_A$ symm.: $m_u = m_d = 0$; then

$$\begin{pmatrix} u \\ d \end{pmatrix} \longmapsto \exp\left(X^V + \gamma^5 Y^A\right) \begin{pmatrix} u \\ d \end{pmatrix}, \text{ or}$$

$$\begin{pmatrix} u_{L/R} \\ d_{L/R} \end{pmatrix} \longmapsto \exp\left(X^{L/R}\right) \begin{pmatrix} u_{L/R} \\ d_{L/R} \end{pmatrix}$$

2. Phase transitions and continuous symmetry breaking in classical lattice systems and FT

2.1 Motivation: See Sect 1.5 ($\lambda|\vec{\phi}|^4$-theory), and following: Consider a canonical QFT with herm. field operators $\varphi(\vec{x},t)$, Hamiltonian $H \geq 0$ and groundstate (vacuum) $\Omega \in \mathcal{H}$, ($\mathcal{H}$: Hilbert space); $H\Omega = 0$. Formally

$$H = \int d^{d-1}x \left\{ \frac{1}{2} \pi(\vec{x},0)^2 + U\left(\vec{\nabla}\varphi(\vec{x},0), \varphi(\vec{x},0)\right) \right\} \quad (1)$$

$$[\pi(\vec{x},0), \varphi(\vec{y},0)] = -i\delta(\vec{x}-\vec{y}) \quad (\hbar = 1) \quad (2)$$

Heisenberg's commutation relations:

$$\varphi(\vec{x},t) = e^{itH} \varphi(\vec{x},0) e^{-itH} \quad (3)$$

$$\pi(\vec{x},t) \overset{(1)}{=} i[H, \varphi(\vec{x},t)] = \dot{\varphi}(\vec{x},t) \quad (4)$$

Let $\varphi(f_t, t) := \int d^{d-1}x \, f_t(\vec{x}) \, \varphi(\vec{x},t), \quad (5)$

where f_t is a real test function.

Then

$$
\begin{aligned}
e^{i\varphi(f_t,t)} H e^{-i\varphi(f_t,t)} &= H + i\,[\varphi(f_t,t), H] \\
&\quad - \frac{1}{2}\,[\varphi(f_t,t),[\varphi(f_t,t),H]] + \cdots \\
&\overset{(2),(4)}{=} H - \pi(f_t,t) + \frac{1}{2}\int d^{d-1}x\,(f_t(\vec{x}))^2
\end{aligned}
$$

Since $H \geq 0$ and $e^{i\varphi(f_t,t)}$ unitary (f_t real-valued!), it follows that $e^{i\varphi(f_t,t)} H e^{-i\varphi(f_t,t)} \geq 0$, hence

$$H - \pi(f_t,t) + \frac{1}{2}\|f_t\|_2^2 \geq 0 \qquad (6)$$

(Araki, Glimm & Jaffe, Herbst)

There is an abundance of such QFT's that can be controlled mathematically, as long as one does not insist on relativistic invariance. Moreover, the Wick rotation, $t \mapsto it$, is meaningful, so that one has a Euclidian field theory:

$$\phi(\vec{x}, t) := \varphi(\vec{x}, it) = e^{-tH} \varphi(\vec{x}, 0) e^{tH}$$

+ associated path integral formalism, path space measure $d\mu(\phi)$, with

$$\left\langle \Omega, \prod_{j=1}^{n} \varphi(\vec{x}_j, it_j)\, \Omega \right\rangle \underset{\uparrow}{=} \int \prod_{j=1}^{n} \phi(\vec{x}_j, t_j)\, d\mu(\phi) \quad (7)$$

$$t_1 < t_2 < \cdots < t_n$$

Choose $\{f_t\}_{t \in \mathbb{R}}$ such that $\|f_t\|_2^2 \underset{|t| \to \infty}{\longrightarrow} 0$, rapidly. We set $f(\vec{x}, t) := f_t(\vec{x})$, with

$$\|f\|_2^2 = \int dt \int d^{d-1}\vec{x}\, (f(\vec{x}, t))^2. \quad (8)$$

<u>Def.</u> $$H\left(\tfrac{k}{N}\right) := H - \pi\left(f_{t=\frac{k}{N}}, \tfrac{k}{N}\right),$$

$k \in \mathbb{Z}$, $N = 1, 2, 3, \cdots$.

Then (6) implies that

$$\left\langle \Omega, \prod_{k=-\infty}^{\infty} e^{-\frac{1}{N}\left(H\left(\frac{k}{N}\right) + \frac{1}{2}\|f_{\frac{k}{N}}\|_2^2\right)}\, \Omega \right\rangle \leqslant 1, \quad (9)$$

or

$$\lim_{N \to \infty} \left\langle \Omega, \prod_{k=-\infty}^{\infty} e^{-\frac{1}{N} H\left(\frac{k}{N}\right)} \Omega \right\rangle \leq e^{\frac{1}{2} \|f\|_2^2}, \qquad (10)$$

with $\|f\|_2^2$ as in (8). The L.S. of (10) is then given by

$$\int e^{-\int d^d x \, \dot{\phi}(x) f(x)} \, d\mu(\phi)$$

$$= \int e^{\int d^d x \, \phi(x) \dot{f}(x)} \, d\mu(\phi),$$

and hence

$$\left\langle e^{\phi(\dot{f})} \right\rangle \leq e^{\frac{1}{2} \|f\|_2^2} \qquad (11)$$

<u>If</u> the theory were <u>relativistic</u> then <u>"Nelson's symmetry"</u> (Euclidian invariance) would readily imply that

$$\left\langle e^{\vec{\nabla}\phi(\vec{f})} \right\rangle \leq e^{\frac{1}{2} \|\vec{f}\|_2^2} \qquad (12)$$

This is the archetypical example of an <u>infrared bound.</u> It implies immediately that

$$\langle \hat{\phi}(k)\hat{\phi}(-k)\rangle \leq \frac{1}{k^2}, \tag{13}$$

which also follows from the Källen–Lehmann rep. of the two-point function.

2.2 Lattice approximation to canonical QFT = classical lattice spin systems

Let $\Lambda = [-L, L]_{1/2} \subset \mathbb{Z}^d$ be a cube with sides of length $2(L+1)$, $L = 0, 1, 2, \cdots$; where

$$[-L, L]_{1/2} := \left\{ k + \tfrac{1}{2} \,\middle|\, -L-1 \leq k \leq L,\ k \in \mathbb{Z} \right\}$$

At each site $x \in \Lambda$, there is a random-variable $\phi_x \in \mathbb{R}^N$ with an a priori distribution

$$d\mu(\phi_x), \tag{14}$$

$d\mu$ a probability measure on $\mathbb{R}^N$.

We introduce an "action functional" or "Hamiltonian"

$$A(\phi_\Lambda) := \sum_{\langle x,y\rangle \subset \Lambda} \frac{1}{2}(\phi_x - \phi_y)^2, \tag{15}$$

where $\langle x,y\rangle$ denotes a nearest-neighbor pair of sites, and periodic b.c. are imposed at $\partial\Lambda$. The Gibbs state of the system at inv. temperature β is given by

$$\underbrace{dP(\phi_\Lambda)}_{\langle(\cdot)\rangle_\Lambda} := \frac{1}{Z_\beta} e^{-\beta A(\phi_\Lambda)} \prod_{x\in\Lambda} d\mu(\phi_x), \tag{16}$$

where
$$Z_\beta = \int e^{-\beta A(\phi_\Lambda)} \prod_{x\in\Lambda} d\mu(\phi_x)$$

Let

$$A(\phi_\Lambda, \vec{h}) := \sum_{\langle x,y\rangle \subset \Lambda} \frac{1}{2}\left(\phi_x - \phi_y + \beta^{-1} h_{xy}\right)^2$$

$$= A(\phi_\Lambda) + \sum_{\langle x,y\rangle\subset\Lambda} \beta^{-1}(\phi_x - \phi_y)\cdot h_{xy}$$

$$+ \frac{1}{2\beta^2}\|\vec{h}\|_2^2, \tag{17}$$

and

$$Z_\beta(\vec{h}) := \int e^{-\beta A(\phi_\Lambda, \vec{h})} \prod_{x\in\Lambda} d\mu(\phi_x) \tag{18}$$

Let

$$\vec{\nabla}\phi(\vec{h}) := \sum_{\langle x,y\rangle \subset \Lambda} (\phi_x - \phi_y)\cdot h_{xy}.$$

Then

$$\left\langle e^{\vec{\nabla}\phi(\vec{h})} \right\rangle = \frac{Z_\beta(\vec{h})}{Z_\beta}\, e^{\frac{1}{2\beta}\|\vec{h}\|_2^2} \tag{19}$$

From (12) we might expect that

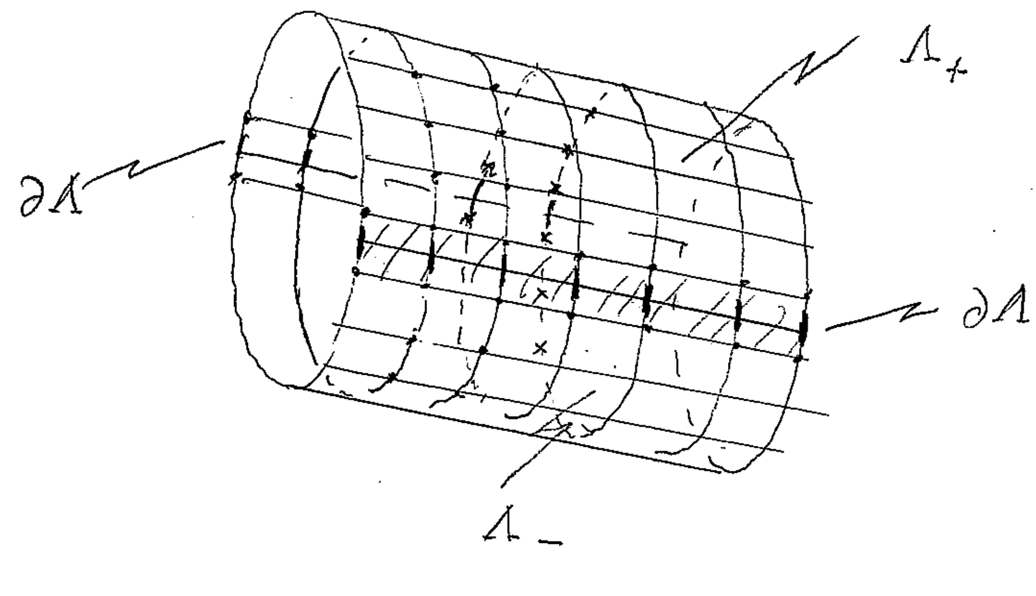
$$\left\langle e^{\vec{\nabla}\phi(\vec{h})} \right\rangle \leq e^{\frac{1}{2\beta}\|\vec{h}\|_2^2} \tag{20}$$

(by a simple rescaling $\phi = \sqrt{\beta}\,\chi$!).

This would follow if we were able to prove the

<u>Theorem 1.</u> $$Z_\beta(\vec{h}) \leq Z_\beta. \tag{21}$$

<u>Proof:</u> We cut Λ into two halves:



$$\Lambda_+ := \left\{ x \in \Lambda \,\middle|\, \tfrac{1}{2} \leq x^\alpha \leq L + \tfrac{1}{2} \right\}$$

$$\Lambda_- := \left\{ x \in \Lambda \,\middle|\, -L - \tfrac{1}{2} \leq x^\alpha \leq -\tfrac{1}{2} \right\} \tag{22}$$

$$\partial\Lambda := \left\{ \langle x, y \rangle \subset \Lambda \,\middle|\, x^\alpha = \tfrac{1}{2},\ y^\alpha = -\tfrac{1}{2}, \text{ or } x^\alpha = L + \tfrac{1}{2},\ y^\alpha = -L - \tfrac{1}{2} \right\}$$

$$A_\pm(\phi, \vec{h}) := \frac{1}{2} \sum_{\langle x,y \rangle \subset \Lambda_\pm} \left(\phi_x - \phi_y + \beta^{-1} h_{xy} \right)^2,$$

(with free b.c. at $\partial\Lambda$), and

$$A_0(\phi, \vec{h}) := \frac{1}{2} \sum_{\langle x,y \rangle \subset \partial\Lambda} \left(\phi_x - \phi_y + \beta^{-1} h_{xy} \right)^2$$

Then

$$A(\phi_\Lambda, \vec{h}) = A_+(\phi, \vec{h}) + A_0(\phi, \vec{h}) + A_-(\phi, \vec{h}), \tag{23}$$

and

$$Z_\beta(\vec{h}) = \iint \Big(\prod_{x \in \Lambda_+} d\mu(\phi_x) \Big) e^{-\beta A_+(\phi, \vec{h})} e^{-\beta A_0(\phi, \vec{h})} \Big(\prod_{x \in \Lambda_-} d\mu(\phi_x) \Big) e^{-\beta A_-(\phi, \vec{h})} \tag{24}$$

After integrating over all ϕ_x, with x such that $\nexists\, y$ with $\langle x, y \rangle \subset \partial\Lambda$, the remaining integral on the R.S. of (24) has the form

$$I := \iint d^n X \, d^n Y \; f(X) \, e^{-\frac{\beta}{2}(X - Y + H)^2} g(Y), \quad (25)$$

where $n = 2(2L+1)^{d-1}$, $X, Y, H \in \mathbb{R}^n$; (if $d\mu(\phi)$ is not absolutely cont. w.r. to $d^N\phi$ one may approximate it on $C(\mathbb{R}^N)$ by a measure that is abs. cont.). By Fourier transformation

$$I = \int d^n K \; \overline{\hat{f}(K)} \, e^{-\frac{1}{2\beta} K^2} e^{i H \cdot K} \hat{g}(K).$$

Thus

$$|I| \leq \int d^n K \, |\hat{f}(K)| \, e^{-\frac{1}{2\beta} K^2} |\hat{g}(K)|$$

Schwarz

$$\leq \sqrt{\iint d^n X \, d^n Y \; \overline{f(X)} \, e^{-\frac{\beta}{2}(X-Y)^2} f(Y)}$$

$$\times \sqrt{\iint d^n X \, d^n Y \; \overline{g(X)} \, e^{-\frac{\beta}{2}(X-Y)^2} g(Y)} \quad (26)$$

Let

$$\vec{h}_{\pm} := \{ h_{xy} \mid h_{xy} = 0 \text{ unless } \langle x, y \rangle \subset \Lambda_{\pm} \}$$

Let:

$$\Theta_\alpha x = (x^1, \dots, -x^\alpha, \dots, x^d)$$

denote the reflection of x at the plane $x^\alpha = 0$, and define

$$\left(\theta_\alpha \vec{h}\right)_{xy} = \vec{h}_{\theta_\alpha x\, \theta_\alpha y}. \tag{27}$$

Then Eqs. (24) and (26) imply that

$$Z_\beta(\vec{h}) \leq \sqrt{Z_\beta(\vec{h}_+ + \theta_\alpha \vec{h}_+)}\,\sqrt{Z_\beta(\vec{h}_- + \theta_\alpha \vec{h}_-)}, \tag{28}$$

with

$$\left(\vec{h}_\pm + \theta_\alpha \vec{h}_\pm\right)_{xy} = 0, \text{ for } \langle xy\rangle \in \partial\Lambda. \tag{29}$$

Since we have imposed periodic b.c. at $\partial\Lambda$, we can repeat the same estimate in both factors on the R.S. of (28), but replacing the plane of reflection, $\{x^\alpha = 0\}$, by any other plane of reflection $\{x^\alpha = k \mid -L \leq k \leq L+1,\ k \neq 0\}$. Moreover, we may let α range over $\{1, \dots, d\}$. Ultimately, this means that $\partial\Lambda$ can be chosen to be an <u>arbitrary</u> double plane of nearest neighbor sites. Thus, iterating (28) many times, we find that

$$Z_\beta(\vec{h}) \leq Z_\beta(\vec{h}=0) = Z_\beta ,$$

which completes the proof of Theorem 1.

<u>Corollary 2.</u> Let Δ be the finite-difference Laplacian. Then

$$\langle \phi(f)\,\phi(-\Delta f)\rangle_\Lambda \leq \beta^{-1} \|f\|_2^2 . \tag{30}$$

<u>Proof.</u>

Let ∂_α be the finite difference derivative in direction α and ∂_α^* its adjoint on $l_2(\Lambda)\otimes\mathbb{C}^N$.

Then
$$-\Delta = \sum_{\alpha=1}^{d} \partial_\alpha^* \partial_\alpha = \sum_{\alpha=1}^{d} \partial_\alpha \partial_\alpha^* . \tag{31}$$

We set

$$\vec{h} = \left(\partial_1^* \Delta^{-1/2} f, \dots, \partial_d^* \Delta^{-1/2} f\right) \tag{32}$$

Then, by (20),

$$\left\langle e^{\vec{\nabla}\phi(\vec{h})} \right\rangle_\Lambda \leq e^{1/2\beta \, \|\vec{h}\|_2^2} ; \tag{33}$$

but $\|\vec{h}\|^2 = \|f\|_2^2$, by (31) and (32)!

Furthermore, by translation invariance,

$$\langle \vec{\nabla}\phi(\vec{h}) \rangle_\Lambda = 0 . \tag{34}$$

Consider $\varepsilon^{-2}\left(\langle e^{\varepsilon \vec{\nabla}\phi(\vec{h})} \rangle_\beta - 1\right)$. By (33),

$$\varepsilon^{-2}\left(\langle e^{\varepsilon \vec{\nabla}\phi(\vec{h})} \rangle_\Lambda - 1\right) \leq \varepsilon^{-2}\left(e^{(\varepsilon^2/2\beta)\|f\|_2^2} - 1\right)$$

Next let ε tend to 0, using (34). Then

$$\langle |\vec{\nabla}\phi(\vec{h})|^2 \rangle_\Lambda \leq \frac{1}{\beta} \|f\|_2^2 ,$$

or, with (32) and (31),

$$\langle \phi(\Delta^{1/2} f)^2 \rangle_\Lambda \leq \frac{1}{\beta} \|f\|_2^2 ,$$

i.e., if $\hat{g}(k=0) = 0$ then

$$\langle \phi(g)^2 \rangle_{\Lambda \nearrow \mathbb{Z}^d} \leq \frac{1}{\beta} \int d^d k \, \frac{|\hat{g}(k)|^2}{2\left(\sum_{\alpha=1}^{d} 1 - \cos k_\alpha\right)} , \tag{35}$$

which is the <u>FSS infrared bound!</u>

Everything else in the theory of R.P. and IR bounds is a rather straightforward generalization of this story!

