e1

Ex 1 $\quad S = \int d\sigma dt\, \mathcal{L}$

$$\mathcal{L} = -\frac{1}{4\pi}\Big(-\dot{X}^i \dot{X}^j G_{ij} + X^{i\prime} X^{j\prime} G_{ij} + \left(-\dot{X}^i X^{j\prime} + X^{\prime i}\dot{X}^j\right) B_{ij} \Big)$$

$$\mathcal{L} = \frac{1}{4\pi}\left(\dot{X}^i \dot{X}^j G_{ij} - X^{i\prime} X^{j\prime} G_{ij} + 2\dot{X}^i X^{j\prime} B_{ij}\right) \quad \cdots (e1)$$

$$P_i = \frac{\partial \mathcal{L}}{\partial \dot{X}^i} = \frac{1}{4\pi}\left(2G_{ij}\dot{X}^j + 2B_{ij}X^{j\prime}\right)$$

$$\longrightarrow \quad 2\pi P_i = G_{ij}\dot{X}^j + B_{ij}X^{j\prime} \quad \checkmark \quad \cdots\cdots (e2)$$

$H = P_i \dot{X}^i - \mathcal{L}$ $\qquad$ Hamiltonian density.

(e2) in matrix notation $\quad 2\pi P = G\dot{X} + BX'$

solve for $\dot{x}$: $\quad \dot{X} = G^{-1}(2\pi P - BX')$

(e1) $\quad \mathcal{L} = \frac{1}{4\pi}\left(\dot{X}^t G \dot{X} - X'^t G X' + 2X'^t B \dot{X}\right)$

$$H = P\, G^{-1}(2\pi P - BX') - \frac{1}{4\pi}\Big(\left(2\pi P^t G^{-1} + X'^t B G^{-1}\right) G G^{-1}\left(2\pi P - BX'\right) - X'^t G X' - 2X'^t B G^{-1}(2\pi P - BX')\Big)$$

e2

$$
\begin{aligned}
4\pi H = {} & 2\,(2\pi P^t)\,G^{-1}(2\pi P) - 2\,(2\pi P^t)\,G^{-1}B\,X' \\
& - \Big((2\pi P^t)G^{-1}(2\pi P) - (2\pi P^t)\,G^{-1}B\,X' \\
& \quad + X'^t B G^{-1}(2\pi P) - X'^t B G^{-1} B\,X' \\
& \quad - X'^t G\, X'^t - 2\, X'^t B G^{-1}(2\pi P) \\
& \quad + 2 X'^t B G^{-1} B\, X' \Big)
\end{aligned}
$$

$$
\begin{aligned}
= {} & (2\pi P^t)\,G^{-1}(2\pi P) - 2\pi P^t\, G^{-1} B\, X' \\
& + X'^t B G^{-1}(2\pi P) \\
& + X'^t \left(G - B G^{-1} B \right) X'
\end{aligned}
$$

$$
= (X', 2\pi P) \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \begin{pmatrix} X' \\ 2\pi P \end{pmatrix}
$$

Ex 2 $\quad \frac{1}{2}\left(D^i D_i - \bar{D}^i \bar{D}_i\right)$

$$= \frac{1}{2} G^{ij}\Big[\left(\partial_i - E_{ik}\tilde{\partial}^k\right)\left(\partial_j - E_{jp}\tilde{\partial}^p\right) - \left(\partial_i + E_{ki}\tilde{\partial}^k\right)\left(\partial_j + E_{pj}\tilde{\partial}^p\right)\Big]$$

$$= \frac{1}{2} G^{ij}\Big[- E_{jp}\partial_i\tilde{\partial}^p - E_{ik}\tilde{\partial}^k\partial_j + E_{ik}E_{jp}\tilde{\partial}^k\tilde{\partial}^p - E_{pj}\partial_i\tilde{\partial}^p - E_{ki}\tilde{\partial}^k\partial_j - E_{ki}E_{pj}\tilde{\partial}^k\tilde{\partial}^p\Big]$$

