Doubled Field Theory, T-duality and Courant Brackets

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Lecture 1

Introduction
String theory in toroidal backgrounds
- Generalized metric
- Constraints
- O(D,D) transformations

Introduction

Closed strings on toroidal backgrounds have momentum and winding. The field theory of closed strings is a doubled field theory (DFT)

$$x^i = \{x^a, x^\mu\}$$

x^a ↑ compact; x^μ ↳ non-compact

x^a dual to p_a (momentum)

need $\tilde{x}_a$ dual to w^a (winding)

Doubled fields: $\phi(x^a, \tilde{x}_a, x^\mu)$

Action $S = \int dx^a \, d\tilde{x}_a \, dx^\mu \, \mathcal{L}(x, \tilde{x}, x^\mu)$

We will focus on the massless sector of closed strings. The familiar low energy effective field theory is given by

$$S_* = \int dx \sqrt{-g}\, e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right] \cdots$$

fields g_{ij}, b_{ij}, ϕ

What does this become for tori, and doubled fields?

Gauge symmetry parameters are

vector fields ξ^i (diffeomorphism) $\in T(M)$

one-forms $\tilde{\xi}_i$ (b_{ij} gauge tr.) $\in T^*(M)$

$\xi + \tilde{\xi} \in T(M) \oplus T^*(M)$, natural setup of generalized geometry (GG)

In GG the Courant bracket is the right extension of the Lie bracket. How does it show up in DFT?

In string theory the field $\mathcal{E}_{ij} = g_{ij} + b_{ij}$ is natural. How does it show in DFT?

In GG and string theory the generalized metric $\mathcal{H}^{MN}$ is a key structure. How does it show up in DFT?

We will write DFT's that are T-duality covariant versions of S_*. Courant brackets, $\mathcal{E}_{ij}$, $\mathcal{H}^{MN}$, will all play a role. There are still geometrical open questions.

Is there a consistent fully doubled field theory using just $g_{ij}(x,\tilde{x})$, $b_{ij}(x,\tilde{x})$ and $\phi(x,\tilde{x})$? Not clear yet.

References:

Hull and Z. 0904.4664
0908.1792

Hohm, Hull and Z. 1003.5027
1006.4823

Siegel 9305073
9302036

Tseytlin NPB350 (1991) 395

Strings in Toroidal backgrounds

Sigma model action :

$$S = -\frac{1}{4\pi}\int_0^{2\pi} d\sigma \int_0 d\tau \Big(\eta^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j G_{ij} + \epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j B_{ij}\Big) \quad \text{--- (1)}$$

$$\eta^{\alpha\beta} = \mathrm{diag}(-1,1) \qquad \epsilon^{01} = -1 ,$$

$$X^i = \{X^a, X^\mu\} \qquad X^a \sim X^a + 2\pi , \qquad i = 0, \ldots D-1$$

G_{ij}, B_{ij} $D\times D$ constant matrices $\quad G^{ij}G_{jk} = \delta^i_k$, X, G, B are unit-free

$$G_{ij} = \begin{pmatrix} \hat{G}_{ab} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix} \qquad B_{ij} = \begin{pmatrix} \hat{B}_{ab} & 0 \\ 0 & 0 \end{pmatrix}$$

$$E_{ij} = G_{ij} + B_{ij} = \begin{pmatrix} \hat{E}_{ab} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix} \quad \text{with } \hat{E}_{ab} = \hat{G}_{ab} + \hat{B}_{ab}$$

From the action

$$2\pi P_i = G_{ij}\dot{X}^j + B_{ij}X^{j\prime} \quad \text{----- (2)}$$

$$4\pi \underset{\sim}{H} = (X', 2\pi P)\, \mathcal{H}(E) \begin{pmatrix} X' \\ 2\pi P \end{pmatrix} \quad \text{-- (3)}$$

with
$$\mathcal{H}(E) = \left(\begin{array}{c|c} G - BG^{-1}B & BG^{-1} \\ \hline -G^{-1}B & G^{-1} \end{array}\right) \quad \text{--- (4)}$$

$2D\times 2D$ matrix

<u>Exercise 1</u> Prove (2), (3), (4).

$\mathcal{H}(E)$ is a $2D \times 2D$ symmetric matrix constructed out of G and B. It is called the "generalized metric". More precisely we will identify it with an object $\underline{\mathcal{H}^{MN}}$ with $M, N = 1, \ldots, 2D$

$$\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \quad \cdots (5)$$

$$\mathcal{H}^{-1} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \quad \cdots (6)$$

It is clearly a rearrangement.

Define $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ another metric (symm) (7)

Then $\eta \mathcal{H} \eta = \mathcal{H}^{-1}$ (A constraint) (8)

Let's put indices here $\mathcal{H} \leftrightarrow \mathcal{H}^{MN}$

$\mathcal{H}^{-1} \Leftrightarrow \mathcal{H}_{MN}$

'' like for metrics

(8) becomes

$$\eta_{PM} \mathcal{H}^{MN} \eta_{NQ} = \mathcal{H}_{PQ}$$

$$\mathcal{H}^{MN} \eta_{MP} \eta_{NQ} = \mathcal{H}_{PQ} \quad \cdots\cdots (9)$$

Lowering the indices of $\mathcal{H}$ with the η metric gives us the $\mathcal{H}^{-1}$)

Oscillator expansions $\alpha_n, \tilde{\alpha}_n$ $\quad (X^i = x^i + w^i\sigma + \tau G^{ij}p_j \ldots$

$$\alpha_0^i = \frac{1}{\sqrt{2}} G^{ij}(p_j - E_{jk}w^k)$$

$$\bar{\alpha}_0^i = \frac{1}{\sqrt{2}} G^{ij}(p_j + E_{kj}w^k)$$

$$\alpha_{0i} = -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial x^i} - E_{ik}\frac{\partial}{\partial \tilde{x}_k}\right) \equiv -\frac{i}{\sqrt{2}} D_i$$

$$\bar{\alpha}_{0i} = -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial x^i} + E_{ki}\frac{\partial}{\partial \tilde{x}_k}\right) \equiv -\frac{i}{\sqrt{2}} \bar{D}_i$$

These are useful derivatives

$$\boxed{\begin{aligned} D_i &= \partial_i - E_{ik}\tilde{\partial}^k & \qquad D^i &\equiv G^{ij}D_j \\ \bar{D}_i &= \partial_i + E_{ki}\tilde{\partial}^k & \qquad \bar{D}^i &\equiv G^{ij}\bar{D}_j \end{aligned}} \quad (10)$$