2.3 Application to the classical Heisenberg model.

$\phi_x \in \mathbb{R}^N$,

$$d\mu(\phi_x) := \text{const.}\, \delta\left(|\phi_x|^2 - 1\right) d^N\phi_x \, ; \qquad (36)$$

everything else as above. Pass directly to T'D limit, with

$$\langle (\cdot) \rangle_\beta = \lim_{\Lambda \nearrow \mathbb{Z}^d} \langle (\cdot) \rangle_\Lambda \, , \; \langle \phi_x \rangle_\beta = 0 \, .$$

Let $\hat{\phi}(k)$, $k \in \mathbb{T}^d$, be F.T. of ϕ_x. Since $\langle (\cdot) \rangle_\beta$ given by pos. measure,

$$\left\langle |\hat{\phi}(k)|^2 \right\rangle_\beta = \left\langle \hat{\phi}(k) \cdot \hat{\phi}(-k) \right\rangle_\beta \geq 0 .$$

By (35)

$$0 \leq \left\langle |\hat{\phi}(k)|^2 \right\rangle_\beta \leq M_\beta^2 \, \delta_0(k) + \frac{N}{2\beta \left(\sum_{\alpha=1}^{d} 1 - \cos k_\alpha \right)} \qquad (37)$$

$M_\beta^2 \geq 0$ is LRO. By F.T. back to x-space

$$\langle \phi_0 \cdot \phi_x \rangle_\beta \xrightarrow[|x| \to \infty]{} M_\beta^2 \, . \qquad (38)$$

By standard high-temperature (cluster) expansions (↗ D. Ueltschi!):

$$M_\beta = 0, \text{ for } \beta \text{ small enough.}$$

By (37)

$$0 \leq \langle \phi_0^2 \rangle_\beta \leq M_\beta^2 + \frac{N}{\beta} \underbrace{\int_{\mathbb{T}^d} \frac{d^d k}{2\left(\sum_{\alpha=1}^{d} 1 - \cos k_\alpha\right)}}_{I(d)} \qquad (39)$$

$I(d)$ diverges, for $d = 1, 2$; but

$$I(d) < \infty, \quad d \geq 3,$$

with

$$I(3) = 0.505462019 7 \cdots$$

(↗ Watson). By (36), $\phi_0^2 = 1$; hence

$$1 \leq M_\beta^2 + \frac{N}{\beta} I(d) \qquad (40)$$

It follows that, for $d \geq 3$,

$$M_\beta^2 > 0, \qquad (41)$$

if $\beta > N\, I(d)$.

<u>Decomposition into pure phases.</u>

It can be shown (see, e.g., Lanford, Les Houches

1970; see also J.F., 1974; J. Yngvason, 1975)

$$\langle (\cdot) \rangle_\beta = \int_\Sigma d\nu(\sigma) \, \langle (\cdot) \rangle_\beta^\sigma \,, \tag{42}$$

where $d\nu$ is a probability measure on a compact measure space Σ, and $\langle (\cdot) \rangle_\beta^\sigma$ is <u>ergodic</u> under lattice translations; hence

$$\frac{1}{|\Lambda|^2} \sum_{\substack{x \in \Lambda \\ y \in \Lambda}} \langle \phi_x \cdot \phi_y \rangle_\beta^\sigma \xrightarrow[\Lambda \nearrow \mathbb{Z}^d]{} |\langle \phi_0 \rangle_\beta^\sigma|^2$$

Then

$$\underbrace{\lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|^2} \sum_{x, y \in \Lambda} \langle \phi_x \cdot \phi_y \rangle_\beta}_{= M_\beta^2} = \int_\Sigma d\nu(\sigma) |\langle \phi_0 \rangle_\beta^\sigma|^2$$

Thus, for $M_\beta^2 > 0$,

$$\langle \phi_0 \rangle_\beta^\sigma \neq 0, \tag{43}$$

for a set of σ's of positive measure, (w.r. to $d\nu$). $\longrightarrow$ <u>Spontaneous symmetry breaking</u>, $\exists$ of <u>Goldstone mode</u>!

(1) <u>Structure of Σ</u>

(1.1) <u>$d=2$, $N=1$ (Ising model):</u> $\Sigma = \{+,-\}$

in broken symm. phase. This result is due to M. Aizenman.

(1.2) <u>$d \geq 3$, $N=1$, $N=2$ (plane rotor model):</u>

Suppose β_0 is such that $M_{\beta_0} > 0$ and free energy density, f_β, is differentiable in β at β_0; (differentiability of f_β holds at all, but possibly countably many values of β, because βf_β is <u>concave</u> in β). Then

for $N=1$, $\Sigma = \{+,-\}$;

for $N=2$, $\Sigma = S^1$.

This result is due to J.F. and C.-E. Pfister.

For $N \geq 3$, similar results are expected, but are not proven rigorously.

(2) "Uniqueness" of $\langle(\cdot)\rangle_{\beta,h}$ in ext. magnetic field, h.

$N = 1, 2$ or 3;

$$d\mu(\phi_x) = \text{const.}\, e^{h\phi_x^1}\, \delta(|\phi_x|^2 - 1)\, d^N\phi_x$$

Then, for $h \neq 0$,

$$\langle \phi_{x_1}^{\alpha_1} \cdots \phi_{x_n}^{\alpha_n} \rangle^c_{\beta,h} \quad \text{is real-analytic}$$

in h and exhibits exponential clustering.

This is a consequence of the Lee-Yang theorem pointed out by J.F. & J.-F. Rodriguez.

(3) $\eta \geq 0$, $\nu \leq 1$.

(4) Decay of correlations in the 2D XY (rotor) model

$\phi_x = (\phi_x^1, \phi_x^2) \in S^1$.

Parametrize S^1 by an angle $\theta \in [0, 2\pi)$,

$d\mu(\phi_x) := d\theta_x$

$$A(\phi_\Lambda) = -\sum_{\langle x,y\rangle \subset \Lambda} \underbrace{\cos(\theta_x - \theta_y)}_{\phi_x \cdot \phi_y} + h \sum_x \underbrace{\cos\theta_x}_{\phi_x^1} \quad (44)$$

No phase transitions and exponential clustering of connected correlations, for $h \neq 0$: See (2), above.

Next, consider $\underline{h = 0,\ d = 2}$:

$$\langle \phi_0 \cdot \phi_x \rangle_\Lambda = \frac{1}{Z_\Lambda} \int \prod_{\langle u,v \rangle \subset \Lambda} e^{\beta \cos(\theta_u - \theta_v)} \, e^{i(\theta_0 - \theta_x)} \prod_{j \in \Lambda} d\theta_j \,, \tag{45}$$

because $\langle \sin(\theta_0 - \theta_x) \rangle_\Lambda = 0$

We shift the θ-integration contours into the complex plane, using analyticity in all θ_j's and periodicity in $\mathrm{Re}\,\theta_j$, $j \in \Lambda$, of the integrand:

$$\theta_j := \psi_j + i\alpha_j \,, \tag{46}$$

$\psi_j \in [0, 2\pi)$, $\alpha_j \in \mathbb{R}$ fixed. We choose α_j as follows:

$$\alpha_j := \begin{cases} \varepsilon\left(\ln(|x|+1) - \ln(|j|+1)\right), & |j| \leq |x|, \\ 0, & |j| > |x|. \end{cases} \tag{47}$$

Then

$$\cos(\theta_u - \theta_v) = \cos(\psi_u - \psi_v)\cosh(\alpha_u - \alpha_v) - i\sin(\psi_u - \psi_v)\sinh(\alpha_u - \alpha_v), \tag{48}$$

for arb. $\langle u, v\rangle \subset \Lambda$, and

$$e^{i(\theta_0 - \theta_x)} \underset{(47)}{\overset{}{=}} e^{i(\psi_0 - \psi_x)} e^{-\varepsilon \ln(|x|+1)} \tag{49}$$

Next, for $\langle u,v\rangle \subset \Lambda$, with $\min(|u|,|v|) \leq |x|$,

$$|\alpha_u - \alpha_v| < \varepsilon \frac{1}{\min(|u|,|v|)+1} \leq \varepsilon$$

Thus

$$\cosh(\alpha_u - \alpha_v) \leq 1 + c\,\varepsilon^2 \left(\frac{1}{\min(|u|,|v|)+1}\right)^2, \tag{50}$$

for some $1 < c < \infty$, with $c \searrow 1$, as $\varepsilon \searrow 0$.

Hence

$$0 \underset{\substack{\uparrow \\ \text{Ginibre}}}{\leqslant} \langle \phi_0 \cdot \phi_x \rangle_\Lambda$$

$$\underset{\substack{\uparrow \\ (48),(49)}}{=} \frac{1}{Z_\Lambda} \int e^{-\beta A(\psi_\Lambda)} \prod_{\langle u,v \rangle} \left\{ e^{\beta \cos(\psi_u - \psi_v)[\cosh(\alpha_u - \alpha_v) - 1]} \times \right.$$

$$\left. \times e^{i\beta \sin(\psi_u - \psi_v) \sinh(\alpha_u - \alpha_v)} \right\} e^{i(\psi_0 - \psi_x)} e^{-\varepsilon \ln(|x|+1)} \times$$

$$\times \prod_{j \in \Lambda} d\psi_j . \tag{51}$$

Taking absolute values under the integral on the R.S. of (51) and using (50) and $|\cos(\psi_u - \psi_v)| \leqslant 1$, we find that

$$0 \leqslant \langle \phi_0 \cdot \phi_x \rangle_\Lambda \leqslant \exp\Big[\beta c \varepsilon^2 \sum_{\substack{\langle u,v \rangle, \\ \min(|u|,|v|) \leqslant |x|}} \Big(\frac{1}{\min(|u|,|v|) + 1}\Big)^2\Big]$$

$$\times e^{-\varepsilon \ln(|x|+1)} \tag{52}$$

Next,

$$c \sum_{\substack{\langle u,v \rangle, \\ \min(|u|,|v|) \leqslant |x|}} \Big(\frac{1}{\min(|u|,|v|)+1}\Big)^2 \underset{\substack{\uparrow \\ \underline{d=2}}}{\leqslant} C \ln(|x|+1) \tag{53}$$

Minimizing the function $C\beta\varepsilon^2 - \varepsilon$ w.r. to ε, we find that

$$0 \leq \langle \phi_0 \cdot \phi_x \rangle_\Lambda \leq e^{-(1/4C\beta)\ln(|x|+1)} = \left(\frac{1}{|x|+1}\right)^{1/4C\beta} \qquad (54)$$

<u>Mc Bryan – Spencer upper bound</u> (1977).

This is a more quantitative version of the Mermin – Wagner theorem.

The idea with the complex translations plays a crucial rôle in the proof of existence of a Kosterlitz – Thouless transition in the 2D XY model, due to J.F. & T. Spencer (1980).

3. General Theory of Reflection Positivity and Infrared Bounds.

3.1 General (quantum) lattice systems

With each site, x, of a lattice $\Gamma \overset{\text{e.g.}}{=} \mathbb{Z}^d$, we associate a C^*-algebra, $\mathcal{A}_x$, of bounded operators, with unit 1_x.

For quantum lattice systems, $\mathcal{A}_x$ is typically a full matrix algebra,

$$\mathcal{A}_x \simeq M_n(\mathbb{C}), \quad n < \infty, \tag{1}$$

for classical lattice systems, $\mathcal{A}_x$ is typically given by

$$\mathcal{A}_x \simeq C(M), \tag{2}$$

where M is a compact Hausdorff space. With an arb. finite subset $X \subset \Gamma$, we associate the C^*-algebra

$$\mathcal{A}_X = \bigotimes_{x \in X} \mathcal{A}_x \tag{3}$$

If $X' \subset X$ we view $\mathcal{A}_{X'}$ as a subalgebra of $\mathcal{A}_X$ by

$$\mathcal{A}_{X'} = \Big(\bigotimes_{x \in X'} \mathcal{A}_x \Big) \otimes \Big(\bigotimes_{x \in X \setminus X'} \mathbb{1}_x \Big);$$

"<u>local algebras</u>".

If a is a transl. in Γ then $X+a$ denotes the translation of X by a, and τ_a denotes the natural isomorphism from $\mathcal{A}_X$ to $\mathcal{A}_{X+a}$.

$$\mathcal{A} := \overline{\bigvee_{X \subset \Gamma} \mathcal{A}_X}^{\|(\cdot)\|} \qquad (4)$$

is the algebra of all "quasi-local observables".

Classically,

$$\mathcal{A} \simeq C\Big(\underbrace{\bigtimes_{x \in \Gamma} M_x}_{\equiv \Omega} \Big), \quad M_x \simeq M.$$

(Stone - Weierstrass).

A <u>state</u> ρ on $\mathcal{A}$ is a positive, linear functional on $\mathcal{A}$ normalized such that

$$\rho(\mathbb{1}) = 1.$$

(Classically: ρ = regular Borel prob. measure on $\bigtimes_{x \in \Gamma} M_x$.)

We define a special state, tr, on $\mathcal{A}$ as follows:

In QM, see (1),

$$tr_x(\cdot) := \frac{1}{n} Tr(\cdot) \tag{5}$$

on $\mathcal{A}_x \simeq M_n(\mathbb{C})$, where Tr is the standard trace on $M_n(\mathbb{C})$, and

$$tr(\cdot) := \bigotimes_{x \in \Gamma} tr_x(\cdot) \tag{6}$$

Classically, see (2), we choose a prob. measure $d\mu$ on M and set

$$tr(\cdot) := \prod_{x \in \Gamma} d\mu\Big|_{M_x \simeq M} \tag{7}$$

Interactions

An interaction is a function on $\mathcal{P}_f(\Gamma)$ (= all finite subsets of Γ) with values in $\mathcal{A}$ such that

(i) $\Phi : X \longmapsto \Phi(X) \in \mathcal{A}_X$

(ii) $\Phi(X+a) = \tau_a(\Phi(X)), \quad \forall a,$

(translation invariance)

and

(iii) $\Phi(X)^* = \Phi(X), \quad \forall X \in \mathcal{P}_f(\Gamma)$.

(iv) $\sum_{X \ni 0} \| \Phi(X) \| =: \| \Phi \|_0 < \infty$.

Interactions form a Banach space, $\mathcal{B}$, with norm $\|(\cdot)\|_0$.

<u>Symmetries.</u> Let G be a compact topological group acting as a group of *automorphisms, σ_g, $g \in G$, on $\mathcal{A}_0$. Then σ_g can be extended to *automorphisms of $\mathcal{A}_x$ by

$$\sigma_g(a) := \tau_x(\sigma_g(a_0)),$$

where $\mathcal{A}_x \ni a = \tau_x(a_0)$, with $a_0 \in \mathcal{A}_0$. Hence $\{\sigma_g \mid g \in G\}$ can be extended to all of $\mathcal{A}$.

G is a <u>symmetry</u> of an <u>interaction</u> $\Phi \in \mathcal{B}$ iff

$$\tau_g(\Phi(X)) = \Phi(X), \qquad (8)$$

$\forall g \in G, \ X \in \mathcal{P}_f(\Gamma)$.

<u>Hamiltonians</u>

Let Λ be a finite subset of Γ. We define

$$H_\Lambda^\Phi := \sum_{X \cap \Lambda \neq \emptyset} \Phi(X), \qquad (9)$$

(possibly with periodic b.c. if Λ is a cube).

Formerly, H_Λ^Φ has been denoted by A.

Let ω_{Λ^c} be a state on $\overline{\bigvee_{X \cap \Lambda = \emptyset} \mathcal{A}_X}^{\|\cdot\|}$;

($\to$ b.c.!). We define an <u>equilibrium state</u> at inverse temperature β to be a state given by

$$\rho_\Lambda^{\beta,\Phi}(a) := \frac{1}{Z_\Lambda^{\beta,\omega}} \, \omega_{\Lambda^c} \otimes \mathrm{tr}\left(e^{-\beta H_\Lambda^\Phi} a\right), \qquad (10)$$

$a \in \mathcal{A}_\Lambda$, where

$$Z_\Lambda^{\beta,\omega} := \omega_{\Lambda^c} \otimes \mathrm{tr}\left(e^{-\beta H_\Lambda^\Phi}\right). \qquad (12)$$

<u>Time evolution</u>

$$\alpha_t^\Lambda(a) := e^{itH_\Lambda^\Phi} a \, e^{-itH_\Lambda^\Phi}, \qquad (13)$$

$a \in \mathcal{A}_\Lambda$.

<u>Lemma 1.</u> If $\sum_{X \ni 0} e^{r|X|} \|\Phi(X)\| =: \|\Phi\|_r < \infty,$ (14)

for some $r > r(d)$, $d = \dim T$, then

$$n\text{-}\lim_{\Lambda \nearrow T} \alpha_t^{\Lambda}(a) =: \alpha_t(a) \quad (15)$$

exists, for all $a \in \mathcal{A}$.

<u>Proof.</u> See, e.g., R. B. Israel (~1975).

In QM, we may then define an <u>infinite-volume</u> $(\Lambda \nearrow \mathbb{Z}^d)$ <u>equilibrium state</u> as a state, $\rho^{\beta,\Phi}$, on $\mathcal{A}$ satisfying the <u>KMS condition</u>:

$$\rho^{\beta,\Phi}(a\,\alpha_t(b)) = \rho^{\beta,\Phi}(\alpha_{t-i\beta}(b)\,a) \quad (15)$$

<u>Proposition 2.</u> (J.F., generalizing Greenberg & Lanford)

If (14) holds and β is small enough then $\exists$ exactly <u>one</u> state $\rho^{\beta,\Phi}$ satisfying (15).

<u>Remark.</u> For transl.-inv. states,

KMS $\approx$ Gibbs variational equality;

($\nearrow$ Araki)

Thermodynamic functions

$$\beta f_\Lambda(\beta, \Phi) := -\frac{1}{|\Lambda|} \ln \operatorname{tr}\left(e^{-\beta H_\Lambda^\Phi}\right)$$

$\rho_X := \rho|_{\mathcal{A}_X}$, ρ transl.-inv. state on $\mathcal{A}$.

$$u_\Lambda(\rho, \Phi) := \frac{1}{|\Lambda|} \rho\left(H_\Lambda^\Phi\right)$$

$$s_\Lambda(\rho) := -\frac{1}{|\Lambda|} \operatorname{tr}\left(\rho_\Lambda \ln \rho_\Lambda\right)$$

Theorem 3. (e.g., Ruelle)

T.D. limits of f_Λ, u_Λ and s_Λ exist; denoted by $f(\beta, \Phi)$, $u(\rho, \Phi)$ and $s(\rho)$, resp., and

(i) $s(\rho)$ is affine and upper semi-continuous

(ii) $s(\rho) \leq \beta u(\rho, \Phi) - \beta f(\beta, \Phi)$,

"Gibbs variational inequality"

(iii) Given $\Phi \in \mathcal{B}$, $\exists$ at least one transl. inv. state, ρ, on $\mathcal{A}$ s.t.

$$s(\rho) = \beta u(\rho, \Phi) - \beta f(\beta, \Phi), \tag{16}$$

"Gibbs variational equality".

Remark. Any cluster point of the states (10) satisfies (16) $\Rightarrow$ (16) characterizes equilibrium states. $\rightarrow$ Transl.-invariant equilibrium states as <u>tangent functionals</u> to f (viewed as a functional on $\mathcal{B}$).

$\Delta^{\beta,\Phi} :=$ convex set of states, ρ, satisfying (16)

<u>Theorem 4.</u> (e.g., Lanford, Les Houches 1970)

$\Delta^{\beta,\Phi}$ is a <u>Choquet simplex</u>, i.e., every $\rho \in \Delta^{\beta,\Phi}$ is the barycenter of a unique probability meas. supported on $\underbrace{\text{extremal elements}}_{\text{"pure phases"}}$ of $\Delta^{\beta,\Phi}$.

<u>Def.</u> We say that a lattice system with interaction $\Phi \in \mathcal{B}$ has a <u>phase transition</u> if $\mathrm{card}\left(\mathrm{Ext}\,\Delta^{\beta,\Phi}\right)$ is <u>not</u> constant in β.

