Emergent Strings, WGC & their Imprint on Geometry

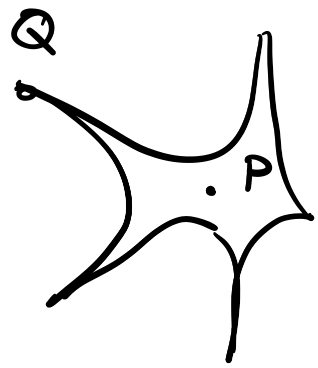
I.) Emergent String Conjecture

Reminder: Distance Conjecture [Ooguri–Vafa '06]

At infinite distance in moduli space of QG

$\exists$ tower of states with

$$m(Q) \simeq m(P)\, e^{-\alpha\, d(P,Q)}$$



$\alpha = \mathcal{O}(1)$ (Refined DC) [Klaewer – Palti]

Questions:

* What is the nature of the tower?
  - → spacing, degeneracy, "origin"
  - → only particles or higher-dim. objects?
* What is the asymptotic theory?
  - → just a mess, a new theory, a known theory?
* Which values of α?
* Relation to other conjectures, e.g. WGC?

Proposal to address these points:

Emergent String Conjecture [Lee, Lerche, TW '19]
1910.01135

In a QG in $\mathbb{R}^{1,d-1}$, $d \geq 3$, all ∞-distance limits are either

i) decompactification limits:
$\leftrightarrow$ lightest tower has (dual) interpretation as a KK tower

ii) emergent string limits:

$\leftrightarrow$ lightest tower $\equiv$ excitations of a <u>unique</u>, <u>critical</u>, asymptotically tensionless (w.r.t. M_{Pl}) & weakly coupled string
+ KK tower at some scale (for $d<9$)

<u>Motivation</u>: So far mainly non-trivial evidence from string & M-theory

<u>Remarks:</u>

i) KK vs string tower:

KK: $M_n \sim n \cdot M_0$ for $n \gg 1$
string: $M_n \sim \sqrt{n}\, M_0$ + exponential degeneracy

ii) KK tower often in disguise $\leftrightarrow$ Duality!

iii) Excluded by ESC:

* For $d<9$, no string tower possible below

any other tower — need KK tower (criticality)

* Emergent string is unique
  — No 2 or more critical strings can become tensionless & leading at same time!

→ Non-trivial tests in explicit theories — sometimes quantum corrections required to reinstall consistency

iv) Tower WGC essentially a direct consequence — at least asymptotically — see Lecture 2

2.) Testing the ESC in M-theory on CY_3

Framework: ∞ distance limits in Kähler moduli space of CY_3 probed by M-theory

→ classically exact, hence pedagogically suitable

→ M-theory not a theory of strings, hence particularly strong test of ESC

2.a) Some background

* M-theory in 11d approximated by 11d SUGRA

- bosonic action:

$$S_{11d} = \underbrace{2\pi M_{11}^9}_{= \frac{2\pi}{\ell_{11}^9}} \int_{\mathbb{R}^{1,10}} R * \mathbb{1} - \frac{1}{2} dC_3 \wedge * dC_3 + \dots$$

- M2-brane $\quad S_{M2} = \frac{2\pi}{\ell_{11}^3} \int_{2+1} \sqrt{-\det g} + 2\pi \int C_3$

M5-brane $\quad S_{M5} = \frac{2\pi}{\ell_{11}^6} \int_{5+1} \sqrt{-\det g} + \dots$

* Compactify M-theory on CY_3 Y (general)
$\longrightarrow$ 5d $N=1$ SuGRA
with 5d Planck scale

$$\frac{M_{pl}^3}{M_{11}^3} = 4\pi \mathcal{V}_Y \qquad \mathcal{V}_Y = \text{volume of } Y \text{ in units of } \ell_{11}$$

2.b) Geometric Analysis

Classification of ∞ dist. limits in <u>classical</u> Kähler moduli space of CY_3

* Kähler form J computes volumes of holomorphic subm.folds:

$$\mathcal{V}_C = \int_C J \,, \qquad \mathcal{V}_D = \frac{1}{2}\int_D J \wedge J \,, \qquad \mathcal{V}_Y = \frac{1}{6}\int_Y J^3$$

curve divisor CY_3

* $\{ J_\alpha \,,\ \alpha = 1, \dots, h^{1,1}(Y) \}$ basis of $H^{1,1}(Y)$ consisting of generators of Kähler cone

$$J = t^\alpha J_\alpha \qquad t^\alpha \in \mathbb{R}^+_0$$

(real) Kähler moduli

$$\mathcal{V}_C = t^\alpha \int_C J_\alpha \longrightarrow t^\alpha = \mathcal{V}_{C^\alpha} \quad \{ C^\alpha \,,\ \alpha = 1, \dots, h^{1,1}(Y) \}$$

$$\int_{C^\alpha} J_\beta = C^\alpha \cdot J_\beta = \delta^\alpha{}_\beta$$

* $\mathcal{V}_Y = \frac{1}{6}\int_Y J^3 = \frac{1}{6} \kappa_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma \qquad \kappa_{\alpha\beta\gamma} = \int_Y J_\alpha \cdot J_\beta \cdot J_\gamma$