Note that

$$L_0 = \frac{1}{2}\alpha_0^i G_{ij}\alpha_0^j + N - 1$$

$$\bar{L}_0 = \frac{1}{2}\bar{\alpha}_0 G_{ij}\bar{\alpha}_0^j + \bar{N} - 1$$

$$L_0 - \bar{L}_0 = N - \bar{N} + \frac{1}{2}\left(-\frac{1}{2}\right)\left(D^i G_{ij} D^j - \bar{D}^i G_{ij}\bar{D}^j\right)$$

$$L_0 - \bar{L}_0 = N - \bar{N} - \frac{1}{2}\frac{1}{2}\left(D^i D_i - \bar{D}^i \bar{D}_i\right)$$

Exercise 2 Show that

$$\boxed{\frac{1}{2}(D^i D_i - \bar{D}^i \bar{D}_i) = -2\,\partial_i \tilde{\partial}^i} \quad (11)$$

Since $L_0 - \bar{L}_0 = 0$ for all states of the theory.

$$N - \bar{N} = \frac{1}{2}\left(-2\partial_i\tilde{\partial}^i\right) = -\partial_i\tilde{\partial}^i = -\partial\cdot\tilde{\partial}$$

For the familiar massless fields

$$\int dp\,dw\; e_{ij}(p,w)\,\alpha^i_{-1}\bar{\alpha}^j_{-1}\,c_1\bar{c}_1\,|p,w\rangle \qquad N=\bar{N}=0$$

$$\int dp\,dw\; d(p,w)\,(c_1c_{-1}-\bar{c}_1\bar{c}_{-1})\,|p,w\rangle \qquad N=\bar{N}=0$$

we have $e_{ij}(x,\tilde{x})$, $d(x,\tilde{x})$ fields.

Soo we must have

$$\partial\cdot\tilde{\partial}\,\{e_{ij}(x,\tilde{x}),\,d(x,\tilde{x})\} = 0 \qquad (12)$$

O(D,D) transformations   Invariance of the physics under background transformations.

From the Hamiltonian density $\underset{\sim}{H}$ in (3):

$$H = \int_0^{2\pi} d\sigma \, \underset{\sim}{H} = \frac{1}{2} Z^t \mathcal{H}(E) Z + N + \bar{N} + \cdots \qquad (13)$$

$$Z = \begin{pmatrix} w^i \\ p_i \end{pmatrix}$$ 2D column vector with winding and momentum integer quantum numbers

$$N - \bar{N} = \frac{1}{2} Z^t \eta Z \qquad \text{Condition on spectrum} \quad (14)$$

Consider a reshuffling of the (w,p) quantum numbers

$Z = h^t Z'$, with some $2D \times 2D$ matrix h

Then need $Z'^t \eta Z' = Z^t \eta Z = Z'^t h \eta h^t Z'$

which requires $h \eta h^t = \eta$ .... (15)

Exercise 3   Show that (15) implies

$$\boxed{h^t \eta h = \eta} \quad \text{------} \quad (16)$$

h is a $2D \times 2D$ matrix of integers

$h \in O(D,D)$ group. $h = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)$ ...(17)

Equation (16) gives

$$
\begin{aligned}
a^t c + c^t a &= 0, \\
b^t d + d^t b &= 0, \\
a^t d + c^t b &= 1.
\end{aligned}
\qquad (18)
$$

Equation (15) gives

$$
\begin{aligned}
a b^t + b a^t &= 0, \\
c d^t + d c^t &= 0, \\
a d^t + b c^t &= 1.
\end{aligned}
\qquad (19)
$$

Exercise 4 Show that $h^{-1} = \begin{pmatrix} d^t & b^t \\ c^t & a^t \end{pmatrix}$. ... (20)

What more is needed for invariance of the spectrum? From (13)

$$Z^t H(E) Z = Z'^t H(E') Z'$$

$$Z'^t h H(E) h^t Z' = Z'^t H(E') Z'$$

Thus we need

$$\boxed{H(E') = h H(E) h^t.} \quad \cdots \quad (21)$$

Is this possible?

Claim: Letting

$$\boxed{E' = h(E) = (aE+b)(cE+d)^{-1}} \quad \text{---- (22)}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} E \quad \text{(notation !)}$$

Will lead to the desired result.

Exercise 5 Show that

$$E'^t = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} E^t \left(= (aE^t-b)(d-cE^t)^{-1} \right) \quad (23)$$

Back to the claim: Create E from the identity background $E = I$

$E = h_E(I)$ what is $h_E \in O(D,D)$?

take $E = G + B$, with $G = AA^t$ (A is like a vielbein)

$$h_E = \begin{pmatrix} A & B(A^t)^{-1} \\ 0 & (A^t)^{-1} \end{pmatrix} \ ? \quad (24)$$

can verify $h_E \in O(D,D)$

check it $h_E(I) = \left(AI + B(A^t)^{-1}\right)\left(0 \cdot I + (A^t)^{-1}\right)^{-1}$

$$= \left(A + B(A^t)^{-1}\right) A^t = AA^t + B = E \checkmark$$

h_E is ambiguous ... $h_E \longrightarrow h_E \cdot g$

where $g(I) = I$ If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow a+b = c+d$

hint: consider also that $g^{-1}(I) = I$

Exercise 6

Show that the elements g that satisfy $g(I) = I$ form an $O(D) \times O(D)$ subgroup of $O(D,D)$ and $g^t g = g g^t = I$

Now let $E' = h(E)$

$$\rightarrow \quad h_{E'}(I) = h\, h_E(I)$$

$$h_{E'} = h\, h_E\, g \qquad g \in O(D) \times O(D); \quad g^t g = g g^t = I$$

Now calculate $h_E h_E^t$

$$h_E h_E^t = \begin{pmatrix} A & B A^{t^{-1}} \\ 0 & (A^t)^{-1} \end{pmatrix} \begin{pmatrix} A^t & 0 \\ -A^{-1} B & A^{-1} \end{pmatrix} \qquad \begin{array}{l} G = AA^t \\ G^{-1} = A^{t^{-1}} A^{-1} \end{array}$$

$$= \left(\begin{array}{c|c} AA^t - B A^{t^{-1}} A^{-1} B & B A^{t^{-1}} A^{-1} \\ \hline -(A^t)^{-1} A^{-1} B & (A^t)^{-1} A^{-1} \end{array}\right)$$

$$= \left(\begin{array}{c|c} G - B G^{-1} B & B G^{-1} \\ \hline -G^{-1} B & G^{-1} \end{array}\right) = \mathcal{H}(E)$$

$$\text{So} \qquad \mathcal{H}(E) = h_E h_E^t \qquad \text{- - - - - - (25)}$$

$$\mathcal{H}(E') = h_{E'} h_{E'}^t = h\, h_E\, g\, (h\, h_E\, g)^t$$

$$= h\, h_E h_E^t\, h^t = h\, \mathcal{H}(E)\, h^t \quad \checkmark$$

this proves (21).