Since, for β small, we know that

$$\mathrm{card}(\Delta^{\beta,\Phi}) = 1$$

(uniqueness!), it is enough to show that, for large β, $\Delta^{\beta,\Phi}$ contains a <u>non-extremal</u> state,

in order to be sure that $\exists$ p.t.,

Let $\rho^{\beta,\Phi} \in \Delta^{\beta,\Phi}$ (fixed). Set

$$\langle a \rangle := \rho^{\beta,\Phi}(a), \quad a \in \mathcal{A}.$$

Given $a \in \mathcal{A}$, let $a_x := \tau_x(a)$.

We define

$$M_a^2 := \lim_{\Lambda \nearrow T} \frac{1}{|\Lambda|^2} \sum_{x,y \in \Lambda} \langle a_x^* a_y \rangle$$

$$\langle a^* a_x \rangle^c := \langle a^* a_x \rangle - M_a^2. \tag{17}$$

Then $\langle\langle(\cdot)\rangle\rangle$ <u>not</u> extremal invariant equ. state

iff, for some $a \in \mathcal{A}$,

$$\langle a^* a \rangle^c < \langle a^* a \rangle - |\langle a \rangle|^2, \tag{18}$$

$$\Leftrightarrow \quad M_a^2 > |\langle a \rangle|^2. \tag{19}$$

$$\Delta_\Lambda a^\# := \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \Big(a_x^\# - \underbrace{\langle a_x^\# \rangle}_{\langle a^\# \rangle} \Big)$$

Then (18), (19) $\Leftrightarrow$

$$\lim_{\Lambda \nearrow T} \langle \Delta_\Lambda a^* \Delta_\Lambda a \rangle = M_a^2 - |\langle a \rangle|^2 > 0 \tag{20}$$

<u>LRO</u>

Proof.

If $\langle(\cdot)\rangle$ extremal-invariant, i.e., ergodic under lattice transl. then

$$\lim_{\Lambda \nearrow \Gamma} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} a_x^\# = \text{const.}$$

$$\Rightarrow \quad \text{hence} = \langle a^\# \rangle \quad (\text{by transl. inv.})$$

Thus,

$$M_a^2 = |\langle a \rangle|^2 \tag{21}$$

Let $d\omega(k)$ denote the Fourier transform of $\langle a^* a_x \rangle$. This is a positive measure on Γ^* (1^{st} Brillouin zone). Then, by (17),

$$d\omega(k) = M_a^2\, \delta(k)\, d^d k + d\omega^c(k),$$

where $d\omega^c(k) = G(k)\, d^d k$ is F.T. of $\langle a^* a_x \rangle^c$.

3.2 Strategy for proof of p.t.

(i) Choice of an "order parameter" $a \in \mathcal{A}$;

(ii) Upper bound on $d\omega^c(k)$

(iii) Lower bound on $\int d\omega(k)$

$\Rightarrow$ (iv) Lower bound, $\tilde{M}_a^2$, on M_a^2

Then if

$$\tilde{M}_a^2 > |\langle a \rangle|^2 \tag{22}$$

$\Longrightarrow$ $\langle (\cdot) \rangle$ not extremal-inv. $\rightarrow \exists$ p.t.!

<u>Connection to spontaneous symmetry breaking</u>

G a compact topological group that is a symm. of Φ; $a \in \mathcal{A}$

$$a_G := a - \int_G dg\, \tau_g(a), \tag{23}$$

where dg is the Haar measure on G.

Let $\langle (\cdot) \rangle$ be a <u>G-invariant state</u>. Then

$$\langle a_G^{\#} \rangle = 0, \tag{24}$$

since $\int_G dg = 1$. Now, if

$$M_{a_G}^2 > 0 \tag{25}$$

then $\langle (\cdot) \rangle$ <u>not</u> extremal. By Theorem 4,

$$\langle (\cdot) \rangle = \int_\Sigma d\nu(\sigma)\, \langle (\cdot) \rangle^\sigma$$

Recall that

$$M_{a_G}^2 = \lim_{\Lambda \nearrow T} \frac{1}{|\Lambda|^2} \sum_{x,y \in \Lambda} \langle a_{G,x}^* a_{G,y} \rangle$$

If $M_{a_G}^2 > 0$ (i.e., (25) holds) then

$$0 < M_{a_G}^2 = \lim_{\Lambda \nearrow \Gamma} \frac{1}{|\Lambda|^2} \sum_{x,y \in \Lambda} \langle a_{G,x}^* \, a_{G,y} \rangle$$

$$\overset{\text{Thm. 4!}}{=} \lim_{\Lambda \nearrow \Gamma} \frac{1}{|\Lambda|^2} \int_{\Sigma} d\nu(\sigma) \sum_{x,y \in \Lambda} \langle a_{G,x}^* \, a_{G,y} \rangle^{\sigma},$$

where $\Sigma = \mathrm{Ext}\, \Delta^{\beta,\Phi}$, $\langle(\cdot)\rangle^{\sigma}$ extremal-inv., for all $\sigma \in \Sigma$. Hence, by <u>ergodic thm.</u>,

$$\lim_{\Lambda \nearrow \Gamma} \frac{1}{|\Lambda|^2} \sum_{x,y \in \Lambda} \langle a_{G,x}^* \, a_{G,y} \rangle^{\sigma} = |\langle a_G \rangle^{\sigma}|^2 .$$

Thus,

$$0 < M_{a_G}^2 = \int_{\Sigma} d\nu(\sigma) \, |\langle a_G \rangle^{\sigma}|^2 ,$$

i.e., $\exists$ set of σ's of positive $d\nu$-measure

s.t. $$\langle a_G \rangle^{\sigma} = \langle a \rangle^{\sigma} - \int_G dg \, \langle \tau_g(a) \rangle^{\sigma} \neq 0, \qquad (26)$$

i.e., $\langle(\cdot)\rangle^{\sigma}$ <u>not</u> G-invariant.

$\longrightarrow$ <u>G spontaneously broken in some pure phases of positive $d\nu$-measure.</u>

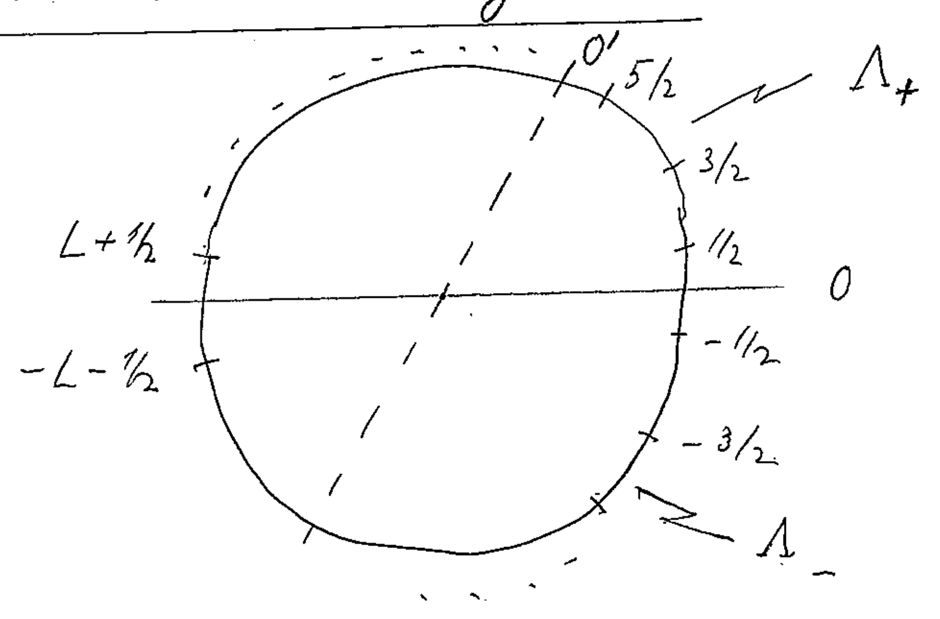
3.3 Implementation of Strategy 3.2: Reflection Positivity and Infrared Bounds

Henceforth, we consider a certain cone of interactions, Φ, satisfying a positivity property called reflection positivity. In order not to be overly abstract (see F-I-L-S), we set

$$\Gamma = \mathbb{Z}^d_{1/2} := \mathbb{Z}^d + (1/2, \dots, 1/2).$$

Let Λ be cube in Γ with sides of length $2(L+1)$; (see Sect. 2.2 and note a slight change of notations). We impose periodic b.c. at $\partial\Lambda$, wrapping Λ on a torus. We define $\Lambda_\pm \equiv \Lambda^\alpha_\pm$ as in Eq. (22) of Sect. 2.

Cross section through Λ:



$$\theta_\alpha x = (x^1, \dots, -x^\alpha, \dots, x^d)$$
$$\theta_\alpha : \Lambda_\pm \longrightarrow \Lambda_\mp \tag{27}$$

We define $\Theta_\alpha : \mathcal{A}_x \longrightarrow \mathcal{A}_{\theta_\alpha x}$

by
$$\Theta_\alpha a = \tau_{\theta_\alpha x - x}(\overline{a}), \tag{28}$$

for $a \in \mathcal{A}_x$, where $\overline{a} = (a^*)^T$ is complex conjugation, so that
$$\Theta_\alpha (a b) = (\Theta_\alpha a)(\Theta_\alpha b), \tag{29}$$

i.e., Θ_α is an anti-linear homomorphism from $\mathcal{A}_x$ to $\mathcal{A}_{\theta_\alpha x}$. Θ_α clearly extends to an anti-linear homomorphism
$$\Theta_\alpha : \mathcal{A}_{\Lambda_\pm} \longrightarrow \mathcal{A}_{\Lambda_\mp}. \tag{30}$$

<u>Lemma 5.</u> For all $a_i \in \mathcal{A}_{\Lambda_+}$, $i = 1, 2, \dots, n$

$$tr\left(\prod_{i=1}^n a_i \, \Theta(a_i)\right) \geq 0; \quad \Theta \equiv \Theta_\alpha . \tag{31}$$

<u>Proof.</u> Since $\Theta a_i \in \mathcal{A}_{\Lambda_-}$, for all i, and $[a, b] = 0$, for arb. $a \in \mathcal{A}_{\Lambda_+}$, $b \in \mathcal{A}_{\Lambda_-}$, we have that

$$\operatorname{tr}\left(\prod_{i=1}^{n} a_i\, \Theta(a_i)\right) = \operatorname{tr}\left(\prod_{i=1}^{n} a_i \prod_{j=1}^{n} \Theta(a_j)\right)$$

$$= \operatorname{tr}\left(\prod_{i=1}^{n} a_i\right) \operatorname{tr}\left(\prod_{j=1}^{n} \Theta(a_j)\right)$$

By (28),

$$\operatorname{tr}\left(\prod_{j=1}^{n} \Theta(a_j)\right) = \operatorname{tr}\left(\prod_{j=1}^{n} \bar{a}_j\right)$$

$$= \overline{\operatorname{tr}\left(\prod_{j=1}^{n} a_j\right)},$$

which completes the proof.

<u>Definition.</u> An interaction Φ satisfies <u>reflection positivity</u> iff, for all $X \in \mathcal{P}_f(\mathbb{Z}^d_{1/2})$,

$$\Theta_\alpha(\Phi(X)) = \Phi(\theta_\alpha X), \qquad (32)$$

and, for all finite cubes Λ,

$$\sum_{\substack{X \cap \Lambda_+ \neq \emptyset \\ X \cap \Lambda_- \neq \emptyset}} \Phi(X) = -\sum_i B_i\, \Theta_\alpha(B_i), \qquad (33)$$

for operators $B_i \in \mathcal{A}_{\Lambda_+}$.

Then

$$H_\Lambda^\Phi = \underbrace{\sum_{X \subset \Lambda_+} \Phi(X)}_{H^\Phi_{\Lambda_+}} + \sum_{X \subset \Lambda_-} \underbrace{\Phi(X)}_{\Theta(\Phi(\underbrace{\theta X}_{\subset \Lambda})),\ \text{by } (32)} - \sum_i B_i\, \Theta(B_i)$$

$$= H^{\Phi}_{\Lambda_+} + \Theta(H^{\Phi}_{\Lambda_+}) - \sum_i B_i \Theta(B_i) \qquad (34)$$

Thus

$$e^{-\beta H^{\Phi}_{\Lambda}} = \lim_{k\to\infty} \Big[e^{-\beta H^{\Phi}_{\Lambda_+}/k}\, \Theta\big(e^{-\beta H^{\Phi}_{\Lambda_+}/k}\big) \prod_i e^{\beta B_i \Theta(B_i)/k} \Big]^k, \qquad (35)$$

by Trotter's product formula.

Let

$$\rho^{\beta,\Phi}_{\Lambda}(a) = \frac{1}{Z^{\beta}_{\Lambda}} \operatorname{tr}\big(e^{-\beta H^{\Phi}_{\Lambda}}\, a\big), \qquad (36)$$

where Z^{β}_{Λ} is such that $\rho^{\beta,\Phi}_{\Lambda}(\mathbb{1}) = 1$.

<u>Theorem 6.</u> Let Φ satisfy reflection positivity (RP); see (32), (33). Let $a_i \in \mathcal{A}_{\Lambda_+}$, $i = 1, \dots, n$. Then

$$\rho^{\beta,\Phi}_{\Lambda}\Big(\prod_{i=1}^{n} a_i\, \Theta(a_i)\Big) \geq 0. \qquad (37)$$

<u>Proof.</u> Use (35) and expand all the exponentials $e^{\beta B_i \Theta(B_i)/k}$ into their power series. Then

$$\rho_\Lambda^{\beta,\Phi}\left(\prod_{i=1}^{n} a_i \ominus (a_i)\right)$$

$$= \lim_k \sum_l \operatorname{tr}\left(\prod_{j=1}^{n_{kl}} c_j^{kl} \ominus (c_j^{kl})\right),$$

for operators $c_j^{kl} \in \mathcal{A}_{\Lambda_+}$. Application of Lemma 5 then completes the proof.

<u>Remark.</u> Note that Lemma 5 is a special case of Theorem 6, because $\rho_\Lambda^{\beta=0,\Phi}(\cdot) = \operatorname{tr}(\cdot)$.

<u>Schwarz inequality.</u>

By Theorem 6,

$$(a,b)_\beta := \rho^{\beta,\Phi}\left(a \ominus (b)\right), \quad a, b \in \mathcal{A}_{\Lambda_+}$$

defines a <u>positive semi-definite inner product</u> on $\mathcal{A}_{\Lambda_+}$ (anti-linear in the <u>second</u> argument).

Thus

$$|(a,b)_\beta|^2 \leqslant (a,a)_\beta \cdot (b,b)_\beta, \tag{38}$$

$\forall\, a, b \in \mathcal{A}_{\Lambda_+}$.

<u>Corollary 7.</u> For A, B, C_i, D_i all in $\mathcal{A}_{\Lambda_+}$

$$\left|\operatorname{tr}\left(e^{A + \ominus B + \sum C_i \ominus (D_i)}\right)\right|^2 \leqslant$$

$$\operatorname{tr}\left(e^{A+\Theta A+\sum_i C_i \Theta(C_i)}\right)\cdot \operatorname{tr}\left(e^{B+\Theta B+\sum D_i \Theta(D_i)}\right) \tag{39}$$

<u>Proof.</u> Same ideas as in proof of Thm. 6, plus applications of (38) (for $\beta = 0$).

<u>Corollary 8.</u> (Chessboard Estimates)

Assume that Φ satisfies RP; see (32), (33). Then

$$\left|\rho_\Lambda^{\beta,\Phi}\left(\prod_{i\in\Lambda} a_i\right)\right| \leq \prod_{i\in\Lambda}\left(\rho_\Lambda^{\beta,\Phi}\left(\prod_{j\in\Lambda}\tau_{j-i}(a_i)^{\#_j}\right)\right)^{1/|\Lambda|}, \tag{40}$$

where
$$\tau_{j-i}(a_i)^{\#_j} = \begin{cases} \tau_{j-i}(a_i), & |j-i| \text{ even} \\ \tau_{j-i}(\overline{a}_i), & |j-i| \text{ odd}, \end{cases}$$

with $|x|$ = parity of # of lattice steps from 0 to x.

This corollary follows from the following general

<u>Lemma 9.</u> (Generalized Hölder inequality)

Let V be some complex vector space, and $-: V \ni A \mapsto \overline{A} \in V$ an anti-linear involution ("complex conjugation"). Let ω be a multilinear functional on $V^{\times 2l}$ with the following three properties:

(A) $\omega(B_1, \cdots, B_{2l}) = \omega(B_2, \cdots, B_{2l}, B_1)$

"cyclic invariance"

(B) for arbitrary $B_1^i, \cdots, B_l^i$ in V, $i = 1, 2, 3, \cdots$,

$$M_{ij} := \omega\left(B_1^i, \cdots, B_l^i, \overline{B}_l^j, \cdots, \overline{B}_1^j\right) \qquad (41)$$

defines a positive semi-definite matrix; in particular

(C) $\overline{\omega(A_1, \cdots, A_l; \overline{B}_l, \cdots, \overline{B}_1)}$

$$= \omega\left(B_1, \cdots, B_l; \overline{A}_l, \cdots, \overline{A}_1\right)$$

Then

$$|\omega(B_1, \cdots, B_{2l})| \leq \prod_{\alpha=1}^{2l} \omega\left(B_\alpha, \overline{B}_\alpha, \cdots, B_\alpha, \overline{B}_\alpha\right)^{1/2l}$$

<u>Proof.</u> By (41)

$$\omega\left(A_1, \cdots, A_l, \overline{B}_l, \cdots, \overline{B}_1\right)$$

defines a positive semi-definite inner product on $V^{\times l}$. Thus

$$|\omega(A_1,\dots,A_\ell;\bar{B}_\ell,\dots,\bar{B}_1)|$$

$$\leq \underbrace{\omega(A_1,\dots,A_\ell;\bar{A}_\ell,\dots,\bar{A}_1)^{1/2}}\,\omega(B_1,\dots,B_\ell;\bar{B}_\ell,\dots,\bar{B}_1)^{1/2}$$

$$\overset{(A)}{=} \omega(A_2,\dots,\underline{A_\ell},\underline{\bar{A}_\ell};\bar{A}_{\ell-1},\dots,\underline{\bar{A}_1},\underline{A_1})$$

$$\uparrow$$

apply Schwarz again!

etc.

Using homogeneity of order 1 in each argument of ω and a simple maximization argument, the proof is completed; (see J. F. & E. H. Lieb, 1977).

Corollary 8 follows from Lemma 9 by recalling that we have imposed periodic b.c. on $\partial\Lambda \rightarrow$ "cyclicity" (A), Theorem 6 $\rightarrow$ (B), (C), and letting α range from 1 to d.

Another corollary of Lemma 9 is the <u>Hölder inequality for traces</u>:

$$|\mathrm{tr}(C_1 \cdots C_n)| \leq \prod_{i=1}^{n} \|C_i\|_{m_i}, \qquad (42)$$

where $\sum_{i=1}^{n} \frac{1}{m_i} = 1$; (see J. F., 1977).

Examples of interactions satisfying RP.