For J_α Kähler cone generators

$$\kappa_{\alpha\beta\gamma} \geq 0$$

* Infinite distance limits :

One or several $t^\alpha \longrightarrow \infty$

i) If $\mathcal{V}_Y \to \infty$, then rescale :

$$\mathcal{V}_Y \sim \mu^3 \to \infty \quad : \quad J' := \mu^{-1} J \quad \text{s.t.}$$

$$\mathcal{V}_Y' = \frac{1}{6}\int (J')^3 \quad \text{finite}$$

$\Rightarrow$ Left to consider finite volume ∞ distance limits & then rescale back

$\Rightarrow$ Omit " $'$ " in J' & focus on

ii) Finite volume ∞ distance limits

If (after rescaling) no $t^\alpha \to \infty$, then no residual infinite distance limit

In such cases $\quad t^\beta \to 0 \iff$ shrinking of contractible cycle at <u>finite</u> distance

Assume therefore:

$t^\alpha \to \infty$ for some α

<u>Def:</u> <u>λ = rate of fastest divergence</u>

i.e. $\quad t^\alpha \sim \lambda \longrightarrow \infty \qquad \forall\, \alpha \in I_\lambda$

$\quad\quad t^\beta \prec \lambda \qquad \forall\, \beta \in I / I_\lambda$

$\qquad I \in \{1, \dots, h^{1,1}(Y)\}$

<u>Immediate Consequence:</u>

$$\int_Y J^3 = \text{const} \implies (J_\alpha)^3 = 0 \quad \forall\, \alpha \in I_\lambda$$

(in fact, also for other diverging directions)

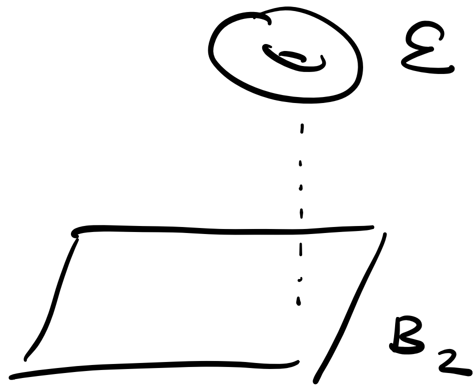
Interlude:

$\Rightarrow$ Fibration structure & Oguiso's theorem

If Y has Kähler cone generator J_0 w/ $J_0^3 = 0$, then:

α) If $J_0^2 \neq 0$: Y admits a fibration by a genus-one curve

$$\begin{array}{ccc} \underbrace{\mathcal{E}}_{\text{generic fibre}} & \hookrightarrow & Y \\ & & \downarrow \pi \\ & & B_2 \; : \text{base} \end{array}$$



& $J_0 = \pi^* D_0^b \qquad D_0^b \in H^{1,1}(B_2)$

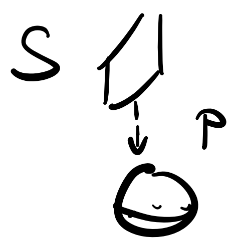
Indeed: $(\pi^* D_0^b)^3 = 0$

but $(\pi^* D_0)^2 = c_0 \mathcal{E} \qquad c_0 = D_0^b \cdot_{B_2} D_0^b$

$\Rightarrow t^0$: Volume of base curve $\neq 0$

ii) if $J_0^2 = 0$, Y admits surface fibration

$$\begin{array}{ccc} S & \hookrightarrow & Y \\ & & \downarrow p \\ & & \mathbb{P}^1 \end{array}$$



S is either K3 surface if $\int_S c_2(Y) = 24$

or abelian surface if $\int_S c_2(Y) = 0$
(T^4)