How are G and G' related?

Exercise 7 Prove that

$$\boxed{(d+cE)^t \, G' \, (d+cE) = G} \quad \cdots \quad (26)$$

Hint: write $G' = \frac{E' + E'^t}{2}$, use (22) and evaluate the LHS.

Prove that:

$$\boxed{(d-cE^t)^t \, G' \, (d-cE^t) = G} \quad \cdots \quad (27)$$

Use (23) and write $G' = \frac{(E')^t + ((E')^t)^t}{2}$.

Define

$$M \equiv (d - cE^t)^t,$$

$$\bar{M} \equiv (d + cE)^t.$$

Then (26), (27) become

$$G = \bar{M} G' \bar{M}^t,$$

$$G = M G' M^t. \qquad (28)$$

Examples of $O(D,D)$ "tensors"

$$G_{\bar{i}\bar{j}} = \bar{M}_{\bar{i}}{}^{\bar{p}} \, \bar{M}_{\bar{j}}{}^{\bar{q}} \, G'_{\bar{p}\bar{q}},$$

$$G_{ij} = M_i{}^p \, M_j{}^q \, G'_{pq}, \qquad (29)$$

barred and unbarred indices.

Relating the fundamental $\Theta = \begin{pmatrix} \tilde{\theta}_i \\ \theta^i \end{pmatrix}$ of $O(D,D)$ to these tensors:

$$\begin{pmatrix} \tilde{\theta}' \\ \theta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{\theta} \\ \theta \end{pmatrix}$$ Definition of fundamental (30)

$$Y_i \equiv -\tilde{\theta}_i + E_{ij}\,\theta^j \quad (Y = -\tilde{\theta} + E\theta)$$
$$\bar{Y}_i \equiv \tilde{\theta}_i + E_{ji}\,\theta^j \quad (\bar{Y} = \tilde{\theta} + E^t\theta)$$ (31)

Then

$$Y_i = M_i{}^j\, Y'_j, \qquad \text{(32)}$$
$$Y_i = \bar{M}_i{}^j\, \bar{Y}_j .$$

Exercise 8 Prove the 1st of (32).
For this you will need the identity:

$$b^t - E a^t = -M E', \text{ prove it!} \qquad \text{(33)}$$

Application $X^M = \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}$ is in the fundamental,

partial derivatives $\partial_M = \begin{pmatrix} \tilde{\partial}^i \\ \partial_i \end{pmatrix}$.

$\rightarrow$ $\partial^M \equiv \eta^{MN}\partial_N = \begin{pmatrix} \partial_i \\ \tilde{\partial}^i \end{pmatrix}$ also in the fund.

Note $\partial^M \partial_M = 2\partial_i \tilde{\partial}^i = 0$ is our constraint

$$-\partial_i + E_{ij}\tilde{\partial}^j = -D_i$$ our "derivatives"

$$\partial_i + E_{ji}\tilde{\partial}^j = \bar{D}_i$$

Transform covariantly!

$$D_i = M_i{}^j D'_j \qquad (DF(X') = MD'F(X'))$$

$$\bar{D}_i = \bar{M}_i{}^j \bar{D}'_j \qquad (34)$$

E is ~~not a~~ tensor, but its variation is!

Let $E' = h(E)$, then

$$E' + \delta E' = h(E + \delta E)$$

$$= (a(E+\delta E) + b)(c(E+\delta E) + d)^{-1}$$

$$= (aE + b + a\delta E)(cE + d + c\delta E)^{-1} \; {}^{**}$$

$$= (aE+b+a\delta E)\left((cE+d)^{-1} - (cE+d)^{-1} c\delta E\, (cE+d)^{-1}\right)$$

$$= E' + a\delta E(cE+d)^{-1} - E'c\,\delta E(cE+d)^{-1}$$

$$\rightarrow \delta E' = (a - E'c)\,\delta E\,(\bar{M}^t)^{-1} \quad (35)$$

** $(A+\epsilon)^{-1} = A^{-1} - A^{-1}\epsilon A^{-1}$

Now claim that

$$a - E'c = M^{-1} \quad \cdots\cdots (36)$$

Indeed multiply by M from the left

$$M(a - E'c) \stackrel{?}{=} 1$$

$$Ma - \underline{ME'}c \stackrel{?}{=} 1 \qquad \text{use } (33)$$

$$(d^t - Ec^t)a + (b^t - Ea^t)c \stackrel{?}{=} 1$$

$$\underbrace{d^t a + b^t c}_{1} - E\underbrace{(c^t a + a^t c)}_{0} \stackrel{?}{=} 1 \qquad \checkmark$$

Therefore; back in (35)

$$\delta E' = M^{-1}\delta E(\overline{M}^t)^{-1}$$

$$\Rightarrow \delta E = M\,\delta E'\,\overline{M}^t \qquad \cdots\cdots (37)$$

$$\boxed{\delta E_{ij} = M_i{}^k\,\overline{M}_j{}^l\,\delta E'_{kl}} \qquad \cdots (37')$$

A tensor!