Let $A = A^* = \overline{A}$, $B_i = \pm B_i^* = \overline{B}_i$, $i = 1, 2, 3, \cdots$, be elements of $\mathcal{A}_{x_0}$. We define an interaction Φ by setting

$$\Phi(\{x\}) := \tau_{x-x_0}(A)$$

$$\Phi(\{x,y\}) := -\sum_i J^{(i)}_{x-y}\, \tau_{x-x_0}(B_i)\, \tau_{y-x_0}(B_i), \qquad (43)$$

$$\Phi(X) = 0, \text{ for } |X| \geq 3.$$

Lemma 10. If

$$\sum_{\substack{x;\, x^\alpha > 0,\\ y;\, y^\alpha > 0}} J^{(i)}_{x-\theta_\alpha y}\, z_x \overline{z}_y \geq 0, \qquad (44)$$

for all i and all $\alpha = 1, \cdots, d$, for arb. sequences $\{z_x\}_{x \in \Lambda_+}$ of complex numbers then Φ, as in (43), satisfies R.P.

Remark. By redefining A we can extend the definition of $\Phi(\{x,y\})$ to $x = y$ and require that

$$\sum_x J^{(i)}_{\hat{x}} = \hat{J}^{(i)}(0) = 0, \qquad (45)$$

$\hat{x} = x - (1/2, \cdots, 1/2) \in \mathbb{Z}^d$.

<u>Proof.</u> We choose $z_x = e^{i\vec{k}\cdot\vec{x}} c_{x^\alpha}$, $x^\alpha > 0$, with $\vec{k} = (k_1, \dots, \check{k}_\alpha, \dots, k_d)$, $\vec{x} = (x^1, \dots, \check{x}^\alpha, \dots, x^d)$, and $c_{x^\alpha} \in \mathbb{C}$. Let $\hat{J}^{(i)}_{x^\alpha}(\vec{k})$ denote the Fourier transform of $J^{(i)}_x$ in $x^1, \dots, \check{x}^\alpha, \dots, x^d$, for fixed $x^\alpha > 0$. Then (44) implies that

$$\sum_{\substack{x^\alpha > 0 \\ y^\alpha > 0}} \hat{J}^{(i)}_{x^\alpha + y^\alpha}(\vec{k})\, c_{x^\alpha} \overline{c_{y^\alpha}} \geq 0 . \qquad (RP)$$

By a variant of Bochner's theorem

$$\hat{J}^{(i)}_n(\vec{k}) = \int \lambda^{|n|-1} \underbrace{d\rho^{(i)}(\lambda, \vec{k})}_{\geq 0 \text{ on } \mathbb{R}.}$$

Thus

$$\sum_{\substack{x^\alpha > 0 \\ y^\alpha > 0}} \hat{J}^{(i)}_{x^\alpha + y^\alpha}(\vec{k})\, c_{x^\alpha} \overline{c_{y^\alpha}}$$

$$= \int \Big(\sum_{x^\alpha = 1/2, 3/2, \dots} \lambda^{x^\alpha - 1/2} c_{x^\alpha} \Big) \Big(\sum_{y^\alpha = 1/2, 3/2, \dots} \lambda^{y^\alpha - 1/2} \overline{c_{y^\alpha}} \Big)$$

$$\times\, d\rho^{(i)}(\lambda, \vec{k}) \qquad (45)$$

which is indeed positive. Now replace $e^{i\vec{k}\cdot\vec{x}} c_{x^\alpha}$ by $\frac{-1}{(2\pi)^{\frac{d-1}{2}}} \sum_{\vec{x}} e^{i\vec{k}\cdot\vec{x}} \tau_{x-x_0}(B_i)$, and plug this into (45). This proves the Lemma!

Lemma 11.

(1) If Φ_1 and Φ_2 are RP then

$$\lambda \Phi_1 + \mu \Phi_2 \text{ is RP},$$

for arb. $\lambda, \mu > 0$.

(2) If J satisfies (44) then so does J^{-1}, (where J^{-1} is the matrix inverse of J).

(3) If J^1 and J^2 satisfy (44) then so does $J^1 \circ J^2$, where

$$(J^1 \circ J^2)_x = J^1_x J^2_x$$

Proof. (1) follows immediately from the definition of RP; (2) and (3) follow from straightforward calculations with Gaussian measures.

Examples of J's satisfying (44)

$$J_x = \delta_{|x|,1}\,, \quad J_x = \frac{1}{|x|^{d-2+\eta}}\,, \quad \eta \geq 0 \ (\to QFT) \quad (46)$$

Infrared Bounds

We consider a quantum or classical lattice system as discussed in Sects. 3.1, 3.2. The interaction Φ is given by

$$\Phi(X) = \Psi(X) + \Phi^{(2)}(X), \tag{47}$$

where Ψ and $\Phi^{(2)}$ (and hence Φ) satisfy RP; see (32), (33), and

$$\begin{aligned} &\Phi^{(2)}(X) = 0, \text{ unless } X = \{x,y\}, \\ &\Phi^{(2)}(\{x,y\}) = -\sum_{x,y} J^{(i)}_{x-y}\, S^i_x\, S^i_y, \end{aligned} \tag{48}$$

with

$$S^i_x = \tau_{x-x_0}(B_i), \quad B_i = \pm B_i^* = \overline{B_i}.$$

and $J^{(i)}$ as in Lemma 10,

$$\sum_x J^{(i)}_{\hat{x}} = \hat{J}^{(i)}(0) = 0; \tag{49}$$

see (45).

Examples.

(1) $B_i = \sigma^i$: i^{th} component of quantum spin

$\underline{\sigma} := (\sigma^1, \sigma^2, \sigma^3)$, with $\underline{\sigma} \cdot \underline{\sigma} = S(S+1)$,

$S = 1/2, 1, 3/2, \dots$ (spin).

(2) $B_i = \phi^i$; i^{th} component of

$\phi = (\phi^1, \dots, \phi^N) \in \mathbb{R}^N$, $\operatorname{supp}(d\mu(\phi))$ compact.

(3) $B_{i=\{k,l\}} := U(g)_{kl}$, $g \in G$, (G = Lie group)

(U a finite dim. rep. of G.

We set

$$\Phi_h(X) := \Psi(X) + \Phi_h^{(2)}(X),$$

where

$$\Phi_h^{(2)}(\{x,y\}) := -\sum_{x,y} J_{x-y}^{(i)} (S_x^i + h_x^i)(S_y^i + h_y^i), \tag{50}$$

where $h_x = (h_x^1, h_x^2, \dots)$, $h_x^i \in \mathbb{R}$, $\forall x, \forall i$,

with $h_x^i \equiv 0$ if $B_i^* = -B_i$; $(\Phi_h^{(2)} = \Phi_h^{(2)*}!)$.

Let $H_\Lambda^{\Phi_h}$ be as in (9), with periodic b.c. at $\partial\Lambda$, and

$$Z_\Lambda^\beta(h) := tr\left(e^{-\beta H_\Lambda^{\Phi_h}}\right) > 0. \tag{51}$$

s.a., for h as above

Theorem 12.

$$0 < Z_\Lambda^\beta(h) \leq Z_\Lambda^\beta(h \equiv \text{const.}) = Z_\Lambda^\beta \tag{52}$$

Proof.

(1) If $h_x^i \equiv h^i = \text{const}$ then

$$\sum_{x-y} J_{x-y}^{(i)} S_x^i h^i = \hat{J}^{(i)}(0) h^i \sum_x S_x^i$$
$$= 0,$$

and

$$\sum_{x-y} J_{x-y}^{(i)} h^i h^i = |\Lambda| (h^i)^2 \hat{J}^{(i)}(0)$$
$$= 0,$$

by (49).

(2) Let $h_\pm := h|_{\Lambda_\pm}$, and

$$(h_\pm^{\theta_\alpha})_x := h_{\pm\, \theta_\alpha x}.$$

Moreover

$$h_\alpha^{(\pm)} := h_\pm + h_\pm^{\theta_\alpha}.$$

Then Corollary 7 readily implies that

$$0 < Z_\Lambda^\beta(h) \leq Z_\Lambda^\beta(h_\alpha^{(+)})^{1/2}\, Z_\Lambda^\beta(h_\alpha^{(-)})^{1/2} \tag{53}$$

(Schwarz inequality).

Iterating this inequality for both factors on the R.S. of (53) after shifting the plane of reflection (see bottom of page 86) and letting α range from $1, \dots, d$, we find, using a simple maximization argument, that

$$Z_\Lambda^\beta(h) \leq \prod_{x \in \Lambda} \underbrace{Z_\Lambda^\beta(h \equiv h_x)}_{= Z_\Lambda^\beta \text{, by step (1)}}{}^{1/|\Lambda|}$$

$$= Z_\Lambda^\beta .$$

The lower bound in (52), (53) follows from (51).

Q.E.D.

Thus

$$\varepsilon^{-2}\left[\frac{Z_\Lambda^\beta(\varepsilon h)}{Z_\Lambda^\beta} - 1 \right] \leq 0, \tag{54}$$

for all $\varepsilon > 0$. Since $\rho_\Lambda^{\beta,\Phi}(\cdot)$ is translation-invariant,

$$\sum_x J_{x-y}^{(i)} \underbrace{\rho_\Lambda^{\beta,\Phi}(S_x^i)}_{\text{indep. of } x} h_y^i \propto \hat{J}^{(i)}(0) = 0. \tag{55}$$

Let

$$J*h_x := \sum_y J_{x-y} h_y \,, \text{ and}$$
$$S(J*h) := \sum_{x\in\Lambda} S_x (J*h)_x . \tag{56}$$

Let

$$(A,B)_\beta := \int_0^\beta d\tau\, \rho_\Lambda^{\beta,\Phi}\left(e^{\tau H_\Lambda^\Phi} A^* e^{-\tau H_\Lambda^\Phi} B\right) \tag{57}$$

"Duhamel two-point function"; (see Sect. 1.3, (30)).

Note that

$$\frac{d}{d\varepsilon} e^{-\beta(A+\varepsilon B)} = \int_0^\beta d\tau\, e^{-(\beta-\tau)(A+\varepsilon B)} B\, e^{-\tau(A+\varepsilon B)} \tag{58}$$

(e.g. from Trotters product formula).

Then (54), (55) and (58) imply that

$$\boxed{\left(S^i(J^{(i)}*h^i),\, S^i(J^{(i)}*h^i)\right)_\beta \le \frac{1}{\beta}\, h^i(J^{(i)}*h^i),} \tag{59}$$

if $S^i = (S^i)^* = \overline{S^i}$.

"<u>Infrared Bound</u>"

The inequality of Falk and Bruch

$$Z \equiv Z^{\beta} = Tr\left(e^{-\beta H}\right)$$

$$\langle A \rangle = Z^{-1} Tr\left(e^{-\beta H} A\right)$$

$$(A, B) = Z^{-1} \int_0^1 ds\, Tr\left(e^{-\beta(1-s)H} A^* e^{-\beta s H} B\right)$$

$$= \frac{1}{\beta} (A, B)_{\beta} .$$

Then

$$\langle A \rangle = \frac{1}{Z} \frac{d}{d\lambda} Tr\left(e^{-\beta H + \lambda A}\right)\Big|_{\lambda=0}$$

$$(A, B) = \frac{1}{Z} \frac{\partial^2}{\partial\lambda\partial\mu} Tr\left(e^{\cancel{-\beta H + \lambda A^* + \mu B}}\right)\Big|_{\lambda=\mu=0} .$$

$$F_A(s) := Z^{-1} Tr\left(e^{-\beta(1-s)H} A^* e^{-\beta s H} A\right). \quad (60)$$

We have seen in Sect. 1.3, (31) that $F_A(s)$ is <u>convex</u>, so that

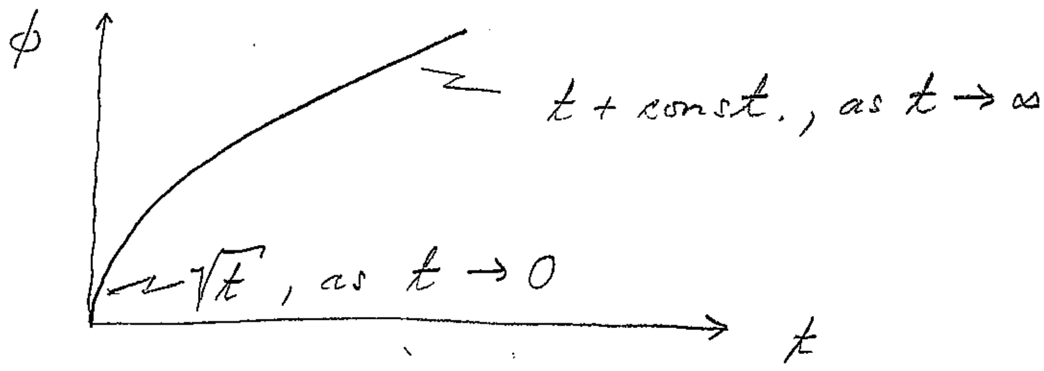
$$\int_0^1 ds\, F_A(s) \leq \frac{1}{2}\left(F_A(0) + F_A(1)\right). \quad (61)$$

(Actually, $F_A(s)$ is log-convex, since

$$F_A'' F_A - (F_A')^2 \geq 0. \nearrow \text{B. Tóth!})$$

Let

$$\phi(t) := \sqrt{t}\,\coth\frac{1}{\sqrt{t}}, \quad t \in [0,\infty)$$



The function ϕ is <u>concave</u>.

<u>Lemma 13</u> (Falk & Bruch, DLS)

$$\frac{1}{2}\langle \underbrace{\{A^*, A\}}_{\text{anti-commutator}} \rangle \leq \frac{\beta\langle [A^*, [H,A]] \rangle}{4} \times \phi\left(\frac{4\,(A,A)}{\beta\langle [A^*,[H,A]] \rangle}\right)$$

<u>Proof.</u> By (60), (61),

$$\frac{1}{2}\langle \{A^*, A\} \rangle = \frac{1}{2}\left(F_A(0) + F_A(1)\right)$$

$$(A,A) = \int_0^1 F_A(s)\,ds$$

<u>Def.</u>

$$a := \beta\,\langle [A^*, [H,A]] \rangle = F_A'(1) - F_A'(0); \quad (62)$$

as follows immediately from (60).

Note that, by the spectral theorem for $\operatorname{ad} H$ (= "Liouvillian"),

$$F_A(s) = \int e^{st}\, d\mu(t),$$

where $d\mu$ is a positive measure on $\mathbb{R}$ with $\int d\mu(t) = F_A(0) = \langle A^*A \rangle$.

$$\left.\begin{aligned} b &:= \underbrace{\int_0^1 F_A(s)\, ds}_{=(A,A)} = \int \frac{e^t - 1}{t}\, d\mu(t), \\ c &:= \underbrace{\frac{1}{2}\big(F_A(0) + F_A(1)\big)}_{=\frac{1}{2}\langle \{A^*, A\}\rangle} = \int \frac{e^t + 1}{2}\, d\mu(t), \\ \text{and} \quad a &= \underbrace{F_A'(1) - F_A'(0)}_{\beta\langle [A^*,[H,A]]\rangle} = \int t(e^t - 1)\, d\mu(t). \end{aligned}\right\} \quad (63)$$

<u>Def.</u>

$$d\nu(t) := \Big[\underbrace{\int t(e^t-1)\, d\mu(t)}_{a}\Big]^{-1} \underbrace{t(e^t-1)}_{\geq 0}\, d\mu(t)$$

$d\nu$ is a probability measure on $\mathbb{R}$.

Then

$$\left.\begin{aligned} \frac{4b}{a} &= \int \frac{4}{t^2}\, d\nu(t), \\ \frac{4c}{a} &= \int \frac{2}{t} \coth\frac{t}{2}\, d\nu(t). \end{aligned}\right\} \quad (64)$$

Next, using concavity of ϕ and Jensen's inequality, we conclude that

$$\phi\left(\frac{4b}{a}\right) = \phi\left(\int \frac{4}{t^2}\, d\nu(t)\right)$$

$$\geq \int \phi\left(\frac{4}{t^2}\right) d\nu(t)$$

$$\text{def. of } \phi \to \; = \int \frac{2}{t} \coth\frac{t}{2}\, d\nu(t) = \frac{4c}{a} \qquad (65)$$

In view of (62), (63), this proves the Lemma.

We can apply this inequality to derive a lower bound on the L.S. of (59). We set $A = S^i\left(J^{(i)} * h^i\right) = A^*$. If $\langle [A^*, [H, A]] \rangle > 0$ (check!), we may replace $\phi(\beta^{-1}\cdot)$ by const. $\sqrt{\beta^{-1}\cdot}$, for β large enough (with const. $\to 1$, as $\beta \nearrow \infty$); by the def. of ϕ.

$\longrightarrow$ done!

More generally, use that

$$\phi(t) \lesssim \sqrt{t} + t.$$

Sketch:

(A) Backbone of proof / reduction to IRB.

$\nu(2)$] we do it for isotropic HAF ($u=-1$) and HF ($u=1$)
jointly] with no transversal field $h=0$

this restriction simplifies computations but it is not essential

(B) Proof of IRB: works for $u \leq 0$ & $h=0$
(i.e. antiferromagnetic coupling and no transversal field)
this restriction is essential

(A) Backbone of proof / reduction to IRB

$$H_1 = -\frac{1}{2}\sum_{|x-y|=1} \left\{ S_1(x) S_1(y) + S_2(x) S_2(y) \mp S_3(x) S_3(y) \right\}$$

$-$ for AF , $+$ for F

in the sequel: upper sign for AF
lower sign for F

correlation fct:

$$(x) = \langle S_1(0) S_1(x) \rangle = \langle S_2(0) S_2(x) \rangle = (\mp 1)^{|x|} \langle S_3(0) S_3(x) \rangle$$

Duhamel correlation fct.:

$$b(x) = \left(S_1(0), S_1(x)\right) = \left(S_2(0), S_2(x)\right) = (\mp 1)^{|x|} \left(S_3(0), S_3(x)\right)$$

Double-commutator correlation fct (Bogoliubov corr fct)

$$g(x) = \langle [S_1(0), [H_\Lambda, S_1(x)]] \rangle = \langle [S_2(0), [H_\Lambda, S_2(x)]] \rangle =$$
$$(\mp 1)^{|x|} \langle [S_3(0), [H_\Lambda, S_3(x)]] \rangle$$

$$\hat{c}(k) = \sum_x e^{ikx} c(x) = \frac{1}{|\Lambda|} \langle \hat{S}_1(k) \hat{S}_1(-k) \rangle$$

$$\hat{b}(k) = \sum_x e^{ikx} b(x) = \frac{1}{|\Lambda|} \left(\hat{S}_1(k), \hat{S}_1(k)\right)$$

$$\hat{g}(k) = \sum_x e^{ikx} g(x) = \frac{1}{|\Lambda|} \langle [\hat{S}_1(-k), [H_\Lambda, \hat{S}_1(k)]] \rangle$$

Apply Falk-Bruch with $A = \hat{S}_1(k)$, $A^* = \hat{S}_1(-k)$

a the notation the proof of F-B: $c = |\Lambda| \hat{c}(k)$, $b = |\Lambda| \hat{b}(k)$, $g = |\Lambda| \hat{g}(k)$

F-B: $$\hat{c}(k) \le \frac{\beta \cdot \hat{g}(k)}{4} \Phi\left(\frac{4 \hat{b}(k)}{\beta \cdot \hat{g}(k)}\right)$$

these computations would be more complicated without the restrictive assumptions

2) compute the double commutator:

$$[S_1(0), [H_1, S_1(x)]] = \begin{cases} \sum\limits_{|e|=1} \{S_2(0)S_2(e) \mp S_3(0)S_3(e)\} & x = 0 \\ -\{S_3(0)S_3(x) \mp S_2(0)S_2(x)\} & |x-0| = 1 \\ 0 & |x-0| > 1 \end{cases}$$

HomeWork: check it! (very instructive!)

assume cubic Λ – not essential, but simplifies computations

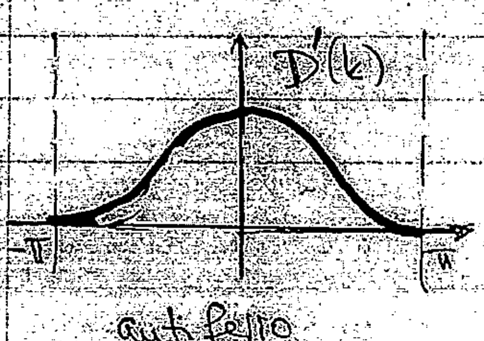
let $t = \langle S_1(0)S_1(e)\rangle = \langle S_2(0)S_2(e)\rangle = \mp\langle S_3(0)S_3(e)\rangle$

$|e| = 1$ this would not hold without the restrictive assumptions

then

$$g(x) = \begin{cases} 4dt & |x-0| = 0 \\ \pm 2t & |x-0| = 1 \\ 0 & |x-0| > 1 \end{cases} ; \quad \hat{g}(k) = 4t\,\underbrace{(2d - D(k))}_{= D'(k)} \quad \text{AF}$$

$$\hat{g}(k) = 4t\,D(k) \quad \text{F}$$



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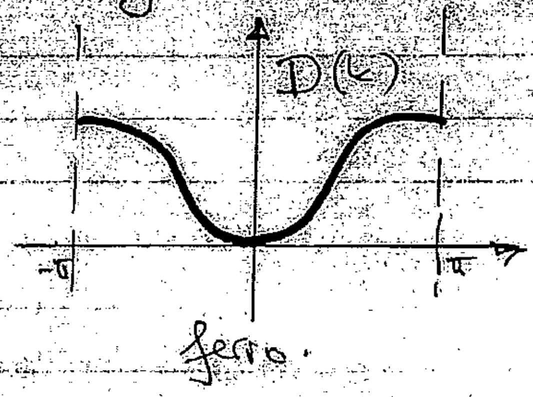


ferro.