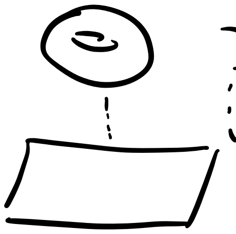
& $J_0 = [S]$ generic fibre

$J_0 \cdot J_0 = 0$ ✓

t^0 : volume of base $\mathbb{P}^1$

⇒ Intuition: For $t^0 \sim \lambda \to \infty$

α)



$$\left.\begin{array}{l} \mathcal{V}_{\mathcal{E}} \sim \lambda^{-2} \\ \mathcal{V}_{B_2} \sim t_0^2 + \dots \sim \lambda^2 \end{array}\right\} \underbrace{\mathcal{V}_Y = \mathcal{V}_{\mathcal{E}} \mathcal{V}_{B_2} + \dots}_{\text{finite}}$$

or

β)

$$\left.\begin{array}{l} \mathcal{V}_S \sim \lambda^{-1} \\ \mathcal{V}_{B_2} \sim t_0 \sim \lambda \end{array}\right\} \mathcal{V}_Y \text{ finite}$$

<u>Can prove rigorously following pattern:</u>

i) Every finite volume ∞ dist limit is of one of the 2 mutually exclusive cases:

1) T^2 - limit: Y has T^2 - fibration w/

$$\mathcal{V}_{B_2} \sim \lambda^2 \to \infty, \quad \mathcal{V}_{T^2} \sim \lambda^{-2} \to 0$$

& if Y also admits a $K3/T^4$ fibration, then

$$\mathcal{V}_{K3/T^4} > \lambda^{-4} \cong (\mathcal{V}_{T^2} \cdot \mathcal{V}_{T^2})$$

T^2 fibre shrinks faster than $K3/T^4$ fibre

2) $K3/T^4$ limit: Y has $K3/T^4$ fibr. w/

$$\mathcal{V}_{\mathbb{P}^1} \sim \lambda, \quad \mathcal{V}_{K3/T^4} \sim \lambda^{-1} \to 0$$

& every curve $C \subset K3/T^4$ w/ $C \cdot C \geq 0$ has $\mathcal{V}_C \sim \lambda^{-1/2}$

ii) For every such limit, the fibre shrinking at fastest rate is unique.

<u>E.g.:</u> Cannot have 2 $K3/T^4$ fibre limits w/ same rate λ

If so, then this is a T^2 or different $K3/T^4$ limit at a faster vanishing rate

Trivial example:

$$Y = \underbrace{T^2_a \times T^2_b}_{S_1} \times T^2_c \qquad T^2_a \times \underbrace{T^2_b \times T^2_c}_{S_2}$$

Suppose $\mathcal{V}_{S_1} \sim \lambda^{-1} \sim \mathcal{V}_{S_2}$

$\Rightarrow \mathcal{V}_{T^2_c} \sim \lambda \sim \mathcal{V}_{T^2_a} \Rightarrow \mathcal{V}_{T^2_b} \sim \lambda^{-2}$

$\Rightarrow$ T^2-limit w/ base $T^2_a \times T^2_c$

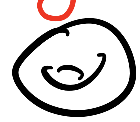
because:

$$\mathcal{V}_{S_1} \sim \lambda^{-1} \gg \lambda^{-4} = \left(\mathcal{V}_{T^2_b}\right)^2$$

This can be prove generally. [LLW '19]

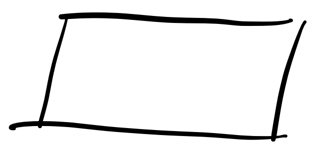
3) Interpretation & Emergent String Conjecture

a) T^2-limit



$\mathcal{V}_{T^2} \sim \lambda^{-2}$

Light towers



$\mathcal{V}_{B_2} \sim \lambda^2$

- M2 brane on T^2 fibre $\Rightarrow$ particle in $\mathbb{R}^{1,4}$

$$M_{M2} \sim \mathrm{Vol}(T^2)\, \underbrace{T_{M2}}_{\sim M_{11}^3} \sim \mathcal{V}_{T^2} \cdot M_{11}$$

$$\Rightarrow \frac{M_{M2}}{M_{11}} \sim \lambda^{-2} \to 0$$

- Tower from M2 multi-wrapped n times on T^2 $\forall\, n \in \mathbb{Z}$

5d BPS index (Gopakumar - Vafa invariants)

$$N^{(g=0)}_{nT^2} = \underbrace{\chi(Y)}_{\text{Euler characteristic}} \qquad Y \text{ genus-one fibration}$$

- SUGRA KK states from large base

$$M_{KK}^4 \sim \frac{1}{\mathrm{Vol}(B_2)}$$

$$\Rightarrow \frac{M_{KK}}{M_{11}} \sim \lambda^{-1/2} \to 0 \qquad \text{but not leading tower!}$$