This applies to any derivative

$$\partial_i E = M\,\partial_i E'\,\overline{M}^t \qquad (38)$$

$$\tilde{\partial}^i E = M\,\tilde{\partial}^i E'\,\overline{M}^t$$

and then

$$D_i E = M D_i E'\,\overline{M}^t \qquad \overline{D}_i E = M\,\overline{D}_i E'\,\overline{M}^t \quad (39)$$

So $D_i E_{jk}$, $\overline{D}_i E_{jk}$ are tensors!! (40)

Lecture 2

Free doubled field theory
Aspects of interactions
Strong constraint.
Courant brackets
Gauge algebra

Free theory: background E_{ij} fields e_{ij}, d

$$S^{(2)} = \int dx\, d\tilde{x} \left[\tfrac{1}{4} e^{ij} \Box e_{ij} + \tfrac{1}{4} (\bar{D}^j e_{ij})^2 + \tfrac{1}{4} (D^i e_{ij})^2 - 2 d D^i \bar{D}^j e_{ij} - 4 d \Box d \right] \tag{41}$$

indices raised by G^{ij}, $\Box = D^i D_i = \bar{D}^i \bar{D}_i$

constraint $D^2 - \bar{D}^2 = 0 \Leftrightarrow \partial \cdot \tilde{\partial} = 0$
on fields and gauge parameters

Exercise 9 Prove that the following are gauge invariances of $S^{(2)}$:

$$\begin{aligned} \delta e_{ij} &= \bar{D}_j \lambda_i , \\ \delta d &= -\tfrac{1}{4} D \cdot \lambda . \end{aligned} \tag{42}$$

In fact there are also the gauge invariances:

$$\begin{aligned} \delta e_{ij} &= D_i \bar{\lambda}_j , \\ \delta d &= -\tfrac{1}{4} \bar{D} \cdot \bar{\lambda} . \end{aligned} \tag{43}$$

Assume $B=0$,

What does the $S^{(2)}$ become ?

$$S^{(2)} = \int dx d\tilde{x} \Big[\frac{1}{4} h^{ij} \partial^2 h_{ij} + \frac{1}{2} (\partial^j h_{ij})^2 - 2d \partial^i \partial^j h_{ij} - 4d \partial^2 d$$
$$\frac{1}{4} h^{ij} \tilde{\partial}^2 h_{ij} + \frac{1}{2} (\tilde{\partial}^i h_{ij})^2 + 2d \tilde{\partial}^i \tilde{\partial}^j h_{ij} - 4d \tilde{\partial}^2 d$$
$$+ \frac{1}{4} b^{ij} \partial^2 b_{ij} + \frac{1}{2} (\partial^j b_{ij})^2$$
$$+ \frac{1}{4} b^{ij} \tilde{\partial}^2 b_{ij} + \frac{1}{2} (\tilde{\partial}^j b_{ij})^2$$
$$+ (\partial_k h^{ik})(\tilde{\partial}^j b_{ij}) + (\tilde{\partial}^k h_{ik})(\partial_j b^{ij}) - 4d \partial^i \tilde{\partial}^j b_{ij} \Big] \tag{44}$$

The gauge parameters are redefined;

$$\epsilon_i = \frac{1}{2}(\lambda_i + \bar{\lambda}_i) \qquad \tilde{\epsilon}_i = \frac{1}{2}(\lambda_i - \bar{\lambda}_i) \tag{45}$$

Then

$$\delta h_{ij} = \partial_i \epsilon_j + \partial_j \epsilon_i ,$$
$$\delta b_{ij} = -(\tilde{\partial}_i \epsilon_j - \tilde{\partial}_j \epsilon_i) , \quad \text{- - - -} \quad (46)$$
$$\delta d = -\frac{1}{2} \partial \cdot \epsilon \qquad \Phi \equiv d + \frac{h}{4}, \quad \delta \Phi = 0$$

$$\tilde{\delta} h_{ij} = \tilde{\partial}_i \tilde{\epsilon}_j + \tilde{\partial}_j \tilde{\epsilon}_i$$
$$\tilde{\delta} b_{ij} = -(\partial_i \tilde{\epsilon}_j - \partial_j \tilde{\epsilon}_i) \quad \text{- - - -} \quad (47)$$
$$\tilde{\delta} d = \frac{1}{2} \tilde{\partial} \cdot \tilde{\epsilon} \qquad \tilde{\Phi} = d - \frac{h}{4} \qquad \tilde{\delta} \tilde{\Phi} = 0$$

no scalar dilaton !!

O(D,D) covariance of the theory in (41)

$e_{i\bar{j}}$ is an O(D,D) tensor, so are $D_i, \bar{D}_j$ and $g^{ij}, (\equiv g^{\bar{i}\bar{j}})$. Since all indices are contracted properly the action is invariant.

Cubic terms exist consistent with a nonlinear extension of the gauge invariance

$$S^{(3)} = \int dx\, d\tilde{x} \Big[\tfrac{1}{4} e_{ij} \big((D^i e_{kl})(\bar{D}^j e^{kl}) - D^i e_{kl}\, \bar{D}^l e^{kj} - D^k e^{il}\, \bar{D}^j e_{kl} \big)$$

$$+ \; d e^2 \text{ terms} \quad + \; d^2 e \text{ terms} \quad + \; d^3 \text{ term.} \Big] \qquad (48)$$

$$\delta_\lambda e_{ij} = \bar{D}_j \lambda_i + \tfrac{1}{2} \left[(D_i \lambda^k) e_{kj} - (D^k \lambda_i) e_{kj} + \lambda_k D^k e_{ij} \right] \ldots \qquad (49)$$

products projected to the kernel of $\partial \cdot \tilde{\partial}$

Construction to higher orders may be very nontrivial, or may not exist (keeping only e_{ij}, d)

<u>Options</u>:
* Continue the construction to higher order
* Extract lessons so far.

So for all fields or gauge parameters $A_1, A_2, A_3, \ldots$ are killed by $\partial_M \partial^M$.

Require now that all products $A_i A_j$ be killed by $\partial_M \partial^M$.

$$\partial_M \partial^M (A_i A_j) = 0 \quad \rightarrow \quad \boxed{\partial_M A_i \partial^M A_j = 0} \quad (50)$$

$O(D,D)$ "strong" constraint: all "fields" and all products are killed by $\partial \cdot \tilde{\partial}$.

<u>Theorem</u>: For a set of fields $A_i(x, \tilde{x})$ that satisfy (50) there is a duality frame $(\tilde{x}'_i, x'^i)$ in which the fields do not depend on $\tilde{x}'_i$.

The duality frame need not be specified explicitly, the constraint (50) does not break $O(D,D)$ and can be imposed without selecting frames..

The resulting theory will have $O(D,D)$ symmetry, will involve $x, \tilde{x}$, but it is an $O(D,D)$ covariantization of a theory that in some dual frame is not doubled. (close relation to Siegel)

The theory can be constructed explicitly

* It shows an interplay of $O(D,D)$ with gauge invariance.

* The gauge algebra will be novel: Courant bracket !!

* The action can be background independent and duality invariant.

* The action can be written in terms of the generalized metric

$\longrightarrow$ Let us begin by considering Courant brackets!