$$= \frac{1}{2} G^{ij}\Big[-2G_{jp}\partial_i\tilde{\partial}^p - 2G_{ik}\tilde{\partial}^k\partial_j\Big] + \underset{\partial^k}{\tilde{\partial}}\left(E^t G^{-1} E - E G^{-1} E^t\right)\underset{\partial^p}{\tilde{\partial}}$$

$$= -\partial_i\partial^i - \partial_i\partial^i + \tilde{\partial}\Big(\left(G-B\right)G^{-1}\left(G+B\right) - \left(G+B\right)G^{-1}\left(G-B\right)\Big)$$

$$= -2\partial_i\partial^i + \tilde{\partial}\,\underbrace{\Big[\left(1 - BG^{-1}\right)\left(G+B\right) - \left(1+BG^{-1}\right)\left(G-B\right)\Big]}_{\substack{G+B-B-BG^{-1}B \\ -\left(G-B+B-BG^{-1}B\right) \,=\, 0}}$$

$$\frac{1}{2}\left(D^i D_i - \bar{D}^i \bar{D}_i\right) = -2\partial_i\tilde{\partial}^i \quad \checkmark$$

Ex 3

$$h \eta h^t = \eta \quad \ldots (1)$$

note $\eta^{-1} = \eta$

Take the inverse of (1)

$$(h^t)^{-1} \eta h^{-1} = \eta$$

$h^t \to \; \leftarrow h \qquad \eta = h^t \eta h$ ✓

Ex 4

From (16) $\quad h^t \eta h = \eta$

multiply η from left $\quad \eta \cdot \eta = 1$

$$\underbrace{\eta h^t \eta}_{\text{must be } h^{-1}} h = 1$$

So $\quad h^{-1} = \eta h^t \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} b^t & d^t \\ a^t & c^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} d^t & b^t \\ c^t & a^t \end{pmatrix}$$ ✓

Ex 5 $\quad E' = h(E) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} E$ $\quad$ **

$$E' = (aE+b)(cE+d)^{-1}$$

Solve for E

$$E'(cE+d) = aE+b$$

$$E'd - b = aE - E'cE$$

$$E'd - b = (a - E'c)E$$

$$(a - E'c)^{-1}(E'd - b) = E$$

transpose now:

$$(d^t E'^t - b^t)(-c^t E'^t + a^t)^{-1} = E^t$$

$$\begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix} E'^t = E^t$$

↑ show it belongs to $O(D,D)$

its inverse is therefore $\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$

So $\quad E'^t = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} E^t$

** one easily finds $\quad E = h^{-1}(E') = \begin{pmatrix} d^t & b^t \\ c^t & a^t \end{pmatrix} E'$

but taking transposes is not consistent with the projective form (E' appears to the left of the matrices)

Ex 6 $\quad g(I) = I \quad$ with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} I = (aI+b)(cI+d)^{-1} = I$$

$$= (a+b)(c+d)^{-1} = I$$

$$\rightarrow \boxed{a+b = c+d} \quad \ldots (i)$$

Now $\quad g(I) = I \rightarrow I = g^{-1}(I) \qquad g^{-1} = \begin{pmatrix} d^t & b^t \\ c^t & a^t \end{pmatrix}$

thus $\quad d^t + b^t = c^t + a^t$

$$\rightarrow \boxed{d+b = c+a} \quad (ii)$$

Adding (i) and (ii) $\quad b = c$ and therefore $a = d$.

$$\rightarrow \quad g = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$O(D,D)$ conditions from (18), (19)

$$a^t b + b^t a = 0$$

$$a b^t + b a^t = 0$$

$$a^t a + b^t b = 1$$

$$a a^t + b b^t = 1$$

$$g^t = \begin{pmatrix} a^t & b^t \\ b^t & a^t \end{pmatrix} \qquad g g^t = \begin{pmatrix} aa^t + bb^t & ab^t + ba^t \\ ba^t + ab^t & bb^t + aa^t \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ✓$$

so $\boxed{g g^t = g^t g = 1}$

Consider the matrix $\quad R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad R = R^t, \; R^2 = 1$

$$R \eta R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = J$$

$$R g R = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix} \equiv \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

$$g^t \eta g = \eta \rightarrow (R g^t R)(R \eta R)(R g R) = R \eta R$$

$$\rightarrow \begin{pmatrix} x^t & 0 \\ 0 & y^t \end{pmatrix} J \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = J \rightarrow \boxed{\begin{matrix} x \in O(D) \\ y \in O(D) \end{matrix}}$$