③ Infrared bound for the Duhamel correlation fct

$$\hat{b}(k) \le \frac{1}{2\beta}\,\frac{1}{D(k)}$$

Proved for $u \le 0$, $h = 0$ only (essential restric

but expected to hold in general (possibly with

$$\frac{K}{2\beta}\,\frac{1}{D(k)}\Big)$$ proof postponed

④ Put things together:

$$\hat{c}(k) \underset{\text{F-B}}{\le} \frac{\beta \hat{g}(k)}{4}\,\phi\left(\frac{4\hat{b}(k)}{\beta\,\hat{g}(k)}\right)$$

$$\underset{\text{IRB}}{\le} \frac{\beta \hat{g}(k)}{4}\,\phi\left(\frac{2}{\beta^2 D(k)\,\hat{g}(k)}\right)$$

$$\underset{\text{computation}}{=} \begin{cases} \beta t\, D'(k)\,\phi\left(\dfrac{1}{2\beta^2 t\, D(k)\, D'(k)}\right) & \text{AF (proved} \\ \beta t\, D(k)\,\phi\left(\dfrac{1}{2\beta^2 t\, D^2(k)}\right) & \text{F (presumed} \end{cases}$$

same in the thermodyn. limit $\Lambda \uparrow \mathbb{Z}^d$

ground state expectations : $\lim_{\beta\to\infty} \langle \cdots \rangle_\beta$ 11.

⑤ LRO in the ground state AF

(the ground states of the F are trivial)

Remark : as long as Λ is finite the HAF in Λ has a unique g.s. (Lieb & Mattis)

$$\beta \to \infty : \quad \hat{C}(k) \le \sqrt{\frac{t}{2}} \left(\frac{D'(k)}{D(k)}\right)^{1/2}$$

$$\text{So: } t = C(e) = \frac{1}{2d} \sum_{|e|=1} C(e) = \frac{1}{|\Lambda|} \sum_{k\in\Lambda^*} \left(1 - \frac{D(k)}{d}\right) \hat{C}(k)$$

$$= \underbrace{\frac{1}{|\Lambda|} \hat{C}(k=0)}_{r_\Lambda^2} + \frac{1}{|\Lambda|} \sum_{\substack{k\in\Lambda^* \\ k\neq 0}} \left(1 - \frac{D(k)}{d}\right) \hat{C}(k) \qquad \forall k:\ \hat{C}(k) \ge 0$$

$$\le r_\Lambda^2 + \frac{1}{|\Lambda|} \sum_{\substack{k\in\Lambda^* \\ k\neq 0}} \left(1 - \frac{D(k)}{d}\right)_+ \hat{C}(k)$$

$$\le r_\Lambda^2 + \frac{1}{|\Lambda|} \sum_{\substack{k\in\Lambda^* \\ k\neq 0}} \left(1 - \frac{D(k)}{d}\right)_+ \cdot \sqrt{\frac{t}{2}} \left(\frac{D'(k)}{D(k)}\right.$$

in the Thermodynamic limit :

$$t \le r^2 + \sqrt{\frac{t}{2}} \underbrace{\int \left(1 - \frac{D(k)}{d}\right)_+ \left(\frac{D'(k)}{D(k)}\right)^{1/2} dk}_{I = I(d)}$$

$$r^2 \geq 2\sqrt{\frac{t}{2}}\left(\sqrt{\frac{t}{2}} - \frac{1}{2} I\right) \qquad (*)$$

5b:

$$\frac{s(s+1)}{3} = \langle S_1(0) S_1(0) \rangle = C(0) = \frac{1}{|\Lambda|} \sum_{k \in \Lambda^*} \hat{C}(k)$$

$$= \underbrace{\frac{1}{|\Lambda|} \hat{C}(k=0)}_{r_\Lambda^2} + \frac{1}{|\Lambda|} \sum_{\substack{k \in \Lambda^* \\ k \neq 0}} \hat{C}(k)$$

$$\leq r_\Lambda^2 + \frac{1}{|\Lambda|} \sum_{\substack{k \in \Lambda^* \\ k \neq 0}} \sqrt{\frac{t}{2}} \left(\frac{D'(k)}{D(k)}\right)^{1/2}$$

in the thermodynamic limit

$$\frac{s(s+1)}{3} \leq r^2 + \sqrt{\frac{t}{2}} \underbrace{\int \left(\frac{D'(k)}{D(k)}\right)^{1/2} dk}_{J \,=\, J(d)}$$

$$r^2 \geq \frac{s(s+1)}{3} - \sqrt{\frac{t}{2}}\, J \qquad (**)$$

$(*)$ & $(**)$ $\Longrightarrow$ $r^2 > 0$ guaranteed if $\frac{3}{2} I J < s(s+1)$

	I	J	$\frac{3}{2} I J$
$d=2$	0.690	1.443	$1.494 < 1(1+1)$
$d=3$	0.350	1.157	$0.607 < \frac{3}{4} = \frac{1}{2}\left(\frac{1}{2}+1\right)$

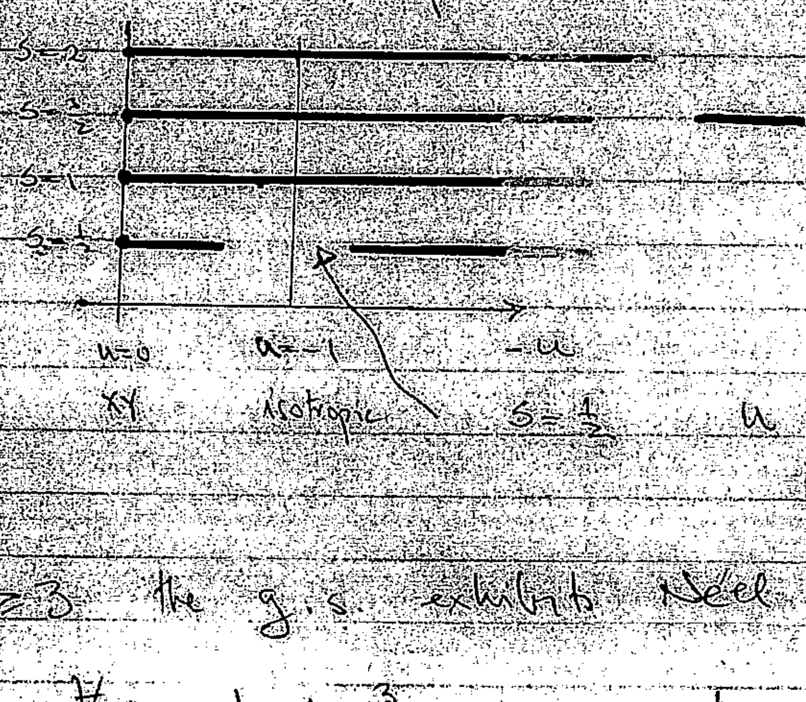
Conclusion :

- in two dimensions the isotropic HAF exhibits Néel order in the ground state (zero temp.) for $s = 1, \frac{3}{2}, 2, \ldots$
  $s = \frac{1}{2}$ remains open

- in three (and more) dimensions the isotropic HAF exhibits Néel order in the ground state (zero temperature) for any spin $s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$

Remark : the anisotropic HAF is more subtle

$d=2$ ground state



existence of g.s. Néel order proved

$s = \frac{1}{2}$ $\quad u \in (-1.78, -0.13)$

open $\qquad$ Kubo-Kishi

in $d \geq 3$ the g.s. exhibits Néel order for

$$\forall s = \frac{1}{2}, 1, \frac{3}{2}, \ldots \quad \text{and} \quad \forall u \leq 0 .$$

$\square$ ground state

⑥ Stability of the ground state LRO for low (but positive) temperatures, in $d \geq 3$: a dominated convergence argument

$$\Gamma_\beta^2 = \frac{\Delta(\Delta+1)}{3} - \int \hat{C}_\beta(k)\, dk$$

$$\hat{C}_\beta(k) \longrightarrow \hat{C}_\infty(k) \quad \text{as } \beta \nearrow \infty$$

$$\hat{C}_\beta(k) \leq \begin{cases} \beta t\, D'(k)\, \phi\left(\dfrac{1}{2\beta^2 t\, D(k)\, D'(k)}\right) & \text{AF (proved)} \\ \beta t\, D(k)\, \phi\left(\dfrac{1}{2\beta^2 t\, D^2(k)}\right) & \text{F (presumed)} \end{cases}$$

$$\sup_{\beta \geq 1} \beta x\, \phi\left(\frac{1}{\beta^2 y}\right) \leq a \frac{x}{\sqrt{y}} + b \frac{x}{y}$$

use $\phi(y) \leq a\sqrt{y} + b\, y$

hence

$$\hat{C}_\beta(k) \leq \left.\begin{cases} \underbrace{a\sqrt{\frac{t}{2}}}_{\leq a'} \sqrt{\dfrac{D'(k)}{D(k)}} + \dfrac{b}{2}\, \dfrac{1}{D(k)} & \text{proved} \\ \underbrace{a\sqrt{\frac{t}{2}}}_{\leq a'} + \dfrac{b}{2}\, \dfrac{1}{D(k)} & \text{presumed} \end{cases}\right\} \text{dominating fct integrable in } d \geq 3$$

$\Rightarrow$ $\Gamma_\beta^2 \longrightarrow \Gamma_\infty^2$ in $d \geq 3$ as $\beta \to \infty$

Conclusion:

in $d \geq 3$

$$\Gamma_\beta^2 \longrightarrow \Gamma_{g.s.}^2 \quad \text{as} \quad \beta \to \infty$$

$\square$ DLS proved modulo IRB. $\square$

B) Proof of IRB:

$$H_\Lambda = -\frac{1}{2} \sum_{|x-y|=1} \left\{ S_1(x) S_1(y) + S_2(x) S_2(y) + u\, S_3(x) S_3(y) \right\} - h \sum_x S_3(x)$$

$$= \frac{1}{4} \sum_{|x-y|=1} \left\{ \left(S_1(x) - S_1(y)\right)^2 + \left(S_2(x) - S_2(y)\right)^2 + u\left(S_3(x) - S_3(y)\right)^2 \right\}$$

$$- \sum_x \left\{ h\, S_3(x) + d\left(S_1^2(x) + S_2^2(x) + u\, S_3^2(x)\right) \right\}$$

$v : \Lambda \longrightarrow \mathbb{R}$ a "scalar field" on Λ

$$H_\Lambda(v) = \frac{1}{4} \sum_{|x-y|=1} \left\{ \left(S_1(x) + v(x) - S_1(y) - v(y)\right)^2 + \left(S_2(x) - S_2(y)\right)^2 + u\left(S_3(x) - S_3(y)\right)^2 \right\}$$

$$- \sum_x \left\{ h\, S_3(x) + d\left(S_1^2(x) + S_2^2(x) + u\, S_3^2(x)\right) \right\}$$

Relativistic Quantum Theory

Jürg Fröhlich*

October 1, 2019

Abstract

The purpose of this paper is to sketch an approach towards a reconciliation of quantum theory with relativity theory. It will actually be argued that these two theories ultimately rely on one another. A general operator-algebraic framework for relativistic quantum theory is outlined. Some concepts of space-time structure are translated into algebra. Following deep results of Buchholz et al., the key role of massless modes, photons and gravitons, and of Huygens' Principle in a relativistic quantum theory well suited to describe "events" and "measurements" is highlighted. In summary, a relativistic version of the "*ETHApproach*" to quantum mechanics is described.

"I leave to several futures (not to all) my garden of forking paths"
(J. L. Borges)

Contents

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1 Topics to be addressed

Anybody who attempts to work on the foundations – or "interpretation" – of quantum theory realizes quickly that this field is in a state of utmost confusion. Whether authorities in this matter or not, *Richard Feynman* once said: "If someone tells you they understand quantum mechanics then all you've learned is that you've met a liar"; and *Sean Carroll*, of the California Institute of Technology, in a popular article that appeared in the 'New York Times' [1], writes: "... quantum mechanics has a reputation for being especially mysterious. What's surprising is that physicists seem to be O.K. with not understanding the most important theory they have. ... Physicists don't understand their own theory any better than a typical smartphone user understands what's going on inside the device. ... The whole thing is preposterous. Why are observations special? What counts as an "observation", anyway? When exactly does it happen? Does it need to be performed by a person? Is *consciousness* somehow involved in the basic rules of reality? Together these questions are known as the "measurement problem" of quantum theory. ... " – Well, obviously a text like this leaves the reader in a state of bewilderment and/or anger! In the same article *Caroll* also writes: "You would naturally think, then, that understanding quantum mechanics would be the absolute highest priority among physicists worldwide. ... Physicists, you might imagine, would stop at nothing until they truly understood quantum mechanics."

Quite some time (perhaps thirty years) ago, I arrived at a conclusion similar to the one *Caroll* reached in the last two sentences quoted above. In 2012, when I retired from my position at *ETH* and did not have to make a career, anymore, I started to consider it to be one of my obligations to help removing some of the confusion surrounding the foundations of quantum mechanics. I do not have any illusions about the chances of success in pursuing this goal,[1] not because it is impossible to understand quantum mechanics – I actually think it **is possible** – but chiefly because people have so many prejudices about it.

Here is my credo in this endeavor:

- Talking of the "interpretation" of a physical theory presupposes implicitly that the theory has reached its final form, but that it is not completely clear, yet, what it tells us about natural phenomena. Otherwise, we had better speak of the "foundations" of the theory. Quantum Mechanics has apparently not reached its final form, yet. Thus,

[1]A recent paper of mine on the foundations of quantum mechanics triggered the following comment from a "colleague": *"Hi, again and again. How many time will you recycle your papers? Cannot see (you?) that no one is interested in your obscure thinking. Adding 'ETH' will not help. You are old and essentially useless. Go fishing. Best, A."*

it is not really just a matter of interpreting it, but of completing its foundations.

- The only form of "interpretion" of a physical theory that I find legitimate and useful is to delineate approximately the ensemble of natural phenomena the theory is supposed to describe and to construct something resembling a "structure-preserving map" from a subset of mathematical symbols used in the theory that are supposed to represent physical quantities to concrete physical objects and phenomena (or events) to be described by the theory. Once these items are clarified the theory is supposed to provide its own "interpretation". (A good example is Maxwell's electrodynamics, augmented by the special theory of relativity.)

- The ontology a physical theory is supposed to capture lies in *sequences of events, sometimes called "histories",* which form the objects of series of observations extending over possibly long stretches of time and which the theory is supposed to describe.

- In discussing a physical theory and mathematical challenges it raises it is useful to introduce clear concepts and basic principles to start from and then use precise and – if necessary – quite sophisticated mathematical tools to formulate the theory and to cope with those challenges.

- To emphasize this last point very explicitly, I am against denigrating mathematical precision and ignoring or neglecting precise mathematical tools in the search for physical theories and in attempts to understand them, derive consequences from them and apply them to solve concrete problems.

In this paper I will sketch some ideas about a formulation of **local relativistic quantum theory** designed to describe "events" and, ultimately, to solve the "measurement problem" alluded to above. (In doing this I try to follow the credo formulated above.) I will specifically address the following topics:

1. Why is it fundamentally impossible to use a physical theory to predict the future? – Sect. 2.

2. Why is quantum theory intrinsically probabilistic? – Sect. 2.

3. How are "locality" and "Einstein causality" expressed in relativistic quantum theory; what is their meaning? – Sect. 3.

4. What are "events" in quantum theory – Sect. 4 – and how does one describe their recording? What is meant by "measuring a physical quantity"? – Sect. 5.

5. How do *states of physical systems* evolve in (space-)time, according to quantum theory? What is the probabilistic law governing their evolution? – Sect. 4.

6. How does quantum theory distinguish between past and future; how does it talk about space-time? Could it be that a consistent "Quantum Theory of Events" must necessarily be relativistic and involve massless modes? Could it be that such a quantum theory could explain why space-time is even-dimensional and that it might incorporate gravitation as an "emergent phenomenon"? – Sect. 6.

Acknowledgements: I am very much indebted to my collaborators on matters related to the results sketched in this paper; among them to Philippe Blanchard, Jérémy Faupin, Martin Fraas, and especially to *Baptiste Schubnel.* I also thank Detlev Buchholz, Gian Michele Graf, Klaus Hepp, Sandu Popescu, Renato Renner, but primarily Detlef Dürr and Shelly Goldstein for many helpful and enjoyable discussions and for serving as patient "sounding boards".

I wish to mention that various ideas related to ones elaborated on in [2, 3] and in this paper have been described in [4, 5]. In particular, many years ago, the late *Rudolf Haag* has emphasized the importance of introducing a clear notion of "events" in quantum theory and to elucidate their role.

This paper is dedicated to the memory of *Gian Carlo Ghirardi.* My approach to the foundations of quantum mechanics (dubbed *"ETH Approach"*) shares some general features with *GRW* [6]; in particular, an important role is played by "state collapse". I wish to thank *Detlef Dürr* for having invited me to present my ideas in this book.

2 Why are we not able to predict the future by using our physical theories, and why is quantum theory intrinsically probabilistic?