Courant brackets

In a theory with a metric $g_{ij}(x)$ and a KR field $b_{ij}(x)$ the symmetries are diffeomorphisms generated by vector fields $V^i(x)$ and KR gauge transformations generated by one-forms $\xi_i(x)$

$V \in T(M)$, $\xi \in T^*(M)$, formally $V+\xi \in T(M) \oplus T^*(M)$

What are the gauge transformations?

$$\delta_{V+\xi}\, g = \mathcal{L}_V g, \qquad \ldots (51)$$

$$\delta_{V+\xi}\, b = \mathcal{L}_V b + d\xi .$$

Recall that on forms $\mathcal{L}_V = i_V d + d i_V$, ... (52)
so that $\mathcal{L}_V d = d\mathcal{L}_V$, ... (53)

On the metric: $\mathcal{L}_V g_{ij} = (\partial_i V^k) g_{kj} + \partial_j V^k g_{ik} + V^k \partial_k g_{ij}$.

on any tensor or form

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]} \qquad \ldots (54)$$

$$[\mathcal{L}_X, i_Y] = i_{[X,Y]} \qquad \ldots (55)$$

$[X,Y]^k \equiv X^p \partial_p Y^k - Y^p \partial_p X^k$

Lie bracket of vectors

Compute the algebra of gauge transf.:

$$[\delta_{V_2+\xi_2}, \delta_{V_1+\xi_1}]g = \delta_{V_2+\xi_2}\delta_{V_1+\xi_1} g - (1,2)$$
$$= \delta_{V_2+\xi_2} \mathcal{L}_{V_1} g - (1,2)$$
$$= \mathcal{L}_{V_1}\mathcal{L}_{V_2} g - (1,2)$$
$$= \mathcal{L}_{[V_1,V_2]} g \quad \cdots\cdots (56)$$

On b it is more nontrivial

$$[\delta_{V_2+\xi_2}, \delta_{V_1+\xi_1}]b = \delta_{V_2+\xi_2}(\mathcal{L}_{V_1} b + d\xi_1) - (12)$$
$$= \mathcal{L}_{V_1}(\mathcal{L}_{V_2} b + d\xi_2) - (12)$$
$$= \mathcal{L}_{[V_1,V_2]} b + d(\mathcal{L}_{V_1}\xi_2 - \mathcal{L}_{V_2}\xi_1) \quad (57)$$

Compare with (51) and conclude that

$$[\delta_{V_2+\xi_2}, \delta_{V_1+\xi_1}] = \delta_{\underbrace{[V_1,V_2] + \mathcal{L}_{V_1}\xi_2 - \mathcal{L}_{V_2}\xi_1}_{\text{defines a bracket}}}$$

$$[V_1+\xi_1, V_2+\xi_2] = [V_1,V_2] + \mathcal{L}_{V_1}\xi_2 - \mathcal{L}_{V_2}\xi_1 \quad \cdots (58)$$

Gauge algebra defined by this bracket (?)
Jacobi identity is satisfied $[[\ ]] + \text{cyc.} = 0$

There is an ambiguity, however, in reading the one-form since generally,

$$\delta_{V+\xi}\, b = \mathcal{L}_V b + d\xi$$
$$= \mathcal{L}_{V+(\xi+d\sigma)} \quad \ldots\ldots (59)$$

The 1-form ξ is ambiguous up to $d\sigma$.
"Symmetry for a symmetry"

Thus reading the one-form from (57) is ambiguous.

$$d\left(\mathcal{L}_{V_1}\xi_2 - (1 \cdot 2)\right)$$
$$= d\left(d\,\iota_{V_1}\xi_2 + \iota_{V_1} d\xi_2 - (12)\right)$$

change this coefficient $1 \to 1 - \frac{\beta}{2}$

$$= d\left(\left(1-\tfrac{\beta}{2}\right) d\,\iota_{V_1}\xi_2 + \iota_{V_1} d\xi_2 - (12)\right)$$
$$= d\left(\mathcal{L}_{V_1}\xi_2 - \tfrac{1}{2}\beta\, d\,\iota_{V_1}\xi_2 - (1,2)\right)$$
$$= d\left(\mathcal{L}_{V_1}\xi_2 - \mathcal{L}_{V_2}\xi_1 - \tfrac{1}{2}\beta\, d\left(\iota_{V_1}\xi_2 - \iota_{V_2}\xi_1\right)\right).$$

$\longrightarrow$ generic bracket

$$[V_1+\xi_1, V_2+\xi_2] = [V_1, V_2] + \mathcal{L}_{V_1}\xi_2 - \mathcal{L}_{V_2}\xi_1$$
$$- \tfrac{1}{2}\beta\, d\left(\iota_{V_1}\xi_2 - \iota_{V_2}\xi_1\right) \quad \ldots\ldots (60)$$

Unavoidable ambiguity $\beta \neq 0 \to$ Failure of Jacobi

T. Courant, 1990 $\quad \beta = 1$

$$[V_1+\xi_1, V_2+\xi_2] = [V_1, V_2] + \mathcal{L}_{V_1}\xi_2 - \mathcal{L}_{V_2}\xi_1 - \frac{1}{2} d\cdot(i_{V_1}\xi_2 - i_{V_2}\xi_1) \quad \cdots\cdots (61)$$

Jacobi identity does not hold.

$$Z_1 = V_1+\xi_1 \quad Z_2 = V_2+\xi_2 \,,\quad Z_3 = V_3+\xi_3$$

$$[Z_1[Z_2, Z_3]] + \text{cyclic} = dN(Z_1, Z_2, Z_3)$$

exact 1-form.

What is special for $\beta = 1$?

There is an automorphism of the bracket, called B-transformation. Given a closed 2-form B ($dB = 0$)

$$\text{B transf: } \quad X+\xi \rightarrow X + (\xi + i_X B) \quad \cdots (62)$$