Ex. 7 $\quad G' = \frac{1}{2}(E' + E'^t)$

$$= \frac{1}{2}\left[(aE+b)(cE+d)^{-1} + (cE+d)^{-1t}(aE+b)^t\right]$$

$$(d+cE)^t G' (d+cE) = \frac{1}{2}\left[(d^t + E^t c^t)(aE+b) + (E^t a^t + b^t)(d+cE)\right]$$

$$= \frac{1}{2}\Big[\overbrace{(d^t a + b^t c)}^{1} E + \underbrace{d^t b + b^t d}_{0}$$

$$+ E^t \underbrace{(c^t a + a^t c)}_{0} E + E^t \underbrace{(c^t b + a^t d)}_{1}\Big]$$

$$= \frac{1}{2}\left[E + E^t\right] = G \quad \checkmark$$

$$G' = \frac{1}{2}\left[(aE^t - b)(a - cE^t)^{-1} + (d - cE^t)^{-1t}(Ea^t - b^t)\right]$$

$$(d - cE^t)^t G' (d - cE^t) = \frac{1}{2}\Big[(d^t - Ec^t)(aE^t - b)$$

$$+ (Ea^t - b^t)(d - cE^t)\Big]$$

$$= \frac{1}{2}\Big[\overbrace{(d^t a + b^t c)}^{1} E^t - \underbrace{(d^t b + b^t d)}_{0}$$

$$- E\underbrace{(c^t a + a^t c)}_{0} E^t + E\underbrace{(c^t b + a^t d)}_{1}\Big]$$

$$= G \quad \checkmark$$

Exercise 8

First, let us prove: $b^t - Ea^t = -ME'$

We have: $-ME' = -(d-cE^t)^t(aE+b)(cE+d)^{-1}$

The identity will be proved if we can show that

$$(b^t - Ea^t)(cE+d) = -(d-cE^t)^t(aE+b) \qquad (*)$$

We have: $LHS(*) = b^t cE - Ea^t d - Ea^t cE + b^t d$

$$RHS(*) = (Ec^t - d^t)(aE+b)$$

$$= -d^t aE + Ec^t b + Ec^t aE - d^t b$$

$$= -\cancel{E} + b^t cE + \cancel{E} - Ea^t d - Ea^t cE + b^t d \quad = LHS$$

Thus, $(*)$ is verified, and we have

$$\underline{b^t - Ea^t = -ME'}$$

Then: $MY' = -(d-cE^t)^t\tilde{\theta}' + ME'\theta'$

$$= -(d-cE^t)^t\tilde{\theta}' - (b^t - Ea^t)\theta'$$

$$= (Ec^t - d)\tilde{\theta}' + (Ea^t - b^t)\theta'$$

$$= -(d^t\tilde{\theta}' + b^t\theta') + E(c^t\tilde{\theta}' + a^t\theta')$$

Remembering that $\begin{pmatrix}\tilde{\theta}'\\ \theta'\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\tilde{\theta}\\ \theta\end{pmatrix}$ and that $\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1} = \begin{pmatrix}d^t & b^t\\ c^t & a^t\end{pmatrix}$

we see that

$$MY' = -\tilde{\theta} + E\theta \quad = Y$$

$\cancel{\text{Thus: } Y_i = -\tilde{\theta}_i + E_{ij}\theta}$ Thus: $\underline{Y_i = M_i{}^j Y'_j}$

Exercise 9

$$\left.\begin{aligned}\delta e_{ij} &= \bar{D}_j \lambda_i \\ \delta d &= -\tfrac{1}{4} D\cdot\lambda\end{aligned}\right\} (*)$$

Let's start with the variation of $(-4d\Box d)$:

$$\delta(-4d\Box d) \underset{\uparrow}{=} -8\left(-\tfrac{1}{4}\right) d\,\Box(D\cdot\lambda) = 2d\,\Box(D\cdot\lambda) \qquad (i)$$

we are using integration by part here.