- (hypothetical) string from M5 on vertical divisor

$$\frac{T_{M5}}{M_{11}^2} \sim \mathcal{V}_{\pi^*(D)} \sim \frac{1}{\lambda^2}\cdot\lambda \sim \frac{1}{\lambda}$$

$$\frac{M_{M5}}{M_{11}} \sim \lambda^{-1/2} \to 0 \qquad \text{but not leading (if even existent)}$$

Leading tower : M2 multi-wrapped on T^2
has interpretation as (dual) KK tower

5d $\to$ 6d

$$M_{nT^2} = n\, M_{T2} \Rightarrow \frac{M_{KK}}{M_{11}} \sim \mathcal{V}_{T^2} \sim \frac{1}{R}\frac{1}{M_{11}}$$

R: KK radius

$\Rightarrow$ Just recovered wellknown 6d F-theory limit

Important: How to match Planck scales?

$$M_5^3 = M_6^4 \cdot R$$

$$\Rightarrow M_6^4 = M_5^3 \cdot \frac{1}{R} = \underbrace{M_5^3}_{\sim M_{11}^3} \cdot \mathcal{V}_{T^2} \cdot M_{11} \sim \frac{1}{\lambda^2}$$

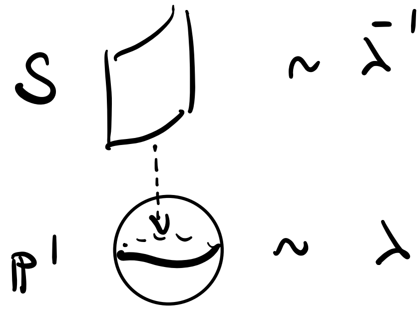
$$\boxed{\frac{M_6}{M_5} \sim \lambda^{-1/2}} \qquad \boxed{\frac{M_{KK}}{M_5} \sim \lambda^{-2}}$$

$\Rightarrow$ extra KK states from base / hypothetical further string states sit at higher-dim. Planck scale after decomp.

$\Rightarrow$ no light states in asympt. theory

Reason: $M_6 \sim \Lambda_{sp,KK}$

b) K3/T^4 limits



Towers

* KK tower from $\mathbb{P}^1$:

$$\frac{M_{kk}^2}{M_{11}^2} \sim \frac{1}{\mathcal{V}_{\mathbb{P}^1}} \sim \frac{1}{\lambda}$$

* M5 brane wrapped on fibre

$\longrightarrow$ string on $\mathbb{R}^{1,4}$ $\nearrow$ $S = K3$: critical heterotic string

$\searrow$ $S = T^4$: critical Type II string

Claim follows from analysis of worldsheet content of the so-called MSW string

- 2d $N = (0,4)$ worldsheet theory
- K3: 16 left-moving fermions

Tension: $\frac{M_{M5}^2}{M_{11}^2} \sim \mathcal{V}_{K3/T^4} \sim \frac{1}{\lambda}$

* Towers of states from M2-branes wrapped on curves $C \subset S$ w/

$$C \cdot_S C \geq 0$$

see general theorem

$$\frac{M_c}{M_{11}} \sim \mathcal{V}_c \overset{\downarrow}{\sim} \frac{1}{\sqrt{\lambda}}$$

Why tower?

General result from BPS counting [Harvey-Moore, ...]:

- if $C \cdot_S C < 0 \implies$ No multi-wrapping, i.e.

$$\exists\, m \text{ s.t. } N^{(g=0)}_{mC} = 0 \quad \forall\, n > m$$

- if $C \cdot_S C \geq 0 \implies N^{(g=0)}_{nC} \neq 0$ for $S = K3$ $\forall\, n$

(Caveat: For $S = T^4$: $N_{nC} = 0$, but due to enhanced SUSY)

Summary: 3 types of states at same scale

$$\frac{M}{M_{11}} \sim \lambda^{-1/2} \longrightarrow 0$$

Interpretation

Go to new duality frame set by "emergent heterotic / Type II string" with various types of KK / winding states at same string scale

For $S = K3$: manifestation of duality

M-theory on $K3 \hookrightarrow Y \downarrow \mathbb{P}^1_b$ $\longleftrightarrow$ Heterotic on $K3_{het} \times S^1 /$ Wilson-lines

Important: No 2 distinct K3 fibres shrink at same time at same rate

— would lead to non-pert. situation w/ 2 "critical strings"

ESC claims this to be general

II.) Weak Gravity Conjecture

Arkani-Hamed, Motl, Nicolis, Vafa '06

Reminder: In a $U(1)$ gauge theory coupled to quantum gravity $\exists$ state w/

$$\frac{g_{U(1)} \cdot q}{m} \geq \left.\frac{g_{U(1)}\, Q}{M}\right|_{\text{extremal BH}}$$

Stronger statement: Tower WGC, more specifically:

Asymptotic tower WGC

In limit "$g_{U(1)} \to 0$" $\exists$ tower of super-extremal charged states.