Imagine that the space-time of our Universe has an event horizon that hides what may happen in causally disconnected regions of space-time. *Figure 1*, below, illustrates the claim that, for fundamental reasons, observers are then unable to use relativistic theories to fully predict their future; for, **never** do they have access to complete knowledge of the initial conditions of the Universe that would be necessary (but not necessarily sufficient) to predict the future.[2] This argument applies to both, classical *and* quantum theories. But quantum theories have an additional feature that makes it impossible to use them to predict the future precisely: *They are fundamentally probabilistic.*

[2]The same is true if there exist waves propagating at the speed of light along *surfaces* of light-cones

Figure 1 is supposed to illustrate, furthermore, that the *"Past"* consists of a "History of Events" or "Facts", while the *"Future"* consists of an ensemble of "Potentialities". In a proper formulation of Quantum Mechanics this dichotomy should be retained! In this paper we will try to find out how to implement it in relativistic quantum theory.

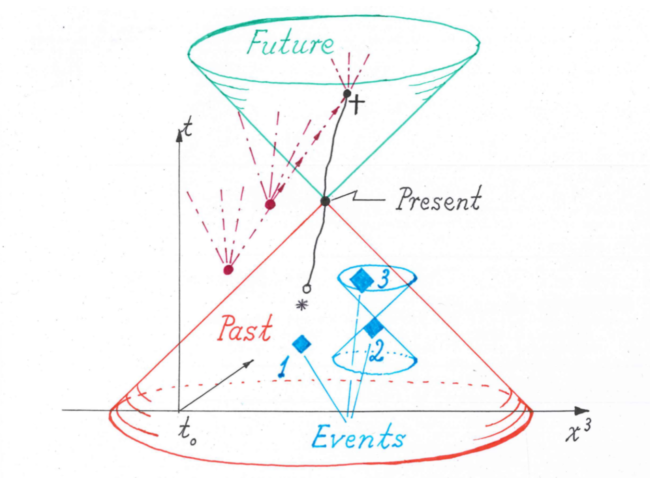

Fig. 1

Caption: The "observer" sits at "Present" and is unaware of the dangers lurking from outside his past light-cone (denoted *"Past"*). He might get killed at †, a space-time point in his future light-cone (denoted *"Future"*). Events are numbered in the figure; events 1 and 2 are space-like separated, event 3 is in the future of event 2.

Let S be an *"isolated physical system"* to be described by a model of relativistic quantum theory. – *Note:* An isolated system has the property that, over some period of time, its evolution does not depend on anything happening in its complement, i.e., in the rest of the Universe, in the sense that, during a certain period of time, the *Heisenberg-picture dynamics* of physical quantities characteristic of S is, *for all practical purposes*, **independent** of the degrees of freedom in the complement of S, (a consequence of cluster properties). It should be noted, however, that the state of S can be entangled with the state of its complement!. –

The concept of an *isolated* physical system is important in quantum mechanics, because, *only for such systems*, we know how to describe the time evolution of operators representing physical quantities in the Heisenberg picture (in terms of conjugation of those operators with the unitary propagator of the system). In order to describe the quantum dynamics of an isolated physical system S, we will allways start from the Heisenberg-picture dynamics of "observables" (i.e., of self-adjoint operators representing physical quantities) referring to S. The dynamics of *states* of S is considerably more subtle to understand and is, in a sense, at the *core of our considerations* in this paper – as it has been in the work of Ghirardi, Rimini and Weber.

In this paper we use (for simplicity) the following pedestrian formulation of the quantum mechanics of an isolated physical system S in the Heisenberg picture: States of S are given by density matrices, Ω, acting on a separable Hilbert space, $\mathcal{H}$, of "pure state vectors" of S. Let $\hat{X}$ be a physical quantity of S, and let $X(t) = X(t)^*$ be the self-adjoint linear operator on $\mathcal{H}$ representing $\hat{X}$ at time t. Then the operators $X(t)$ and $X(t')$ representing $\hat{X}$ at two *different* times t and t', respectively, are unitarily conjugated to one another:

$$X(t) = U(t', t)\, X(t') U(t, t')\,, \tag{1}$$

where, for each pair of times t, t', $U(t, t')$ is the propagator (from t' to t) of the system S, which is a unitary operator acting on $\mathcal{H}$, and $\big\{U(t, t')\big\}_{t,t' \in \mathbb{R}}$ satisfy

$$U(t, t') \cdot U(t', t'') = U(t, t''),\ \forall \text{ pairs } t, t', \quad U(t, t) = \mathbf{1}\,,\ \forall\, t\,.$$

It is often said that, in the Heisenberg picture, states of S are *independent* of time; and that the Heisenberg picture is equivalent to the Schrödinger picture, where physical quantities are time-independent, but states evolve according to the propagator $U(t, t')$, solving a *deterministic* Schrödinger equation. Even if quantum mechanics were put under the auspices of the so-called "Copenhagen interpretation", this is, of course, nonsense, as has been amply demonstrated on many examples; (see [10, 11, 8], and refs. given there)! For, whenever a "measurement" is made, at some time t, say – we will later speak, more accurately, of an **"event"** happening at approximately time t – the deterministic unitary evolution of the state of S in the Schrödinger picture is **interrupted** at this time, and the state "jumps", or "collapses" into an eigenspace of the "observable" that is measured – more accurately: the state jumps into the image of an orthogonal projection representing the "event" that actually happens at time t, with jumping probabilities as given by *Born's Rule*; (see also [4, 3]). Expressed in the Heisenberg picture, one can say that, while operators representing physical quantities referring to an isolated physical system S evolve in time according to Eq. (1), the *state* of S changes *randomly* whenever an "event" happens; it thus exhibits a *non-trivial, stochastic evolution* in time, a kind of *stochastic branching process* described in [12, 2, 3, 13] and in Sect. 4 of this paper. In order to avoid paradoxes [7, 8, 9], it is crucial to assume that the occurrence of an event (for example, the successful completion of a measurement) has an **objective** meaning independent of the "observer" – and independent of whether an "observer" is actually present or not.

One should think that, by now, these things are exceedingly well-known and appreciated, and hence I won't dwell on them any further. – It might be added, however, that, in *Bohmian mechanics*, randomness enters in a way that differs from the one in other formulations of quantum mechanics: Randomness is due, in Bohmian mechanics, to incomplete knowledge of initial conditions; see [14].[3]

3 The meaning of "locality" or "Einstein causality" in relativistic quantum theory

In this section, I sketch remarks on "locality" or "Einstein causality". For, there appears to exist a certain amount of confusion concerning the question in which sense quantum mechanics is *"non-local"* and in which sense it is perfectly *"local"*. Let us consider an isolated system, S, consisting of two spin-$\frac{1}{2}$ particles, p and p', and of equipment serving to measure components of their spins along two directions given by unit vectors $\vec{n}$ and $\vec{n}'$, respectively. We imagine that, after preparation of the initial state, Ω, of S, particle p propagates into a cone, C, opening in the direction of the negative x-axis, while p' propagates into a cone, C', opening in the direction of the positive x-axis, with only tiny probabilities for sojourn outside C and C', respectively. Let us assume that the measurement of the spin of p takes place inside a region $B \subset C$ in an interval $[t_1, t_2]$ of times, while the measurement of the spin of p' takes place in a region $B' \subset C'$ within a time-interval $[t'_1, t'_2]$, and let us imagine that the space-time regions $B \times [t_1, t_2]$ and $B' \times [t'_1, t'_2]$ are *space-like separated.* The results of the two measurements are described by two orthogonal projection operators, $\Pi^{p}_{\vec{n},\sigma}$, $\sigma = \pm$, and $\Pi^{p'}_{\vec{n}',\sigma'}$, $\sigma' = \pm$, where "$\sigma = +$" means that the spin of p is aligned with $\vec{n}$ after the measurement has been completed, while "$\sigma = -$" means that the spin of p is anti-parallel to $\vec{n}$ after its measurement, and similarly for p'. The operators $\Pi^{p}_{\vec{n},\sigma}, \sigma = \pm$, have the following properties:

$$\Pi^{p}_{\vec{n},+} \cdot \Pi^{p}_{\vec{n},-} = 0, \; \Pi^{p}_{\vec{n},+} + \Pi^{p}_{\vec{n},-} = \mathbf{1}\,, \tag{2}$$

and similarly for the operators $\Pi^{p'}_{\vec{n}',\sigma'}, \sigma' = \pm$. Moreover, the operators $\Pi^{p}_{\vec{n},\sigma}$ and $\Pi^{p'}_{\vec{n}',\sigma'}$ are localized in *space-like separated* regions, $B \times [t_1, t_2]$ and $B' \times [t'_1, t'_2]$, respectively, of space-time, for all choices of σ and of σ'. We would like to make an educated guess of the state used by a localized observer, $\mathcal{O}$, to predict his future if $\mathcal{O}$ has the property that the past light-cones of all points inside $\mathcal{O}$ contain both regions, $B \times [t_1, t_2]$ and $B' \times [t'_1, t'_2]$. The answer to the question which of the two spin measurements was initiated or completed *first* then obviously depends on the past "world-tube" of the observer $\mathcal{O}$. This is because $B \times [t_1, t_2]$ and $B' \times [t'_1, t'_2]$ are space-like separated. Let us suppose that, for an observer $\mathcal{O}$, the spin of p was measured first, that the state of S before any of these measurements were carried out was given

[3] The Bohmian point of view cannot be discussed any further in this paper

by a density matirx Ω, and that between the preparation of the state Ω of S and further observations by $\mathcal{O}$ *only* the measurements of the spins of p and of p' happened. According to the standard "projection postulate" (of the Copenhagen interpretation), the state used by $\mathcal{O}$ to predict future measurement outcomes is then given by

$$\Omega_{\mathcal{O}} = [\mathcal{N}_{(\vec{n},\sigma),(\vec{n}',\sigma')}]^{-1}\, \Pi^{p'}_{\vec{n}',\sigma'} \cdot \Pi^{p}_{\vec{n},\sigma}\, \Omega\, \Pi^{p}_{\vec{n},\sigma} \cdot \Pi^{p'}_{\vec{n}',\sigma'}\,, \tag{3}$$

where $\mathcal{N}_{(\vec{n},\sigma),(\vec{n}',\sigma')} := \mathrm{tr}\Big(\Pi^{p'}_{\vec{n}',\sigma'} \cdot \Pi^{p}_{\vec{n},\sigma}\, \Omega\, \Pi^{p}_{\vec{n},\sigma} \cdot \Pi^{p'}_{\vec{n}',\sigma'}\Big)$ is a normalization factor. Imagine now that $\mathcal{O}'$ is an observer localized in the *same* space-time region as $\mathcal{O}$, but for whom the spin of p' is measured *before* the spin of p. He then proposes to use the state $\Omega_{\mathcal{O}'}$ given by a formula arising form (3) by exchanging the order of $\Pi^{p}_{\vec{n},\sigma}$ and $\Pi^{p'}_{\vec{n}',\sigma'}$. We want to impose the requirement that *the predictions made by* $\mathcal{O}$ *and* $\mathcal{O}'$ *concerning future measurements* (i.e., ones localized in their common future light-cone) *must be compatible.* This implies that the two states $\Omega_{\mathcal{O}}$ and $\Omega_{\mathcal{O}'}$ must agree on the algebra of all "observables" potentially measureable in the future of $\mathcal{O}$ = future of $\mathcal{O}'$. This would be guaranteed if (but does **not** imply that)

$$\Pi^{p'}_{\vec{n}',\sigma'} \cdot \Pi^{p}_{\vec{n},\sigma} = \Pi^{p}_{\vec{n},\sigma} \cdot \Pi^{p'}_{\vec{n}',\sigma'}\,, \tag{4}$$

for arbitrary choices of $(\vec{n},\sigma)$ and $(\vec{n}',\sigma')$, assuming, as stated above, that the localization regions $B \times [t_1, t_2]$ and $B' \times [t'_1, t'_2]$ are space-like separated. Equation (4) is what is called *"locality"* or *"Einstein causality"* in relativistic quantum field theory. This is a sufficient (but not necessary) condition to eliminate ambiguities in the predictions of possible future measurement outcomes made by different observers that are due to the impossibility of unambiguously ordering measurements according to the times at which they are initiated (or completed). But Eq. (4) does **not** imply that quantum mechanics is "local" in the following sense: Consider the state

$$\Omega_{(\vec{n},\sigma)} := [\mathcal{N}_{(\vec{n},\sigma)}]^{-1} \Pi^{p}_{\vec{n},\sigma}\, \Omega\, \Pi^{p}_{\vec{n},\sigma},$$

where $\mathcal{N}_{(\vec{n},\sigma)}$ is a normalization factor chosen such that $\mathrm{tr}(\Omega_{(\vec{n},\sigma)}) = 1$. Let A be an "observable" localized in a space-time region space-like separated from $B \times [t_1, t_2]$; (for example $A = \Pi^{p'}_{\vec{n}',\sigma'}$). One might expect that

$$\mathrm{tr}\big(\Omega\, A\big) = \mathrm{tr}\big(\Omega_{(\vec{n},\sigma)}\, A\big)\,,$$

for any operator A with these properties. But, of course, this equality does **not** hold! This fact is what people call the "non-locality" of quantum theory. In quantum field theory, this kind of "non-locality" is neatly reflected in the Reeh-Schlieder theorem [15]. It results from entanglement.

One major purpose of this paper is to render the "projection postulate" (or "collapse postulate" – see Eq. (3)) more precise, to explain its origin and to find out *under what conditions it is applicable.* In contrast to the ideas described in [6], we will not invoke any mechanism extraneous to quantum mechanics that produces "state collapse".

4 Relativistic quantum theory, and the notion of "events"

In this section we propose an algebraic definition of local relativistic quantum theory and then introduce a precise notion of "events". We require some rudimentary knowledge of the theory of operator algebras. In particular, the reader might profit from knowing what a C^*- and what a von Neumann algebra is and what, for example, the Gel'fand-Naimark-Segal (GNS) construction is. What will be used from the theory of operator algebras, in this paper, can be learned in a few hours! A useful reference may be [16].

For the time being, we will consider space-time, $\mathcal{M}$, to be given; but we do not equip $\mathcal{M}$ with a Lorentzian metric. Later, we will try to clarify how properties of algebras of operators representing localized potentialities equip $\mathcal{M}$ with a causal structure. But to start with, we assume $\mathcal{M}$ to be given by Minkowski space, $\mathbb{M}^d$, with $d = 4$.

In relativistic quantum theory, all operators representing physical quantities characteristic of an isolated physical system S are localized in space-time regions. Given a region $\mathcal{O} \subset \mathcal{M}$, we denote by $\mathcal{A}(\mathcal{O})$ the algebra generated by all bounded operators localized in $\mathcal{O}$ that represent physical quantities. The family $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\subset\mathcal{M}}$ is called a "net of local algebras". For an introduction to these concepts and to algebraic quantum field theory the reader is advised to consult [17]. In the following considerations, the regions $\mathcal{O}$ are usually taken to be forward or backward light-cones with apex in an arbitrary space-time point $P \in \mathcal{M}$.

<u>A general formulation of local relativistic quantum theory</u>:

We consider an isolated physical system S to be described with the help of a model of local relativistic quantum theory.

<u>Definition 1</u>: By $\mathcal{F}_P$ we denote the *algebra generated by all operators representing physical quantities referring to S (such as potential events) localized in the "future" of the space-time point P, while $\mathcal{P}_P$ denotes the algebra generated by all operators representing physical quantities localized in the "past" of P. $\square$

We assume that all the algebras $\mathcal{F}_P$ are contained in a C^*-algebra $\mathcal{E}$, and

$$\mathcal{E} = \overline{\bigvee_{P\in\mathcal{M}} \mathcal{F}_P}\,, \tag{5}$$

where the closure on the right side is taken in the operator norm of $\mathcal{E}$. We assume that all these algebras are represented on a common separable Hilbert space $\mathcal{H}$ and that all *"states of physical interest"* of S can be identified with density matrices (non-negative trace-class operators normalized to have trace $= 1$) acting on $\mathcal{H}$.[4] In our notation, we will not distinguish between an abstract element of the algebra $\mathcal{E}$ and the linear operator on $\mathcal{H}$ representing it.

Definition 2: We define $\mathcal{E}_P$ to be the von Neumann algebra obtained by closure of the algebra $\mathcal{F}_P$ in the weak operator topology of the algebra, $B(\mathcal{H})$, of all bounded operators on $\mathcal{H}$. □

If S is a physical system in a state of *finite* energy describing only excitations of strictly positive rest mass then

$$\mathcal{E}_P \simeq B(\mathcal{H})\,, \text{ for any point } P \in \mathcal{M}\,. \tag{6}$$

It is expected that this equality always holds in a space-time of *odd* dimension, *even* if massless particles are present. This is because Huygens' Principle does not hold in space-times of odd dimension. (It also does not hold in certain even-dimensional space-times with non-vanishing curvature. But that's another story, which, for reasons that I will not explain in any detail, is not expected to invalidate the following considerations.) The property expressed in Eq. (6) is one most people sub-consciously consider to be always valid. But this is actually *not* the case! (If it were we would probably be unable to introduce a reasonable notion of "events" in quantum theory, and we would never solve the "measurement problem".)

If there exist massless particles, in particular photons and/or gravitons and Dark-Energy modes, and if Huygens' Principle holds in an appropriate sense ($\mathcal{M}$ even-dimensional, specifically $\mathcal{M} = \mathbb{M}^4$),[5] the algebra $\mathcal{E}_P$ tends to have an infinite-dimensional commutant, $\mathcal{E}'_P$. (The commutant, $\mathfrak{M}'$, of an algebra $\mathfrak{M}$ contained in $B(\mathcal{H})$ is the algebra of all bounded operators on $\mathcal{H}$ commuting with *all* operators in $\mathfrak{M}$.) More specifically, within an algebraic framework of local relativisitic quantum field theory over four-dimensional Minkowski space-time, *Detlev Buchholz* has shown [18] that, in the presence of massless particles, $\mathcal{E}'_{P_t} \cap \mathcal{E}_{P_{t_0}}$ is an *infinite-dimensional, non-commutative* algebra, whenever P_{t_0} is a space-time point in the *past* of the space-time point P_t, as indicated in Figure 2.

In his proof, *Huygens' Principle* is exploited in the form that asymptotic out-fields creating on-shell massless particles escaping to infinity do not propagate into the *interior* of forward light-cones contained in the future of the space-time region (denoted by $\mathcal{O}$ in Figure 2) where they are localized, but

[4]It is sometimes advantageous to formulate this assumption in a more abstract, algebraic way involving, among other ingredients, the *GNS*-construction; see, e.g., [17].

[5]or in the presence of blackholes in space-time

propagate along the *surface* of forward light-cones with apices in $\mathcal{O}$. Such asymptotic out-fields are then shown to commute with all operators in the algebra $\mathcal{E}_{P_t}$.