Now: $$\delta\left(-2d D^i\bar{D}^j e_{ij}\right) = \tfrac{1}{2}(D\cdot\lambda) D^i\bar{D}^j e_{ij} - 2d D^i\bar{D}^j\bar{D}_j\lambda_i$$
$$= \tfrac{1}{2}(D\cdot\lambda) D^i\bar{D}^j e_{ij} - 2d\,\Box(D\cdot\lambda) \qquad (ii)$$

$$\delta\left(\tfrac{1}{4}\left(D^i e_{ij}\right)^2\right) = \tfrac{1}{2}\left(D^i\bar{D}_j\lambda_i\right)\left(D_k e^{kj}\right)$$
$$= -\tfrac{1}{2}\left(D^i\bar{D}^j e_{ij}\right)(D\cdot\lambda) \qquad \text{(after integration by part)} \qquad (iii)$$

$$\delta\left(\tfrac{1}{4}\left(\bar{D}^j e_{ij}\right)^2\right) = \tfrac{1}{2}\left(\bar{D}^j\bar{D}_j\lambda_i\right)\left(\bar{D}_k e^{ik}\right) = \tfrac{1}{2}(\Box\lambda_i)\bar{D}_j e^{ij} \qquad (iv)$$

$$\delta\left(\tfrac{1}{4} e^{ij}\Box e_{ij}\right) = \tfrac{1}{2}\left(\bar{D}_j\lambda_i\right)\Box e^{ij} = \tfrac{1}{2} e^{ij}\Box\bar{D}_j\lambda_i \quad \text{(by part)}$$
$$= -\tfrac{1}{2}(\Box\lambda_i)\bar{D}_j e^{ij} \quad \text{(by part)} \qquad (v)$$

Then we see that $(i)+(ii)+(iii)+(iv)+(v) = 0$

Thus, $(*)$ is a gauge invariance of $S^{(2)}$

Exercise 10

We will need: $\quad \mathcal{L}_X i_Y - i_Y \mathcal{L}_X = i_{[X,Y]} \qquad (i)$

and: $\quad d i_X + i_X d = \mathcal{L}_X \qquad (ii)$

$$[X+\xi+i_X B,\ Y+\eta+i_Y B]$$

$$= [X,Y] + \mathcal{L}_X(\eta + i_X B) - \mathcal{L}_Y(\xi + \mathcal{L}_X B) - \frac{\beta}{2} d\left(i_X(\eta + i_Y B) - i_Y(\xi + i_X B)\right)$$

$$= [X+\xi, Y+\eta] + \mathcal{L}_X i_Y B - \mathcal{L}_Y i_X B - \frac{\beta}{2} d\left(i_X i_Y B - i_Y i_X B\right)$$

$$= [X+\xi, Y+\eta] + \mathcal{L}_X i_Y B - \mathcal{L}_Y i_X B - \frac{\beta}{2}\left(-i_X d i_Y B + \mathcal{L}_X i_Y B + i_Y d i_X B - \mathcal{L}_Y i_X B\right)$$

$$= [X+\xi, Y+\eta] + \mathcal{L}_X i_Y B - \mathcal{L}_Y i_X B - \frac{\beta}{2}\left(i_X i_Y \underbrace{\cancel{dB}}_{0} - i_X \mathcal{L}_Y B + \mathcal{L}_X i_Y B - i_Y i_X \underbrace{\cancel{dB}}_{0} + i_Y \mathcal{L}_X B - \mathcal{L}_Y i_X B\right)$$

$$= [X+\xi, Y+\eta] + \left(1-\frac{\beta}{2}\right)[\mathcal{L}_X, i_Y]B + (1-\beta)\, i_Y \mathcal{L}_X B - \left(1-\frac{\beta}{2}\right)[\mathcal{L}_Y, i_X]B - (1-\beta)\, i_X \mathcal{L}_Y B$$

$$= [X+\xi, Y+\eta] + (2-\beta)\, i_{[X,Y]} B + (1-\beta)\left(i_Y \mathcal{L}_X B - i_X \mathcal{L}_Y B\right)$$