More precise formulation:

$$* \; S = \frac{M_{pl}^{D-2}}{2} \int \sqrt{-g}\, R - \frac{1}{2} \int_{\mathbb{R}^{1,d-1}} f_{\alpha\beta}\, F^{\alpha} \wedge * F^{\beta}$$

$*$ Combination $\; U(1)_c = c_\alpha\, U(1)^\alpha$

$$(\; F_c = c_\alpha F^\alpha \;)$$

w/ $\; g_c^2 := c_\alpha f^{\alpha\beta} c_\beta$

$*$ Def.: $\; \Lambda^2_{WGC,c} := g_c^2\, M_{pl}^{D-2}$

Asymptotic Tower WGC

If i) $\frac{\Lambda^2_{WGC,c}}{M_{pl}^2} \to 0$ &

ii) $\frac{\Lambda^2_{WGC,c}}{M_{pl,\infty}^2} \longrightarrow 0$

[Cota, Mininno, Wieser, TW '22 + wip]
2212.09758

where $M_{pl,\infty}$ is the Planck scale of asymptotic theory in the infinite distance limit leading to i), then $\exists$ tower of super-extremal <u>particle</u> states.

<u>Intuition / Claim:</u>

$U(1)_c$ that allow for limits w/ i) & ii) are either

- * KK $U(1)$s of a (dual) decomp. limit
- * $U(1)$s from asymptotically weakly coupled heterotic string

Then, existence of super-extremal tower is a statement about spectra in these gauge theories.

This claim can be proven e.g. in M-theory on CY_3.

1.) Weakly coupled $U(1)$s in M-theory on CY_3

* $C_3 = (2\pi M_{11})^{-1} \underbrace{A^\alpha}_{\text{abelian gauge fields in } \mathbb{R}^{1,4}} \wedge J_\alpha$ J_α: basis of $H^{1,1}(Y)$

* $\frac{M_{pl}}{2\pi (4\pi)^{1/3}} \cdot f_{\alpha\beta} = \frac{1}{\mathcal{V}_Y} \int_Y J_\alpha \wedge * J_\beta$

$$= \hat{\mathcal{V}}_\alpha \hat{\mathcal{V}}_\beta - \hat{\mathcal{V}}_{\alpha\beta}$$

$$\mathcal{V}_\alpha =: \mathcal{V}_{J_\alpha} \,, \qquad \mathcal{V}_{\alpha\beta} = \mathcal{V}_{J_\alpha \cdot J_\beta}$$

$$\hat{\mathcal{V}}_\alpha = \frac{1}{\mathcal{V}^{1/3}} \mathcal{V}_\alpha \qquad \hat{\mathcal{V}}_{\alpha\beta} = \frac{1}{\mathcal{V}_Y^{2/3}} \mathcal{V}_{\alpha\beta}$$

* Given curve $C \subset H_2(Y_3)$:

$$U(1)_C := (2\pi M_{11})^{-1} \int_C C_3 = c_\alpha A^\alpha \qquad C = c_\alpha C^\alpha$$

"integrate C_3 over curve C" $\qquad C^\alpha \cdot J_\beta = \delta^\alpha{}_\beta$

<u>Result:</u> A $U(1)_C$ can admit a weak coupling limit i) & ii) iff either

a) $C = \mathcal{E}$ = genus-one fibre of $\mathcal{E} \hookrightarrow Y \to B_2$

or

b) $C \subset S$ $\qquad S$: generic $K3/T^4$ fibre of Y or fibre degenerating at finite distance in fibre moduli space

Strategy for proof:

* $g_c^2 M_{pl} \to 0$ is at ∞ distance
* Since rescaling of $\mathcal{V}_y$ does not change result, can focus on finite volume ∞ dist. limits