changes the 1-form

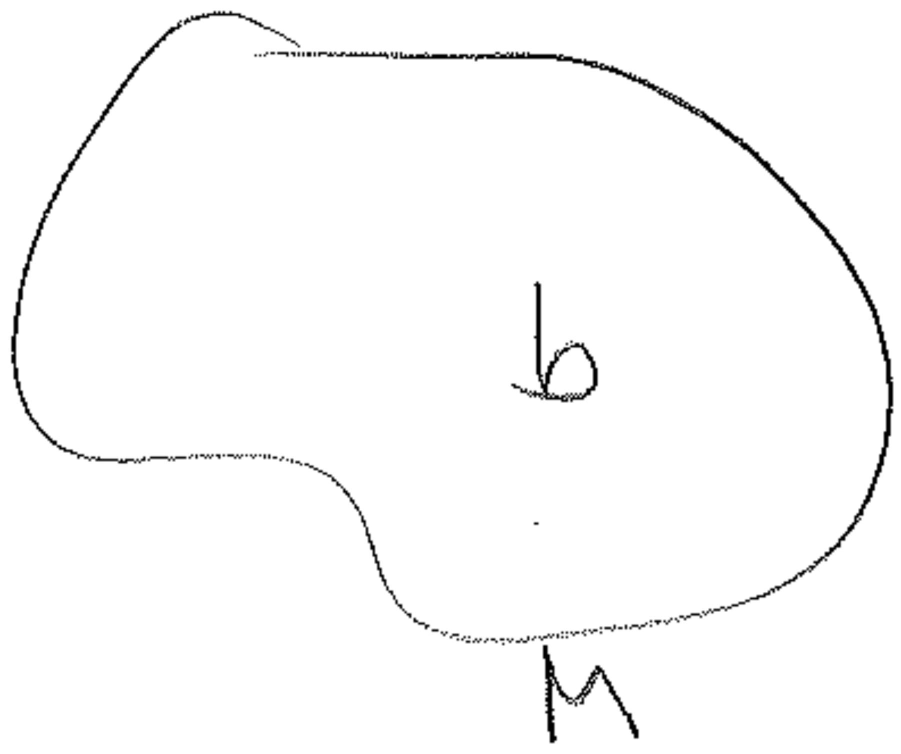
Autom. means

$$[X+\xi+i_X B, Y+\eta+i_Y B] = [X+\xi, Y+\eta] + i_{[X,Y]}B \quad \cdots (63)$$

Exercise 10 Show that the existence of this automorphism selects $\beta=1$ in (60), thus giving (61).

Why was this a natural thing to require, for a mathematician?

Consider the symmetry of b-fields on a manifold M



$$\boxed{\begin{array}{l} \underline{V}+\underline{\underline{\xi}} \text{ is a symmetry of } b \\ \Longleftrightarrow \mathcal{L}_{\underline{V}} b = d\underline{\underline{\xi}} \end{array}} \quad (64)$$

Consider a 2-form B with $dB=0$

Claim that the B transform of $V+\xi$ is a symmetry of $\underline{b+B}$

$$\boxed{V+(\xi+i_V B) \text{ is a symmetry of } b+B} \quad (65)$$

Verify:

$$\begin{aligned} \mathcal{L}_V(b+B) &= \mathcal{L}_V b + (d i_V + \cancel{i_V d})B \\ &= d\xi + d i_V B \\ &= d(\xi + i_V B) \end{aligned} \quad \ldots (66)$$

B transformations (changes of b) do not change the symmetries of the theory
$\longrightarrow$ promote them to automorphisms

What is the gauge algebra in the strongly constrained theory?

As will be derived later, ...

$$\xi^M = \begin{pmatrix} \tilde{\xi}_i \\ \xi^i \end{pmatrix} \quad \text{assemble gauge parameters} \qquad (67)$$

The gauge algebra governed by a C-bracket $[\ ,\ ]_C$

$$\left([\xi_1, \xi_2]_C \right)^M \xleftarrow{} \text{component}$$

$$\left([\xi_1, \xi_2]_C \right)^M = \xi^P_{[1} \partial_P \xi^M_{2]} - \frac{1}{2} \eta^{MN} \eta_{PQ} \xi^P_{[1} \partial_N \xi^Q_{2]} \quad \cdots (68)$$

$$= \xi_{[1} \cdot \partial \xi^M_{2]} - \frac{1}{2} \xi_{P[1} \partial^M \xi^P_{2]}$$

Writing $\xi^M \sim \xi_1 + \tilde{\xi}_1$ in the GG language

$$\boxed{\begin{aligned} [\xi_1 + \tilde{\xi}_1, \xi_2 + \tilde{\xi}_2] = {} & [\xi_1, \xi_2] + \mathcal{L}_{\tilde{\xi}_1} \xi_2 - \mathcal{L}_{\tilde{\xi}_2} \xi_1 \\ & - \frac{1}{2} \tilde{d} \left(\tilde{\iota}_{\tilde{\xi}_1} \xi_2 - \tilde{\iota}_{\tilde{\xi}_2} \xi_1 \right) \\ & + [\tilde{\xi}_1, \tilde{\xi}_2] + \mathcal{L}_{\xi_1} \tilde{\xi}_2 - \mathcal{L}_{\xi_2} \tilde{\xi}_1 \\ & - \frac{1}{2} d \left(\iota_{\xi_1} \tilde{\xi}_2 - \iota_{\xi_2} \tilde{\xi}_1 \right) \quad \cdots (69) \end{aligned}}$$

$$[\tilde{\xi}_1, \tilde{\xi}_2]_j = \tilde{\xi}_{1i} \tilde{\partial}^i \tilde{\xi}_{2j} - (1,2) \qquad \tilde{d} f = \tilde{\partial}^i f (d\tilde{x}_i)$$

(69) is a Courant bracket on a Lie bi-algebroid.

What happens if the gauge parameters are independent of $\tilde{x}$

then $\mathcal{L}_{\tilde{\xi}} \to 0 \quad \tilde{\partial} \to 0 \quad [\tilde{\xi}, \tilde{\xi}] \to 0$

and get

$$[\xi_1 + \tilde{\xi}_1, \xi_2 + \tilde{\xi}_2] = [\xi_1, \xi_2] + \mathcal{L}_{\xi_1}\tilde{\xi}_2 - \mathcal{L}_{\xi_2}\tilde{\xi}_1 - \frac{1}{2} d\left(i_{\xi_1}\tilde{\xi}_2 - i_{\xi_2}\tilde{\xi}_1 \right) \quad \cdots (70)$$

the Courant bracket!

Comment: the β-parameter would break the $O(D,D)$ covariance of the C-bracket

$$[\xi_1, \xi_2]^M = \cdots + \left(\frac{\beta-1}{4}\right) \partial^M \left(\Omega_{PQ}\, \xi_{[1}^P\, \xi_{2]}^Q \right)$$

with $\Omega_{PQ} = \begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix}$

↑

not an $O(D,D)$ invariant tensor

What are the B-transformations?