So, for $\underline{\beta = 1}$, we have the automorphism

$$\cancel{[X+\xi, Y+\eta]}\quad [X+\xi+i_X B,\ Y+\eta+i_Y B] = [X+\xi, Y+\eta] + i_{[X,Y]} B$$

Ex 11

$$\hat{\mathcal{L}}_\xi A_M = \xi^P \partial_P A_M + (\partial_M \xi^P - \partial^P \xi_M) A_P$$

$$\xi = X^i \partial_i + \xi_i dz^i$$

$$[\hat{\mathcal{L}}_{\xi_1}, \hat{\mathcal{L}}_{\xi_2}] A_M = \xi_1^Q \partial_Q \left(\xi_2^P \partial_P A_M + (\partial_M \xi_2^P - \partial^P \xi_{2M}) A_P\right)$$

$$+ (\partial_M \xi_1^Q - \partial^Q \xi_{1M}) \left\{ (\partial_Q \xi_2^P - \partial^P \xi_{2Q}) A_P + \xi_2^P \partial_P A_Q \; \cancel{- (1 \leftrightarrow 2)} \right\}$$

$$= \underbrace{\xi_1^Q \partial_Q \xi_2^P}_{1} \partial_P A_M + \xi_1^Q \xi_2^P \cancelto{0}{\partial_Q \partial_P} A_M + \xi_1^Q (\underbrace{\partial_Q \partial_M \xi_2^P}_{2} - \underbrace{\partial^P \partial_Q \xi_{2M}}_{3}) A_P$$

$$+ \xi_1^Q (\partial_M \xi_2^P - \cancel{\partial^P \xi_{2M}}) \partial_Q A_P + (\underbrace{\partial_M \xi_1^Q}_{4} - \partial^Q \xi_{1M})(\partial_Q \xi_2^P - \partial^P \xi_{2Q}) A_P$$

$$+ \xi_2^Q (\partial_M \xi_1^P - \cancel{\partial^P \xi_{1M}}) \partial_Q A_P$$

$$- (1 \leftrightarrow 2)$$

$$+\hat{\mathcal{L}}_{[\xi_1, \xi_2]_C} A_M = +[\xi_1, \xi_2]_C^P \partial_P A_M + (\partial_M [\xi_1, \xi_2]_C^P - \partial^P [\xi_1, \xi_2]_{C M}) A_P$$

with

$$[\xi_1, \xi_2]_C^M = \xi_1^Q \partial_Q \xi_2^M - \frac{1}{2} \xi_1^Q \partial^M \xi_{2Q} - (1 \leftrightarrow 2)$$

$$= +(\xi_1^Q \partial_Q \xi_2^P - \frac{1}{2} \xi_1^Q \partial^P \xi_{2Q}) \partial_P A_M + \partial_M (\xi_1^Q \partial_Q \xi_2^P - \frac{1}{2} \xi_1^Q \partial^P \xi_{2Q}) A_P$$

$$- \partial^P (\xi_1^Q \partial_Q \xi_{2M} - \frac{1}{2} \xi_1^Q \partial_M \xi_{2Q}) A_P - (1 \leftrightarrow 2)$$

$$= (\underbrace{\xi_1^Q \partial_Q \xi_2}_{1} - \frac{1}{2} \xi_1^Q \cancelto{0}{\partial^P \xi_{2Q}}) \partial_P A_M + \xi_1^Q (\underbrace{\partial_M \partial_Q \xi_2^P}_{2} - \frac{1}{2} \cancel{\partial_M \partial^P \xi_{2Q}}) A_P$$

$$+ \partial_M \xi_1^Q (\underbrace{\partial_Q \xi_2^P}_{4} - \frac{1}{2} \underbrace{\partial^P \xi_{2Q}}_{6}) A_P - \xi_1^Q (\underbrace{\partial^P \partial_Q \xi_{2M}}_{3} - \frac{1}{2} \cancel{\partial_M \partial^P \xi_{2Q}}) A_P$$

$$- \partial^P \xi_1^Q (\underbrace{\partial_Q \xi_{2M}}_{5} - \frac{1}{2} \underbrace{\partial_M \xi_{2Q}}_{6}) A_P - (1 \leftrightarrow 2)$$