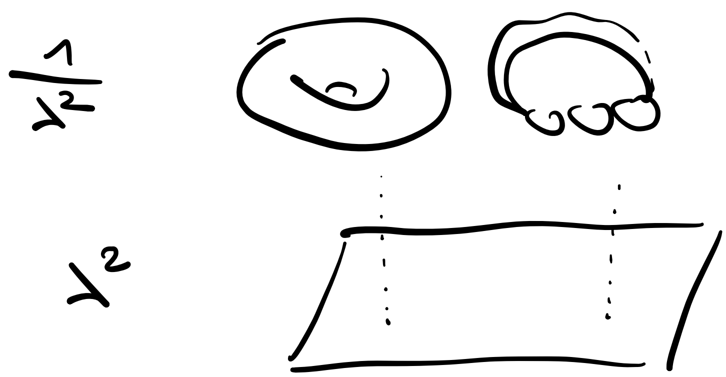
$\to$ T^2 or $K3/T^4$ limits

Case 1: T^2-limits

$$J = t^\alpha J_\alpha = \lambda \tilde{v}^a J_a + \frac{1}{\lambda^2} \tilde{v}^i J_i \quad \text{for } \lambda \to \infty$$

a : base curves

i : fibral curves



$$\mathcal{E} = \sum_i c_i C^i$$

Rough intuition:

$$f_{ij} = \lambda^4 \hat{\mathcal{V}}_i \hat{\mathcal{V}}_j - \lambda \cdot \kappa_{ija} \hat{v}^a$$

$$f_{ab} \sim \mathcal{O}\left(\frac{1}{\lambda^2}\right) \quad \left(\text{since } \hat{\mathcal{V}}_a \sim \frac{1}{\lambda}\right)$$

$$f_{ai} \sim \lambda$$

Closer inspection: $c_i \hat{V}_j - c_j \hat{V}_i = 0$ as $t \to \infty$

$\rightarrow \; U(1)^i_- := c_i\, U(1)^i - c_{i+1}\, U(1)^{i+1}$: $M_{pl}\, g_c^2 \sim \frac{1}{\lambda}$

$U(1)_\varepsilon = \sum_i c_i\, U(1)^i$: $M_{pl}\, g_c^2 \sim \frac{1}{\lambda^4}$

& $U(1)_a$ strongly coupled !

Test of weak coupling criteria:

$M_{pl} \sim \mathcal{O}(1)$ but $\frac{M_{pl,\infty}}{M_{pl}} \sim \lambda^{-1/2}$ $\left(M_{pl,\infty} \sim M_6\right)$

see first lecture

$$\Rightarrow \quad \frac{g_-^2\, M_{pl}^3}{M_{pl,\infty}^2} \sim {}^{1/\lambda}\!/_{1/\lambda} \sim 1$$

$U(1)^i_-$ not weakly coupled in asymptotic theory

$$\text{but} \quad \frac{g_\varepsilon^2\, M_{pl}^3}{M_{pl,\infty}^2} \sim {}^{1/\lambda^4}\!/_{1/\lambda} \longrightarrow 0 \quad \checkmark$$

$\Rightarrow$ $\boxed{\text{only asymptotically weakly coupled } U(1) \text{ is } U(1)_\varepsilon = U(1)_{KK} \text{ of } 5d \to 6d \text{ limit}}$ $\checkmark$

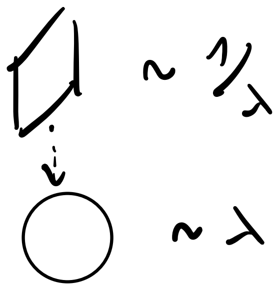
Is there a WGC tower ?

Yes: KK tower

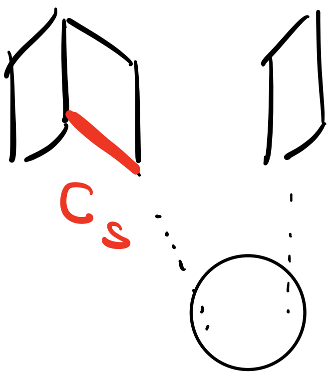
Here: BPS states & hence extremal $\checkmark$

Case 2: K3/T^4 limit

Result of analogous analysis:



All $U(1)_C$ for $C \subset S$ weakly coupled except if C localised in components of C



$U(1)_{C_S}$ not satisfying weak coupling conditions

Note: $M_{pl,\infty} = \Lambda_{sp.} = M_{het/II}$

Reason: Again linear relations in intersection form

Interpretation & Tower WGC

* Consider K3 fibr & suppose there are no special fibres (for simplicity of presentation)

Lattice of curve classes in fibre non-trivial in $H_2(Y)$:

$$\Lambda = \Lambda_+ \oplus \Lambda_- \qquad \Lambda_\pm : \begin{Bmatrix} \text{self} \\ \text{anti-self} \end{Bmatrix}\text{-dual}$$

If $Q_\pm \in \Lambda_\pm \Rightarrow Q_\pm \cdot Q_\pm \gtrless 0$

$$\Lambda \subseteq \Gamma^{3,19} = \underset{\substack{\updownarrow \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}}{U} \oplus U \oplus U \oplus \underset{\substack{\downarrow \\ \text{e.g. } E_8 \oplus E_8}}{\Gamma^{0,16}}$$

Results from BPS counting:

If $\Sigma \in \Lambda$: $\Sigma \cdot \Sigma \geq 0$ then

$$N^{(g=0)}_{m\Sigma} \neq 0 \quad \forall\, m \in \mathbb{N} \implies \exists \text{ BPS tower}$$

$\implies$ automatically (super-)extremal ✓

If $\Sigma \in \Lambda$: $\Sigma \cdot \Sigma < 0$, then no BPS tower

$\longrightarrow$ Is Tower WGC failing?