$h = \left(\begin{array}{c|c} 1\!\!1 & b \\ \hline 0 & 1\!\!1 \end{array} \right)$ $\quad$ b antisymmetric, constant

$$E' = h(E) = (E+b)(\mathbb{1})^{-1} = E+b$$

So indeed
$$B \to B+b$$

How does it act on gauge parameters $\xi^M = \begin{pmatrix} \tilde{\xi}_i \\ \xi^i \end{pmatrix}$

$$\begin{pmatrix} \tilde{\xi}_i \\ \xi \end{pmatrix} \to \left(\begin{array}{c|c} 1 & b \\ \hline 0 & \mathbb{1} \end{array}\right) \begin{pmatrix} \tilde{\xi} \\ \xi \end{pmatrix}$$

$$\xi \to \xi \qquad\qquad \xi + \tilde{\xi} \to \xi + \tilde{\xi} + i_\xi b \ \checkmark$$
$$\tilde{\xi} \to \tilde{\xi} + b\xi$$

$\tilde{\partial} \to \tilde{\partial}$ fields remain restricted

The B-transformations of the mathematicians have a counterpart in part of the T-duality group. T-duality invariance thus brings in the Courant bracket.

Lecture 3

Passing from E_{ij} to $\mathcal{E}_{ij}$

Action in terms of $\mathcal{E}_{ij}, d$

Duality invariance

Gauge invariance

Action in terms of the Generalized metric

Generalized Lie derivatives

Gauge algebra

Curvature scalar

In the perturbative action above we had a background E_{ij}, constant, and a fluctuation $e_{ij}(x,\tilde{x})$. Just like for $g_{ij} = \eta_{ij} + h_{ij}$ in relativity, one suspects that the action constructed earlier could be rewritten in terms of

$$\mathcal{E}_{ij}(x,\tilde{x}) = E_{ij} + e_{ij}(x,\tilde{x}) + \Theta(e_{ij}^2(x,x))$$

$$\mathcal{E}_{ij}(X) = E_{ij} + e_{ij}(X) + \dots \qquad (71)$$

$$X^M = \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}$$

What could be duality $\quad X \to X' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} X$

$$\mathcal{E}'(X') = (a\,\mathcal{E}(X) + b)(c\,\mathcal{E}(X) + d)^{-1}$$

$$d'(X') = d(X) \qquad (72)$$

The derivatives in (10)

$$D_i = \partial_i - E_{ik}\tilde{\partial}^k \longrightarrow \mathcal{D}_i \equiv \partial_i - \mathcal{E}_{ik}(X)\,\tilde{\partial}^k$$

$$\bar{D}_i = \partial_i + E_{ki}\tilde{\partial}^k \longrightarrow \bar{\mathcal{D}}_i \equiv \partial_i + \mathcal{E}_{ki}(X)\,\tilde{\partial}^k \qquad (73)$$

All we learned about
O(D,D) transformations remains true

$$\mathcal{E}_{ij} = g_{ij} + b_{ij}$$

From (4) $$\mathcal{H}(\mathcal{E}) = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix} \equiv \mathcal{H}^{MN}(\mathcal{E}) \qquad (74)$$

$$\begin{aligned} M &= (d - cE^t)^t \longrightarrow M(X) = (d - c\mathcal{E}^t)^t \\ \bar{M} &\equiv (d + cE)^t \longrightarrow \bar{M}(X) = (d + c\mathcal{E})^t \end{aligned} \qquad (75)$$

g is an O(D,D) tensor, from (28)

$$\begin{aligned} g(X) &= \bar{M}(X)\, g'(X')\, \bar{M}^t(X) \\ g(X) &= M(X)\, g'(X')\, M^t(X) \end{aligned} \qquad (76)$$

(21) becomes

$$\mathcal{H}(E'(X')) = h\, \mathcal{H}(E(X))\, h^t \qquad (77)$$

(34) becomes

$$\begin{aligned} \mathcal{D}_i &= M_i{}^j(X)\, \mathcal{D}'_j \\ \bar{\mathcal{D}}_i &= \bar{M}_i{}^j(X)\, \bar{\mathcal{D}}'_j \end{aligned} \qquad (78)$$

(37) becomes

$$\delta E(X) = M(X)\, \delta E'\, \bar{M}^t(X) \qquad (79)$$

From (40) $\mathcal{D}_i \mathcal{E}_{jk}$, $\bar{\mathcal{D}}_i \mathcal{E}_{jk}$ are tensors!

Transformation of the dilaton under gauge transformation

$$\delta d = -\tfrac{1}{2}\partial_M \xi^M + \xi^M \partial_M d \qquad \xi^M = \begin{pmatrix} \tilde{\xi}_i \\ \xi^i \end{pmatrix} \quad (80)$$

this implies

$$\delta e^{-2d} = \partial_M \left[\xi^M e^{-2d}\right] \quad (81)$$

e^{-2d} is a density
it is identified with
$\sqrt{-g}\, e^{-2\phi}$

$$S_{\mathcal{E},d} = \int dx\, d\tilde{x}\, e^{-2d} \Big[-\tfrac{1}{4} g^{ik} g^{jl} \mathcal{D}^p \mathcal{E}_{kl} \mathcal{D}_p \mathcal{E}_{ij}$$
$$+ \tfrac{1}{4} g^{kl} \left(\mathcal{D}^j \mathcal{E}_{ik} \mathcal{D}^i \mathcal{E}_{jl} + \bar{\mathcal{D}}^j \mathcal{E}_{ki} \bar{\mathcal{D}}^i \mathcal{E}_{lj} \right) \quad (82)$$
$$+ \left(\mathcal{D}^i d\, \bar{\mathcal{D}}^j \mathcal{E}_{ij} + \bar{\mathcal{D}}^i d\, \mathcal{D}^j \mathcal{E}_{ji} \right) + 4 \mathcal{D}^i d\, \mathcal{D}_i d \Big]$$

Each term independently O(D,D) invariant

Expanding about E reproduces $S^{(2)} + S^{(3)}$ given before. No O(D,D) invariant term (with two derivatives) can be added to $S_{\mathcal{E},d}$ without spoiling this agreement.

Taking $\tilde{\partial} = 0$ $S_{\mathcal{E},d}$ reduces to an action that is identical to S_* (p. 2) when $\sqrt{-g}\, e^{-2\phi} = e^{-2d}$!

Gauge transformations

$$\delta_\xi \mathcal{E}_{ij} = \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i$$
$$+ \mathcal{L}_\xi \mathcal{E}_{ij} + \mathcal{L}_{\tilde{\xi}} \mathcal{E}_{ij} \quad (83)$$
$$- \mathcal{E}_{ik} \left(\tilde{\partial}^k \xi^l - \tilde{\partial}^l \xi^k \right) \mathcal{E}_{lj}$$

Proving directly the gauge invariance is hard.