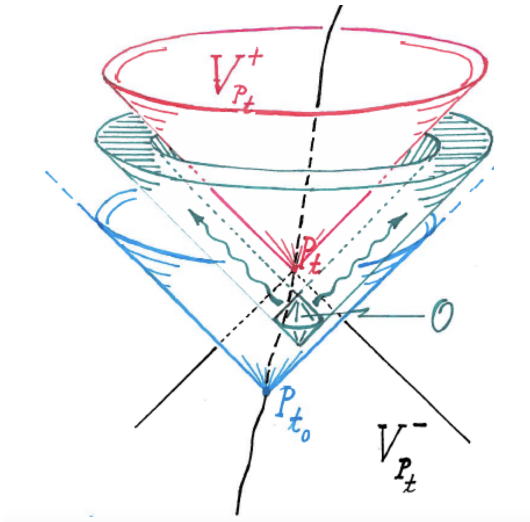

Fig. 2

Caption: The black line is the world-line of an "observer" who, at time t, is localized near P_t. Operators representing physical quantities potentially observable by the "observer" in the future of P_t are localized inside the forward light-cone $V^+_{P_t}$. They generate the algebra $\mathcal{E}_{P_t}$. Asymptotic out-field operators describing the emission of (on-shell) photons or gravitons in the region $\mathcal{O}$ propagate along the light-cones contained in $V^+_{P_{t_0}}$ but *not* contained in $V^+_{P_t}$.

One expects that, if space-time is even-dimensional and in the presence of massless particles, the algebras $\mathcal{E}_P$ have the property that all non-zero orthogonal projections belonging to $\mathcal{E}_P$ have an infinite-dimensional range. This implies that there do **not** exist any normal *pure* states on these algebras. Furthermore, they are expected to be isomorphic to a certain "universal" von Neumann algebra,[6] $\mathfrak{R}$, i.e., $\mathcal{E}_P \simeq \mathfrak{R}$, $\forall\, P \in \mathcal{M}$.

We now use these insights to extract a **general algebraic formulation** of *local relativistic quantum theory* compatible with the appearence of "events" and promising a solution of the "measurement problem". We assume that space-time $\mathcal{M}$ is a topological space with the property that, with every point $P \in \mathcal{M}$, one can associate a von Neumann algebras, $\mathcal{E}_P$, the "algebra of potential events that might possibly happen in the future of P", with the property that $\mathcal{E}_P$ is contained in a C^*-algebra $\mathcal{E}$, for all $P \in \mathcal{M}$.

The family of algebras $\{\mathcal{E}_P\}_{P\in\mathcal{M}}$ equips space-time $\mathcal{M}$ with the following *causal structure*:

[6] a von Neumann algebra of type III_1

Definition 3: A space-time point P' is in the future of a space-time point P, written as $P' \succ P$, (or, equivalently, P is in the past of P', written as $P \prec P'$) iff

$$\boxed{\mathcal{E}_{P'} \subsetneqq \mathcal{E}_P\,, \quad \mathcal{E}'_{P'} \cap \mathcal{E}_P \text{ is an } \infty - \text{dim. non-commutative algebra}} \tag{7}$$

$\square$

Equation (7) expresses what I call the

"Principle of Diminishing Potentialities" (PDP)

This principle is a **theorem** in an axiomatic formulation of quantum electrodynamics over four-dimensional Minkowski space proposed by D. Buchholz and the late J. Roberts [20].

Henceforth, the *Principle of Diminishing Potentialities will always be* **assumed** *to hold*; and, within our formulation of relativistic quantum theories, (a model of) an *isolated physical system* S is defined by specifying the following data:

$$S = \left\{\mathcal{M}, \mathcal{E}, \mathcal{H}, \left\{\mathcal{E}_P\right\}_{P\in\mathcal{M}} \text{ satisfying } PDP\right\}, \tag{8}$$

where $\mathcal{M}$ is a model of space-time, $\mathcal{E}$ is a C^*-algebra represented on a Hilbert space $\mathcal{H}$, and $\left\{\mathcal{E}_P\right\}_{P\in\mathcal{M}}$ is a family of von Neumann algebras satisfying the "Principle of Diminishing Potentialities" introduced in Eq. (7).

Definition 4: If a space-time point P' is neither in the future of a space-time point P nor in the past of P we say that P and P' are *space-like separated*, written as $P \times P'$. $\square$

Let Σ be a space-like subset of $\mathcal{M}$. If $\mathcal{M} = \mathbb{M}^4$ we imagine that Σ is a subset of a space-like hypersurface of co-dimension 1 in $\mathcal{M}$. Since all the algebras $\mathcal{E}_P, p \in \mathcal{M}$, are assumed to be contained in the C^*-algebra $\mathcal{E}$, the following definition is meaningful:

$$\mathcal{E}_\Sigma := \overline{\bigvee_{P\in\Sigma} \mathcal{E}_P}\,, \tag{9}$$

where the closure is taken in the weak topology of $B(\mathcal{H})$. A state, ω_Σ, on the algebra $\mathcal{E}_\Sigma$ is a normalized, positive linear functional on $\mathcal{E}_\Sigma$.

Remark: At this point we should comment on the question of what the operational meaning of a "state" of an isolated system S is, and how one can *prepare* S in a specific state. Obviously these are important questions, which, however, cannot be discussed here; but see [21].

Definition 5: Let $\mathfrak{M}$ be a von Neumann algebra, and let ω be a normal state on $\mathfrak{M}$. For an operator $X \in \mathfrak{M}$, we define $ad_X(\omega)$ to be the linear functional on $\mathfrak{M}$ defined by

$$ad_X(\omega)(Y) := \omega([Y, X]), \quad \forall\, Y \in \mathfrak{M}.$$

We define the *centralizer*, $\mathcal{C}_\omega(\mathfrak{M})$, of the state ω by

$$\mathcal{C}_\omega(\mathfrak{M}) := \big\{X\,|X \in \mathfrak{M}, ad_X(\omega) = 0\big\}. \tag{10}$$

It is easy to verify that $\mathcal{C}_\omega(\mathfrak{M})$ is a (von Neumann) subalgebra of $\mathfrak{M}$, and that ω is a normalized trace on $\mathcal{C}_\omega(\mathfrak{M})$. (This property implies that centralizers are completely classified!)
Given an algebra $\mathfrak{N}$, the *center*, $\mathcal{Z}(\mathfrak{N})$, is the abelian subalgebra of $\mathfrak{N}$ consisting of all operators in $\mathfrak{N}$ commuting with all other operators in $\mathfrak{N}$. We set

$$\mathcal{Z}_\omega(\mathfrak{M}) := \mathcal{Z}(\mathcal{C}_\omega(\mathfrak{M})) \tag{11}$$

□

Motivation underlying the following notions and definitions is provided in [2, 3, 13].

Definition 6: Given a point $P \in \mathcal{M}$, a *potential event* in the future of P is a family, $\big\{\pi_\xi|\, \xi \in \mathfrak{X}\big\}$, ($\mathfrak{X}$ a countable set of indices[7]), of orthogonal projections belonging to $\mathcal{E}_P$ with the properties

$$\pi_\xi \cdot \pi_\eta = \delta_{\xi\eta}\pi_\xi,\ \forall\ \xi, \eta \in \mathfrak{X}, \quad \sum_{\xi \in \mathfrak{X}} \pi_\xi = \mathbf{1}\,. \tag{12}$$

It is expected that events usually have a *finite duration*. This would imply that operators $\big\{\pi_\xi|\xi \in \mathfrak{X}\big\}$ representing a potential event in the future of the point P would be localized in a *compact* region of space-time contained in the future of P (the future light-cone with apex in P). □

Definition 7: Given a state ω_P on the algebra $\mathcal{E}_P$, we say that an *event happens in the future of the space-time point* P iff the algebra

$$\mathcal{Z}_{\omega_P} := \mathcal{Z}\big(\mathcal{C}_{\omega_P}(\mathcal{E}_P)\big)$$

is generated by the projections $\big\{\pi_\xi|\, \xi \in \mathfrak{X}\big\} \subset \mathcal{Z}_{\omega_P} \subset \mathcal{E}_P$ of a potential event in the future of P with the properties that the cardinality of $\mathfrak{X}$ is at least 2 and that there exist projections $\pi_{\xi_1}, ..., \pi_{\xi_n}, n \geq 2$, such that

$$\omega(\pi_{\xi_j}) > 0, \qquad \forall j = 1, ..., n,\ n \geq 2\,. \tag{13}$$

[7]Here it is assumed that potential events can be identified with the spectral projections of self-adjoint operators with *discrete* spectrum ($\simeq \mathfrak{X}$); more generally, one could identify potential events with spectral projections of families (abelian algebras) of commuting self-adjoint operators that may have continuous spectrum

(The quantity $\omega(\pi_\xi)$ will turn out to be the *Born probability* for π_ξ to occur in the future of P.) $\square$

Let ω_P be the state of S on the algebra $\mathcal{E}_P$. It is easy to see that if an event described by the family $\{\pi_\xi | \, \xi \in \mathfrak{X}\} \subset \mathcal{Z}_{\omega_P}$ of projections happens in the future of the point P then

$$\omega_P(X) = \sum_{\xi \in \mathfrak{X}} \omega\big(\pi_\xi \, X \, \pi_\xi\big), \quad \forall X \in \mathcal{E}_P \, , \tag{14}$$

i.e., the state ω_P on the algebra $\mathcal{E}_P$ is a *mixture* of the states

$$\omega_{P,\xi} := \big[\omega_P(\pi_\xi)\big]^{-1} \omega\big(\pi_\xi(\cdot)\pi_\xi\big) \tag{15}$$

labelled by the points $\xi \in \mathfrak{X}$.

The following is a crucial axiom.

Axiom 1 ("State-collapse" postulate): If an event happens in the future of a point $P \in \mathcal{M}$, in the sense of Definition 7, then the state to be used to make predictions of further events possibly happening in the future of P is given by ω_{P,ξ_*}, for some $\xi_* \in \mathfrak{X}$ with $\omega_P(\pi_{\xi_*}) > 0$, where $\omega_{P,\xi_*}, \xi_* \in \mathfrak{X}$, is defined in Eq. (15).

The probability that ω_{P,ξ_*} is selected among the states $\{\omega_{P,\xi} | \, \xi \in \mathfrak{X}\}$ is given by *Born's Rule*, namely it is given by $\omega_P(\pi_{\xi_*})$. The projection π_{ξ_*} is called the *"actual event"* happening in the future of P. $\square$

Next, we consider two points, P and P', in a subset Σ of $\mathcal{M}$, with $P \times P'$, (i.e., P and P' are space-like separated), We assume that the state ω_Σ defined in Eq. (9) is given, so that the states $\omega_P = \omega_\Sigma|_{\mathcal{E}_P}$ and $\omega_{P'} = \omega_\Sigma|_{\mathcal{E}'_P}$ are known, too. We suppose that, given ω_Σ, events happen in the future of P and of P'. Let $\mathcal{Z}_{\omega_P}$ denote the center of the centralizer of the state ω_P on the algebra $\mathcal{E}_P$, which describes the event $\{\pi^P_\xi | \xi \in \mathfrak{X}^P\}$ happening in the future of P, and let $\mathcal{Z}_{\omega_{P'}}$ be the algebra describing the event happening in the future of the point P'. We require the following axiom.

Axiom 2 (Events in the future of space-like separated points commute): Let $P \times P'$. Then all operators in $\mathcal{Z}_{\omega_P}$ commute with all operators in $\mathcal{Z}_{\omega_{P'}}$. In particular,

$$\big[\pi^P_\xi, \pi^{P'}_\eta\big] = 0, \;\; \forall \, \xi \in \mathfrak{X}^P \text{ and all } \; \eta \in \mathfrak{X}^{P'}. \qquad \square$$

This axiom may be one reflection of what people sometimes interpret as the fundamental **non-locality** of quantum theory: Projection operators representing events in the future of two *space-like separated* points P and P' in space-time are **constrained** to commute with each other! Actually, this implies what in quantum field theory is understood to express **locality** or Einstein causality.

Next, we assume that some slice, $\mathfrak{F}$, in space-time $\mathcal{M}$ is foliated by space-like hypersurfaces, Σ_τ: $\mathfrak{F} := \{\Sigma_\tau | \tau \in [0,1]\}$, where τ is a time coordinate in the space-time region filled by $\mathfrak{F}$. Let P be an arbitrary space-time point in the leaf Σ_1, and let the "recent past" of P, $V_P^-(\mathfrak{F})$, consist of all points in $\bigcup_{\tau<1} \Sigma_\tau$ that are in the *past* of P, in the sense specified in Definition 3, above. The task we propose to tackle is the following one: We suppose that we know the state ω_{Σ_0} on the algebra $\mathcal{E}_{\Sigma_0}$, (see Eq. (9)). Assuming that Axioms 1 and 2 hold, we propose to determine the state ω_P on $\mathcal{E}_P$, for the given point $P \in \Sigma_1$. Let $\{P_\iota | \iota \in \mathfrak{I}(\mathfrak{F})\}$ denote the subset of points in $V_P^-(\mathfrak{F})$ in whose future events happen (see Definition 7), and let

$$\{\pi_{\xi_\iota}^{P_\iota} | \iota \in \mathfrak{I}(\mathfrak{F})\} \subset \mathcal{E}_{\Sigma_0}$$

be the *actual events* (see Axiom 1) that happen in the future of the points P_ι, $\iota \in \mathfrak{I}(\mathfrak{F})$; (here $\mathfrak{I}(\mathfrak{F})$ is a set of indices labelling the points in $V_P^-(\mathfrak{F})$ in whose future events happen; it is here assumed to be countable). We define a so-called *"History Operator"*

$$H\big(V_P^-(\mathfrak{F})\big) := \vec{\Pi}_{\iota \in \mathfrak{I}(\mathfrak{F})}\, \pi_{\xi_\iota}^{P_\iota}\,, \tag{16}$$

where the ordering in the product $\vec{\Pi}$ is such that a factor $\pi_{\xi_\kappa}^{P_\kappa}$ corresponding to a point P_κ stands to the right of a factor $\pi_{\xi_\iota}^{P_\iota}$ corresponding to a point P_ι iff $P_\kappa \prec P_\iota$, (i.e., if P_κ is in the past of P_ι). But if $P_\iota \times P_\kappa$, i.e., if P_ι and P_κ are space-like separated the order of the two factors is *irrelevant – thanks to* Axiom 2!

The state on the algebra $\mathcal{E}_P$ relevant to make predictions about events happening in the future of P, in the sense of Definition 7, is then given by

$$\omega_P(X) \equiv \omega_P^{\mathfrak{F}}\big(X\big) = \big[\mathcal{N}_P^{\mathfrak{F}}\big]^{-1} \omega_{\Sigma_0}\big(H(V_P^-(\mathfrak{F}))^*\, X\, H(V_P^-(\mathfrak{F}))\big)\,,\ X \in \mathcal{E}_P\,, \tag{17}$$

where the normalization factor $\mathcal{N}_P^{\mathfrak{F}}$ is given by

$$\mathcal{N}_P^{\mathfrak{F}} = \omega_{\Sigma_0}\big(H(V_P^-(\mathfrak{F}))^* \cdot H(V_P^-(\mathfrak{F}))\big)\,. \tag{18}$$

We recall that, according to Definition 7, an event happens in the future of a point $P \in \Sigma_1$ iff the center, $\mathcal{Z}_{\omega_P}$, of the centralizer of the state ω_P on the algebra $\mathcal{E}_P$, defined in (17), contains at least two disjoint orthogonal projections of strictly positive probability, as given by *Born's Rule*; (see Axiom 1).

The quantities $\mathcal{N}_P^{\mathfrak{F}}$ can be used to equip the *tree-like* space (the so-called "non-commutative spectrum" of S) of all possible histories of events in the future of points belonging to the foliation $\mathfrak{F}$ with a *probability measure*; see [3].

The ideas and results discussed here are illustrated in Figure 3, below.

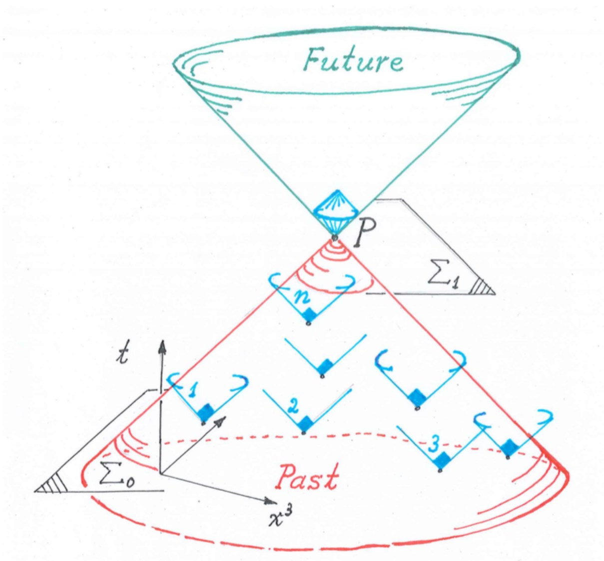

Fig. 3

Caption: It is tacitly assumed here that all events that happened in the past of the point P have a strictly finite duration. They are marked by small "diamonds" and are numbered from 1 to n. Notice that 1×2 and $2 \prec n$.

To conclude this discussion, in the approach to relativistic quantum theory presented in this paper (called "*ETHApproach*"), the **evolution** (along the foliation $\mathfrak{F}$) of the **state** of an isolated physical system S, given the initial state ω_{Σ_0} on the algebra $\mathcal{E}_{\Sigma_0}$ defined in Eq.(9),[8] can be viewed as a *generalized stochastic branching process*, whose state space is what I have called the "non-commutative spectrum" of the system S, (see [3], and Eq. (27), Sect. 6, for a definition), and with *branching rules* derived from Definition 7, Axioms 1 and 2 and Eqs. (16) - (18).[9]

Mathematical details can be made precise if space-time is discretized. Additional information can be found in [3, 23, 24].

[8]and assuming the axiom of choice

[9]This picture has reminded my former student *P.-F. Rodriguez* of the sentence from the short story *"The Garden of Forking Paths"*, by *Jorge Luis Borges*, that I have appended to the abstract of this paper

5 Monitoring events by measuring physical quantities

Let $S = \{\mathcal{M}, \mathcal{E}, \mathcal{H}, \{\mathcal{E}_P\}_{P\in\mathcal{M}} \text{ satisfying } PDP\}$ be the data defining an isolated physical system, with the properties specified in Sect. 4, Eq. (8), and assumed to satisfy Axioms 1 and 2. In Sect. 4, we have introduced a precise notion of **"events"** featured by S. In this section, we propose to explain how events can be recorded/monitored by measuring physical quantities referring to S.

For the purposes of the present exposition it is convenient to define a *"physical quantity"* to be an abstract self-adjoint linear operator $\hat{X}$ with the property that, for every point $P \in \mathcal{M}$, there exists a concrete self-adjoint linear operator $X(P) \in \mathcal{E}_P$ acting on the Hilbert space $\mathcal{H}$ of S and representing the quantity $\hat{X}$; (see [3] for a somewhat more general and abstract notion of physical quantities).

Remark: If space-time $\mathcal{M}$ is given by Minkowski space $\mathbb{M}^4$ the operator $X(P)$ is conjugated to the operator $X(P')$ by a unitary operator on the Hilbert space $\mathcal{H}$ representing the space-time translation from P to P'. But on general space-times there isn't any simple relation between $X(P)$ and $X(P')$.

We define

$$\mathcal{O}_S := \left\{ \hat{X}_\iota = \hat{X}_\iota^* \,|\, \iota \in \mathfrak{I}(S) \right\} \tag{19}$$

to be a list of all physical quantities available, at present, to characterize properties of S for which there exists a prescription of how they can be measured.[10] The list $\mathcal{O}_S$ is not intrinsic to the theoretical description of the system S; rather it specifies those physical quantities referring to S that, during a given era, can be expected to be measurable in real experiments. In quantum theory, this list is *not* an algebra (unless all operators belonging to $\mathcal{O}_S$ commute with one another), and it is usually not even a real linear vector space. The question to be addressed in the following is what we mean by saying that some quantity $\hat{X} \in \mathcal{O}_S$ is measured in the future of a space-time point P, and how such a measurement can be used to record an event that happens in the future of P.