Resolution: • $\exists$ non-BPS towers from excitations of heterotic string in 5d

• Subset of them turns out to satisfy the WGC

<u>Sketch of proof:</u>

For simplicity, assume $\Sigma \in \Lambda_-$

Exploit M-theory – IIA duality :

M-theory on $Y \times S^1_M$	Type IIA on Y
M5 brane	D4 - D2 - D0 bound state
* r times on S^1_M	w/ charges
* KK mom. n	$\gamma = (r, Q, n)$
* bound to M2 - brane on $Q \in \Lambda_-$	

Here understood that M5 wrapped on some surface $S \subset Y$

$\longrightarrow$ BPS particles in 4d effective theory

Can count 4d BPS index (Donaldson - Thomas invariants)

For $r = 1$ (single wrapped) find in particular:

$$N^{(4d)}(\gamma) \neq 0 \quad \text{for} \quad \underbrace{n = -\tfrac{1}{2}\, Q \cdot Q}_{\text{specific value will be important later}} =: -\tfrac{1}{2} Q^2$$

Dual interpretation via heterotic string

For S : K3 fibre, view this as heterotic state with

* winding number $w = r = 1$
* KK momentum $n_{KK} = 1$

Reminder: Level-matching condition

$$n_L + \underbrace{\tilde{\omega}}_{=1} \cdot n_{KK} = \underbrace{-\delta_I}_{\text{internal \& } E_8 \times E_8}$$

$\rightarrow$ can swap left-moving number n_L vs n_{KK}

$\Rightarrow$ 4d BPS state with $\left(n_{KK} = -\frac{1}{2} Q^2\right)$ implies existence also of 5d excitation of 5d, unwrapped heterotic string with

$$n_L = -\frac{1}{2} Q^2 \quad (\& \; n_R \text{ matched accordingly})$$

Such states are <u>not</u> BPS in 5d

Relevance for Tower WGC:

These states can be shown to be just marginally super-extremal in 5d $\Rightarrow$ super-extremal tower ✓

<u>Derivation:</u>

For $g_{YM} \to 0$, super-extremility coincides w/ "self-repulsiveness"

$$F_{\text{Coulomb}} \geq F_{\text{Grav.}} + F_{\text{Yukawa}}$$

$$\frac{M_{\text{Pl}} g_{\text{YM}}^2}{M_k^2/M_{\text{Pl}}^2} \equiv \frac{(M_{\text{Pl}} g_5^2)(Q_\alpha f^{\alpha\beta} Q_\beta)}{M_k^2/M_{\text{Pl}}^2} \geq \left.\frac{d-3}{d-2}\right|_{d=5} + \frac{1}{4}\frac{M_{\text{Pl}}^4}{M_k^4} \mathfrak{g}^{AB} \frac{\partial}{\partial \Phi^A}\left(\frac{M_k^2}{M_{\text{Pl}}^2}\right) \frac{\partial}{\partial \Phi^B}\left(\frac{M_k^2}{M_{\text{Pl}}^2}\right)$$

$$\frac{(M_{\rm Pl}g_5^2)(Q_\alpha f^{\alpha\beta}Q_\beta)}{M_k^2/M_{\rm Pl}^2} \geq \left.\frac{d-3}{d-2}\right|_{d=5} + \frac{1}{4}\frac{M_{\rm Pl}^4}{M_k^4}\mathfrak{g}^{AB}\frac{\partial\hat{v}^\alpha}{\partial\Phi^A}\frac{\partial\hat{v}^\beta}{\partial\Phi^B}\partial_\alpha\left(\frac{M_k^2}{M_{\rm Pl}^2}\right)\partial_\beta\left(\frac{M_k^2}{M_{\rm Pl}^2}\right)$$

$$\geq \left.\frac{d-3}{d-2}\right|_{d=5} + \frac{1}{2}\frac{M_{\rm Pl}^4}{M_k^4}\left(f^{\alpha\beta} - \frac{1}{3}\hat{v}^\alpha\hat{v}^\beta\right)\partial_\alpha\left(\frac{M_k^2}{M_{\rm Pl}^2}\right)\partial_\beta\left(\frac{M_k^2}{M_{\rm Pl}^2}\right)$$