Formulation with Generalized Metric :

Recall constraint $\partial^M A\, \partial_M B = 0 \quad \forall A, B$

(of course $\partial^M \partial_M A = \partial^M \partial_M B = 0$)

$X' = hX \longrightarrow X'^M = h^M{}_N X^N$ with indices

from $\mathcal{H}(\mathcal{E}') = h\, \mathcal{H}(\mathcal{E})\, h^t$

$$\longrightarrow \mathcal{H}(\mathcal{E}')^{MN} = h^M{}_P\, h^N{}_Q\, \mathcal{H}^{PQ}(\mathcal{E}) \qquad (84)$$

How about $h^t \eta h = \eta$?

$$h^M{}_P\, h^N{}_Q\, \eta_{MN} = \eta_{PQ} \quad (85) \quad \text{(Invariant tensor)}$$

raise Q $\quad h^M{}_P\, h_M{}^Q = \delta^Q_P$. (86)

For a lowered indexed object:

$$(87) \quad Y'_M = h_M{}^P\, Y_P \ ; \quad h_M{}^P \equiv \eta_{MN}\, h^N{}_R\, \eta^{RP}$$

Indeed,

$$X'^M Y'_M = \underbrace{h^M{}_N\, h_M{}^P}\, X^N Y_P = \delta^P_N X^N Y_P = X^P Y_P .$$

Index contraction is $O(D,D)$ invariant.

Recall (8), (9) $\quad \eta \mathcal{H} \eta = \mathcal{H}^{-1}$

$$(88) \quad \mathcal{H}^{MN}\, \eta_{MP}\, \eta_{NQ} = \mathcal{H}_{PQ} \qquad (\mathcal{H}^{PM} \mathcal{H}_{MQ} = \delta^P_Q)$$

$$\boxed{\begin{aligned} S_{\mathcal{H}} = \int dx\, d\tilde{x}\; e^{-2d} \Big(&\tfrac{1}{8}\mathcal{H}^{MN}\partial_M \mathcal{H}^{KL}\, \partial_N \mathcal{H}_{KL} \\ &- \tfrac{1}{2}\mathcal{H}^{MN}\partial_N \mathcal{H}^{KL}\, \partial_L \mathcal{H}_{MK} \\ &- 2\,\partial_M d\, \partial_N \mathcal{H}^{MN} \\ &+ 4\,\mathcal{H}^{MN}\partial_M d\, \partial_N d \Big). \end{aligned}} \tag{89}$$

How do we write the gauge symmetry?

Conventional Lie derivatives L_ξ

$$L_\xi A_M = \xi^P \partial_P A_M + \partial_M \xi^P A_P, \tag{90}$$

$$L_\xi B^N = \xi^P \partial_P B^N - \partial_P \xi^N B^P. \tag{91}$$

<u>Generalized</u> Lie derivatives $\hat{L}$

$$\hat{L}_\xi A_M = \xi^P \partial_P A_M + \left(\partial_M \xi^P - \underline{\partial^P \xi_M}\right) A_P, \tag{92}$$

$$\hat{L}_\xi B^N = \xi^P \partial_P B^N - \left(\partial_P \xi^N - \underline{\partial^N \xi_P}\right) B^P. \tag{93}$$

with the help of the η^{MN} we have introduced a new term (each index is transformed covariantly plus contravariantly).

What changes? Our theory has redundant gauge symmetries

$$\xi^M = \partial^M \chi, \tag{94}$$

$$\begin{pmatrix} \tilde{\xi}_i \\ \xi^i \end{pmatrix} = \begin{pmatrix} \partial_i \chi \\ \tilde{\partial}^i \chi \end{pmatrix},$$ should not generate gauge transformations

$$L_{\xi=\partial\chi} A_M = \cancel{\partial^P\chi\,\partial_P A_M}^{\,0} + \partial_M(\partial^P\chi)\,A_P^{\;N} \neq 0$$

$$\hat{L}_{\xi=\partial\chi} A_M = \partial^P\chi\,\partial_P A_M + \big(\underbrace{\partial_M\partial^P\chi - \partial^P\partial_M\chi}_{0}\big) A_P = 0 \;!$$

Nice, generalized Lie derivatives generate no transformations for "trivial gauge symmetries" $\hat{\mathcal{L}}_{\xi=\partial\chi} = 0$ (95)

The gauge symmetry of $S_{\mathcal{H}}$ is:

$$\boxed{\delta\mathcal{H}^{MN} = \hat{\mathcal{L}}_\xi \mathcal{H}^{MN}} \qquad (96)$$

$$\delta e^{-2d} = \partial_M\left[\xi^M e^{-2d}\right] \quad (97)$$ vanishes for $\xi^M = \partial^M\chi$

What is the algebra of Generalized $\hat{L}_\xi$?

Claim:

$$\left[\hat{\mathcal{L}}_{\xi_1}, \hat{\mathcal{L}}_{\xi_2}\right] = -\hat{\mathcal{L}}_{[\xi_1,\xi_2]_C} \;! \qquad (98)$$

Exercise 11

Use (92) to prove (98) when acting on A_M

$$[\hat{\mathcal{L}}_{\xi_1}, \hat{\mathcal{L}}_{\xi_2}] A_M = \cdots\cdots \quad \cdots = -\mathcal{L}_{[\xi_1,\xi_2]_C} A_M \;!$$

Not too hard to verify that

$$L_{e,d} \equiv L_{\mathcal{H}}$$ equality of Lagrangians

Is there a generalized Scalar curvature ??

Yes! It is $\mathcal{R}$, with:

$$\mathcal{R} = 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN}$$
$$-4\mathcal{H}^{MN}\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d$$
$$+\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL}$$
$$-\frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL}\,.$$

A simple rearrangement of total derivatives in $S_\mathcal{H}$ shows that

$$S_\mathcal{H} = \int dx\,d\tilde{x}\,e^{-2d}\,\mathcal{R}(\mathcal{H},d)\,.$$

To prove gauge invariance, we can show that

$$\delta_\xi\mathcal{R} = \hat{\mathcal{L}}_\xi\mathcal{R} = \xi^M\partial_M\mathcal{R}\,.$$

$\mathcal{R}$ is a generalized scalar.

(there is also a Generalized Ricci, but perhaps no generalized Riemann)

$$\mathcal{R}\Big|_{\tilde{\partial}=0} = R + 4(\Box\phi - (\partial\phi)^2) - \frac{1}{12}H^2$$

This is the combination of familiar scalars that is an O(D,D) invariant generalized scalar!

THE END