Suppose that, for some point $P \in \mathcal{M}$, the *center* $\mathcal{Z}_{\omega_P}$ (of the centralizer $\mathcal{C}_{\omega_P}(\mathcal{E}_P) \subset \mathcal{E}_P$ of the state ω_P on the algebra $\mathcal{E}_P$) is non-trivial and is generated by a family $\{\pi_\xi | \, \xi \in \mathfrak{X}\}$ of disjoint orthogonal projections describing an event happening in the future of P. Let ε be a positive number; (it will turn out to be a measure of the "resolution" of the recording of this event in a measurement of a physical quantity $\hat{X} \in \mathcal{O}_S$). We let $\{\pi_1, \dots, \pi_N\}$ be a finite number of disjoint orthogonal projections contained in $\mathcal{Z}_{\omega_P}$ with the

[10]For simlicity, we assume that all operators in $\mathcal{O}_S$ have discrete spectrum

property that

$$\omega(\pi_j) \geq \varepsilon,\ \forall\, j = 1, ..., N, \qquad \omega(\mathbf{1} - \sum_{i=1}^{N} \pi_i) < \varepsilon\,. \tag{20}$$

The projections $\{\pi_1, \ldots, \pi_N\}$ form the basis of an N-dimensional vector space, $\mathcal{V}^{(\varepsilon)}_{\omega_P}$, equipped with a (positive-definite) scalar product, $\langle \cdot, \cdot \rangle$, given by

$$\langle \pi_i, \pi_j \rangle := \omega(\pi_i \cdot \pi_j) = \omega(\pi_i)\, \delta_{ij} \geq \varepsilon\, \delta_{ij}, \quad \text{for } i, j = 1, ..., N\,. \tag{21}$$

Every vector $Z \in \mathcal{V}^{(\varepsilon)}_{\omega_P}$ can be represented as a linear combination,

$$Z = \sum_{j=1}^{N} z_j \pi_j \in \mathcal{Z}_{\omega_P}\,, \text{ for complex numbers } z_1, ..., z_N\,. \tag{22}$$

We can thus identify $\mathcal{V}^{(\varepsilon)}_{\omega_P}$ with an N-dimensional subspace, actually an N-dimensional *subalgebra* of $\mathcal{Z}_{\omega_P}$.

Let $\mathcal{H}_{\omega_P}$ be the Hilbert space and Ω_P the cyclic vector in $\mathcal{H}_{\omega_P}$ obtained by applying the Gel'fand-Naimark-Segal construction to the pair $(\mathcal{E}_P, \omega_P)$; (see. e.g., [16]). There is a bijection between the vector space $\mathcal{V}^{(\varepsilon)}_{\omega_P}$ and the subspace $\mathcal{W}^{(\varepsilon)}_{\omega_P} \subset \mathcal{H}_{\omega_P}$ spanned by the vectors

$$\{Z\, \Omega_P |\, Z \in \mathcal{V}^{(\varepsilon)}_{\omega_P}\}\,.$$

By $Q^{(\varepsilon)}$ we denote the orthogonal projection onto $\mathcal{W}^{(\varepsilon)}_{\omega_P}$.

Let $\hat{X} \in \mathcal{O}_S$ be a physical quantitiy characteristic of S, and let $X(P) \in \mathcal{E}_P$ denote the self-adjoint operator representing $\hat{X}$. We consider the spectral decomposition of $X(P)$:

$$X(P) = \sum_{k=1}^{M} x_j\, \Pi_j(P)\,, \tag{23}$$

where the operators $\Pi_k(P) \in \mathcal{E}_P, k = 1, ..., M \leq \infty$, are the spectral projections of $X(P)$, with

$$\Pi_k(P) = \Pi_k(P)^*\,, \Pi_j(P) \cdot \Pi_k(P) = \delta_{jk}\, \Pi_j(P), \forall j, k\,, \sum_{k=1}^{M} \Pi_k(P) = \mathbf{1},$$

and $x_1, ..., x_M$ are the eigenvalues of $X(P)$ (= eigenvalues of $\hat{X}$), ordered in such a way that the sequence $\big(\omega_P(\Pi_k(P))\big)_{k=1}^{M}$ is *decreasing*. Let $L \leq M$ be such that

$$\omega_P(\mathbf{1} - \sum_{k=1}^{L} \Pi_k) < \varepsilon\,.$$

Given an operator $A \in \mathcal{E}_P$, we denote by $\epsilon_{\omega_P}(A)$ the *unique* operator in the algebra $\mathcal{V}^{(\varepsilon)}_{\omega_P} \subset \mathcal{Z}_{\omega_P}$ given by

$$Q^{(\varepsilon)} A \Omega_P =: \epsilon_{\omega_P}(A)\Omega_P \,, \qquad \epsilon_{\omega_P}(A) \in \mathcal{V}^{(\varepsilon)}_{\omega_P} \,. \tag{24}$$

The map

$$\epsilon_{\omega_P} : \mathcal{E}_P \to \mathcal{V}^{(\varepsilon)}_{\omega_P}$$

is called a *"conditional expectation"*; (see [25] for a systematic theory). Claiming that a measurement of the physical quantity $\hat{X}$ can be expected to be possible and to record the event $\{\pi_\xi \,|\, \xi \in \mathfrak{X}\}$ generating $\mathcal{Z}_{\omega_P}$ with a resolution of order ε relies on the validity of the following

Basic Assumption:

$$\|\Pi_k(P) - \epsilon_{\omega_P}\big(\Pi_k(P)\big)\| < \varepsilon, \quad \forall\, k = 1, ..., L \,. \tag{25}$$

It is not hard to verify (but see [3], Eqs. (22), (23), for a proof) that this Assumption implies that

$$\omega_P(A) = \sum_{k=1}^{L} \omega\big(\Pi_k(P)\, A\, \Pi_k(P)\big) + \mathcal{O}\big(L\,\varepsilon\, \|A\|\big) \,, \qquad \forall\, A \in \mathcal{E}_P \,, \tag{26}$$

i.e., the state ω_P is an **incoherent** superposition of eigenstates of the operator $X(P)$, up to an error of order ε. In this very precise sense, one can say that Assumption (25) implies that there is an approximate measurement of the physical quantity $\hat{X}$ in the future of the point P.

Using a simple lemma (see [22], Lemma 8 and Appendix C), one can show that if ε is sufficiently small Assumption (25) implies that there are orthogonal projections $\pi_k(\hat{X}) \in \mathcal{Z}_{\omega_P}$ with the property that

$$\|\Pi_k(P) - \pi_k(\hat{X})\| < \mathcal{O}(\varepsilon),$$

and

$$\omega_P(A) = \sum_{k=1}^{L} \omega\big(\pi_k(\hat{X})\, A\, \pi_k(\hat{X})\big) + \mathcal{O}\big(L\,\varepsilon\, \|A\|\big) \,, \qquad \forall\, A \in \mathcal{E}_P \,.$$

In this precise sense, if $L \geq 2$ a measurement of the quantity $\hat{X}$ in the future of P yields non-trivial information about the **event** described by $\mathcal{Z}_{\omega_P}$ happening in the future of P. If $L = N$ the projections $\{\pi_k(\hat{X}) | k = 1, ..., L\}$ must coincide with the projections $\{\pi_j | j = 1, ..., N\}$ introduced right before (20), provided $\varepsilon \ll 1$ is sufficiently small. In this case, a measurement of $\hat{X}$ yields very precise information about the event happening in the future of P.

For further discussion of these matters see [3], (Sect. 3, V.).

6 Conclusions and outlook

In this last section, some scattered remarks and speculations that grow out of the results sketched in Sections 4 and 5 are presented.

1. In our attempt to cast *local relativistic quantum theory* in a form compatible with the manifestation of what we have defined to be *"events"* and with a solution of the *"measurement problem"*, the *"Principle of Diminishing Potentialities" (PDP)*, (see Definition 3, Sect. 4, Eq. (7), and [3]), plays a fundamental role. We have seen that if space-time is *even-dimensional* (e.g., $\mathcal{M} = \mathbb{M}^4$) and if there exist massless particles – photons, gravitons and, possibly, Dark-Energy modes – satisfying some form of Huygens' Prinicple, (see [18]), then (PDP) holds. One may argue that (PDP) also holds in space-times containing blackholes. From a very general point of view, it appears that a quantum theory satisfying (PDP) is necessarily "relativistic", and the dimension of its space-time must be even.

2. In Definitions 3 and 4 of Sect. 4, we have seen that there is a purely algebraic way to equip space-time $\mathcal{M}$ with a *causal structure*: A space-time point P is in the past of a space-time point P' (written as $P \prec P'$) iff
$$\mathcal{E}_{P'} \subsetneqq \mathcal{E}_P,$$
and the relative commutant, $\mathcal{E}'_{P'} \cap \mathcal{E}_P$, of the algebra $\mathcal{E}_{P'}$ in $\mathcal{E}_P$ is a non-commutative algebra. Two points P and P' are space-like separated (written as $P \times P'$) iff P is not in the past of P' and P' is not in the past of P. It would be desirable to further elucidate the relationship of the algebras $\mathcal{E}_P$ and $\mathcal{E}_{P'}$ in case the points P and P' are space-like separated.

   Ultimately, we would like to **reconstruct** space-time from purely algebraic data concerning a family (or families) of operator algebras equipped with certain relations, in particular inclusions and statements about relative commutants, given a state on these algebras. A (presumably not entirely successful) attempt in this direction has been made in [26].

3. In the formalism described in Sect.4, **"events"** are localized in the future of certain space-time points, P; in the sense that they are described in terms of the abelian algebras $\mathcal{Z}_{\omega_P} \subset \mathcal{E}_P$, where, for a given point P, $\mathcal{Z}_{\omega_P}$ is the center of the centralizer of the state ω_P on the algebra $\mathcal{E}_P$, with $\mathcal{E}_P$ describing all potentialities in the *future* of P. The *actual event* happening in the future of some point P is an orthogonal projection, π_ξ^P, belonging to $\mathcal{Z}_{\omega_P}$, for some point ξ in an index set $\mathfrak{X}^P$, and having a strictly positive probability as predicted by *Born's*

Rule. In view of Axiom 2, Sect. 4, it would be important to have a more precise idea about the space-time regions where the operators $\pi_\xi^P, \xi \in \mathfrak{X}^P$, are localized. This might actually yield information about the **geometry** of space-time and, ultimately, support the view that gravitation is an "emergent" (or "derived") phenomenon.

To render these remarks a little more precise, we recall that one expects that all the algebras $\mathcal{E}_P$ are isomorphic to a "universal" von Neumann algebra $\mathfrak{N}$. One would like to know more about properties of states, ω, on $\mathfrak{N}$ for which the centers, $\mathcal{Z}_\omega(\mathfrak{N})$, of the centralizers $\mathcal{C}_\omega(\mathfrak{N})$ of ω are non-trivial, in the sense of Definition 7, Sect. 4. In [3],

$$\mathfrak{Z}_S := \bigcup_\omega \mathcal{Z}_\omega(\mathfrak{N}), \tag{27}$$

where ω ranges over all "states of physical interest", has been dubbed the *"non-commutative spectrum"* of the system S. It is the "state space" of the stochastic branching process defined by Eqs. (16), (17) and (18) of Sect. 4, which describes the *stochastic evolution of states* of S. Unfortunately, we have very little insight into the structure of the non-commutative spectrum $\mathfrak{Z}_S$.

It would be important to equip the algebra $\mathfrak{N}$ (and hence $\mathcal{E}_P$, for $P \in \mathcal{M}$) with a local structure, (in the sense that $\mathfrak{N}$ is generated by a net of local sub-algebras), and to attempt to show that *events*, i.e., elements of one of the algebras $\mathcal{Z}_\omega(\mathfrak{N})$, with ω a "state of physical interest", *are typically contained in sub-algebras of* $\mathfrak{N}$ *corresponding to what can be considered a "bounded region" of space-time*. This would help to introduce a more precise version of Axiom 2. But this topic, too, remains to be clarified.

4. One would expect that, for initial conditions given by states, ω_{Σ_0}, of S of "physical interest", (see Eq. (9), Sect. 4), the ensemble of events happening in the future of the points belonging to a foliation $\{\Sigma_\tau | \tau \in [0,1]\}$ of some slab of space-time (see Sect. 4, after Axiom 2) is *countable*, and that these events are localizable in bounded regions of space-time. One would expect, moreover, that the metric extension of a space-time region within which an event can be localized is constrained by *space-time uncertainty relations* of a kind discussed, e.g., in [27]. This ought to be a consequence of time-energy uncertainty relations and of the possibility that blackholes form in the aftermath of energetic events, which, afterwards, would evaporate.

   Alas, I don't know how to even start to derive these expectations from a more precise formalism of local relativistic quantum theory. Yet, the results reviewed in this paper and in [24] suggest that, once we truly understand what is meant by a local relativistic quantum theory of events, we will view **events** as the **basic building blocks** weaving the fabric of space-time and the *relations between events* as determining the **geometry of space-time**.

To conclude, I want to express the hope that the results, problems and speculations reviewed in this paper might challenge colleagues with more technical knowledge and strength than I am able to muster to go further towards the goal of truly understanding the miracles of quantum theory.

References

[1] Sean Carroll, in the 'New York Times'

[2] Ph. Blanchard, J. Fröhlich and B. Schubnel, *A 'Garden of Forking Paths' – the Quantum Mechanics of Histories of Events*, Nucl. Phys. B**912** (2016), 463-484

[3] J. Fröhlich, *A brief review of the ETH Approach to Quantum Mechanics*, to appear in: "Frontiers in Analysis and Probability", N. Anantharaman and A. Nikeghbali (eds.), Springer-Verlag (2020) [arXiv:1905.06603]

[4] R. Haag, *Fundamental Irreversibility and the Concept of Events*, Commun. Math. Phys. **132** (1990), 245-251;
R. Haag, *Events, Histories, Irreversibility*, in: *Quantum Control and Measurement*, Proc. ISQM, ARL Hitachi, H. Ezawa and Y. Murayama (eds.), North Holland, Amsterdam 1993;
Ph. Blanchard and A. Jadczyk, *Event-Enhanced Quantum Theory and Piecewise Deterministic Dynamics*, Annalen der Physik **4** (1995), 583-599

[5] Bernard S. Kay and Varqa Abyaneh, *Expectation values, experimental predictions, events and entropy in quantum gravitationally decohered quantum mechanics*, arXiv:0710.0992 (v1), unpublished;
Bernard S. Kay, *The Matter-Gravity Entanglement Hypothesis*, Found Phys **48** (2018), 542-557

[6] G. C. Ghirardi, A. Rimini, and T. Weber, *Unified dynamics for microscopic and macroscopic systems*, Phys. Rev. D **34** (1986), 470

[7] E. P. Wigner, *Remarks on the mind-body question*, in: I. J. Good, "The Scientist Speculates", Heinemann, London 1961

[8] L. Hardy, *Quantum mechanics, local realistic theories, and Lorentz-invariant realistic theories*, Phys. Rev. Letters. **68**, (20) (1992), 2981–2984; and
Nonlocality for two particles without inequalities for almost all entangled states, Phys. Rev. Letters. **71**, (11) (1993), 1665–1668.

[9] D. Frauchiger and R. Renner, *Quantum Theory Cannot Consistently Describe the Use of Itself*, Nature Communications **9** (2018), # 3711

[10] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics*, Cambridge University Press, Cambridge (UK) 1987.
See also:
J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement*, Princeton University Press, Princeton NJ, 1983;
K. Hepp, *Quantum Theory of Measurement and Macroscopic Observables*, Helv. Phys. Acta **45** (1972), 237-248;
H. Primas, *Asymptotically Disjoint Quantum States*, in: *Decoherence: Theoretical, experimental and Conceptual Problems*, pp 161-178, Ph. Blanchard, D. Giulini, E. Joos, C. Kiefer and I.-O. Stamatescu (eds.), Springer-Verlag, Berlin 2000

[11] J. Faupin, J. Fröhlich and B. Schubnel, *On the Probabilistic Nature of Quantum Mechanics and the Notion of 'Closed' Systems*, Ann. Henri Poincaré **17** (2016), 689-731

[12] J. Fröhlich and B. Schubnel, *Quantum Probability Theory and the Foundations of Quantum Mechanics*, arXiv:1310.1484, in: *The Message of Quantum Science – Attempts Towards a Synthesis*, Ph. Blanchard and J. Fröhlich (eds.), Springer-Verlag, Berlin-Heidelberg-New York 2015

[13] J. Fröhlich, *Quantum Theory and Causality*, Talks at the University of Leipzig (2018), TU-Stuttgart (2019), IHES (2019) and at Vietri sul Mare (Italy) (2019) – slides available on 'ResearchGate'.

[14] D. Dürr and S. Teufel, *Bohmian Mechanics – The Physics and Mathematics of Quantum Theory*, Springer-Verlag, Berlin-Heidelberg-New York 2009

[15] R. Jost, *The General Theory of Quantized Fields*, AMS Publ., Providence R.I., 1965.

[16] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics, Vol. 1*, 2nd edition, Springer-Verlag, Berlin-Heidelberg-New York, 1997.

[17] R. Haag, *Local Quantum Physics – Fields, Particles, Algebras*, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[18] D. Buchholz, *Collision Theory for Massless Bosons*, Commun. Math. Phys. **52**, (1977), 147-173; (see Theorem 9)

[19] D. Buchholz and S. Doplicher, *Exotic infrared representations of interacting systems*, Ann. lnst. H.Poincaré (Physique théorique) **40**, no. 2 (1984), 175-184

[20] D. Buchholz and J. E. Roberts, *New Light on Infrared Problems: Sectors, Statistics, Symmetries and Spectrum*, Commun. Math. Phys. **330** (2014), 935-972

[21] J. Fröhlich and B. Schubnel, *The Preparation of States in Quantum Mechanics*, J. Math. Phys. **57** (2016), 042 101

[22] J. Fröhlich and B. Schubnel, *Do We Understand Quantum Mechanics – Finally?*, arXiv:1203.3678, in: Proceedings of conference in memory of Erwin Schrödinger, Vienna, January 2011, publi. in 2012

[23] B. Schubnel, *Mathematical Results on the Foundations of Quantum Mechanics*, PhD thesis 2014, available at https://doi.org/10.3929/ethz-a-010428944

[24] J. Fröhlich, *'ETH' in Quantum Mechanics*, Notes of Lectures on the Foundations of Quantum Mechanics, Les Diablerets, January 9 - January 14, 2017

[25] M. Takesaki, *Conditional Expectations in von Neumann Algebras*, J. Funct. Anal. **9** (1972), 306-321;
F. Combes, *Poids et Espérances Conditionnelles dans les Algèbres de von Neumann*, Bull. Soc. Math. France **99** (1971), 73-112

[26] U. Bannier, *Intrinsic Algebraic Characterization of Space-Time Structure*, Intl. J. of Theor. Physics **33** (1994), 1797-1809

[27] S. Doplicher, K. Fredenhagen and J. E. Roberts, *Spacetime quantization induced by classical gravity*, Physics Letters **331** (1994), 39-44