- Mass of heterotic excitations:

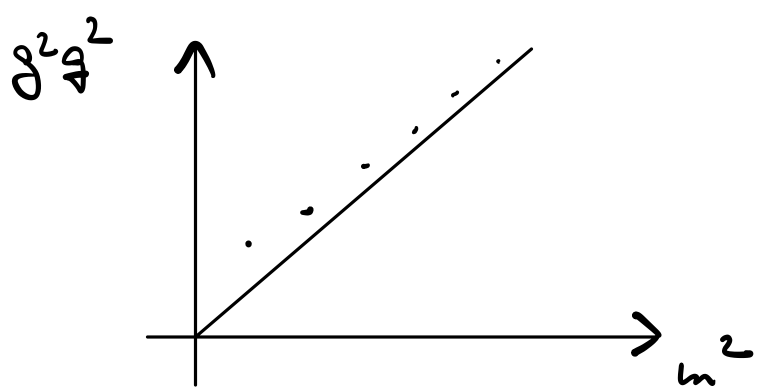
$$M_{n_{\rm L},\boldsymbol{Q}}^2 = \underbrace{8\pi(n_{\rm L}-a)T_s}_{\text{excitations}} + \underbrace{\Delta_{\rm CB}}_{\text{from Coulomb branch in 5d}} = 16\pi^2(4\pi)^{-2/3}\left((n_{\rm L}-a)\widehat{\mathcal{V}}_{\mathbf{S}} + \frac{1}{4}Q_iQ_j\hat{v}^i\hat{v}^j\right)M_{\rm Pl}^2$$

String tension: $T_s = 2\pi\mathcal{V}_{\mathbf{S}}M_{\rm 11d}^2 = 2\pi(4\pi)^{-2/3}\widehat{\mathcal{V}}_{\mathbf{S}}M_{\rm Pl}^2\,.$

- Explicit evaluation of both sides for

$\rightarrow$ $n_L = -\frac{1}{2}Q^2$ & in ∞ distance limit

$$\text{RHS}|_{(5.34)} \simeq \text{LHS}|_{(5.34)} - \frac{\widehat{\mathcal{V}}_{\rm K3}^2}{\left(\frac{1}{4n_{\rm L}}Q_iQ_j\hat{v}^i\hat{v}^j + \widehat{\mathcal{V}}_{\rm K3}\right)^2}\frac{a}{n_{\rm L}} + \mathcal{O}\left(\left(\frac{a}{n_{\rm L}}\right)^2\right)\,,$$



— Finally, let me point out another importance of strings:

III. Bounding degrees of freedom in QG

Goal: Can we bound # of d.o.f. — eg rank of gauge group — in QG?

Yes — by exploiting
* completeness conjecture $\leftrightarrow$ QG
+ * consistency of probe objects
  ↳ places constraints on theory

Schematic Argument

Example: Axionic / EFT strings in 4d N=1 SUGRA

$$S_{gauge} = -\frac{1}{4\pi} \underbrace{C_i}_{\text{numerical coefficient}} \int \big(\underbrace{s^i}_{\text{saxions}} F \wedge * F + \underbrace{a^i}_{\text{axions}} F \wedge F\big) + \ldots$$

Consider string magnetically coupled to axion a^i

Completeness Conj. $\longrightarrow$ Must exist as dynamical object in theory

$\Longrightarrow$ Not a critical string (in general), but a solitonic string

Consistency of WS gives general bound

"rk of gauge group $\leq$ gravitational couplings"

Higher curvature corrections:

$$S_{grav} = -\frac{1}{96\pi} \boxed{\tilde{C}_i} \int (b^i \, \mathrm{tr}\, R \wedge * R + a^i \, \mathrm{tr}\, R \wedge R)$$

gravitational Anomaly inflow on string must be cancelled by WS d.o.f. — which carry charge!

$\Rightarrow$ • Quantisation conditions on $\tilde{C}_i$

• Rank bounds:

$$\boxed{r(e) \leq 2\, \tilde{C}_i e^i - 2}$$

$\uparrow$ measured by string of charge e^i

[Martucci, Risso, TW '22
2210.10797]

Application to mathematics

Prediction e.g. for upper rank of MW group of elliptic fibration

6d: $\quad \mathrm{rk}\,(U(1)) \leq 28$

$\rightarrow$ Swampland Conj. for Mathematics